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Asymptotics of multiple orthogonal polynomials for a system of two measures supported on a starlike set

A. López García*,[1]

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

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Abstract

For a system of two measures supported on a starlike set in the complex plane, we study the asymptotic properties of the associated multiple orthogonal polynomials Q_n and their recurrence coefficients. These measures are assumed to form a Nikishin-type system, and the polynomials Q_n satisfy a three-term recurrence relation of order three with positive coefficients. Under certain assumptions on the orthogonality measures, we prove that the sequence of ratios $\{Q_{n+1}/Q_n\}$ has four different periodic limits, and we describe these limits in terms of a conformal representation of a compact Riemann surface. Several relations are found involving these limiting functions and the limiting values of the recurrence coefficients. We also study the nth root asymptotic behavior and zero asymptotic distribution of Q_n.

Keywords: Higher-order three-term recurrences; Nikishin systems; Ratio asymptotics; nth root asymptotics; Zero asymptotic distribution

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1. Introduction and statement of main results

This work was motivated by recent investigations of Aptekarev et al. [2] on asymptotic properties of monic polynomials Q_n generated by the higher-order three-term recurrence relation

$$zQ_n = Q_{n+1} + a_n Q_{n-p}, \quad n \geq p, \ p \in \mathbb{N}, \ a_n > 0, \tag{1}$$

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* Corresponding address: Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B, B-3001, Leuven, Belgium.

E-mail addresses: abey.lopezgarcia@wis.kuleuven.be, abey.lopez@gmail.com.

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with initial conditions

$$Q_j(z) = z^j, \quad j = 0, \ldots, p. \tag{2}$$

In [2], strong asymptotics of Q_n was studied by assuming that the recurrence coefficients satisfy

$$\sum_{n=p}^{\infty} |a_n - a| < \infty, \quad a > 0. \tag{3}$$

An important element in the asymptotic analysis of the polynomials Q_n is the starlike set

$$\widetilde{S}_0 := \bigcup_{k=0}^{p} \exp(2\pi \mathrm{i} k/(p+1))[0, \alpha], \quad \alpha := [(p+1)/p^{p/(p+1)}]a^{1/(p+1)}.$$

In fact, [2, Theorem 7.2] asserts that

$$\lim_{n\to\infty} \frac{Q_n(z)}{w_0^n(z)} = F_0(z), \quad \text{uniformly on compact subsets of } \overline{\mathbb{C}} \setminus \widetilde{S}_0,$$

where $w_0(z)$ is the unique branch of the algebraic equation $w^{p+1} - zw^p + a = 0$ that is meromorphic at infinity and has an analytic continuation in $\mathbb{C} \setminus \widetilde{S}_0$.

We remark that notable families of polynomials satisfy (1) in the constant coefficients case, for example the classical monic Chebyshev polynomials of the second kind $U_n(x) = \sin((n+1)\cos^{-1}(x/2))/\sin(\cos^{-1}(x/2))$ for the segment $[-2, 2]$ ($p = 1, a_n = 1$ for all n). It was shown by He and Saff [9] that the Faber polynomials associated with the closed domain bounded by the $(p+1)$-cusped hypocycloid with parametric equation

$$z = \exp(\mathrm{i}\theta) + \frac{1}{p}\exp(-p\mathrm{i}\theta), \quad 0 \le \theta < 2\pi, \; p \ge 2,$$

are also generated by the recurrence relation (1) with constant coefficients $a_n = a = 1/p$, and their zeros are contained in $\widetilde{S}_0$. Many other properties of the zeros of these Faber polynomials were obtained in [9,6].

Using operator theoretic techniques, in [3] it was proved that the polynomials Q_n generated by (1)–(2) are in fact multiple orthogonal polynomials with respect to a system of p measures supported on

$$\bigcup_{k=0}^{p} \exp(2\pi \mathrm{i} k/(p+1))[0, \infty).$$

Moreover, if (3) holds then the orthogonality measures have a specific hierarchy structure; they form a Nikishin-type system (see Section 8 and Theorem 9.1 in [2]). This system is the system of spectral measures of the banded Hessenberg operator (with only two nonzero diagonals) associated with (1).

In this paper we study, among other topics, ratio and nth root asymptotics of multiple orthogonal polynomials associated with a Nikishin-type system of two measures supported on a starlike set, starting from assumptions on these orthogonality measures. For simplicity we assume that these measures are given by weights. Under similar assumptions, analogous results can be obtained for general measures. We introduce next the Nikishin-type system.

Let

$$S_0 := \bigcup_{k=0}^{2} \exp(2\pi \mathrm{i} k/3)[0,\alpha], \quad 0<\alpha<\infty.$$

We emphasize that α is arbitrary here. Assume that s_1 is a complex-valued function defined on S_0, such that

$$\begin{aligned} &s_1 \geq 0 \quad \text{on } (0,\alpha),\ s_1 \in L^1(0,\alpha),\\ &s_1\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} z\right) = \mathrm{e}^{\frac{4\pi \mathrm{i}}{3}} s_1(z), \quad z \in S_0 \setminus \left\{0,\alpha,\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}\alpha, \mathrm{e}^{\frac{4\pi \mathrm{i}}{3}}\alpha\right\}. \end{aligned}$$

Set

$$f(z) := z^2 \int_{-b}^{-a} \frac{s_2(t)}{z^3-t^3}\mathrm{d}t, \quad 0<a<b<\infty,$$

where s_2 is a non-negative integrable function defined on $[-b,-a]$. Note that f is analytic in $\mathbb{C}\setminus S_1$, where

$$S_1 := \bigcup_{k=0}^{2} \exp(2\pi \mathrm{i} k/3)[-b,-a].$$

We may assume that $s_2 \equiv 0$ on $(-\infty,0]\setminus[-b,-a]$, and we extend s_2 to the set $\bigcup_{k=0}^{2} \exp(2\pi \mathrm{i} k/3)(-\infty,0]$ through the symmetry property

$$s_2\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} z\right) = \mathrm{e}^{\frac{4\pi \mathrm{i}}{3}} s_2(z), \quad z \in \cup_{k=0}^{2} \exp(2\pi \mathrm{i} k/3)(-\infty,0].$$

Then

$$f(z) = \frac{1}{3}\int_{S_1} \frac{s_2(t)}{t-z}\mathrm{d}t = \frac{z^2}{3}\int_{-b^3}^{-a^3} \frac{s_2(\sqrt[3]{\tau})}{(z^3-\tau)\tau^{2/3}}\mathrm{d}\tau, \quad z\in\mathbb{C}\setminus S_1. \tag{4}$$

The Nikishin-type system is then the system of measures $\{s_1(t)\mathrm{d}t, f(t)s_1(t)\mathrm{d}t\}$ defined on S_0.

Let $\{Q_n\}_{n=0}^{\infty}$ be the sequence of *monic* polynomials of lowest degree that satisfy the following conditions:

$$\begin{cases} \int_{S_0} Q_{2n}(t)t^k s_1(t)\mathrm{d}t = 0, & k=0,\ldots,n-1,\\ \int_{S_0} Q_{2n}(t)t^k f(t) s_1(t)\mathrm{d}t = 0, & k=0,\ldots,n-1,\\ \int_{S_0} Q_{2n+1}(t)t^k s_1(t)\mathrm{d}t = 0, & k=0,\ldots,n,\\ \int_{S_0} Q_{2n+1}(t)t^k f(t) s_1(t)\mathrm{d}t = 0, & k=0,\ldots,n-1. \end{cases} \tag{5}$$

These are the polynomials whose algebraic and asymptotic properties we investigate.

