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Combinatorial Decompositions of a Class of Rings

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1. INTRODUCTION

In this paper we introduce some combinatorial methods for the study of graded rings and use these methods to analyze a class of rings that are of importance in combinatorics. Our methods are based on a "unique representation theorem" which is shown (Section 2) to hold for any graded algebra and which in a sense is an extension of the direct sum decomposition for Cohen–Macaulay algebras. This unique representation is reminiscent of a similar result obtained by Rees [20]. Both results yield a decomposition of the algebra. However, our result differs from that of Rees in that it carries more information about the algebra.

Our main results are concerned with a class of rings (see Section 3 for the definition) associated to simplicial complexes. There are several natural operations that can be performed on simplicial complexes to obtain new complexes which in turn have an interpretation for the corresponding rings. We show (Sections 4, 5, and 6) that our basic decompositions can be transferred during these operations from the original rings to the newly constructed ones. The operations studied here are: the *chain transform* or barycentric subdivision (Section 4), *rank-selection* both for partially ordered sets and for simplicial complexes (Section 5), and *localization* (Section 6).

As we develop these tools we give some applications. In Section 5 we give a new topological characterization of the Cohen–Macaulay property for partially ordered sets, an immediate consequence of which is the Rank-Selection Theorem (Baclawski [2]). This characterization is closely related to those obtained by Garsia in [14].

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A fundamental result in the theory of Cohen–Macaulay complexes is a theorem first proved by Reisner [21] which gives a topological characterization. A great deal of combinatorial information can be extracted from this result. In particular it played a crucial role in Stanley's proof [24] of the Upper Bound Conjecture. Unfortunately, the original proof is very difficult and relies on rather sophisticated algebraic machinery. One of the applications we give here is a new proof of Reisner's Theorem. Our proof is considerably less intricate, and though it is by no means elementary, it makes more transparent the inner relationship among the ring-theoretical, the topological and the combinatorial aspects of the Cohen–Macaulay property.

Reisner's Theorem is composed of two parts. Perhaps we should point out that the more interesting part (the one used by Stanley) is easier for us to prove than the other part. Furthermore, the other part is relatively easy to prove when the simplicial complex is the chain complex of a partially ordered set, the special case which is of most interest combinatorially.

There now exist numerous applications of our techniques. In commutative algebra one can analyze many important classes of rings, for example, the coordinate rings of Grassmannians, flag varieties, Schubert cycles and certain determinantal varieties. All these rings are examples of "algebras with lexicographic straightening law." See Baclawski [5, 7]. For another approach see DeConcini *et al.* [11]. One can also analyze Diophantine rings using our approach. See Baclawski and Garsia [8]. All the rings mentioned above are important not only in commutative algebra but also in combinatorics and in invariant theory (as in, for example, Doubilet *et al.* [12], Hochster [16] and Stanley [23]).

We also have applications that have a more topological flavor. Since a simplicial complex that triangulates a compact manifold is usually not Cohen–Macaulay, one must work with decompositions of the most general kind (as described in Section 2) in order to analyze these complexes. Such decompositions are computed explicitly by Baclawski in [6] for "almost Cohen–Macaulay complexes," which include triangulations of compact manifolds as a special case. For other applications see [3, 4].

The following conventions are employed in this paper. We use the terms "Lemma," "Theorem," and "Corollary" for our own results, and we reserve the term "Proposition" for results of other authors that have been included for the sake of completeness or for which we give a new proof that is of independent interest. We write $\mathbb{N}$ for the natural numbers (nonnegative integers), and K for a field which is arbitrary but fixed throughout the paper. No special properties of K are used except that it be a field (indeed in some cases it would suffice for K to be a principal ideal domain). All rings considered in this paper are finitely generated graded K-algebras, and all simplicial complexes and partially ordered sets are finite. Lastly, we will write $[n]$ for the set $\{1, 2,..., n\}$.

2. Frames and Privileged Frames

A K-algebra is said to be $\mathbb{N}^m$-*graded* if it can be written as a direct sum $R = \bigoplus_{\nu \in \mathbb{N}^m} \mathscr{H}_\nu R$, such that

(1) $\mathscr{H}_0 R$ is the field K,

(2) for $\nu, \mu \in \mathbb{N}^m$, $(\mathscr{H}_\nu R)(\mathscr{H}_\mu R) \subseteq \mathscr{H}_{\nu+\mu} R$.

The elements of $\mathscr{H}_\nu R$ are said to be *homogeneous of multidegree* ν. We will only consider $\mathbb{N}^m$-graded K-algebras which are finitely generated over K. For such a ring the *Hilbert series* is defined by

$$H(R; t_1, ..., t_m) = \sum_{\nu \in \mathbb{N}^m} \dim_K(\mathscr{H}_\nu R) t^\nu,$$

where t^ν is defined to be $t_1^{\nu_1} \cdots t_m^{\nu_m}$. By the Hilbert Syzygy Theorem, if $f_1, ..., f_m$ are a set of homogeneous generators of R, and if the multidegree of f_i is $\nu(i)$, then the Hilbert series is a rational function of the form

$$\frac{p(t_1, ..., t_m)}{\prod_{i=1}^m (1 - t^{\nu(i)})},$$

where $p(t_1, ..., t_m)$ is a polynomial with integral coefficients [1, 15].

The case of an ($\mathbb{N}$-) graded K-algebra R is of particular importance, and every $\mathbb{N}^m$-graded K-algebra may be regarded as a graded K-algebra by defining $\mathscr{H}_n R$ to be $\bigoplus_{|\nu|=n} \mathscr{H}_\nu R$, where $|\nu| = \nu_1 + \cdots + \nu_m$. We call this the *associated graded K-algebra* of R. The Hilbert series of this graded K-algebra is given by

$$H(R; t) = H(R; t, ..., t).$$

For a graded K-algebra R, the *Krull dimension* of R (written K-dim R), is the order of the pole of $H(R, t)$ at $t = 1$. A *homogeneous system of parameters* for R is a set of $r = K\text{-dim}\, R$ homogeneous elements $\theta_1, ..., \theta_r$ of positive degree such that $K\text{-dim}\, R/(\theta_1, ..., \theta_r)$ is 0, i.e., $R/(\theta_1, ..., \theta_r)$ is finite dimensional over K. A *frame* for R is an *ordered* homogeneous system of parameters for R. We call θ_i the ith *parameter* of the frame $(\theta_1, ..., \theta_r)$. By the Noether Normalization Lemma, if K is infinite, then R has a frame; moreover, if R is generated by $\mathscr{H}_1 R$, then R has a *linear* frame, i.e., a frame whose parameters are all in $\mathscr{H}_1 R$. In general, it may not be possible to choose the parameters of a frame to be homogeneous with respect to an $\mathbb{N}^m$-grading on R for $m > 1$.

We can now state our basic decomposition result.

Theorem 2.1. *Let R be a finitely generated graded K-algebra of Krull* *dimension r. Then there is a frame* $(\theta_1, \ldots, \theta_r)$, *a finite sequence of homogeneous elements* $(\eta_1, \ldots, \eta_N)$ *and a function* $k\colon [N] \to \{0, 1, \ldots, r\}$ *such that*

(1) *the images of* $\eta_1, \ldots, \eta_N$ *in* $R/(\theta_1, \ldots, \theta_r)$ *form a basis over K,*

(2) *every element of R may be expressed in a unique fashion as a sum of the form*

$$\sum_{j=1}^{N} \eta_j p_j(\theta_1, \ldots, \theta_{k(j)}),$$

where p_j *is a polynomial in* $K[X_1, \ldots, X_{k(j)}]$,

(3) *for every j,* $\eta_j(\theta_{k(j)+1}, \ldots, \theta_r) \subseteq (\theta_1, \ldots, \theta_{k(j)})$.

We will say that a frame is *privileged* if $\{\eta_1, \eta_2, \ldots\}$ can be found so that properties (1) and (2) of the above theorem are satisfied. We call $\{\eta_j \mid j \in [N]\}$ a set of *separators* for the privileged frame $(\theta_1, \ldots, \theta_r)$, and for a separator η_j, we call $k(j)$ the *level* of η_j.

To prove the theorem we will use the following graded version of a result of Kaplansky [18]. For a graded K-algebra R, we will write $\mathscr{H}_+ R$ or simply R_+ for $\bigoplus_{n>0} \mathscr{H}_n R$.

LEMMA 2.2. *Let R be a finitely generated graded K-algebra such that every homogeneous element of* $\mathscr{H}_+ R$ *is a zero-divisor. Then there exists a nonzero homogeneous element of R that annihilates* $\mathscr{H}_+ R$.

Proof. For a homogeneous element η of $R \backslash \{0\}$, we write $A(\eta)$ for the *annihilator* of η, $A(\eta) = \{f \in R \mid f\eta = 0\}$. The annihilators are all homogeneous ideals, and the set of all of them will be denoted $\mathscr{A}$, i.e., $\mathscr{A} = \{A(\eta) \mid \eta \text{ is homogeneous and } \eta \neq 0\}$. For an ideal $I \subseteq R$, we write I' for the *set* of homogeneous elements of I.

We first observe that every annihilator in $\mathscr{A}$ is contained in a maximal annihilator. To see this let $A(\eta_1) \subset A(\eta_2) \subset \cdots$ be a strictly ascending chain of annihilators. For each i, let $x_i \in A(\eta_i) \backslash A(\eta_{i-1})$. Thus $x_i \eta_i = 0$ but $x_i \eta_j \neq 0$ for $j < i$. By the Hilbert Basis Theorem, we can find k so that for all $m > k$, $x_m \in (x_1, \ldots, x_k)$. For example, $x_{k+1} = \sum_{i=1}^{k} h_i x_i$ for some $h_i \in R$. Now $x_{k+1} \eta_k \neq 0$, but $x_{k+1} \eta_k = \sum_{i=1}^{k} h_i x_i \eta_k$ and $x_i \in A(\eta_i) \subseteq A(\eta_k)$ imply that $x_{k+1} \eta_k = 0$. We thus have a contradiction. So in fact $\mathscr{A}$ satisfies the ascending chain condition.

We next claim that every maximal annihilator in $\mathscr{A}$ is a prime ideal. Let $A(\eta) \in \mathscr{A}$ be maximal and suppose that $a, b \notin A(\eta)$ but that $ab \in A(\eta)$. Let a_i, b_i, and $(ab)_i$ be the components of a, b, and ab, respectively, of degree i. Define j and k to be the smallest integers such that $a_j \eta \neq 0$ and $b_k \eta \neq 0$, respectively. Then $(ab)_{j+k} \eta = a_j b_k \eta + (a_0 b_{j+k} + \cdots + a_{j-1} b_{k+1}) \eta +$

$(a_{j+1}b_{k-1} + \cdots + a_{j+k}b_0)\eta = 0$, since $ab \in A(\eta)$. But the last two terms above vanish because of the choice of j and k, and so $a_j b_k \eta = 0$. Therefore $b_k \in A(a_j\eta)\backslash A(\eta)$, and in particular $A(\eta) \subsetneq A(a_j\eta)$. Now $a_j\eta$ is homogeneous and $a_j\eta \neq 0$, so $A(a_j\eta) \in \mathscr{A}$. Thus we have a contradiction to the maximality of $A(\eta)$, and it follows that $A(\eta)$ is a prime ideal.

We now show that there are only finitely many maximal elements in $\mathscr{A}$. Suppose that $A(\eta_1), A(\eta_2),...$ were an infinite sequence of these elements. By another application of the Hilbert Basis Theorem, there is a k such that $\eta_m \in (\eta_1,..., \eta_k)$ for all $k > m$. This immediately implies that for $m > k$

$$A(\eta_1)A(\eta_2) \cdots A(\eta_k) \subseteq A(\eta_m),$$

but since $A(\eta_m)$ is prime we must have $A(\eta_i) \subseteq A(\eta_m)$ for some $i \leqslant k$. By the maximality of $A(\eta_i)$, we then have that $A(\eta_i) = A(\eta_m)$. So we again come to a contradiction. So we can only have a finite number, say, $A(\eta_1), A(\eta_2),..., A(\eta_k)$, of maximal elements of $\mathscr{A}$. The hypothesis on R then implies that $\mathscr{H}_+ R' \subseteq A(\eta_1)' \cup \cdots \cup A(\eta_k)'$. To see this, let x be in $\mathscr{H}_+ R'$. Then $x\eta = 0$ for some nonzero $\eta \in R$. Since x is homogeneous, we have $x\eta_i = 0$ for every component η_i of η and one of these must be nonzero. Therefore x is in some $A(\eta) \in \mathscr{A}$, and by our first observation it is then contained in a maximal element of $\mathscr{A}$.