Proposition 1.1. *The degree of each polynomial Q_n is maximal, i.e.,* deg $Q_n = n$. *Moreover, if $n=3j$, then Q_n has exactly j simple zeros on the interval* $(0,\alpha)$. *If $n=3j+1$, then Q_n has a simple zero at the origin and j simple zeros on* $(0,\alpha)$. *Finally, if $n=3j+2$, then Q_n has a* *double zero at the origin and j simple zeros on* $(0, \alpha)$. *The remaining zeros of Q_n are located on the rays* $\exp(2\pi i/3)(0, \alpha)$, $\exp(4\pi i/3)(0, \alpha)$, *and are rotations of the zeros on* $(0, \alpha)$.

Proposition 1.2. *The monic polynomials Q_n satisfy the following three-term recurrence relation*

$$zQ_n = Q_{n+1} + a_n Q_{n-2}, \quad n \geq 2,\ a_n \in \mathbb{R}, \tag{6}$$

where

$$Q_j(z) = z^j, \quad j = 0, 1, 2. \tag{7}$$

The coefficients a_n are given by the formulas

$$a_{2n} = \frac{\int_0^\alpha t^n Q_{2n}(t) s_1(t) \mathrm{d}t}{\int_0^\alpha t^{n-1} Q_{2n-2}(t) s_1(t) \mathrm{d}t}, \qquad a_{2n+1} = \frac{\int_0^\alpha t^n Q_{2n+1}(t) f(t) s_1(t) \mathrm{d}t}{\int_0^\alpha t^{n-1} Q_{2n-1}(t) f(t) s_1(t) \mathrm{d}t}. \tag{8}$$

Moreover, $a_n > 0$ for all $n \geq 2$.

Propositions 1.1 and 1.2 are proved in Section 2. Let

$$\Psi_n(z) := \int_{S_0} \frac{Q_n(t)}{t - z} s_1(t) \mathrm{d}t.$$

The functions Ψ_n (usually called *functions of second type*) satisfy:

$$\begin{cases} \Psi_n \in H(\overline{\mathbb{C}} \setminus S_0), \\ \Psi_{2n}(z) = O(1/z^{n+1}), & z \to \infty, \\ \Psi_{2n+1}(z) = O(1/z^{n+2}), & z \to \infty. \end{cases} \tag{9}$$

It is important for our analysis to determine the exact number of zeros of Ψ_n outside S_0, and their location. The following result, proved in Section 3, gives the answers to these questions.

Proposition 1.3. *For each $j \in \{0, 1, 2, 3, 5\}$, the function Ψ_{6l+j} has exactly $3l$ simple zeros in $\mathbb{C} \setminus S_0$, of which l zeros are located in $(-b, -a)$, and the remaining $2l$ zeros are rotations of these l zeros by angles of $2\pi/3$ and $4\pi/3$; Ψ_{6l+j} has no other zeros in $\mathbb{C} \setminus S_0$. The function Ψ_{6l+4} has exactly $3l + 3$ simple zeros in $\mathbb{C} \setminus S_0$, of which $l + 1$ zeros are located in $(-b, -a)$, and the remaining $2l + 2$ zeros are rotations of these $l + 1$ zeros by angles of $2\pi/3$ and $4\pi/3$; Ψ_{6l+4} has no other zeros in $\mathbb{C} \setminus S_0$.*

Let us define $Q_{n,2}$ as the *monic* polynomial whose zeros coincide with the zeros of Ψ_n in $\mathbb{C} \setminus S_0$.

The following result asserts that for consecutive values of n, the zeros of Q_n interlace, and the same is true for the zeros of $Q_{n,2}$.

Theorem 1.4. *For every $n \geq 0$, the polynomials Q_n and Q_{n+1} do not have any common zeros in $(0, \alpha)$. Moreover, there is exactly one zero of Q_{n+1} between two consecutive zeros of Q_n in $(0, \alpha)$. Conversely, there is exactly one zero of Q_n between two consecutive zeros of Q_{n+1} in $(0, \alpha)$.*

Additionally, for every $n \geq 0$, the functions Ψ_n and Ψ_{n+1} do not have any common zeros in $(-b, -a)$. There is exactly one zero of Ψ_{n+1} between two consecutive zeros of Ψ_n in $(-b, -a)$, and vice versa.

Theorem 1.4 is proved in Section 4. We can determine exactly how the zeros of Q_n interlace, thanks to the fact that the recurrence coefficients a_n are all positive (see Proposition 4.2 in Section 4).

We next describe the ratio asymptotics of the polynomials Q_n and $Q_{n,2}$, and the limiting behavior of the recurrence coefficients a_n. By Propositions 1.1 and 1.3, for some polynomials P_n and $P_{n,2}$ we may write:

$$Q_{3k}(\tau) = P_{3k}(\tau^3), \qquad Q_{3k+1}(\tau) = \tau P_{3k+1}(\tau^3), \qquad Q_{3k+2}(\tau) = \tau^2 P_{3k+2}(\tau^3), \tag{10}$$

$$Q_{n,2}(\tau) = P_{n,2}(\tau^3). \tag{11}$$

Theorem 1.5. *Assume that $s_1 > 0$ a.e. on $[0,\alpha]$ and $s_2 > 0$ a.e. on $[-b,-a]$. Then, for each $i \in \{0,\dots,5\}$, the following limits hold:*

$$\lim_{k\to\infty} \frac{P_{6k+i+1}(z)}{P_{6k+i}(z)} = \widetilde{F}_1^{(i)}(z), \qquad z \in \mathbb{C} \setminus [0,\alpha^3], \tag{12}$$

$$\lim_{k\to\infty} \frac{P_{6k+i+1,2}(z)}{P_{6k+i,2}(z)} = \widetilde{F}_2^{(i)}(z), \qquad z \in \mathbb{C} \setminus [-a^3,-b^3], \tag{13}$$

where convergence is uniform on compact subsets of the indicated regions. Moreover (cf. (6)),

$$\lim_{k\to\infty} a_{6k+i} = \begin{cases} -C_1^{(i)}, & \text{for } i \in \{0,1,3,4\}, \\ -C_0^{(i)}, & \text{for } i \in \{2,5\}, \end{cases} \tag{14}$$

where

$$\widetilde{F}_1^{(i)}(z) = \begin{cases} 1 + C_1^{(i)}/z + O(1/z^2), & \text{for } i \in \{0,1,3,4\}, \\ z + C_0^{(i)} + O(1/z), & \text{for } i \in \{2,5\}, \end{cases} \tag{15}$$

is the Laurent expansion at ∞ of $\widetilde{F}_1^{(i)}$. Consequently, the limits

$$\lim_{k\to\infty} \frac{Q_{6k+i+1}(z)}{Q_{6k+i}(z)} = z\widetilde{F}_1^{(i)}(z^3), \quad z \in \mathbb{C} \setminus S_0,\ i \in \{0,1,3,4\},$$

$$\lim_{k\to\infty} \frac{Q_{6k+i+1}(z)}{Q_{6k+i}(z)} = \frac{\widetilde{F}_1^{(i)}(z^3)}{z^2}, \quad z \in \mathbb{C} \setminus S_0,\ i \in \{2,5\},$$

$$\lim_{k\to\infty} \frac{Q_{6k+i+1,2}(z)}{Q_{6k+i,2}(z)} = \widetilde{F}_2^{(i)}(z^3), \quad z \in \mathbb{C} \setminus S_1,\ i \in \{0,\dots,5\},$$

hold uniformly on compact subsets of the indicated regions.

We also describe in Proposition 5.8 (Section 5) the ratio asymptotic behavior of the functions of second type Ψ_n, as well as the ratio asymptotic behavior of the polynomials p_n, $p_{n,2}$ defined in (67) (these polynomials are "orthonormal versions" of the polynomials P_n, $P_{n,2}$ defined in (10)–(11), see Proposition 5.3) and their leading coefficients.

Several relations can be established among the limiting functions $\widetilde{F}_1^{(i)}$, $\widetilde{F}_2^{(i)}$, and the limiting values of the recurrence coefficients (see also the boundary value properties described in Proposition 5.5).

Let us define

$$a^{(i)} := \lim_{k\to\infty} a_{6k+i}, \qquad 0 \le i \le 5.$$

Proposition 1.6. *The following relations among the functions $\widetilde{F}_j^{(i)}$ are valid:*

$$\widetilde{F}_1^{(2)}(z) = z\widetilde{F}_1^{(0)}(z), \qquad \widetilde{F}_1^{(5)}(z) = z\widetilde{F}_1^{(3)}(z), \tag{16}$$

$$\widetilde{F}_1^{(0)}\widetilde{F}_1^{(1)} = \widetilde{F}_1^{(3)}\widetilde{F}_1^{(4)}, \qquad \widetilde{F}_1^{(1)}\widetilde{F}_1^{(2)} = \widetilde{F}_1^{(4)}\widetilde{F}_1^{(5)}, \qquad \widetilde{F}_1^{(2)}\widetilde{F}_1^{(3)} = \widetilde{F}_1^{(5)}\widetilde{F}_1^{(0)}, \tag{17}$$

$$\frac{1-\widetilde{F}_1^{(3)}}{1-\widetilde{F}_1^{(0)}} = \frac{a^{(3)}}{a^{(0)}}, \qquad \frac{1-\widetilde{F}_1^{(4)}}{1-\widetilde{F}_1^{(1)}} = \frac{a^{(4)}}{a^{(1)}}, \qquad \frac{z-\widetilde{F}_1^{(5)}(z)}{z-\widetilde{F}_1^{(2)}(z)} = \frac{a^{(5)}}{a^{(2)}}, \tag{18}$$

$$\widetilde{F}_2^{(0)} = \widetilde{F}_2^{(2)}, \qquad \widetilde{F}_2^{(3)} = \widetilde{F}_2^{(5)}, \tag{19}$$

$$\widetilde{F}_2^{(0)}\widetilde{F}_2^{(1)} = \widetilde{F}_2^{(3)}\widetilde{F}_2^{(4)}, \qquad \widetilde{F}_2^{(1)}\widetilde{F}_2^{(2)} = \widetilde{F}_2^{(4)}\widetilde{F}_2^{(5)}, \qquad \widetilde{F}_2^{(2)}\widetilde{F}_2^{(3)} = \widetilde{F}_2^{(5)}\widetilde{F}_2^{(0)}. \tag{20}$$

Furthermore, the functions $\widetilde{F}_1^{(i)}$, $i \in \{0,\dots,5\}$, are all distinct, and the functions $\widetilde{F}_2^{(i)}$, $i \in \{0,1,3,4\}$, are also distinct.

For every $i \in \{0,\dots,5\}$, $a^{(i)} > 0$, and the following relations hold:

$$a^{(0)} = a^{(2)}, \qquad a^{(3)} = a^{(5)}, \qquad a^{(0)} + a^{(1)} = a^{(3)} + a^{(4)}. \tag{21}$$

The following inequalities also hold:

$$a^{(0)} \neq a^{(3)}, \qquad a^{(0)} \neq a^{(4)}, \qquad a^{(1)} \neq a^{(3)}, \qquad a^{(1)} \neq a^{(4)}.$$

In fact, we will show that $a^{(4)} > a^{(1)}$, and therefore (21) implies that $a^{(0)} > a^{(3)}$ (see Remark 6.2). Theorem 1.5 and Proposition 1.6 are proved in Section 5.

We next describe the limiting functions $\widetilde{F}_j^{(i)}$ in terms of a conformal representation of a compact Riemann surface. Let $\Delta_1 := [0, \alpha^3]$, and $\Delta_2 := [-b^3, -a^3]$. Consider the three-sheeted Riemann surface

$$\mathcal{R} = \overline{\mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2},$$

formed by the consecutively "glued" sheets

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus \Delta_1, \qquad \mathcal{R}_1 := \overline{\mathbb{C}} \setminus (\Delta_1 \cup \Delta_2), \qquad \mathcal{R}_2 := \overline{\mathbb{C}} \setminus \Delta_2. \tag{22}$$

Since $\mathcal{R}$ has genus zero, there exists a unique conformal representation ψ of $\mathcal{R}$ onto $\overline{\mathbb{C}}$ satisfying:

$$\begin{cases} \psi(z) = -2z/a^3 + O(1), & z \to \infty^{(1)} \in \mathcal{R}_1, \\ \psi(z) = B/z + O(1/z^2), & z \to \infty^{(2)} \in \mathcal{R}_2,\ B \neq 0. \end{cases} \tag{23}$$

Here $-a^3$ is the right endpoint of Δ_2. Let $\{\psi_k\}_{k=0}^2$ denote the branches of ψ.

Finally, given an arbitrary function $H(z)$ that has in a neighborhood of infinity a Laurent expansion of the form $H(z) = Cz^k + O(z^{k-1})$, $C \neq 0$, $k \in \mathbb{Z}$, we denote by $\widetilde{H}$ the function H/C.