We can clearly choose a subset $\{i_1 < i_2 < \cdots < i_l\} \subseteq [k]$ so that $\mathscr{H}_+ R' \subseteq A(\eta_{i_1})' \cup \cdots \cup A(\eta_{i_l})'$ and no proper subset of $\{i_1,..., i_l\}$ has this property. We claim that l must be equal to 1. Suppose that $l > 1$. By the choice of $\{i_1,..., i_l\}$ we can for each s pick $x_s \in A(\eta_{i_s})'$ such that $x_s\eta_{i_t} \neq 0$ for $t \neq s$. Consider $x = x_1^p + (x_2 \cdots x_l)^q$, where p and q are chosen so that x is homogeneous. (Here is where we use $l > 1$.) Again by the choice of $\{i_1,..., i_l\}$ we must have $x\eta_{i_s} = 0$ for some s. Now if $s = 1$, we get $(x_2 \cdots x_l)^q\eta_{i_1} = 0$, and by the primality of $A(\eta_{i_1})$, this implies that $x_j\eta_{i_1} = 0$ for some $j \neq 1$. This contradicts the choice of x_j. On the other hand, if $s \neq 1$, then we get $x_1^p\eta_{i_s} = 0$ and by the primality of $A(\eta_{i_s})$, we get $x_1\eta_{i_s} = 0$. This contradicts the choice of x_1. Thus we have a contradiction either way and it follows that $l = 1$. Hence $\mathscr{H}_+ R' \subseteq A(\eta)'$ for some nonzero homogeneous η. This is precisely what we wanted to prove. ∎

Proof of Theorem 2.1. We construct $\eta_1,..., \eta_N$ and $\theta_1,..., \theta_r$ inductively as follows. If some homogeneous element of R annihilates R_+, we choose one and call it η_1. In this case the ideal (η_1) has dimension 1 over K. Thus $R/(\eta_1)$ has the same Krull dimension as R. We now replace R by $R/(\eta_1)$ and proceed exactly as before. In this way we choose a sequence $\eta_1, \eta_2,...$ such that η_i annihilates $\mathscr{H}_+ R/(\eta_1,..., \eta_{i-1})$ for all i. Clearly $(\eta_1,..., \eta_i)$ has dimension i over K so $(\eta_1) \subsetneq (\eta_1, \eta_2) \subsetneq \cdots$ is a properly ascending chain of ideals of R. Since R is noetherian, this chain eventually stops, say, at $(\eta_1,..., \eta_m)$. Then $\mathscr{H}_+ R/(\eta_1,..., \eta_m)$ has no homogeneous annihilators. By

Lemma 2.2, we can choose a homogeneous element θ_1 of R_+ such that θ_1 is not a zero-divisor of $R/(\eta_1,..., \eta_m)$.

Now η_1 annihilates R_+, so in particular, $\eta_1\theta_1 = 0$. Similarly, η_2 annihilates $\mathscr{H}_+ R/(\eta_1)$ so $\eta_2\theta_1 = c\eta_1$ for some $c \in R$, so $\eta_2\theta_1^2 = 0$. Continuing in this way, we find that $(\eta_1,..., \eta_m)\theta_1^m = 0$. Conversely, suppose that $\eta\theta_1^m = 0$. Then since θ_1 is not a zero-divisor of $R/(\eta_1,..., \eta_m)$, we conclude that $\eta \in (\eta_1,..., \eta_m)$. Thus $(\eta_1,..., \eta_m)$ is precisely the annihilator of θ_1^m. In a similar fashion we see that the mth power of any element of R_+ will annihilate $(\eta_1,..., \eta_m)$. Indeed it is easy to see that $(\eta_1,..., \eta_m)$ is precisely $\{\eta \mid \eta g^M = 0 \text{ for all } g \in R_+\}$, where M is sufficiently large. Thus the ideal $(\eta_1,..., \eta_m)$ does not depend on the particular choices of $\eta_1,..., \eta_m$ made in the above definition.

Now any power of a non-zero-divisor is again a non-zero-divisor, so we can, by the above computations, choose θ_1 so that $(\eta_1,..., \eta_m)$ is the annihilator of θ_1, and that $\deg(\theta_1) > \deg(\eta_j)$ for $j \in [m]$. We now replace R by $R/(\eta_1,..., \eta_m, \theta_1)$. Since $(\eta_1,..., \eta_m)$ has dimension m over K and θ_1 is not a zero-divisor of $R/(\eta_1,..., \eta_m)$, we see that $R/(\eta_1,..., \eta_m, \theta_1)$ has Krull dimension $r - 1$. We then proceed exactly as above to choose homogeneous elements $\eta_{m(1)+1},..., \eta_{m(2)}$ of R_+ (where $m(1) = m$). We can then find a non-zero-divisor θ_2 of $R/(\eta_1,..., \eta_{m(2)}, \theta_1)$. As before, by replacing θ_2 by a power of θ_2 if necessary, we can choose θ_2 so that $\deg(\theta_2) > \deg(\eta_j)$ for $j \in [m(2)]$, so that $(\eta_{m(1)+1},..., \eta_{m(2)})$ is the annihilator of θ_2 in $R/(\eta_1,..., \eta_{m(1)}, \theta_1)$ and so that θ_2 annihilates $(\eta_1,..., \eta_{m(1)})$ in R.

We can actually ensure that we have one more property, namely, that θ_2 may be chosen so that $\eta_j\theta_2 \in (\theta_1)$ for all j such that $m(1) < j \leqslant m(2)$. Now we already know that $\eta_j\theta_2 = 0$ in $R/(\eta_1,..., \eta_m, \theta_1)$ for $j = m(1) + 1,..., m(2)$. Thus $\eta_j\theta_2 = \sum_{i=1}^m f_i\eta_i + g\theta_1$ for some f_i and g in R. Hence $\eta_j(\theta_2)^2 = (g\theta_2)\theta_1$. Thus if we replace θ_2 by θ_2^2, we may arrange that $\eta_j\theta_2 \in (\theta_1)$ as desired.

Continuing the above procedure for r steps, we construct sequences $\eta_1,..., \eta_{m(r)}$ and $\theta_1,..., \theta_r$ of homogeneous elements of R_+. We then take $\{\eta_{m(r)+1},..., \eta_{m(r+1)}\}$ to be a homogeneous basis of $R/(\eta_1,..., \eta_{m(r)}, \theta_1,..., \theta_r)$. The number N is $m(r+1)$ and the function k: $[N] \to \{0,..., r\}$ is given by

$$k(j) = i, \qquad \text{if} \quad m(i) < j \leqslant m(i+1),$$

where $m(0)$ is taken to be 0.

The sequences $\eta_1,..., \eta_N$ and $\theta_1,..., \theta_r$ have the following properties. Write $R(i)$ for $R/(\eta_1,..., \eta_{m(i)}, \theta_1,..., \theta_i)$. Then for $i = 1,..., r$ we have:

(a) $\{\eta_{m(i-1)+1},..., \eta_{m(i)}\}$ is a basis of the ideal $(\eta_{m(i-1)+1},..., \eta_{m(i)}) R(i-1)$.

(b) $(\eta_{m(i-1)+1},..., \eta_{m(i)}) R(i-1)$ is the annihilator of θ_i in $R(i-1)$.

(c) $\eta_j\theta_l \in (\theta_1,..., \theta_{k(j)})$ for all $l > k(j)$.

(d) θ_i is not a zero-divisor of $R(i-1)/(\eta_{m(i-1)+1},\ldots,\eta_{m(i)})$.

(e) $\deg(\theta_i) > \deg(\eta_j)$ for all i, j such that $j \leqslant m(i)$.

Note that property (3) of the theorem coincides with (c) above.

We now show that property (2) holds. We use induction on the Krull dimension of R (the result for Krull dimension 0 being trivial). Replacing R by $R(1)$ we see that $\eta_{m(1)+1},\ldots,\eta_{m(r+1)}$ and $\theta_2,\ldots,\theta_r$ satisfy the same conditions for $R(1)$ as do the two original sequences for R. Thus they have properties (1) and (2) by the induction hypothesis. Let $f \in R' = R/(\eta_1,\ldots,\eta_m)$ be homogeneous of degree d. The image $\bar{f}$ of f in $R(1) = R'/(\theta_1)$ may be expressed as the image of

$$\sum_{j=m+1}^{N} \eta_j q_j^{(0)}(\theta_2,\ldots,\theta_{k(j)}) \tag{*}$$

in $R(1)$. Since the η_j's and θ_i's are homogeneous, we claim that all terms appearing in the above sum can be chosen to have degree d. For if not, we simply throw out all terms not of degree d. The result is the homogeneous component of (*) of degree d. Since $\bar{f}$ is homogeneous of degree d, it will still coincide with the image of (*) in $R(1)$. Now form the difference between (*) and f in R'. The result is homogeneous of degree d and is in the ideal (θ_1), hence is of the form $\theta_1 g$, where g is homogeneous of degree $d-\deg(\theta_1)$. Write g as the image of $\sum_{j=m+1}^{N} \eta_j q_j^{(1)}(\theta_2,\ldots,\theta_{k(j)})$ in $R(1)$ such that all terms in the sum are homogeneous of degree $d-\deg(\theta_1)$. Continuing this process inductively, we find that after at most $l = [(d+1)/\deg(\theta_1)]$ steps, we may express f in the form $\sum_{i=0}^{l}\sum_{j=m+1}^{N} \eta_j q_j^{(i)}(\theta_2,\ldots,\theta_{k(j)})\theta_1^i$. If we set $p_j(\theta_1,\ldots,\theta_{k(j)})$ equal to $\sum_{i=0}^{l} q_j^{(i)}(\theta_2,\ldots,\theta_{k(j)})\theta_1^i$, then we have the desired expansion in R'. The result for R follows from the fact that the ideal $(\eta_1,\ldots,\eta_m)$ has dimension m over K (and $\{\eta_1,\ldots,\eta_m\}$ is a basis over K).

To establish property (2) it remains to show uniqueness of the expansion. This immediately follows from the fact that the expansion in $R(1)$ is unique and that the Hilbert series of R is given by

$$H(R,t) = \sum_{j=1}^{m} t^{\deg(\eta_j)} + H(R(1),t)(1-t^{\deg(\theta_1)})^{-1}.$$

To show this identity, observe that by (a), $H(R',t) = H(R,t) - \sum_{j=1}^{m} t^{\deg(\eta_j)}$ and by (d), $H(R(1),t) = H(R'/(\theta_1),t) = H(R',t)(1-t^{\deg(\theta_1)})$.

We now consider property (1). We proceed by contradiction. Suppose there are constants $c_j \in K$, not all zero such that $\sum_{j=1}^{N} c_j\eta_j \in (\theta_1,\ldots,\theta_r)$. Now by our inductive hypothesis, $\{\eta_j \mid j > m\}$ are linearly independent in $R(1)/(\theta_2,\ldots,\theta_r)$. Now the image of $\sum_{j=1}^{N} c_j\eta_j$ in $R(1)/(\theta_2,\ldots,\theta_r) = R(\eta_1,\ldots,\eta_m,\theta_1,\ldots,\theta_r)$ is $\sum_{j=m+1}^{N} c_j\bar{\eta}_j$. These being linearly independent, we must have $c_j = 0$ for $j > m$. Thus we have $\sum_{j=1}^{m} c_j\eta_j \in (\theta_1,\ldots,\theta_r)$. But by property (e), we have $\deg(\theta_i) > \deg(\eta_j)$ for all i and all $j \in [m]$. Thus we have a contradiction and the result follows. ▮

Rees [20] showed the existence of θ_i's and η_j's satisfying property (2) of Theorem 2.1 when K is an infinite field. It is easy to give an example to show that property (1) does not follow from property (2). Let R be the K-algebra $K[X]$. Take θ_1 to be the polynomial X. Then θ_1 is a privileged frame with $\eta_1 = 1$ (and $k(1) = 1$). However, if we define $\eta_1 = 1$, $\eta_2 = X$ and $k(1) = 0$, $k(2) = 1$, then $(\theta_1; \eta_1, \eta_2)$ will have property (2) but not property (1) of Theorem 2.1. In this example, property (3) also fails to hold.[1]

If R is a graded K-algebra for which there is a privileged frame all of whose separators have level r (such a frame is said to be *basic*), then we say that R is *Cohen–Macaulay* (abbreviated CM). In this case, every frame is basic. See [9] for an elementary proof of this fact. The decomposition of R given by Theorem 2.1 therefore expresses in a quantitative way how far a given ring fails to the Cohen–Macaulay.

We end this section by giving several equivalent conditions for a K-algebra to be Cohen–Macaulay. The decomposition for a Cohen–Macaulay ring is sometimes called "Hironaka's criterion."

PROPOSITION 2.3. *Let R be a finitely generated graded K-algebra of K-dim r. The following are equivalent (d_i denotes $\deg(\theta_i)$):*

(1) *R is Cohen–Macaulay;*

(2) *for some (every) frame $(\theta_1, ..., \theta_r)$,*

$$H(R; t) = \frac{H(R/(\theta_1, ..., \theta_r); t)}{\prod_{i=1}^r (1 - t^{d_i})};$$

(3) *for some frame $(\theta_1, ..., \theta_r)$ and some set $\{\eta_j \mid j \in [N]\}$ of homogeneous elements of R, we have*

(a) *every element of R may be written in the form*

$$\sum_{j=1}^N \eta_j p_j(\theta_1, ..., \theta_r),$$

where the p_j are polynomials in $K[X_1, ..., X_r]$ and

(b) $H(R; t) = (\sum_j t^{\deg(\eta_j)}) / \prod_{i=1}^r (1 - t^{d_i})$;

(4) *for some frame $(\theta_1, ..., \theta_r)$ of R,*

$$\dim_K R/(\theta_1, ..., \theta_r) = \lim_{t \to 1} \prod_{i=1}^r (1 - t^{d_i}) H(R; t);$$

(5) *for some (every) frame $(\theta_1, ..., \theta_r)$ of R, the sequence $\theta_1, ..., \theta_r$ is a*

[1] *Added in proof.* Property (1) follows from properties (2) and (3), [6].

regular sequence *of* R, *i.e., the image of* θ_i *is not a zero-divisor of* $R/(\theta_1,..., \theta_{i-1})$ *for all* i.

(6) *for some (every) frame* $(\theta_1,..., \theta_r)$ *of* R, *the set* $\{\theta_1,..., \theta_r\}$ *satisfies the* **cancellation property** *in* R, *i.e., if* $\sum_{i=1}^r \theta_i h_i = 0$ *in* R, *then there exists* $a_{i,j} \in R$, $1 \leqslant i, j \leqslant r$, *such that* $a_{i,j} = -a_{j,i}$ *for all* i *and* j, *and* $h_i = \sum_{j=1}^r a_{i,j}\theta_j$ *for all* i.