Theorem 1.7. *The following representations are valid:*

$$\widetilde{F}_1^{(0)} = \frac{a^{(0)} - a^{(3)}}{a^{(0)}\widetilde{\psi}_0 - a^{(3)}}, \qquad \widetilde{F}_1^{(1)} = \frac{(a^{(4)} - a^{(1)})\widetilde{\psi}_0}{a^{(4)}\widetilde{\psi}_0 - a^{(1)}}, \qquad \widetilde{F}_1^{(2)}(z) = \frac{z(a^{(0)} - a^{(3)})}{a^{(0)}\widetilde{\psi}_0(z) - a^{(3)}},$$

$$\widetilde{F}_1^{(3)} = \frac{(a^{(0)} - a^{(3)})\widetilde{\psi}_0}{a^{(0)}\widetilde{\psi}_0 - a^{(3)}}, \qquad \widetilde{F}_1^{(4)} = \frac{a^{(4)} - a^{(1)}}{a^{(4)}\widetilde{\psi}_0 - a^{(1)}}, \qquad \widetilde{F}_1^{(5)}(z) = \frac{z(a^{(0)} - a^{(3)})\widetilde{\psi}_0(z)}{a^{(0)}\widetilde{\psi}_0(z) - a^{(3)}},$$

$$\widetilde{F}_2^{(0)}(z) = \widetilde{F}_2^{(2)}(z) = \frac{a^{(0)}(a^{(0)} - a^{(3)})z\widetilde{\psi}_0(z)\widetilde{\psi}_2(z)}{(a^{(0)} - a^{(3)}\omega_1^{(3)}\widetilde{\psi}_0(z)\widetilde{\psi}_2(z)/\omega_1^{(0)})(a^{(0)}\widetilde{\psi}_0(z) - a^{(3)})},$$

$$\widetilde{F}_2^{(3)}(z) = \widetilde{F}_2^{(5)}(z) = \frac{a^{(0)}(a^{(0)} - a^{(3)})z\widetilde{\psi}_0(z)}{(a^{(0)} - a^{(3)}\omega_1^{(3)}\widetilde{\psi}_0(z)\widetilde{\psi}_2(z)/\omega_1^{(0)})(a^{(0)}\widetilde{\psi}_0(z) - a^{(3)})},$$

$$\widetilde{F}_2^{(1)} = \frac{a^{(4)} - a^{(1)}}{\widetilde{\psi}_2(a^{(4)}\widetilde{\psi}_0 - a^{(1)})(\widetilde{\psi}_1 - (\omega_1^{(1)} - 1)/\omega_1^{(4)})},$$

$$\widetilde{F}_2^{(4)} = \frac{a^{(4)} - a^{(1)}}{(a^{(4)}\widetilde{\psi}_0 - a^{(1)})(\widetilde{\psi}_1 - (\omega_1^{(1)} - 1)/\omega_1^{(4)})}.$$

The constants $\omega_1^{(l)}$ are the reciprocals of the right-hand sides in the boundary value Eqs. (92)–(94). *They can be written in terms of the limiting values $a^{(i)}$ as follows:*

$$\omega_1^{(0)} = \omega_1^{(2)} = \frac{a^{(4)} - a^{(1)}}{a^{(0)}a^{(4)}}, \qquad \omega_1^{(3)} = \omega_1^{(5)} = \frac{a^{(0)}}{a^{(0)} - a^{(3)}}, \qquad \omega_1^{(1)} = \frac{a^{(4)}}{a^{(4)} - a^{(1)}},$$

$$\omega_1^{(4)} = \frac{a^{(0)} - a^{(3)}}{(a^{(0)})^2}.$$

Using Theorem 3.1 from [11], we can easily describe the cubic algebraic equation solved by ψ. The coefficients of this equation can be computed exclusively in terms of the endpoints of the intervals Δ_1 and Δ_2.

Proposition 1.8. *Let*

$$\lambda := \frac{2b^3}{a^3} - 1, \qquad \mu := \frac{2\alpha^3}{a^3} + 1, \tag{24}$$

and let β and γ be the unique solutions of the algebraic system

$$\begin{cases} 2(\beta+\gamma)(3-\beta\gamma-\beta-\gamma)(3-\beta\gamma+\beta+\gamma) + (\lambda-\mu)(\beta-\gamma)^3 = 0, \\ (\lambda+\mu)^2(\beta-\gamma)^6 = 4(3+\beta\gamma)^3(1-\beta\gamma)(2+\beta+\gamma)(2-\beta-\gamma), \end{cases}$$

satisfying the conditions $-1 < \gamma < \beta < 1$. Then $w = \psi(z)$ is the solution of the cubic equation

$$w^3 + \left[\frac{2z}{a^3} + 1 + \frac{3 + h + \Theta_2 - \Theta_1}{H(\beta)}\right] w^2 + \left[\frac{4z}{a^3 H(\beta)} + \frac{2}{H(\beta)} + \frac{2 + 2h + \Theta_2 - 3\Theta_1}{H(\beta)^2}\right] w - \frac{2\Theta_1}{H(\beta)^3} = 0, \tag{25}$$

where

$$H(z) = h + z + \frac{\Theta_1 z}{1 - z} + \frac{\Theta_2 z}{1 + z}, \qquad h = \frac{1}{4}(\beta+\gamma)\left(2\beta\gamma - \frac{(\beta-\gamma)^2}{1 - \beta\gamma}\right),$$

$$\Theta_1 = \frac{1}{4}(1-c)(1-d)(1-\beta)(1-\gamma), \qquad \Theta_2 = \frac{1}{4}(1+c)(1+d)(1+\beta)(1+\gamma),$$

c and d are the solutions of the equation

$$x^2 + (\beta+\gamma)x + \frac{(\beta-\gamma)^2}{1-\beta\gamma} - 3 = 0,$$

satisfying $c < -1, d > 1$.

Remark 1.9. Using (25) and Theorem 1.7, it is easy to deduce that

$$a^{(0)}-a^{(3)}=-\frac{a^3\Theta_2}{4H(\beta)}=a^{(4)}-a^{(1)}.$$

Theorem 1.7 and Proposition 1.8 are proved in Section 6. We now describe the results on nth root asymptotics and zero asymptotic distribution for the polynomials Q_n and $Q_{n,2}$. First, we introduce certain definitions and notations.

Given a compact set $E\subset\mathbb{C}$, let $\mathcal{M}_1(E)$ denote the space of all probability Borel measures supported on E. If P is a polynomial of degree n, we indicate by μ_P the associated normalized zero counting measure, i.e.,

$$\mu_P:=\frac{1}{n}\sum_{P(x)=0}\delta_x,$$

where δ_x is the Dirac measure with unit mass at x (in the sum the zeros are repeated according to their multiplicity). If $\mu\in\mathcal{M}_1(E)$, let

$$V^{\mu}(z):=\int\log\frac{1}{|z-t|}\mathrm{d}\mu(t),$$

and for a sequence $\{\mu_n\}\subset\mathcal{M}_1(E)$, $\mu_n\xrightarrow{*}\mu$ refers to the convergence of μ_n in the weak-star topology to μ.

Let E_1,E_2 be compact subsets of $\mathbb{C}$, and let $M=[c_{j,k}]$ be a real, positive definite, symmetric matrix of order two. Given a vector measure $\boldsymbol{\mu}=(\mu_1,\mu_2)\in\mathcal{M}_1(E_1)\times\mathcal{M}_1(E_2)$, we define the combined potential

$$W_j^{\boldsymbol{\mu}}:=\sum_{k=1}^{2}c_{j,k}V^{\mu_k},\qquad j=1,2,$$

and the constants

$$\omega_j^{\boldsymbol{\mu}}:=\inf\{W_j^{\boldsymbol{\mu}}(x):x\in E_j\},\qquad j=1,2.$$

It is well known (see [12, Chapter 5]) that if E_1,E_2 are regular with respect to the Dirichlet problem, and $c_{j,k}\geq 0$ in case $E_j\cap E_k\neq\emptyset$, then there exists a unique vector measure $\overline{\boldsymbol{\mu}}=(\overline{\mu}_1,\overline{\mu}_2)\in\mathcal{M}_1(E_1)\times\mathcal{M}_1(E_2)$ satisfying the properties $W_j^{\overline{\boldsymbol{\mu}}}(x)=\omega_j^{\overline{\boldsymbol{\mu}}}$ for all $x\in\operatorname{supp}(\overline{\mu}_j)$, $j=1,2$. The measure $\overline{\boldsymbol{\mu}}$ is called the vector equilibrium measure determined by the interaction matrix M on the system of compact sets (E_1,E_2), and $\omega_1^{\overline{\boldsymbol{\mu}}},\omega_2^{\overline{\boldsymbol{\mu}}}$ are called the equilibrium constants.

Let λ_1 be the positive, rotationally invariant measure on S_0 whose restriction to the interval $[0,\alpha]$ coincides with the measure $s_1(x)\mathrm{d}x$, and let λ_2 be the positive, rotationally invariant measure on S_1 whose restriction to the interval $[-b,-a]$ coincides with the measure $s_2(x)\mathrm{d}x$.

Let **Reg** denote the space of regular measures in the sense of Stahl and Totik (see definition in [15, pg. 61]). The zero asymptotic distribution and nth root asymptotics for the polynomials P_n and $P_{n,2}$ can be described as follows:

Theorem 1.10. *Assume that the measures λ_1 and λ_2 are in the class* **Reg***, and suppose that* supp(λ_1) *and* supp(λ_2) *are regular for the Dirichlet problem. Then*

$$\mu_{P_n}\xrightarrow{*}\overline{\mu}_1\in\mathcal{M}_1(\Delta_1),\qquad \mu_{P_{n,2}}\xrightarrow{*}\overline{\mu}_2\in\mathcal{M}_1(\Delta_2),\tag{26}$$

where $\overline{\boldsymbol{\mu}} = (\overline{\mu}_1, \overline{\mu}_2)$ is the vector equilibrium measure determined by the interaction matrix

$$\begin{bmatrix} 1 & -1/4 \\ -1/4 & 1/4 \end{bmatrix} \tag{27}$$

on the system of intervals (Δ_1, Δ_2). Therefore, the limits

$$\begin{aligned} \lim_{n\to\infty} |P_n(z)|^{1/n} &= \mathrm{e}^{-\frac{1}{3}V^{\overline{\mu}_1}(z)}, \quad z \in \mathbb{C} \setminus \Delta_1, \\ \lim_{n\to\infty} |P_{n,2}(z)|^{1/n} &= \mathrm{e}^{-\frac{1}{6}V^{\overline{\mu}_2}(z)}, \quad z \in \mathbb{C} \setminus \Delta_2, \end{aligned} \tag{28}$$

hold uniformly on compact subsets of the indicated regions. Moreover,

$$\begin{aligned} \lim_{n\to\infty} \left(\int_0^{\alpha^3} P_n^2(\tau)\mathrm{d}\nu_n(\tau) \right)^{1/n} &= \mathrm{e}^{-\frac{2}{3}\omega_1^{\overline{\boldsymbol{\mu}}}}, \\ \lim_{n\to\infty} \left(\int_{-b^3}^{-a^3} P_{n,2}^2(\tau)\mathrm{d}\nu_{n,2}(\tau) \right)^{1/n} &= \mathrm{e}^{-\frac{4}{3}\omega_2^{\overline{\boldsymbol{\mu}}}}, \end{aligned} \tag{29}$$

where $(\omega_1^{\overline{\boldsymbol{\mu}}}, \omega_2^{\overline{\boldsymbol{\mu}}})$ is the corresponding vector of equilibrium constants, and the varying measures $\mathrm{d}\nu_n$ *and* $\mathrm{d}\nu_{n,2}$ *are defined in* (69).

Corollary 1.11. *Under the same assumptions of Theorem* 1.10*, let $\overline{\boldsymbol{\mu}} = (\overline{\mu}_1, \overline{\mu}_2)$ be the vector equilibrium measure determined by the interaction matrix* (27) *on the system of intervals* $[0, \alpha^3]$, $[-b^3, -a^3]$, *and let $(\omega_1^{\overline{\boldsymbol{\mu}}}, \omega_2^{\overline{\boldsymbol{\mu}}})$ be the corresponding vector of equilibrium constants. Consider the probability measures $\vartheta_1 \in \mathcal{M}_1([0, \alpha])$ and $\vartheta_2 \in \mathcal{M}_1([-b, -a])$, defined as follows:*

$$\vartheta_1(E) := \overline{\mu}_1(E^3), \quad E \subset [0, \alpha], \qquad \vartheta_2(E) := \overline{\mu}_2(E^3), \quad E \subset [-b, -a],$$

where $E^3 = \{x^3 : x \in E\}$. If we denote by Z_{Q_n} the set of all roots of Q_n on $(0, \alpha)$, and by $Z_{Q_{n,2}}$ the set of all roots of $Q_{n,2}$ on $(-b, -a)$, then

$$\frac{1}{n} \sum_{x \in Z_{Q_n}} \delta_x \xrightarrow{*} \frac{1}{3}\vartheta_1, \qquad \frac{1}{n} \sum_{x \in Z_{Q_{n,2}}} \delta_x \xrightarrow{*} \frac{1}{6}\vartheta_2.$$

The limits

$$\begin{aligned} \lim_{n\to\infty} |Q_n(z)|^{1/n} &= \mathrm{e}^{-\frac{1}{3}V^{\overline{\mu}_1}(z^3)}, \quad z \in \mathbb{C} \setminus S_0, \\ \lim_{n\to\infty} |Q_{n,2}(z)|^{1/n} &= \mathrm{e}^{-\frac{1}{6}V^{\overline{\mu}_2}(z^3)}, \quad z \in \mathbb{C} \setminus S_1, \end{aligned}$$

hold uniformly on compact subsets of the indicated regions. Finally, we have

$$\begin{aligned} \lim_{k\to\infty} \left(\int_0^{\alpha} Q_{3k}^2(t) \frac{s_1(t)}{Q_{3k,2}(t)}\mathrm{d}t \right)^{1/k} &= \mathrm{e}^{-2\omega_1^{\overline{\boldsymbol{\mu}}}}, \\ \lim_{k\to\infty} \left(\int_0^{\alpha} Q_{3k+1}^2(t) \frac{t s_1(t)}{Q_{3k+1,2}(t)}\mathrm{d}t \right)^{1/k} &= \mathrm{e}^{-2\omega_1^{\overline{\boldsymbol{\mu}}}}, \\ \lim_{k\to\infty} \left(\int_0^{\alpha} Q_{3k+2}^2(t) \frac{s_1(t)}{t Q_{3k+2,2}(t)}\mathrm{d}t \right)^{1/k} &= \mathrm{e}^{-2\omega_1^{\overline{\boldsymbol{\mu}}}}, \end{aligned}$$

$$\lim_{n\to\infty}\left(\int_{-b}^{-a} Q_{n,2}^2(t)\frac{|h_n(t)|}{|Q_n(t)|}s_2(t)\mathrm{d}t\right)^{1/n} = \mathrm{e}^{-\frac{4}{3}\omega_2^{\overline{\mu}}},$$

where the functions h_n are defined in (68) *(see also* (70)*).*

The following proposition provides a link between the results on ratio and nth root asymptotics.