Proof. For two power series $f(t) = \sum_{i=0}^{\infty} a_i t^i$ and $g(t) = \sum_{i=0}^{\infty} b_i t^i$, we write $f(t) \leqslant_c g(t)$ to mean that $a_i \leqslant b_i$ for all i. Now (1) $\Rightarrow$ (2) follows immediately from the definition of a privileged frame and the fact that every frame of a CM K-algebra is basic. (See [9], for example.)

We next show (2) $\Rightarrow$ (3). Choose the set $\{\eta_j \mid j \in [N]\}$ to be a homogeneous basis of $R/(\theta_1,..., \theta_r)$. Then (b) is immediate. To show (a), let f be a homogeneous element of R. By choice of $\{\eta_j\}$, there are constants $a_j \in K$ such that $f - \sum_{j=1}^N a_j\eta_j \in (\theta_1,..., \theta_r)$, i.e., $f - \sum_{j=1}^N a_j\eta_j = \sum_{i=1}^r \theta_i f_i$, where each f_i is homogeneous and has strictly smaller degree than f. By induction on $\deg(f)$, we get the representation required for f.

Now we consider (3) $\Rightarrow$ (1). We first show that $\{\eta_j \mid j \in [N]\}$ is a basis of $R/(\theta_1,..., \theta_r)$. As in the proof of (2) $\Rightarrow$ (3) above, any homogeneous basis of $R/(\theta_1,..., \theta_r)$ will satisfy (a). Conversely, it is easy to see that (a) implies that $\{\eta_j\}$ spans $R/(\theta_1,..., \theta_r)$ as a vector space over K. Thus if $\{\eta_j\}$ is not a basis of $R/(\theta_1,..., \theta_r)$ we can choose a proper subset $S \subsetneq [N]$ such that $\{\eta_j \mid j \in S\}$ is a basis. We then have

$$H(R; t) = \frac{\sum_{j=1}^N t^{\deg(\eta_j)}}{\prod_{i=1}^r (1 - t^{d_i})} <_c \frac{\sum_{j \in S} t^{\deg(\eta_j)}}{\prod_{i=1}^r (1 - t^{d_i})} \leqslant_c H(R; t),$$

where the last inequality comes from condition (a) for $\{\eta_j \mid j \in S\}$. Thus we get a contradiction. It follows that $\{\eta_j \mid j \in [N]\}$ is a basis of $R/(\theta_1,..., \theta_r)$. It is easy to see that condition (b) implies that the representation in (a) is unique.

The fact that (1) $\Leftrightarrow$ (4) is not used in the sequel; we refer the reader to [9] for a proof.

The regular sequence property (5) is the most commonly seen definition of a CM K-algebra. The equivalence of (2) and (5) follows immediately from the fact that

$$H(R; t) \leqslant_c \frac{H(R/(\theta); t)}{1 - t^{\deg(\theta)}}$$

holds for any homogeneous element θ of R of positive degree, with equality if and only if θ is not a zero-divisor. The exact formula is

$$H(R; t) = \frac{H(R/(\theta); t) - t^{\deg(\theta)} H(A(\theta); t)}{1 - t^{\deg(\theta)}},$$

where $A(\theta)$ is the annihilator of θ. See [9] for details.

The equivalence of (5) and (6) is usually expressed using the Koszul complex. See Serre [22]. However, it is possible to give an elementary proof [9]. As with (4), we will not make use of (6). ∎

COROLLARY 2.4. *Let R be an $\mathbb{N}^m$-graded R-algebra possessing a frame $(\theta_1, \dots, \theta_r)$. Then R is Cohen–Macaulay if and only if*

$$H(R; t_1, \dots, t_m) = \frac{H(R/(\theta_1, \dots, \theta_r); t_1, \dots, t_m)}{\prod_{i=1}^{r} (1 - t^{d(i)})},$$

where $d(i) = \deg(\theta_i)$.

3. SIMPLICIAL COMPLEXES AND PARTIALLY ORDERED SETS

The class of rings we will be studying is based on simplicial complexes and on partially ordered sets (posets). We introduce here some of the terminology of the homology of simplicial complexes and of the rings associated to them.

A (*finite*) *simplicial complex* Δ is a collection of subsets (called *simplices*) of a finite vertex set V such that any subset of a simplex is also a simplex. We always regard $\varnothing$ as being one of the simplices. We do not require that $\{v\}$ be a simplex whenever $v \in V$. A simplicial complex is said to be *pure* if every maximal simplex has the same number of vertices. The *rank* of a simplicial complex Δ, denoted $r(\Delta)$, is the highest cardinality of any simplex.

A simplicial complex is a special kind of poset. We will, however, distinguish between the two concepts. For a simplicial complex Δ, we will write $P(\Delta)$ for $\Delta \backslash \{\varnothing\}$, regarded only as a poset. On the other hand, if P is a finite poset, we define the *order complex* or *chain transform* $\Delta(P)$ to be the simplicial complex whose vertex set is P and whose simplices are the chains $(x_1 < \cdots < x_n)$ of P. If Δ is a simplicial complex then $\Delta(P(\Delta))$ corresponds to (the triangulation of) the barycentric subdivision of (the polyhedron associated to) Δ. If $\Delta(P)$ is pure, then we say P is *ranked* (or *graded*). The *rank* of P is defined to be the rank of $\Delta(P)$.

For a poset P, we write $\hat{P}$ for the poset obtained by adjoining two new elements $\hat{0}$ and $\hat{1}$ to P such that $\hat{0} < x < \hat{1}$ for all $x \in P$. If x is an element of P, we define the *rank* of x, denoted $r(x)$, to be the rank of the half-open interval $(\hat{0}, x]$ of $\hat{P}$. It is easy to see that the elements of P of a given rank form an antichain of P. One could also define the rank of an element of P as follows. The elements of rank 1 are the minimal elements of P. If we remove these, the minimal elements of the resulting poset have rank 2, and so on.

The rank function on a poset P is a special case of a "coloring" of a simplicial complex. Let Δ be a simplicial complex of rank r on vertex set V.

A *coloring* of Δ is a function $c\colon V \to \{1,\dots,r\}$ such that for all $\sigma \in \Delta$, $\{c(v) \mid v \in \sigma\}$ has cardinality $|\sigma|$. All of our results on simplicial complexes of the form $\Delta(P)$ also hold for colorable complexes. A pure, colorable complex is also called a *completely balanced complex*. See Stanley [25].

We use the rank function on a poset P to define an important class of subposets. Let $r = r(P)$, and suppose that $S \subseteq [r]$. The *rank-selected subposet of P with respect to S* is

$$P_S = \{x \in P \mid r(x) \in S\}.$$

We will make use of the concept of the simplicial cohomology of a simplicial complex as well as some of its well-known properties. For a simplicial complex Δ, we recall that the *reduced cochain complex* of Δ is defined as follows. Choose some total order on the vertex set V of Δ. Let $\tilde{C}^i(\Delta, K)$ be the vector space over K on the simplices $\{x_0,\dots,x_i\}$ of rank $i+1$ of Δ. In particular, $\tilde{C}^{-1}(\Delta, K)$ is one-dimensional. We next define a linear map $\delta^i\colon \tilde{C}^i(\Delta, K) \to \tilde{C}^{i+1}(\Delta, K)$ for all $i \geqslant -1$, on basis elements by the formula

$$\delta^i\{x_0,\dots,x_i\} = \sum_{x\in V} (-1)^j\{x_0,\dots,x_i,x\},$$

where the sum is over all x such that $x \notin \{x_0,\dots,x_i\}$ but $\{x_0,\dots,x_i,x\} \in \Delta$, and where j is the number of elements of $\{x_0,\dots,x_i\}$ which precede x in the total order on V. It is easy to check that $\delta^i \circ \delta^{i-1} = 0$. We define the ith *reduced cohomology* of Δ to be the vector space

$$\tilde{H}^i(\Delta, K) = \operatorname{Ker}(\delta^i)/\operatorname{Im}(\delta^{i-1}).$$

This vector space does not depend on the choice of total order chosen for V. Indeed, it is a topological invariant of the polyhedron $|\Delta|$ associated to Δ. This is not that easy to prove; however, we only need that $\tilde{H}^i$ be invariant under barycentric subdivision, i.e., $\tilde{H}^i(\Delta, K) \cong \tilde{H}^i(\Delta(P(\Delta)), K)$ for all i, which is relatively easy to prove.

The reduced cohomology of Δ allows one to compute several important invariants of Δ. The ith *reduced Betti number* of Δ, $\tilde{h}_i(\Delta, K)$, is the dimension of $\tilde{H}_i(\Delta, K)$ over K. The alternating sum of the reduced Betti numbers is the *reduced Euler characterstic* of Δ, denoted $\mu(\Delta) = \sum_{i=-1}^{\infty} (-1)^i \tilde{h}_i(\Delta, K)$. This number is independent of the field K. We will say that Δ is a *bouquet* if $\tilde{H}^i(\Delta, K) = 0$ for all $i \neq r(\Delta) - 1$. Since every simplicial complex trivially satisfies $\tilde{H}^i(\Delta, K) = 0$ for $i > r(\Delta) - 1$, to say that Δ is a bouquet is to restrict all its nonzero cohomology to be in $\tilde{H}^{r-1}(\Delta, K)$, where $r = r(\Delta)$. Thus Δ is a bouquet if and only if $\tilde{h}_i(\Delta, K) = 0$ for $i < r - 1$. A bouquet Δ will therefore satisfy $\mu(\Delta) = (-1)^{r-1}\tilde{h}_{r-1}(\Delta, K)$. When we apply these concepts to the case of a simplicial complex $\Delta(P)$, we will often abbreviate by replacing the symbol $\Delta(P)$ by P. Thus we write $\mu(P)$ for $\mu(\Delta(P))$, etc.

We will require the following two well-known results from algebraic topology, and we give brief proofs for the sake of completeness.

PROPOSITION 3.1. *Let Δ be a simplicial complex. Define Δ_i to be the subcomplex $\{\sigma \in \Delta \mid |\sigma| \leqslant i\}$. Then*

$$\tilde{H}^j(\Delta, K) \cong \tilde{H}^j(\Delta_i, K), \qquad \textit{for} \quad j < i-1.$$

Proof. By the obvious induction, we may assume that $i = r-1$, where $r = r(\Delta)$. Write Δ' for Δ_{r-1}, P for $P(\Delta)$ and P' for $P(\Delta')$. Observe that we have a natural projection

$$\Pi_j\colon \tilde{C}^j(\Delta(P), K) \to \tilde{C}^j(\Delta(P'), K),$$

for every j, defined by

$$\begin{aligned} \Pi_j(\sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_j) &= \sigma_0 \subset \cdots \subset \sigma_j, \qquad && \text{if } \sigma_j \in \Delta', \\ &= 0, && \text{otherwise.} \end{aligned}$$

The kernel is isomorphic to $\bigoplus_{\sigma \in \mathscr{M}} \tilde{C}^{j-1}(\Delta'(\sigma), K)$, where $\mathscr{M} = \Delta \backslash \Delta'$ and $\Delta'(\sigma)$ is the simplicial complex of proper subsets of σ. Now apply the snake lemma to the short exact sequence of complexes whose jth component is

$$0 \to \operatorname{Ker}(\Pi_j) \to \tilde{C}^j(\Delta(P), K) \to \tilde{C}^j(\Delta(P'), K) \to 0.$$

The result is a long exact sequence part of which is

$$\begin{aligned} \cdots &\to \bigoplus_{\sigma \in \mathscr{M}} \tilde{H}^{j-1}(\Delta'(\sigma), K) \to \tilde{H}^j(\Delta(P), K) \to \tilde{H}^j(\Delta(P'), K) \\ &\to \bigoplus_{\sigma \in \mathscr{M}} \tilde{H}^j(\Delta'(\sigma), K) \to \cdots. \end{aligned}$$

Now $\Delta'(\sigma)$ is a standard triangulation of the $(r-2)$-sphere so that $\Delta'(\sigma)$ is a bouquet, i.e., $\tilde{H}^j(\Delta'(\sigma), K) = 0$ for $j < r-2$. The long exact sequence then implies that $\tilde{H}^j(\Delta(P), K) \cong \tilde{H}^j(\Delta(P'), K)$ for $j < r-2$. Since $\tilde{H}^j(\Delta(P), K) \cong \tilde{H}^j(\Delta, K)$ and $\tilde{H}^j(\Delta(P'), K) \cong \tilde{H}^j(\Delta', K)$, the result follows. ∎

Let Δ_1 and Δ_2 be simplicial complexes on disjoint vertex sets V_1 and V_2, respectively. The *join* of Δ_1 and Δ_2 is the simplicial complex on $V_1 \cup V_2$ given by

$$\Delta_1 * \Delta_2 = \{\sigma \cup \tau \mid \sigma \in \Delta_1, \ \tau \in \Delta_2\}.$$

Proposition 3.2. *If Δ_1 and Δ_2 are bouquets, then so is $\Delta_1 * \Delta_2$.*

Proof. The basis elements of $\tilde{C}^j(\Delta_1 * \Delta_2, K)$ are of the form $\sigma \cup \tau$, where $|\sigma \cup \tau| = j+1$ and $\sigma \in \Delta_1$, $\tau \in \Delta_2$. Thus we may "split apart" $\tilde{C}^j(\Delta_1 * \Delta_2, K)$ according to the cardinalities of σ and τ:

$$\tilde{C}^j(\Delta_1 * \Delta_2, K) \cong \bigoplus_{k+l=j-1} (\tilde{C}^k(\Delta_1, K) \otimes \tilde{C}^l(\Delta_2, K)).$$

By the (homological) Künneth formula, we conclude that

$$\tilde{H}^j(\Delta_1 * \Delta_2, K) \cong \bigoplus_{k+l=j-1} (\tilde{H}^k(\Delta_1, K) \otimes \tilde{H}^l(\Delta_2, K)).$$

The result now follows immediately. ∎

Let Δ be a simplicial complex on the vertex set V. We write $K[X_v \mid v \in V]$ for the (free) polynomial ring on indeterminates corresponding to the vertices in V. We define the ideal I_Δ to be the one generated by all square-free monomials $X_{v_1} \cdots X_{v_n}$ such that $\{v_1, ..., v_n\} \notin \Delta$. The quotient ring $K[\Delta] = K[X_v \mid v \in V]/I_\Delta$ is called the *Stanley–Reisner ring* after Stanley and Reisner, who introduced it independently (first by Stanley [24] and later by Reisner [21]). The ring $K[\Delta]$ has a natural grading which is defined by $\deg(X_v) = 1$ for all $v \in V$. Furthermore, it has a natural frame $(\alpha_1, ..., \alpha_r)$, where $r = r(\Delta)$, defined as follows. We first introduce some notation. For $\sigma \in \Delta$ we write X^σ for the monomial $\prod_{v \in \sigma} X_v$. If $\mathscr{A}$ is a statement, the symbol $\chi(\mathscr{A})$ denotes 1 if $\mathscr{A}$ is true and 0 if $\mathscr{A}$ is false. Finally we define $\alpha_i = \alpha_i(\Delta)$ to be the sum $\sum_{\sigma \in \Delta} X^\sigma \chi(|\sigma| = i)$. We will show that $(\alpha_1, ..., \alpha_r)$ is a frame in Proposition 3.2 below.