Proposition 1.12. *Under the same assumptions of Theorem 1.5, the following relations hold:*

$$\begin{aligned} V^{\overline{\mu}_1}(z) &= -\frac{1}{2}\sum_{i=0}^{5}\log|\widetilde{F}_1^{(i)}(z)|, \quad z\in\mathbb{C}\setminus[0,\alpha^3], \\ V^{\overline{\mu}_2}(z) &= -\sum_{i=0}^{5}\log|\widetilde{F}_2^{(i)}(z)|, \quad z\in\mathbb{C}\setminus[-b^3,-a^3], \end{aligned} \tag{30}$$

where $(\overline{\mu}_1,\overline{\mu}_2)$ is the vector equilibrium measure determined by the interaction matrix (27) *on the system of intervals* $[0,\alpha^3]$, $[-b^3,-a^3]$.

Theorem 1.10 and Proposition 1.12 are proved in Section 7. Corollary 1.11 follows immediately from Theorem 1.10, so we omit its proof.

2. The polynomials Q_n

Observe that the functions Ψ_n satisfy the orthogonality conditions

$$0=\int_{S_1} t^\nu\,\Psi_{2n+i}(t)s_2(t)\mathrm{d}t, \quad \nu=0,\ldots,n-1,\ i=0,1. \tag{31}$$

This follows directly from the definition of Ψ_{2n+i}, (4) and (5), since

$$\begin{aligned}\int_{S_1} t^\nu\,\Psi_{2n+i}(t)s_2(t)\mathrm{d}t &= \int_{S_0} Q_{2n+i}(x)s_1(x)\int_{S_1}\frac{t^\nu-x^\nu+x^\nu}{x-t}s_2(t)\mathrm{d}t\mathrm{d}x \\ &= \int_{S_0} Q_{2n+i}(x)(p_\nu(x)-3x^\nu f(x))s_1(x)\mathrm{d}x,\end{aligned}$$

where p_ν is a polynomial of degree at most $n-2$.

Proposition 2.1. *Let Q_n be the monic polynomial of smallest degree satisfying* (5). *If $d_n :=$ $\deg Q_n$, then*

$$Q_n\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}z\right)=\mathrm{e}^{\frac{2\pi\mathrm{i}d_n}{3}}Q_n(z), \qquad Q_n(z)=\overline{Q_n(\overline{z})}. \tag{32}$$

Furthermore, for each $0\le k\le n-1$,

$$0=\int_0^\alpha t^k Q_{2n}(t)(1+\mathrm{e}^{2\pi\mathrm{i}(k+d_{2n})/3}+\mathrm{e}^{4\pi\mathrm{i}(k+d_{2n})/3})s_1(t)\mathrm{d}t, \tag{33}$$

$$0=\int_0^\alpha t^k Q_{2n}(t)(1+\mathrm{e}^{2\pi\mathrm{i}(k+2+d_{2n})/3}+\mathrm{e}^{4\pi\mathrm{i}(k+2+d_{2n})/3})s_1(t)f(t)\mathrm{d}t, \tag{34}$$

$$0=\int_0^\alpha t^k Q_{2n+1}(t)(1+\mathrm{e}^{2\pi\mathrm{i}(k+2+d_{2n+1})/3}+\mathrm{e}^{4\pi\mathrm{i}(k+2+d_{2n+1})/3})s_1(t)f(t)\mathrm{d}t, \tag{35}$$

and for each $0 \leq k \leq n$*,*

$$0 = \int_0^\alpha t^k Q_{2n+1}(t)(1 + \mathrm{e}^{2\pi \mathrm{i}(k+d_{2n+1})/3} + \mathrm{e}^{4\pi \mathrm{i}(k+d_{2n+1})/3}) s_1(t)\mathrm{d}t. \tag{36}$$

Proof. It is easy to check that $Q_n(z)$, $Q_n\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}} z\right)/\mathrm{e}^{\frac{2\pi \mathrm{i} d_n}{3}}$ and $\overline{Q_n(\overline{z})}$ satisfy the same orthogonality conditions. By the uniqueness of the definition of Q_n, these polynomials must be equal to each other, so (32) holds. If we write (5) in terms of $[0, \alpha]$, we obtain (33)–(36). □

Lemma 2.2. *Let* n_1, n_2 *be non-negative integers, and assume that* P_1, P_2 *are polynomials, not both identically equal to zero, such that* $\deg P_1 \leq n_1 - 1$ *and* $\deg P_2 \leq n_2 - 1$*. Then the functions*

$$H_1(t) := P_1(t) + P_2(t)\sqrt[3]{t} f(\sqrt[3]{t}), \quad t > 0,$$
$$H_2(t) := P_1(t)t + P_2(t)\sqrt[3]{t} f(\sqrt[3]{t}), \quad t > 0,$$

have at most $n_1 + n_2 - 1$ *zeros on* $(0, \infty)$*, counting multiplicities.*

Proof. Let σ be a finite positive measure with compact support in $\mathbb{R}$, and let

$$\widehat{\sigma}(z) := \int \frac{\mathrm{d}\sigma(x)}{z - x}.$$

Lemma 5 in [8] asserts that $\{1, \widehat{\sigma}\}$ forms an AT system on any closed interval $\Delta \subset \mathbb{R}$ disjoint from $\mathrm{Co}(\mathrm{supp}(\sigma))$, the convex hull of $\mathrm{supp}(\sigma)$. This means that for any multi-index $(n_1, n_2) \in \mathbb{Z}_+^2$, and any pair of polynomials π_1, π_2 with $\deg \pi_1 \leq n_1 - 1, \deg \pi_2 \leq n_2 - 1$, not both identically equal to zero, the function $\pi_1 + \pi_2 \widehat{\sigma}$ has at most $n_1 + n_2 - 1$ zeros on Δ, counting multiplicities. By (4) we know that $H_2(t) = t(P_1(t) + P_2(t)\widehat{\sigma}(t))$, where σ denotes now the measure $(s_2(\sqrt[3]{\tau})/3\tau^{2/3})\mathrm{d}\tau$ supported on $[-b^3, -a^3]$, so the assertion concerning H_2 is valid.

Let $n_1 \geq n_2$, and suppose that there exist polynomials P_1, P_2, not both identically equal to zero, such that H_1 has at least $n_1 + n_2$ zeros on $(0, \infty)$, counting multiplicities. We may assume that $P_2 \not\equiv 0$. Let T be a polynomial of degree $n_1 + n_2$ that vanishes at $n_1 + n_2$ zeros of H_1 on $(0, \infty)$. H_1 can be analytically extended onto $\mathbb{C} \setminus [-b^3, -a^3]$,

$$\frac{H_1(z)}{T(z)} = \frac{P_1(z)}{T(z)} + \frac{z P_2(z)}{3T(z)} \int_{-b^3}^{-a^3} \frac{s_2(\sqrt[3]{\tau})}{z - \tau} \frac{\mathrm{d}\tau}{\tau^{2/3}} = O\left(\frac{1}{z^{n_2+1}}\right), \quad z \to \infty.$$

By a standard argument this implies that

$$0 = \int_{-b^3}^{-a^3} \frac{\tau^{\nu+1} P_2(\tau) s_2(\sqrt[3]{\tau})}{T(\tau)\tau^{2/3}} \mathrm{d}\tau, \quad 0 \leq \nu \leq n_2 - 1,$$

contradicting the fact that $\deg P_2 \leq n_2 - 1$. If $n_1 < n_2$, we use again this argument by contradiction, but now we divide $H_1(z)$ by $T(z)\widehat{\sigma}(z)$ instead of $T(z)$, and use the fact that $1/\widehat{\sigma}(z) = l(z) + \widehat{\mu}(z)$, where $l(z)$ is a polynomial of degree one and μ is a measure of constant sign supported on $[-b^3, -a^3]$ (see the Appendix of [10]). □

Proof of Proposition 1.1. Assume first that $n = 3l$, $d_{2n} = 3j$. Then (33)–(34) reduce to

$$0 = \int_0^\alpha t^{3k} Q_{2n}(t) s_1(t)\mathrm{d}t = \int_0^\alpha t^{3k} Q_{2n}(t) t f(t) s_1(t)\mathrm{d}t, \quad 0 \leq k \leq l - 1.$$

From (32) and the assumption $d_{2n} = 3j$, we deduce that $Q_{2n}(t) = \widetilde{Q}_{2n}(t^3)$, for a polynomial $\widetilde{Q}_{2n}$ of degree j. Therefore,

$$\begin{aligned} 0 &= \int_0^{\alpha^3} \tau^k \widetilde{Q}_{2n}(\tau) s_1(\sqrt[3]{\tau}) \frac{\mathrm{d}\tau}{\tau^{2/3}} \\ &= \int_0^{\alpha^3} \tau^k \widetilde{Q}_{2n}(\tau) \sqrt[3]{\tau} f(\sqrt[3]{\tau}) s_1(\sqrt[3]{\tau}) \frac{\mathrm{d}\tau}{\tau^{2/3}}, \quad 0 \le k \le l-1. \end{aligned} \tag{37}$$

Suppose that $\widetilde{Q}_{2n}$ has $N < 2l$ sign change knots on $(0, \alpha^3)$. Let P_1, P_2 be polynomials of degree at most $l-1$, $(P_1, P_2) \neq (0, 0)$, such that $H_1(t) = P_1(t) + P_2(t)\sqrt[3]{t} f(\sqrt[3]{t})$ has a zero at each point where $\widetilde{Q}_{2n}$ changes sign on $(0, \alpha^3)$, and a zero of order $2l - 1 - N$ at α^3. By Lemma 2.2, H_1 has no zeros on $(0, \alpha^3]$ other than the $2l - 1$ prescribed. Combining the two orthogonality conditions in (37) we obtain

$$\int_0^{\alpha^3} H_1(\tau) \widetilde{Q}_{2n}(\tau) s_1(\sqrt[3]{\tau}) \frac{\mathrm{d}\tau}{\tau^{2/3}} \mathrm{d}\tau = 0.$$

This contradicts the fact that $H_1 \widetilde{Q}_{2n}$ is real valued and has constant sign on $[0, \alpha^3]$. Applying (32) we conclude that Q_{2n} has exactly $2n$ simple zeros on S_0, $2n/3$ of them are located on $(0, \alpha)$, and the remaining zeros are rotations of the zeros on $(0, \alpha)$ by angles of $2\pi/3$ and $4\pi/3$.

Suppose now that $n = 3l$ and $d_{2n} = 3j + 1$. We will reach a contradiction. In this case $Q_{2n}(t) = t\widetilde{Q}_{2n}(t^3)$, for some polynomial $\widetilde{Q}_{2n}$ of degree j. From (33) and (34) we deduce that

$$\begin{aligned} 0 &= \int_0^{\alpha^3} \tau^k \widetilde{Q}_{2n}(\tau) \tau s_1(\sqrt[3]{\tau}) \frac{\mathrm{d}\tau}{\tau^{2/3}} \\ &= \int_0^{\alpha^3} \tau^k \widetilde{Q}_{2n}(\tau) \sqrt[3]{\tau} f(\sqrt[3]{\tau}) s_1(\sqrt[3]{\tau}) \frac{\mathrm{d}\tau}{\tau^{2/3}}, \quad 0 \le k \le l-1. \end{aligned} \tag{38}$$

The polynomial $\widetilde{Q}_{2n}$ has $N \le j$ sign change knots on $(0, \alpha^3)$. Since $d_{2n} \le 2n$, we have $j \le 2l - 1$. Let P_1, P_2 be polynomials of degree at most $l - 1$, not both simultaneously zero, such that $H_2(t) = P_1(t)t + P_2(t)\sqrt[3]{t} f(\sqrt[3]{t})$ has a zero at each point where $\widetilde{Q}_{2n}$ changes sign on $(0, \alpha^3)$ and has a zero of order $2l - 1 - N$ at α^3. The same argument used before but now applied to H_2 shows that Lemma 2.2 and (38) yield a contradiction. Therefore $d_{2n} = 3j + 1$ is impossible if n is a multiple of 3. Similarly one proves that the assumptions $n = 3l$ and $d_{2n} = 3j + 2$ are not compatible.

The cases $n = 3l + 1$ and $n = 3l + 2$ are handled in an identical manner, showing in the first case that d_{2n} is of the form $3j + 2$ and Q_{2n} has $2l$ sign change knots on $(0, \alpha)$, and in the second case by showing that d_{2n} is of the form $3j + 1$ and Q_{2n} has $2l + 1$ sign change knots on $(0, \alpha)$.

The analysis for the polynomials Q_{2n+1} is similar. The details are left to the reader. □

Corollary 2.3. *The polynomials Q_n and the functions Ψ_n satisfy the symmetry conditions*

$$Q_n\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}} z\right) = \mathrm{e}^{\frac{2\pi \mathrm{i} n}{3}} Q_n(z), \tag{39}$$

$$\Psi_n\left(e^{\frac{2\pi i}{3}} z\right) = e^{-\frac{2\pi i}{3}(1+2n)}\, \Psi_n(z), \tag{40}$$

for all $n \geq 0$.

Proof. (39) follows from (32) and $d_n = n$. (40) is an immediate consequence of (39) and the definition of Ψ_n. □

Proof of Proposition 1.2. The initial conditions (7) are immediate to check. For $n \geq 1$, we write

$$zQ_{2n} = Q_{2n+1} + b_{2n}Q_{2n} + b_{2n-1}Q_{2n-1} + b_{2n-2}Q_{2n-2} + \cdots + b_1Q_1 + b_0Q_0, \tag{41}$$

and let us show that

$$b_{2n-3} = b_{2n-4} = \cdots = b_1 = b_0 = 0, \qquad b_{2n} = b_{2n-1} = 0. \tag{42}$$

We prove (42) by induction. Let $n \geq 2$. If we integrate (41) term by term with respect to $s_1(t)\mathrm{d}t$, the orthogonality relations (5) imply that $b_0 = 0$. The fact that $b_1 = 0$ follows now by integrating (41) term by term with respect to $f(t)s_1(t)\mathrm{d}t$. Assume now that $0 = b_0 = b_1 = \cdots = b_{2k} = b_{2k+1} = 0$ for some $k \leq n-3$. After multiplying (41) by z^{k+1} and integrating the resulting equation first with respect to $s_1(t)\mathrm{d}t$, and then with respect to $f(t)s_1(t)\mathrm{d}t$, we get $b_{2k+2} = b_{2k+3} = 0$ (observe that $\int_{S_0} t^{k+1}Q_{2k+2}(t)s_1(t)\mathrm{d}t \neq 0$ and $\int_{S_0} t^{k+1}Q_{2k+3}(t)f(t)s_1(t)\mathrm{d}t \neq 0$), so the first chain of equalities in (42) follows. The fact that $b_{2n} = b_{2n-1} = 0$ is immediate from (39).