Now suppose that P is a poset. The ring $K[\Delta(P)]$ may now be given a finer structure than just the grading mentioned above. Let $r = r(P)$, and write $e_i \in \mathbb{N}^r$ for the ith standard basis element of $\mathbb{N}^r$, $i \in [r]$. Then $K[\Delta(P)]$ has an $\mathbb{N}^r$-grading defined by $\deg(X_v) = e_{r(v)}$, for $v \in P$.

For a homogeneous element in the $\mathbb{N}^r$-grading on $K[\Delta(P)]$, we will refer to its degree as its *multidegree* to distinguish it from the ordinary degree mentioned earlier. We will often regard a multidegree as a multisubset of $[r]$. The K-algebra $K[\Delta(P)]$ has two choices for a frame. In addition to the frame $(\alpha_1, ..., \alpha_r)$ defined above for arbitrary simplicial complexes, we have the frame $(\beta_1, ..., \beta_r)$ given by $\beta_i = \sum_{v \in P} X_v \chi(r(v) = i)$. Unlike the frame $(\alpha_1, ..., \alpha_r)$, the frame $(\beta_1, ..., \beta_r)$ is homogeneous with respect to the $\mathbb{N}^r$-grading on $K[\Delta(P)]$.

We now compute the Hilbert series of $K[\Delta]$ and $K[\Delta(P)]$.

PROPOSITION 3.3 (Stanley). *For a simplicial complex Δ and a poset P,*

$$H(K[\Delta],t)=\sum_{\sigma\in\Delta} t^{|\sigma|}(1-t)^{-|\sigma|};$$

$$H(K[\Delta(P)];t_1,\ldots,t_r)=\sum_{T\subseteq[r]}(-1)^{|T|-1}\mu(P_T)\frac{t^T}{\prod_{i=1}^r(1-t_i)},$$

where $t^T=\prod_{i\in T}t_i$ and $r=r(P)$.

Proof. To show the first formula, one simple classifies the nonzero monomials of $K[\Delta]$ by their *support*: the support of a monomial w of $K[\Delta]$ is defined by

$$\square(w)=\{v\in V\mid X_v \text{ is a factor of } w\}.$$

To show the second formula, we start with the multigraded version of the first formula and put it over a common denominator:

$$\begin{aligned}H(K[\Delta(P)];t_1,\ldots,t_r)&=\sum_{\sigma\in\Delta(P)}\prod_{v\in\sigma}\frac{t_{r(v)}}{1-t_{r(v)}}\\&=\frac{\sum_{\sigma\in\Delta(P)}\prod_{i\in r(\sigma)}t_i\prod_{i\notin r(\sigma)}(1-t_i)}{\prod_{i=1}^r(1-t_i)},\end{aligned}$$

where $r(\sigma)=\{r(v)\mid v\in\sigma\}$ is called the *rank set* of σ. For a subset $S\subseteq[r]$, we write $c(S)$ for the number of $|S|$-element chains of P_S, i.e., the number of chains of P whose rank set is S. The sum $\sum_{T\subseteq S}(-1)^{|T|}c(T)$ is the same as the alternating sum $\sum_{i=0}^{|S|}(-1)^i c_i(P_S)$, where $c_i(P_S)$ is the number of chains of P_S of size i. By a theorem of Philip Hall, this sum coincides with $-\mu(P_S)$. Therefore,

$$\begin{aligned}H(K[\Delta(P)];t_1,\ldots,t_r)&=\frac{\sum_{\sigma\in\Delta(P)}\prod_{i\in r(\sigma)}t_i\prod_{i\notin r(\sigma)}(1-t_i)}{\prod_{i=1}^r(1-t_i)}\\&=\sum_{\sigma\in\Delta(P)}\sum\frac{(-1)^{|T|-|\sigma|}t^T\chi(r(\sigma)\subseteq T\subseteq[r])}{\prod_{i=1}^r(1-t_i)}\\&=\sum_{S\subseteq[r]}c(S)\frac{\sum(-1)^{|T|-|S|}t^T\chi(S\subseteq T\subseteq[r])}{\prod_{i=1}^r(1-t_i)}\\&=\sum_{T\subseteq[r]}(-1)^{|T|}\sum_{S\subseteq T}(-1)^{|S|}c(S)\frac{t^T}{\prod_{i=1}^r(1-t_i)}\\&=\sum_{T\subseteq[r]}(-1)^{|T|-1}\mu(P_T)\frac{t^T}{\prod_{i=1}^r(1-t_i)}.\quad\blacksquare\end{aligned}$$

An immediate consequence of the first formula in Proposition 3.3 is that the Krull dimension of $K[\Delta]$ is $r(P)$. Thus we have the correct number of polynomials in $(\alpha_1,\ldots,\alpha_r)$ and $(\beta_1,\ldots,\beta_r)$ for these to be frames of $K[\Delta]$ and $K[\Delta(P)]$, respectively. We now show that they are frames.

PROPOSITION 3.4. *If Δ is a simplicial complex and P is a poset, both of rank r, then $(\alpha_1,\ldots,\alpha_r)$ is a frame for $K[\Delta]$ and $(\beta_1,\ldots,\beta_r)$ is a frame for $K[\Delta(P)]$.*

Proof. We first show the result for $(\alpha_1,\ldots,\alpha_r)$. Let V be the set of vertices of Δ, and write $B(V)$ for the simplicial complex of all subsets of V. Then Δ is a subset of $B(V)$ and $K[\Delta]$ is a quotient ring of $K[B(V)]=K[X_v \mid v\in V]$. The image of the parameter $\alpha_i(B(V))$ in $K[\Delta]$ is the parameter α_i (or zero if $i>r$). Thus $K[\Delta]/(\alpha_1,\ldots,\alpha_r)$ is a quotient ring of $K[B(V)]/(\alpha_1(B(V)),\ldots)$. Now $\alpha_i(B(V))$ is easily seen to be the ith elementary symmetric function. An explicit basis of $K[B(V)]/(\alpha_1(B(V)),\ldots)$ was found by Garsia in [14]; hence this vector space is finite-dimensional. Therefore $K[\Delta]/(\alpha_1,\ldots,\alpha_r)$ is also finite-dimensional.

Next consider the sequence $(\beta_1,\ldots,\beta_r)$ of elements of $K[\Delta(P)]$. Since we can express the α_i's in terms of the β_j's,

$$\alpha_i=\sum_{j_1,\ldots,j_i}\beta_{j_1}\cdots\beta_{j_i}\chi(1\leqslant j_1<\cdots<j_i\leqslant r),$$

it follows that $K[\Delta(P)]/(\beta_1,\ldots,\beta_r)$ is a quotient ring of $K[\Delta(P)]/(\alpha_1,\ldots,\alpha_r)$. Thus this case follows from the first one. ∎

Remark 3.5. Suppose that Δ' is a *subcomplex* of Δ, i.e., $\Delta'\subseteq\Delta$ and Δ' is itself a simplicial complex on the vertex set V of Δ. Then we may view $K[\Delta']$ in a natural way as a subset of $K[\Delta]$. However, $K[\Delta']$ will not generally be a subalgebra or submodule of $K[\Delta]$. On the other hand, there is a natural surjective homomorphism $\pi_{\Delta'}\colon K[\Delta]\to K[\Delta']$, defined by

$$\begin{aligned}\pi_{\Delta'}(w)&=w &&\text{if } \square(w)\in\Delta'\\ &=0 &&\text{otherwise.}\end{aligned}$$

This homomorphism endows $K[\Delta']$ with the structure of a $K[\Delta]$-module, and it is this structure that we have in mind whenever we refer to $K[\Delta']$ as a module.

4. The Chain Transform

It is well known that if the chain transform of a simplicial complex Δ is CM then so is Δ; however, the original proof [2, Proposition 3.3] relied on a topological characterization of the CM property. In this section we give a ring-theoretic proof by showing that one can use a basic system for the chain transform of Δ to construct a basic system for Δ itself. As a result, the problem of finding a basic system of $K[\Delta]$ can be accomplished by finding one for the ring $K[\Delta(P(\Delta))]$, which has a simpler structure.

Let Δ be a simplicial complex of rank r on the vertex set V. For the rest of this section we write P for $P(\Delta)$. The ring $K[\Delta]$ has a natural frame $(\alpha_1,..., \alpha_r)$, where $\alpha_i = \sum_{\sigma\in\Delta} X^\sigma \chi(|\sigma| = i)$. On the other hand, the ring $K[\Delta(P)]$ also has a natural frame $(\theta_1,..., \theta_r)$, where $\theta_i = \sum_{\sigma\in P} X_\sigma \chi(|\sigma| = i)$, the symbol X_σ denoting the indeterminate corresponding to $\sigma \in P$. The first fact that one notices is that θ_i and α_i are formally equivalent. Indeed, there is a linear map φ: $K[\Delta(P)] \to K[\Delta]$, defined on monomials by setting $\varphi(X_{\sigma_1} \cdots X_{\sigma_n})$ equal to $X^{\sigma_1} \cdots X^{\sigma_n}$, where $\sigma_1 \leqslant \cdots \leqslant \sigma_n$ is a chain of elements of P; and the map φ carries θ_i to α_i. Moreover, if we define $\mathbb{N}$-gradings on $K[\Delta]$ and on $K[\Delta(P)]$ by $\deg(X_v) = 1$ for $v \in V$ and $\deg(X_\sigma) = |\sigma|$ for $\sigma \in P$, respectively, then φ is degree-preserving. However, φ is *not* a homomorphism of rings. For example, let $\{v, w\} \in \Delta$ be a simplex. Then $\varphi(X_{\{v\}})\, \varphi(X_{\{w\}}) = X_v X_w \neq 0$ in $K[\Delta]$, but $X_{\{v\}} X_{\{w\}} = 0$ in $K[\Delta(P)]$. On the other hand, as a linear map φ is an isomorphism.

LEMMA 4.1. *For a pure simplicial complex Δ, φ: $K[\Delta(P)] \to K[\Delta]$ is a linear degree-preserving isomorphism.*

Proof. We define an inverse ψ: $K[\Delta] \to K[\Delta(P)]$. By definition of $K[\Delta]$, if w is a nonzero monomial of $K[\Delta]$, then its support $\square(w)$ is a simplex of Δ. Let σ_1 be $\square(w)$. Then X^{σ_1} is a factor of w and the quotient $w_1 = w/X^{\sigma_1}$ is a well-defined nonzero monomial of $K[\Delta]$; indeed $w_1 = \prod_{v\in\sigma_1} X_v^{m_v - 1}$, where m_v is the multiplicity of X_v as a factor of w. Thus $\square(w_1)$ is also a simplex of Δ, which we denote by σ_2. Continuing in this manner, we see that we can define a sequence of monomials w_i and simplices σ_i which satisfy $\sigma_i = \square(w_{i-1})$, $\sigma_i \subseteq \sigma_{i-1}$ and $w = \prod_i X^{\sigma_i}$. We call this the *standard factorization* of w. We define ψ: $K[\Delta] \to K[\Delta(P)]$ on monomials by $\psi(w) = \prod_i X_{\sigma_i}$. Since the sequence $\sigma_1, \sigma_2,...$ defined by a nonzero monomial w of $K[\Delta]$ is a chain $\sigma_1 \supseteq \sigma_2 \supseteq \cdots$ of elements of P, $\psi(w)$ is a well-defined nonzero monomial of $K[\Delta(P)]$. It is obvious that φ and ψ are inverse maps. ∎

It follows that $K[\Delta]$ and $K[\Delta(P)]$ have the same Hilbert series and have a very similar structure, but they are generally not isomorphic even as ungraded rings. For example, if Δ has two vertices $V = \{1, 2\}$ and one simplex of rank 2, then $K[\Delta] = K[X_1, X_2]$ and $K[\Delta(P)] = K[X_1, X_2, X_{12}]/(X_1 X_2)$ are not isomorphic (one ring is an integral domain, the other is not). However, the two rings are related by a "straightening law." This idea was inspired by Garsia's work in [14], where this method was used to show that "partition rings," are CM. For a general treatment of rings with straightening law see [5], from which the following discussion was derived.

The key structure possessed by the monomials of $K[\Delta]$ and $K[\Delta(P)]$ is a "lexicographic" partial order. Let $X_{\sigma_1}X_{\sigma_2}\cdots X_{\sigma_k}$ be a monomial of $K[\Delta(P)]$, where $\sigma_1 \supseteq \sigma_2 \supseteq \cdots \supseteq \sigma_k$. The *shape* of $X_{\sigma_1}X_{\sigma_2}\cdots X_{\sigma_k}$ is the descending sequence (partition) $|\sigma_1| \geqslant |\sigma_2| \geqslant \cdots \geqslant |\sigma_k|$. The shape of a monomial is equivalent to its rank multiset, except for the way we have written it. We endow the set of decreasing sequences of positive integers with a total order as follows. Let $\lambda = (a_1 \geqslant \cdots \geqslant a_k)$ and $\nu = (b_1 \geqslant \cdots \geqslant b_l)$ be two such sequences. We say λ *precedes* ν if either $k < l$ or $k = l$ and for some i, $a_1 = b_1$, $a_2 = b_2, \ldots, a_{i-1} = b_{i-1}$, and $a_i > b_i$. We will say that a monomial w precedes a monomial u if the shape of w precedes the shape of u. We induce a partial order on the monomials of $K[\Delta]$ by means of the standard factorization map ψ. Shapes have a natural sum: if λ and μ are as above, then $\lambda + \mu$ is the (disjoint) union of the multisets $\{a_i\}$ and $\{b_j\}$ put into descending order. This operation has the property that the sum of the shapes of monomials w and u of $K[\Delta(P)]$ is the shape of the product wu when $wu \neq 0$. This is not true of monomials of $K[\Delta]$.

We now show that φ is a "perturbation" of a ring isomorphism in the following sense:

STRAIGHTENING LEMMA 4.2. *Let $f_1, f_2, \ldots, f_k$ be homogeneous polynomials in the multigraded K-algebra $K[\Delta(P)]$. Then $\varphi(f_1 f_2 \cdots f_k) - \varphi(f_1)\varphi(f_2)\cdots\varphi(f_k)$ is a linear combination of monomials whose shapes strictly precede the sum of the shapes of $f_1, f_2, \ldots, f_k$.*

Proof. Since φ is a linear map, we may assume without loss of generality that $f_1, \ldots, f_k$ are monomials, say, $f_i = \prod_{j=1}^{l_i} X_{\sigma_{i,j}}$, where $\sigma_{i,1} \supseteq \sigma_{i,2} \supseteq \cdots \supseteq \sigma_{i,l_i}$. Now $\varphi(f_i) = \prod_{j=1}^{l_i} \varphi(X_{\sigma_{i,j}})$. Thus $\prod_{i=1}^{k} \varphi(f_i) = \prod_{i=1}^{k} \prod_{j=1}^{l_i} \varphi(X_{\sigma_{i,j}})$. We may therefore also assume that each f_i has the form X_{σ_i} for some $\sigma_i \in P$. Finally, when $\bigcup \sigma_i \notin P$, we have that $\varphi(f_1 \cdots f_k) = \varphi(f_1) \cdots \varphi(f_k) = 0$. Thus we may assume that $\bigcup \sigma_i \in P$.

We now show that the ordering on shapes satisfies a property we call *admissibility*. More precisely, if we have $\lambda_1 < \lambda_2$ and $\lambda_1' \leqslant \lambda_2'$, then $\lambda_1 + \lambda_1' < \lambda_2 + \lambda_2'$. To see this let $\lambda_1 = (a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k)$, $\lambda_2 = (b_1 \geqslant b_2 \geqslant \cdots \geqslant b_l)$ and use primes to denote the terms of λ_1' and λ_2'. Now if either $k < k'$ or $l < l'$, then we trivially have $\lambda_1 + \lambda_1' < \lambda_2 + \lambda_2'$. So we may assume that $k = k'$ and $l = l'$. If we have $\lambda_1' = \lambda_2'$, then it is easy to see that $\lambda_1 + \lambda_1' < \lambda_2 + \lambda_2'$. Thus we assume that $\lambda_1' < \lambda_2'$. Suppose that $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}$, $a_i > b_i$ and $a_1' = b_1', \ldots, a_{j-1}' = b_{j-1}'$, $a_j' > b_j'$. Now let $\nu = (a_1 \geqslant \cdots \geqslant a_{i-1}) = (b_1 \geqslant \cdots \geqslant b_{i-1})$, $\mu_1 = (a_i \geqslant a_{i+1} \geqslant \cdots)$, $\mu_2 = (b_i \geqslant b_{i+1} \geqslant \cdots)$, and use similar formulas in the primed cases. Then $\lambda_1 = \nu + \mu_1$, $\lambda_2 = \nu + \mu_2$, $\lambda_1' = \nu' + \mu_1'$, and $\lambda_2 = \nu' + \mu_2'$. It is trivial to see that $\mu_1 + \mu_1' < \mu_2 + \mu_2'$. Hence we have $\nu+\nu'+\mu_1+\mu_1' < \nu+\nu'+\mu_2+\mu_2'$, which is the same as $\lambda_1+\lambda_1' < \lambda_2+\lambda_2'$.

Next suppose that $\sigma_1 \supseteq \cdots \supseteq \sigma_l$ is a chain of P and that $\tau \in P$ is such that $\{\sigma_1,\ldots,\sigma_l,\tau\}$ is not in $\Delta(P)$. The standard factorization of $\varphi(X_{\sigma_1}\cdots X_{\sigma_l})\,\varphi(X_\tau) = X^{\sigma_1}\cdots X^{\sigma_l}X^{\tau}$ is easily seen to be $X^{\sigma_1\cup\tau}X^{\sigma_2\cup\tau\cap\sigma_1}\cdots X^{\sigma_l\cup\tau\cap\sigma_{l-1}}X^{\tau\cap\sigma_l}$, whose shape is $\lambda = (|\sigma_1\cup\tau| \geqslant |\sigma_2\cup\tau\cap\sigma_1| \geqslant\cdots\geqslant |\sigma_l\cup\tau\cap\sigma_{l-1}| \geqslant |\tau\cap\sigma_l|)$. We wish to show that this shape precedes the sum $\nu = (|\sigma_1|\geqslant\cdots\geqslant|\sigma_l|) + (|\tau|)$. Choose i so that this sum is $(|\sigma_1|\geqslant\cdots\geqslant|\sigma_i|\geqslant|\tau|>|\sigma_{i+1}|\geqslant\cdots\geqslant|\tau_l|)$, the cases $i=0$, l having the obvious interpretation. Now compare the first i terms of λ and ν: $|\sigma_1\cup\tau|\geqslant|\sigma_1|$, $|\sigma_2\cup\tau\cap\sigma_1|\geqslant|\sigma_2|,\ldots,|\sigma_i\cup\tau\cap\sigma_{i-1}|\geqslant|\sigma_i|$. If any of these are strict, then λ precedes ν and we are done. Thus we may assume they are all equalities. These imply that $\tau\subseteq\sigma_i$. The $(i+1)$st term of λ is then $|\sigma_{i+1}\cup\tau|$ while the $(i+1)$st term of ν is $|\tau|$. Since $|\tau|>|\sigma_{i+1}|$, we conclude that either $|\sigma_{i+1}\cup\tau|>|\tau|$ or $\sigma_{i+1}\subseteq\tau$. In the latter case, we would have that $\{\sigma_1,\ldots,\sigma_l,\tau\}\in\Delta(P)$, since $\tau\subseteq\sigma_i$. Thus the former case holds, and we conclude that λ precedes ν as desired.

We now show, by induction on k, that if each f_i has the form X_{σ_i} for some $\sigma_i\in P$, then $\varphi(f_1)\,\varphi(f_2)\cdots\varphi(f_k)$ either coincides with $\varphi(f_1f_2\cdots f_k)$ or else $\varphi(f_1f_2\cdots f_k)$ vanishes and $\varphi(f_1)\,\varphi(f_2)\cdots\varphi(f_k)$ is a monomial whose shape precedes $\lambda(f_1)+\lambda(f_2)+\cdots+\lambda(f_k)$. By induction this is true for $f_1,\ldots,f_{k-1}$. If $\varphi(f_1)\cdots\varphi(f_{k-1})$ coincides with $\varphi(f_1\cdots f_{k-1})$, then we simply apply the argument above to this monomial and $\varphi(f_k)$. If $\varphi(f_1\cdots f_{k-1})$ vanishes, then so does $\varphi(f_1\cdots f_k)$. Now in this case, $\varphi(f_1)\cdots\varphi(f_{k-1})$ is a monomial whose shape precedes $\lambda(f_1)+\cdots+\lambda(f_{k-1})$. By admissibility and the argument above, we conclude that $\varphi(f_1)\cdots\varphi(f_k)$ is a monomial whose shape precedes $\lambda(f_1)+\cdots+\lambda(f_k)$. The result now follows. ∎

We may now state the main result of this section.

THEOREM 4.3. *Let Δ be a simplicial complex of rank r, and let P be $P(\Delta)$. Define φ: $K[\Delta(P)]\to K[\Delta]$ on monomials by $\varphi(X_{\sigma_1}\cdots X_{\sigma_k}) = X^{\sigma_1}\cdots X^{\sigma_k}$, whenever $\sigma_1\supseteq\cdots\supseteq\sigma_k$ is a chain of elements of P. Suppose that $(\theta_1,\ldots,\theta_r;\ \eta_1,\ldots,\eta_N)$ is a sequence of homogeneous elements in the multigraded algebra $K[\Delta(P)]$ and that k: $[N]\to\{0,\ldots,r\}$ is a function such that every element of $K[\Delta(P)]$ is expressible in the form*

$$\sum_{j=1}^{N} \eta_j\, p_j(\theta_1,\ldots,\theta_{k(j)}), \tag{*}$$

for suitable polynomials p_j. Then $(\varphi(\theta_1),\ldots,\varphi(\theta_r);\ \varphi(\eta_1),\ldots,\varphi(\eta_N))$ satisfies the corresponding condition for $K[\Delta]$. Moreover if (*) is unique then such expressions using the $\varphi(\theta_i)$'s and $\varphi(\eta_j)$'s will also be unique in $K[\Delta]$. In particular, if $K[\Delta(P)]$ is CM, then so is $K[\Delta]$.

Although the statement of the theorem seems quite involved, the essential idea is that frames, basic frames and sets of separators for $K[\Delta(P)]$ may all be "transferred" via φ to produce frames, basic frames and sets of separators, respectively, for $K[\Delta]$. As we will see in Example 4.4, this process is only one-way: we cannot, in general, transfer sets of separators from $K[\Delta]$ to $K[\Delta(P)]$ via ψ.

Proof. Let w be a (nonzero) monomial of $K[\Delta]$. By assumption we may write $\psi(w)$ in the form (*). Since $\psi(w)$, the θ_i's and the η_j's are all homogeneous, we may, by taking the homogeneous component of multidegree $r(\psi(w))$ in each term of (*), assume that all terms of (*) are homogeneous and have the same multidegree as $\psi(w)$. By Lemma 4.2, we have that

$$w - \sum_j \varphi(\eta_j)\, p_j(\varphi(\theta_1),..., \varphi(\theta_{k(j)}))$$

is a linear combination of monomials that precede w. Thus by induction on shapes, we conclude that every element of $K[\Delta]$ may be written in the desired form.

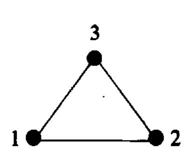
The part of the theorem, which is concerned with uniqueness of expressions of the form (*), is an immediate consequence of the fact that $K[\Delta]$ and $K[\Delta(P)]$ have the "same" Hilbert series, by Lemma 4.1. The last part of the theorem follows from Proposition 2.3. ∎

One immediate consequence of Theorem 4.3 is that

$$H(K[\Delta(P)]/(\theta_1,..., \theta_r); t) \geqslant_c H(K[\Delta]/(\alpha_1,..., \alpha_r); t),$$

where $\geqslant_c$ is coefficientwise inequality (see the proof of Proposition 2.3), with equality when $K[\Delta(P)]$ is CM. We do not know of any example of a simplicial complex Δ for which these two Hilbert series do not coincide. See Theorem 5.4 for a partial result in this direction.

EXAMPLE 4.4. The converse to the first part of Theorem 4.3 is false, as the following example shows. Let Δ be the simplicial complex on three vertices given by the solid triangle



Then $K[\Delta] = K[X_1, X_2, X_3]$ has basic frame

$$\alpha_1 = X_1 + X_2 + X_3, \quad \alpha_2 = X_1X_2 + X_2X_3 + X_1X_3, \quad \alpha_3 = X_1X_2X_3$$

and separators $\eta_1 = 1$, $\eta_2 = X_2$, $\eta_3 = X_3$, $\eta_4 = X_3^2$, $\eta_5 = X_1 X_3$, and $\eta_6 = X_2 X_3^2$. To check this we need only show that the separators are a basis of the K-algebra $K[X_1, X_2, X_3]/(\alpha_1, \alpha_2, \alpha_3)$. On the other hand, the ring $K[\Delta(P)] = K[X_1, X_2, X_3, X_{12}, X_{12}, X_{23}, X_{123}]/(X_1X_2, X_1X_3, X_2X_3, X_1X_{23}, X_2X_{13}, X_3X_{12}, X_{12}X_{13}, X_{12}X_{23}, X_{13}X_{23})$ has frame $\theta_1 = X_1 + X_2 + X_3$, $\theta_2 = X_{12} + X_{13} + X_{23}$, $\theta_3 = X_{123}$, and the monomials corresponding to the above η_j's are $\psi(\eta_1) = 1$, $\psi(\eta_2) = X_2$, $\psi(\eta_3) = X_3$, $\psi(\eta_4) = X_3^2$, $\psi(\eta_5) = X_{13}$, $\psi(\eta_6) = X_3 X_{23}$. It is easy to see that the graded part of $K[\Delta(P)]/(\theta_1, \theta_2, \theta_3)$ of multidegree $\{2\}$ is not spanned by elements of the form $\sum_{j=1}^{6} \psi(\eta_j)\, p_j(\theta_1, \theta_2, \theta_3)$. Thus the converse to the first part of Theorem 4.3 does not hold. There is, however, a partial converse. See Baclawski [5]. Moreover, the converse to the last assertion of Theorem 4.3 does hold as we will show in Corollary 6.3.

5. Rank-Selection and Cohomology

We show in this section that the Cohen–Macaulay property of the ring $K[\Delta(P)]$ for a poset P can be characterized by a topological property of the poset P (or more precisely of its rank-selected subposets). This characterization is new, although it bears some similarity to characterizations found by Reisner [21], Hochster [17], and Munkres [19]. To avoid overly cumbersome notation we will abbreviate $K[\Delta]/(\alpha_1, \ldots, \alpha_r)$ to $K[\Delta]/(\alpha)$, where $r = r(\Delta)$, and similarly $K[\Delta(P)]/(\theta_1, \ldots, \theta_r) = K[\Delta(P)]/(\theta)$.

Our principal tool is the following result which gives an explicit isomorphism between certain cohomology modules of rank-selected subposets of P and modules defined ring-theoretically in terms of $K[\Delta(P)]$.

Theorem 5.1. *Let P be a poset of rank r. Let S be a multisubset of $[r]$. Then*

$$\begin{aligned} \mathscr{H}_S K[\Delta(P)]/(\theta) &\cong \tilde{H}^{|S|-1}(P_S, K), && \textit{if } S \textit{ is a set,} \\ &\cong 0, && \textit{otherwise.} \end{aligned}$$

Proof. Suppose that S is not a set, say that $i \in S$ occurs with multiplicity $m_i > 1$. Let $x_1^{m_1} \cdots x_r^{m_r}$ be a monomial of $K[\Delta(P)]$ of multidegree S. Then $x_1^{m_1} \cdots x_r^{m_r} = \theta_i x_1^{m_1} \cdots x_i^{m_i - 1} \cdots x_r^{m_r}$. Hence in this case $\mathscr{H}_S K[\Delta(P)]/(\theta) = 0$.

Now let $S = \{l_1 < l_2 < \cdots < l_n\}$ be a set, and let $\prod_{j \in S} x_j$ be a monomial of $K[\Delta(P)]$ such that $r(x_j) = j$ for all j. Define a map g: $\mathscr{H}_S K[\Delta(P)] \to \tilde{C}^{|S|-1}(P_S, K)$ so that $g(\prod_{j \in S} x_j)$ is the basis element of $\tilde{C}^{|S|-1}(P_S, K)$ corresponding to the chain $(x_{l_1} < x_{l_2} < \cdots < x_{l_n})$. Extending g linearly, it is clear that g is an isomorphism of K-vector spaces. In a similar manner, we define an isomorphism h: $\bigoplus_{j \in S} \mathscr{H}_{S \setminus \{j\}} K[\Delta(P)] \to \tilde{C}^{|S|-2}(P_S, K)$.

We claim that the following diagram commutes,

$$\begin{array}{ccc} \mathscr{H}_S K[\Delta(P)] & \xrightarrow{g} & \tilde{C}^{|S|-1}(P_S, K) \\ \uparrow{\scriptstyle\theta} & & \uparrow{\scriptstyle\delta} \\ \bigoplus_{j \in S} \mathscr{H}_{S\setminus\{j\}} K[\Delta(P)] & \xrightarrow{h} & \tilde{C}^{|S|-2}(P_S, K), \end{array} \tag{*}$$

where δ is the coboundary map and θ denotes the direct sum $\bigoplus_{j\in S}(-1)^{n(j)}\theta_j$, the symbol θ_j being used to denote multiplication by the element θ_j of $K[\Delta(P)]$ and $n(j)$ denoting the number of elements of S that precede j. To show commutativity, let $\prod_{k\in S\setminus\{j\}} x_k$ be a monomial of $\mathscr{H}_{S\setminus\{j\}}K[\Delta(P)]$. Then $h(\prod_{k\neq j} x_k)$ is the chain $(x_{l_1} < \cdots < x_{l_{i-1}} < x_{l_{i+1}} < \cdots < x_{l_n})$, where $l_i = j$. The image of this chain under δ is the sum $\sum_x (-1)^{i-1}(x_{l_1} < \cdots < x_{l_{i-1}} < x < x_{l_{i+1}} < \cdots < x_{l_n})$, where x varies over the open interval $(x_{l_{i-1}}, x_{l_{i+1}})$ of P_S. It is easy to see that g^{-1} applied to the above sum is precisely $(-1)^{i-1}\theta_j \prod_{k\neq j} x_k$. Thus commutativity of (*) follows.

We now observe that $\tilde{H}^{|S|-1}(P_S, K) = \operatorname{Coker}(\delta)$, while $\operatorname{Coker}(\theta) = \mathscr{H}_S K[\Delta(P)]/(\theta)$. Thus the theorem follows. ∎

We now combine this result with Proposition 3.1 to give the desired topological characterization of the Cohen–Macaulay property.

COROLLARY 5.2. *Let P be a poset of rank r. Then $K[\Delta(P)]$ is Cohen–Macaulay if and only if for every subset $S \subseteq [r]$,*

$$(-1)^{|S|-1}\mu(P_S) = \tilde{h}_{|S|-1}(P_S, K).$$

Proof. By Theorem 5.1, the multivariate Hilbert series of the quotient $K[\Delta(P)]/(\theta)$ is given by

$$H(K[\Delta(P)]/(\theta); t_1,..., t_r) = \sum_{S\subseteq[r]} \tilde{h}_{|S|-1}(P_S, K)t^S,$$

where $t^S = \prod_{i\in S} t_i$. By Proposition 3.3,

$$H(K[\Delta(P)]; t_1,..., t_r) = \sum_{S\subseteq[r]} (-1)^{|S|-1}\mu(P_S)t^S \sum_{i=1}^{r} (1-t_i)^{-1}.$$

By Corollary 2.4, $K[\Delta(P)]$ is CM if and only if

$$\sum_{S\subseteq[r]} \tilde{h}_{|S|-1}(P_S, K)t^S = \sum_{S\subseteq[r]} (-1)^{|S|-1}\mu(P_S)t^S. \quad ∎$$

This characterization of the CM property has many nice features. For example, it is easy to show that if $K[\Delta(P)]$ is CM, then P is ranked. Suppose that P was not ranked. Then for some pair of adjacent ranks, $S=\{n, n+1\}$, we have P_S is of rank 2 but possesses a one-element maximal chain. Thus P_S is not connected, and so $\tilde{h}_0(P_S) \neq 0$. Now P_S is a nonempty poset of rank 2 so $\mu(P_S)=\tilde{h}_0(P_S)-\tilde{h}_1(P_S)$, but by Corollary 5.2, $\mu(P_S)=-\tilde{h}_1(P_S)$. We thus have a contradiction. It follows that if $K[\Delta(P)]$ is CM, then P is ranked.

Another immediate consequence of Corollary 5.2 is the

RANK-SELECTION THEOREM 5.3. *If $K[\Delta(P)]$ is CM of rank r, then for any $S \subseteq [r]$, $K[\Delta(P_S)]$ is also CM.*

We now prove a version of Theorem 5.1 valid for simplicial complexes. The result is much less powerful than Theorem 5.1 because a rank-selected subposet of Δ is not necessarily a simplicial complex. It does, however, allow us to give a direct and natural interpretation of the highest cohomology $\tilde{H}^{r-1}(\Delta, K)$ in terms of ring-theoretical concepts defined by $K[\Delta]$.

THEOREM 5.4. *Let Δ be a simplicial complex of rank r. Let N be $\binom{r+1}{2}$. Then there is an isomorphism $\mathscr{H}_N K[\Delta]/(\alpha) \cong \tilde{H}^{r-1}(\Delta, K)$.*

Proof. Let P be the poset $P(\Delta)$. Since $\tilde{H}^{r-1}(\Delta, K) \cong \tilde{H}^{r-1}(\Delta(P), K)$, we have, by Theorem 5.1, an isomorphism $\tilde{H}^{r-1}(\Delta, K) \cong \mathscr{H}_{[r]} K[\Delta(P)]/(\theta)$. Thus we wish to find an isomorphism $\mathscr{H}_N K[\Delta]/(\alpha) \cong \mathscr{H}_{[r]} K[\Delta(P)]/(\theta)$.

We first define a map $f\colon \mathscr{H}_N K[\Delta] \to \mathscr{H}_{[r]} K[\Delta(P)]$ on a monomial w to be $\psi(w)$ if $\psi(w)$ is in $\mathscr{H}_{[r]} K[\Delta(P)]$ and to be zero otherwise, where $\psi(w)$ is the standard factorization of w as given by Lemma 4.1. Thus $f(w)=0$ unless the multidegree of $\psi(w)$ is $[r]$. Clearly f is surjective.

Now let w be a monomial in $\mathscr{H}_{N-j} K[\Delta]$, where $j \in [r]$. We wish to compute the component of $\psi(\alpha_j w)$ in $\mathscr{H}_{[r]} K[\Delta(P)]$. This component will be zero unless $\psi(w)$ has $r-1$ or r factors, since $\psi(\alpha_j w)$ has either the same number of factors or one more. Therefore we may assume that w has the form $X^{\sigma_1} \cdots X^{\sigma_r}$, where $\sigma_1 \subseteq \cdots \subseteq \sigma_r$ in Δ and we allow σ_1 to be empty but $\sigma_2 \neq \varnothing$. We then compute $\psi(\alpha_j w)=\psi(\sum_{|\tau|=j} X^\tau X^{\sigma_1} \cdots X^{\sigma_r})=\psi(\sum_{|\tau|=j} X^{\sigma_1 \cap \tau} X^{\sigma_1 \cup \tau \cap \sigma_2} \cdots X^{\sigma_r \cup \tau})=\sum_{|\tau|=j} X_{\sigma_1 \cap \tau} X_{\sigma_1 \cup \tau \cap \sigma_2} \cdots X_{\sigma_r \cup \tau}$. The condition that one of these terms have multidegree $[r]$ is that $|\sigma_i \cup \tau \cap \sigma_{i+1}|=i$ for all i such that $0 \leqslant i \leqslant r$, where $\sigma_0=\varnothing$ and $\sigma_{r+1}=V$. These conditions immediately imply that $|\sigma_i| \leqslant i$ and $|\sigma_{i+1}| \geqslant i$ for all i such that $0 \leqslant i \leqslant r$. Equivalently, $i=1 \leqslant |\sigma_i| \leqslant i$ for $i \in [r]$. Let S be $\{i \in [r] \mid |\sigma_i|=i-1\}$. Then $\deg(w)=\sum_i |\sigma_i|=N-|S|$. Hence $|S|=j$. Since $j \neq 0$, S must be nonempty. Thus for a monomial w in $\mathscr{H}_{N-j} K[\Delta]$, $\psi(\alpha_j w)$ can have a nonzero component in $\mathscr{H}_{[r]} K[\Delta(P)]$ only if there is a nonempty subset $S \subseteq [r]$ and chain $\sigma_1 \subseteq \cdots \subseteq \sigma_r$ in Δ such that

$$\begin{aligned} |\sigma_i| &= i && \text{if} \quad i \notin S \\ &= i-1 && \text{if} \quad i \in S, \end{aligned} \qquad |S|=j, \quad w=X^{\sigma_1} \cdots X^{\sigma_r}.$$

Now assume that $w = X^{\sigma_1} \cdots X^{\sigma_r}$ is of the form described above. If $i \in S$, then $|\sigma_i| = i-1$. Hence $|\sigma_i \cup \tau \cap \sigma_{i+1}|$ can be i if and only if τ contains precisely one element of $\sigma_{i+1} \backslash \sigma_i$. On the other hand, since $|S| = j$ and $|\tau| = j$, these represent all the elements of τ. Define T to be $\{i \in S \mid i+1 \notin S\}$. If $i \in S \backslash T$, then $|\sigma_{i+1} \backslash \sigma_i| = 1$ so that in this case τ contains $\sigma_{i+1} \backslash \sigma_i$ and $\sigma_i \cup \tau \cap \sigma_{i+1} = \sigma_{i+1}$. If $i \in T$, then there are two possibilities. When $i \neq r$, we have $|\sigma_{i+1} \backslash \sigma_i| = 2$, say $\sigma_{i+1} \backslash \sigma_i = \{v_1, v_2\}$. Then τ contains exactly one of v_1 or v_2 and $\sigma_i \cup \tau \cap \sigma_{i+1} = \sigma_i \cup \{v_l\}$, where $v_l \in \tau$. When $i = r$, we have $|\sigma_{i+1} \backslash \sigma_i| = |V| - (r-1)$. In this case we note that we have the added condition $\sigma_r \cup \tau \in \Delta$ so that τ contains precisely one vertex of $\mathrm{link}_\Delta(\sigma_r)$; and if this vertex is v, then $\sigma_r \cup \tau = \sigma_r \cup \{v\}$. The last case to consider is $i \notin S$. Here $\sigma_i \cup \tau \cap \sigma_{i+1}$ coincides with σ_i. If $i+1 \in S$ also, then $\sigma_i \cup \tau \cap \sigma_{i+1}$ coincides with both σ_i and σ_{i+1}. We can summarize the above discussion in this table:

$i \in S, i+1 \in S$	$i \in S, i+1 \notin S$
$\sigma_i \cup \tau \cap \sigma_{i+1} = \sigma_{i+1}$ $\sigma_{i+1} \backslash \sigma_i \subseteq \tau$	$\sigma_i \cup \tau \cap \sigma_{i+1} = \sigma_i \cup \{v\}$ $v \in \sigma_{i+1} \backslash \sigma_i$

$i \notin S, i+1 \in S$	$i \notin S, i+1 \notin S$
$\sigma_i \cup \tau \cap \sigma_{i+1} = \sigma_i = \sigma_{i+1}$	$\sigma_i \cup \tau \cap \sigma_{i+1} = \sigma_i$

From this table, we first notice that for every $i \in [r]$, X_{σ_i} is a factor of every monomial in the component of $\psi(a_j w)$ in $\mathscr{H}_{[r]} K[\Delta(P)]$. The other factors are all due to the second case in the table above. Let u be the monomial $\prod_{\sigma \in \square\phi(w)} X_\sigma$. Then the component of $\psi(a_j w)$ in $\mathscr{H}_{[r]} K[\Delta(P)]$ is $(\prod_{i \in T} \theta_i) u$. This follows from the fact that no two elements of T are adjacent and from the table above. In other words, $f(a_j w)$ is either zero or has the form $(\prod_{i \in T} \theta_i) u$ for some monomial u and some nonempty subset $T \subseteq [r]$. Hence

$$f(\mathscr{H}_N K[\Delta] \cap (a_1, ..., a_r)) \subseteq (\theta_1, ..., \theta_r).$$

Therefore f induces a homomorphism:

$$\bar{f}: \mathscr{H}_N K[\Delta]/(a) \to \mathscr{H}_{[r]} K[\Delta(P)]/(\theta).$$

Since f is surjective, so is $\bar{f}$.

It remains to show that $\bar{f}$ is injective. Choose a monomial basis $\{\eta_j\}$ of $K[\Delta(P)]/(\theta)$. The η_j's will necessarily be square-free by Theorem 5.1. As in the proof of (2) $\Rightarrow$ (3) in Proposition 2.3, every element of $K[\Delta(P)]$ may be written in the form $\sum_j \eta_j p_j(\theta_1, ..., \theta_r)$ for suitable polynomials p_j, although this representation may not be unique. By Theorem 4.3, it follows that

$\{\varphi(\eta_j)\}$ spans $K[\Delta]/(\alpha)$ as a vector space over K. Since the η_j's are squarefree monomials, the only $\varphi(\eta_j)$'s having degree $N = \binom{r+1}{2}$ are those for which η_j has multidegree $[r]$. Therefore $\dim_K \mathscr{H}_N K[\Delta]/(\alpha) \leqslant \dim_K \mathscr{H}_{[r]} K[\Delta(P)]/(\theta)$, and the result follows. ∎

Although rank-selection applied to a simplicial complex Δ does not in general produce another simplicial complex, there is one obvious case in which it does. Namely, if Δ has rank r and if $i \in [r]$, define Δ_i to be the subcomplex $\{\sigma \in \Delta \mid |\sigma| \leqslant i\}$ of Δ as in Proposition 3.1. Topologists refer to Δ_i as the $(i-1)$-skeleton of Δ. Theorem 5.4 immediately implies that $\tilde{H}^{i-1}(\Delta_i, K)$ is isomorphic to $\mathscr{H}\binom{i+1}{2} K[\Delta_i]/(\alpha)$. We now show the analog of the Rank-Selection Theorem for simplicial complexes.

THEOREM 5.5. *Let Δ be a simplicial complex of rank r. If Δ is CM, then one can choose a basic set of monomial separators $\{\eta_1, \ldots, \eta_N\}$ for the frame $(\alpha_1, \ldots, \alpha_r)$ of $K[\Delta]$ in such a way that for all $i \in [r]$, $\{\eta_j \mid \Box(\eta_j) \in \Delta_i\}$ is a basic set of monomial separators for the frame $(\alpha_1, \ldots, \alpha_i)$ of $K[\Delta_i]$. In particular, Δ_i is then CM for all $i \in [r]$.*

Proof. Let Δ' be the subcomplex Δ_{r-1}, and let $\mathscr{M} = \Delta \backslash \Delta'$ be the set of simplices of Δ of rank r. We will show that (1) a monomial basis of $K[\Delta']/(\alpha)$ can be extended to a monomial basis of $K[\Delta]/(\alpha)$ using monomials whose support is in $\mathscr{M}$, and that (2) if $K[\Delta]$ is CM then so is $K[\Delta']$. Clearly the theorem follows from (1) and (2) by induction on i. We will, in fact, show a stronger result which relates $K[\Delta]/(\alpha)$ to $K[\Delta']/(\alpha)$ in a precise manner and for any simplicial complex Δ.

We begin by recalling (from Section 3) that there is a natural projection π: $K[\Delta] \to K[\Delta']$ which gives $K[\Delta']$ the structure of a $K[\Delta]$-algebra. The kernel of π is easily identified as being the direct sum $\bigoplus_{\sigma \in \mathscr{M}} X^\sigma K[\sigma]$, where $K[\sigma]$ denotes the Stanley–Reisner ring of the simplicial complex of all subsets of σ. Thus we have the following short exact sequence of $K[\Delta]$-modules:

$$0 \to \bigoplus_{\sigma \in \mathscr{M}} X^\sigma K[\sigma] \to K[\Delta] \to K[\Delta'] \to 0.$$

This short exact sequence immediately gives rise to an exact sequence:

$$\bigoplus_{\sigma \in \mathscr{M}} X^\sigma K[\sigma]/(\alpha_1, \ldots, \alpha_{r-1}) \to K[\Delta]/(\alpha_1, \ldots, \alpha_{r-1}) \to K[\Delta']/(\alpha) \to 0. \qquad (*)$$

Now let $(g(\sigma) X^\sigma \mid \sigma \in \mathscr{M}) \in \bigoplus_{\sigma \in \mathscr{M}} X^\sigma K[\sigma]$ have the property that $\sum_{\sigma \in \mathscr{M}} g(\sigma) X^\sigma = \sum_{i=1}^{r-1} \alpha_i h_i$, for some $h_i \in K[\Delta]$. In other words, $(g(\sigma) X^\sigma)$ is a representative of an element of the kernel of the first map in (*). Fix a simplex $\tau \in \mathscr{M}$. Next apply the homomorphism π_τ: $K[\Delta] \to K[\tau]$ to $\sum_{\sigma \in \mathscr{M}} g(\sigma) X^\sigma$:

$$\begin{aligned} g(\tau)X^\tau &= \pi_\tau\left(\sum_{\sigma\in\mathscr{M}} g(\sigma)X^\sigma\right) = \pi_\tau\left(\sum_{i=1}^{r-1} \alpha_i h_i\right) \\ &= \sum_{i=1}^{r-1} \alpha_i(\tau)\,\pi_\tau(h_i), \end{aligned}$$

where $\alpha_i(\tau) = \sum_{v\subseteq\tau} X^v\chi(|v| = i) = \pi_\tau(\alpha_i)$. Now in $K[\tau]$, there is just one simplex of rank r. Thus $\alpha_r(\tau) = X^\tau$. The equation above may therefore be written as

$$\sum_{i=1}^{r-1} \alpha_i(\tau)\,\pi_\tau(h_i) = \alpha_r(\tau)\, g(\tau).$$

Since $K[\tau] = K[X_v \mid v \in \tau]$ is a free polynomial ring, it is CM. Hence by Proposition 2.3, $\alpha_r(\tau)$ is not a zero-divisor modulo $(\alpha_1(\tau),..., \alpha_{r-1}(\tau))$. Therefore there exist $g_i(\tau) \in K[\tau]$, $i \in [r-1]$, such that $g(\tau) = \sum_{i=1}^{r-1} \alpha_i(\tau)\, g_i(\tau)$. This holds for every $\tau \in \mathscr{M}$. Therefore $(g(\sigma)X^\sigma \mid \sigma \in \mathscr{M})$ represents the zero element of $\bigoplus_{\sigma\in\mathscr{M}} X^\sigma K[\sigma]/(\alpha_1,..., \alpha_{r-1})$. Thus the following sequence is exact:

$$0 \to \bigoplus_{\sigma\in\mathscr{M}} X^\sigma K[\sigma]/(\alpha_1,..., \alpha_{r-1}) \to K[\Delta]/(\alpha_1,..., \alpha_{r-1}) \to K[\Delta']/(\alpha) \to 0.$$

We now go one step further in the above exact sequence and mod out by α_r as well. By the Ker–Coker lemma, we have an exact sequence

$$\begin{aligned} &0 \to \mathrm{Ann}_{K[\Delta]/(\alpha_1,...,\alpha_{r-1})}(\alpha_r) \to K[\Delta']/(\alpha) \\ &\xrightarrow{\oplus X^\sigma} \bigoplus_{\sigma\in\mathscr{M}} X^\sigma K[\sigma]/(\alpha) \to K[\Delta]/(\alpha) \to K[\Delta']/(\alpha) \to 0. \end{aligned} \tag{**}$$

We have used above that α_r annihilates all of $K[\Delta']/(\alpha)$ and that $K[\sigma]$ is CM, so that α_r is not a zero-divisor of $K[\sigma]/(\alpha_1,..., \alpha_{r-1})$. The exact sequence (**) is the basic exact sequence which relates $K[\Delta]/(\alpha)$ to $K[\Delta']/(\alpha)$.

We now show how the theorem follows from (**). Property (1) follows immediately from the exact sequence (**). To show property (2), we note that if $K[\Delta]$ is CM, then α_r is not a zero-divisor of $K[\Delta]/(\alpha_1,..., \alpha_{r-1})$. Hence (**) becomes the exact sequence $0 \to K[\Delta']/(\alpha) \to \bigoplus_{\sigma\in\mathscr{M}} X^\sigma K[\sigma]/(\alpha) \to K[\Delta]/(\alpha) \to K[\Delta']/(\alpha) \to 0$, which is the following relationship among the Hilbert series of these algebras:

$$(1 - t^r)\, H(K[\Delta']/(\alpha); t) = H(K[\Delta]/(\alpha); t) - \sum_{\sigma\in\mathscr{M}} t^r H(K[\sigma]/(\alpha); t).$$

We know that $K[\Delta]$ and all the $K[\sigma]$'s are CM and that $H(K[\sigma]; t)$ is

$1/(1-t)^r$ because $K[\sigma]$ is the free polynomial ring in r variables. We now divide the equation above by $\prod_{i=1}^{r}(1-t^i)$ and use Propositions 2.3 and 3.3 to obtain

$$\begin{aligned}
\frac{H(K[\Delta']/(\alpha);t)}{\prod_{i=1}^{r-1}(1-t^i)} &= \frac{H(K[\Delta]/(\alpha);t)}{\prod_{i=1}^{r}(1-t^i)} - \sum_{\sigma\in\mathscr{A}} \frac{t^r H(K[\sigma]/(\alpha);t)}{\prod_{i=1}^{r}(1-t^i)} \\
&= H(K[\Delta];t) - \sum_{\sigma\in\mathscr{A}} t^r H(K[\sigma];t) \\
&= \sum_{\sigma\in\Delta} t^{|\sigma|}(1-t)^{-|\sigma|} - \sum_{\sigma\in\mathscr{A}} t^r(1-t)^r \\
&= \sum_{\sigma\in\Delta'} t^{|\sigma|}(1-t)^{-|\sigma|} \\
&= H(K[\Delta'];t).
\end{aligned}$$

By Proposition 2.3, $K[\Delta']$ is CM and the theorem follows. ∎

6. Localization

Roughly speaking, localization is a tool for the close examination of a small part of a larger structure. The object of this section is to show that the CM property is local. We then show as a consequence that the CM property for a simplicial complex may be characterized by a local topological condition. More precisely, if Δ is a simplicial complex, then its local structure near $\sigma\in\Delta$ is defined by the subcomplex

$$\operatorname{link}_\Delta(\sigma) = \{\tau\in\Delta \mid \tau\cup\sigma\in\Delta, \tau\cap\sigma=\varnothing\}.$$

Our main result is a method for finding basic sets of separators for the rings $K[\operatorname{link}_\Delta(\sigma)]$ from a basic set for $K[\Delta]$. Namely,

THEOREM 6.1. *Let Δ be a simplicial complex and $\sigma\in\Delta$. Suppose that $(\alpha_1,\ldots,\alpha_r;\ \eta_1,\ldots,\eta_N)$ is a basic frame for $K[\Delta]$. Then some subset of $\{\eta_j \mid \square\eta_j\in\operatorname{link}_\Delta(\sigma)\}$ is a basic set of separators for the frame $(\alpha_1,\ldots,\alpha_l)$ of $K[\operatorname{link}_\Delta(\sigma)]$, where $l=r(\operatorname{link}_\Delta(\sigma))$. In particular, if $K[\Delta]$ is CM, then so is $K[\operatorname{link}_\Delta(\sigma)]$.*

Proof. There is natural projection π: $K[\Delta]\to K[\operatorname{link}_\Delta(\sigma)]$, given by

$$\begin{aligned}
\pi(\tau) &= \tau && \text{if } \tau\in\operatorname{link}_\Delta(\sigma)\\
&= 0 && \text{otherwise.}
\end{aligned}$$

It is easy to see that π is a homomorphism which maps each α_i in $K[\Delta]$ to the analogous element α_i in $K[\operatorname{link}_\Delta(\sigma)]$ if $i\leqslant l$ and to 0 if $i>l$.

Thus π induces a surjective homomorphism $K[\Delta]/(a_1,\dots,a_r)\to K[\mathrm{link}_\Delta(\sigma)]/(a_1,\dots,a_l)$. Thus the first part of the theorem follows immediately from the second part and Proposition 2.3.

Assume that $K[\Delta]$ is CM. Since we have $\mathrm{link}_\Delta(\sigma\cup\tau)=\mathrm{link}_{\mathrm{link}_\Delta(\sigma)}(\tau)$ when $\sigma\cap\tau=\varnothing$, it is no loss of generality to assume that σ consists of a single vertex v. The *star* of v is the subcomplex of Δ given by

$$\mathrm{star}_\Delta(v)=\{\sigma\in\Delta\mid\{v\}\cup\sigma\in\Delta\}.$$

Clearly, $\mathrm{star}_\Delta(v)\supseteq\mathrm{link}_\Delta(v)$ and $K[\mathrm{link}_\Delta(v)]\cong K[\mathrm{star}_\Delta(v)]/(X_v)$. Moreover, it is easy to see that $K[\mathrm{link}_\Delta(v)]$ is a subalgebra of $K[\mathrm{star}_\Delta(v)]$ and that $K[\mathrm{star}_\Delta(v)]$ is a subalgebra of $K[\Delta]$. We propose to show that $(\tilde a_1,\dots,\tilde a_l,X_v)$ is a basic frame for $K[\mathrm{star}_\Delta(v)]$, where $\tilde a_i=\sum_\sigma X^\sigma\chi(|\sigma|=i)\,\chi(\sigma\in\mathrm{link}_\Delta(v))$. Since $K[\mathrm{link}_\Delta(v)]$ is the quotient of $K[\mathrm{star}_\Delta(v)]$ by X_v, this will give the theorem.

We first observe that $K[\mathrm{star}_\Delta(v)]$ has an $\mathbb{N}^2$-grading given by $\deg(X_u)=(\chi(u\neq v),\ \chi(u=v))$. This $\mathbb{N}^2$-grading has the property that its associated grading is the usual grading on $K[\mathrm{star}_\Delta(v)]$ as a subalgebra of $K[\Delta]$. Using this $\mathbb{N}^2$-grading, we will show that X_v is not a zero-divisor modulo $(\tilde a_1,\dots,\tilde a_k)$ for any $k\leqslant l$. Accordingly, let $fX_v\in(\tilde a_1,\dots,\tilde a_k)$, where f is homogeneous with respect to the $\mathbb{N}^2$-grading, say, $fX_v=\sum_{j=1}^k g_j\tilde a_j$. We may assume that each g_j is homogeneous since the $\tilde a_j$ are so. Since $\deg(\tilde a_j)=(j,0)$, it follows that each g_j is divisible by X_v, and since X_v is not a zero-divisor of $K[\mathrm{star}_\Delta(v)]$, we have $f=\sum_{j=1}^k g_j'\tilde a_j$, where $g_j'X_v=g_j$. Thus $f\in(\tilde a_1,\dots,\tilde a_k)$ as desired.

We now show that $(\tilde a_1,\dots,\tilde a_l,X_v)$ is a basic frame for $K[\mathrm{star}_\Delta(v)]$. We do this by showing that this sequence is a regular sequence. Let f be a homogeneous element of $K[\mathrm{star}_\Delta(v)]$ such that $f\tilde a_{k+1}\in(\tilde a_1,\dots,\tilde a_k)$, say, $f\tilde a_{k+1}=\sum_{j=1}^k g_j\tilde a_j$. Since $K[\mathrm{star}_\Delta(v)]$ is a subalgebra of $K[\Delta]$, we may interpret this as an equation in $K[\Delta]$. Now multiply by X_v: $f\tilde a_{k+1}X_v=\sum_{j=1}^k g_j\tilde a_jX_v$. Since X_v annihilates every monomial whose support is not in $\mathrm{star}_\Delta(v)$, we have that $\tilde a_iX_v=a_iX_v$ for all i. Thus $fa_{k+1}X_v=\sum_{j=1}^k g_ja_jX_v$ holds in $K[\Delta]$. By assumption $K[\Delta]$ is CM. Thus by Proposition 2.3, $(a_1,\dots,a_r)$ is a regular sequence. In particular, this implies that $fX_v\in(a_1,\dots,a_k)$, say, that $fX_v=\sum_{j=1}^k h_ja_j$. Multiply once more by X_v: $fX_v^2=\sum_{j=1}^k h_ja_jX_v=\sum_{j=1}^k h_j\tilde a_jX_v$. Now each h_jX_v may be regarded as an element of $K[\mathrm{star}_\Delta(v)]$. Hence $fX_v^2\in(\tilde a_1,\dots,\tilde a_k)$. We already showed that X_v is not a zero-divisor modulo $(\tilde a_1,\dots,\tilde a_k)$. Thus $f\in(\tilde a_1,\dots,\tilde a_k)$. Thus for all k, $\tilde a_{k+1}$ is not a zero-divisor modulo $(\tilde a_1,\dots,\tilde a_k)$. Since X_v is not a zero-divisor modulo $(\tilde a_1,\dots,\tilde a_l)$, we conclude that $(\tilde a_1,\dots,\tilde a_l,X_v)$ is a regular sequence and the theorem follows, by Proposition 2.3. ∎

Our combinatorial decomposition results enable us to put together a reasonable proof of a fundamental result in the theory of CM complexes.

The original proof of this result, due to Reisner [17], is very difficult and uses quite sophisticated machinery. It may be stated as follows.

PROPOSITION 6.2. *Let Δ be a simplicial complex. The ring $K[\Delta]$ is CM if and only if for every $\sigma \in \Delta$, $\operatorname{link}_\Delta(\sigma)$ is a bouquet.*

Proof. We will say that Δ satisfies the *bouquet condition* if for every $\sigma \in \Delta$, $\operatorname{link}_\Delta(\sigma)$ is a bouquet. For a poset P we will say that P satisfies the *interval bouquet condition* if for every $x < y$ in $\hat{P}$, $\Delta(x, y)$ is a bouquet. Let Δ be a simplicial complex of rank r. Henceforth we will write P for $P(\Delta)$.

Suppose that Δ satisfies the bouquet condition. Then P satisfies the interval bouquet condition, for the open intervals (x, y) of $\hat{P}$ are of two types: if $y \neq \hat{1}$, then (x, y) is the boundary of a simplex and hence a bouquet; if $y = \hat{1}$, then (x, y) is isomorphic to $\operatorname{link}_\Delta(x)$ and hence also a bouquet. We claim that if $S \subseteq [r]$ satisfies $|S| = r - 1$ (say, $S = [r] \backslash \{l\}$), then P_S satisfies the interval bouquet condition. Now open intervals of $\hat{P}_S$ are either open intervals of $\hat{P}$ or are obtained by deleting one rank of an open interval of $\hat{P}$. Thus by the obvious induction on r, we need only show that P_S is a bouquet. This follows by essentially the same proof as that of Proposition 3.1. We again have a natural projection $\pi_j\colon \tilde{C}^j(\Delta(P), K) \to \tilde{C}^j(\Delta(P_S), K)$, but now the kernel of π_j is $\bigoplus_{x \in P_l} \tilde{C}^{j-1}(\Delta(\hat{0}, x) * \Delta(x, \hat{1}), K)$, where $P_l = \{x \in P \mid r(x) = l\}$ and the open intervals are taken in $\hat{P}$. Since P satisfies the interval bouquet condition, $\Delta(\hat{0}, x)$ and $\Delta(x, \hat{1})$ are bouquets. By Proposition 3.2, $\Delta(\hat{0}, x) * \Delta(x, \hat{1})$ is a bouquet of dimension $r - 2$. Now, as in the proof of Proposition 3.1, $\Delta(P_S)$ is also a bouquet, and hence P_S satisfies the interval bouquet condition.

We now repeat the above argument inductively to conclude that $\Delta(P_S)$ is a bouquet for all subsets $S \subseteq [r]$. In particular, for every $S \subseteq [r]$ we have that $\mu(\Delta(P_S)) = (-1)^{|S|-1} \tilde{h}_{|S|-1}(\Delta(P_S), K)$. By Corollary 5.2, $K[\Delta(P)]$ is CM. By Theorem 4.3, $K[\Delta]$ is also CM.

Conversely, suppose that $K[\Delta]$ is CM and that the theorem is true for simplicial complexes of smaller rank. By Theorem 6.1, $K[\operatorname{link}_\Delta(\sigma)]$ is CM for all $\sigma \in \Delta$. Hence by the inductive assumption, $\operatorname{link}_\Delta(\sigma)$ is a bouquet for all $\sigma \in \Delta \backslash \{\varnothing\}$. It remains to show that Δ itself is a bouquet. By Theorem 5.5, $K[\Delta']$ is CM, where $\Delta' = \{\sigma \in \Delta \mid |\sigma| < r\}$. By the inductive hypothesis, Δ' satisfies the bouquet condition. By Proposition 3.1, $\tilde{h}_i(\Delta, K) = \tilde{h}_i(\Delta', K)$ for $i < r - 2$. But $\tilde{h}_i(\Delta', K) = 0$ for $i < r - 2$ since Δ' is a bouquet of dimension $r - 2$. Therefore $\tilde{h}_i(\Delta, K) = 0$ for $i \neq r - 1$, $r - 2$. In particular $\mu(\Delta) = (-1)^{r-1} \tilde{h}_{r-1}(\Delta, K) + (-1)^{r-2} \tilde{h}_{r-2}(\Delta, K)$. Thus to show that Δ is a bouquet we need only prove that $\mu(\Delta) = (-1)^{r-1} \tilde{h}_{r-1}(\Delta, K)$.

We now apply Proposition 3.3 to $K[\Delta(P)]$:

$$H(K[\Delta(P)]; t_1, \ldots, t_r) = \sum_{T \subseteq [r]} \frac{(-1)^{|T|-1} \mu(P_T) t^T}{\prod_{i=1}^r (1 - t_i)}.$$

By Lemma 4.1, we can use this to compute $H(K[\Delta]; t)$:

$$H(K[\Delta]; t) = \frac{\sum_{T \subseteq [r]} (-1)^{|T|-1} \mu(P_T) t^{\|T\|}}{\prod_{i=1}^{r} (1-t_i)},$$

where $\|T\| = \sum_{i \in T} i$. Now $K[\Delta]$ is CM, so by Proposition 2.3,

$$H(K[\Delta]; t) = \frac{H(K[\Delta]/(\alpha_1, ..., \alpha_r); t)}{\prod_{i=1}^{r} (1-t^i)}.$$

Thus the two equations above give us

$$H(K[\Delta]/(\alpha_1, ..., \alpha_r); t) = \sum_{T \subseteq [r]} (-1)^{|T|-1} \mu(P_T) t^{\|T\|}.$$

Now the only $T \subseteq [r]$ for which $\|T\| = \binom{r+1}{2}$ is $T = [r]$ so it follows that $\dim_K \mathscr{H}_N K[\Delta]/(\alpha_1, ..., \alpha_r) = (-1)^{r-1} \mu(P)$, where $N = \binom{r+1}{2}$. By Theorem 5.4, the left-hand side above is $\tilde{h}_{r-1}(\Delta, K)$. Since $\mu(P) = \mu(\Delta)$, we have $\mu(\Delta) = (-1)^{r-1} \tilde{h}_{r-1}(\Delta, K)$, and the result now follows. ∎

We now give a partial converse to Theorem 4.3.

COROLLARY 6.3. *Let Δ be a simplicial complex of rank r and let $P = P(\Delta)$. Then $K[\Delta]$ is CM if and only if $K[\Delta(P)]$ is CM. Moreover, in this case $K[\Delta]/(\alpha_1, ..., \alpha_r)$ and $K[\Delta(P)]/(\theta_1, ..., \theta_r)$ are isomorphic as graded vector spaces.*

Proof. We already know by Theorem 4.3 that if $K[\Delta(P)]$ is CM, then $K[\Delta]$ is also. Suppose that $K[\Delta]$ is CM. Then by Proposition 6.2, Δ satisfies the bouquet condition. We show that $\Delta(P)$ also satisfies the bouquet condition. Let $\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k$ be an element of $\Delta(P)$. It is easy to see that $\mathrm{link}_{\Delta(P)}(\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k)$ is isomorphic to the join $\Delta(\hat{0}, \sigma_1) * \Delta(\sigma_1, \sigma_2) * \cdots * \Delta(\sigma_k, \hat{1})$, where all open intervals are computed in $\hat{P}$. Now $(\hat{0}, \sigma_1)$, (σ_1, σ_2),..., (σ_{k-1}, σ_k) are all standard triangulations of spheres and hence are bouquets. The open interval $(\sigma_k, \hat{1})$ is isomorphic to $\mathrm{link}_\Delta(\sigma_k)$, so it too is a bouquet. Thus by Proposition 3.2, $\mathrm{link}_{\Delta(P)}(\sigma_1 \subset \sigma_2 \subset \cdots \subset \sigma_k)$ is a bouquet. By Proposition 6.2, $K[\Delta(P)]$ is CM. For the rest of the Corollary, see the remarks following the proof of Theorem 4.3. ∎

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