Analogously one shows that for $n \geq 1$, $zQ_{2n+1} = Q_{2n+2} + a_{2n+1}Q_{2n-1}$, $a_{2n+1} \in \mathbb{R}$, so (6) is justified. The formulas (8) follow directly from (6). The positivity of the recurrence coefficients is proved later in Proposition 3.6. □

3. The functions of second type Ψ_n and associated polynomials $Q_{n,2}$

Proposition 3.1. *The following formula holds:*

$$\Psi_n(z) = \int_0^{\alpha} \left(\frac{1}{t-z} + \frac{e^{\frac{2\pi i n}{3}}}{e^{\frac{2\pi i}{3}}t - z} + \frac{e^{\frac{4\pi i n}{3}}}{e^{\frac{4\pi i}{3}}t - z} \right) Q_n(t)s_1(t)\mathrm{d}t, \qquad z \notin S_0. \tag{43}$$

In particular, for any integer $k \geq 0$,

$$\begin{aligned}
\Psi_{3k}(z) &= 3z^2 \int_0^{\alpha} \frac{Q_{3k}(t)s_1(t)}{t^3 - z^3}\mathrm{d}t = z^2 \int_0^{\alpha^3} \frac{Q_{3k}(\sqrt[3]{\tau})s_1(\sqrt[3]{\tau})}{\tau - z^3}\frac{\mathrm{d}\tau}{\tau^{2/3}},\\
\Psi_{3k+1}(z) &= 3\int_0^{\alpha} \frac{t^2Q_{3k+1}(t)s_1(t)}{t^3 - z^3}\mathrm{d}t = \int_0^{\alpha^3} \frac{Q_{3k+1}(\sqrt[3]{\tau})s_1(\sqrt[3]{\tau})}{\tau - z^3}\mathrm{d}\tau,\\
\Psi_{3k+2}(z) &= 3z\int_0^{\alpha} \frac{tQ_{3k+2}(t)s_1(t)}{t^3 - z^3}\mathrm{d}t = z\int_0^{\alpha^3} \frac{Q_{3k+2}(\sqrt[3]{\tau})s_1(\sqrt[3]{\tau})}{\tau - z^3}\frac{\mathrm{d}\tau}{\tau^{1/3}}.
\end{aligned} \tag{44}$$

Proof. The definition of Ψ_n and the symmetry property (39) give directly (43). □

If we apply carefully the orthogonality conditions in Proposition 2.1 and the fact that $d_n = n$, we obtain:

$$0 = \int_0^{\alpha^3} \tau^k Q_{6l+1}(\sqrt[3]{\tau})s_1(\sqrt[3]{\tau})\mathrm{d}\tau, \quad 0 \leq k \leq l-1,$$

$$0=\int_0^{\alpha^3}\tau^k Q_{6l+3}(\sqrt[3]{\tau})s_1(\sqrt[3]{\tau})\frac{\mathrm{d}\tau}{\tau^{2/3}},\quad 0\le k\le l,\tag{45}$$
$$0=\int_0^{\alpha^3}\tau^k Q_{6l+5}(\sqrt[3]{\tau})s_1(\sqrt[3]{\tau})\frac{\mathrm{d}\tau}{\tau^{1/3}},\quad 0\le k\le l.$$

Consequently, we can improve the estimate at infinity $\Psi_{2n+1}(z)=O(1/z^{n+2})$ given in (9) to $\Psi_{2n+1}(z)=O(1/z^{n+3})$. To see this, observe that from (45) we deduce:

$$\int\frac{Q_{6l+1}(\sqrt[3]{\tau})s_1(\sqrt[3]{\tau})}{\tau-z}\mathrm{d}\tau=O\left(\frac{1}{z^{l+1}}\right),\qquad \int\frac{Q_{6l+3}(\sqrt[3]{\tau})s_1(\sqrt[3]{\tau})}{\tau-z}\frac{\mathrm{d}\tau}{\tau^{2/3}}=O\left(\frac{1}{z^{l+2}}\right),$$
$$\int\frac{Q_{6l+5}(\sqrt[3]{\tau})s_1(\sqrt[3]{\tau})}{\tau-z}\frac{\mathrm{d}\tau}{\tau^{1/3}}=O\left(\frac{1}{z^{l+2}}\right).$$

If we take into account now the representations (44) of the functions Ψ_n, the claim is justified. In conclusion, the following estimates are valid as $z\to\infty$:

$$\begin{array}{lll}\Psi_{6l}(z)=O(1/z^{3l+1}), & \Psi_{6l+2}(z)=O(1/z^{3l+2}), & \Psi_{6l+4}(z)=O(1/z^{3l+3}),\\ \Psi_{6l+1}(z)=O(1/z^{3l+3}), & \Psi_{6l+3}(z)=O(1/z^{3l+4}), & \Psi_{6l+5}(z)=O(1/z^{3l+5}).\end{array}\tag{46}$$

It is convenient to rewrite the orthogonality conditions in (31) in terms of the interval $(-b^3, -a^3)$. Applying the symmetry properties of Ψ_n (cf. (40)) and s_2, we obtain:

Proposition 3.2. *The functions Ψ_n satisfy:*

$$0=\int_{-b}^{-a}t^\nu\,\Psi_{2n}(t)\left(1+\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}(\nu-4n-1)}+\mathrm{e}^{\frac{4\pi\mathrm{i}}{3}(\nu-4n-1)}\right)s_2(t)\mathrm{d}t,\quad \nu=0,\dots,n-1,$$
$$0=\int_{-b}^{-a}t^\nu\,\Psi_{2n+1}(t)\left(1+\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}(\nu-n)}+\mathrm{e}^{\frac{4\pi\mathrm{i}}{3}(\nu-n)}\right)s_2(t)\mathrm{d}t,\quad \nu=0,\dots,n-1.$$

In particular, for any integer $l\ge 0$,

$$0=\int_{-b^3}^{-a^3}\tau^k\,\Psi_{6l+j}(\sqrt[3]{\tau})s_2(\sqrt[3]{\tau})\frac{\mathrm{d}\tau}{\tau^{1/3}},\quad 0\le k\le l-1,\ j=0,3,$$
$$0=\int_{-b^3}^{-a^3}\tau^k\,\Psi_{6l+2+j}(\sqrt[3]{\tau})s_2(\sqrt[3]{\tau})\mathrm{d}\tau,\quad 0\le k\le l-1,\ j=0,3,$$
$$0=\int_{-b^3}^{-a^3}\tau^k\,\Psi_{6l+1}(\sqrt[3]{\tau})s_2(\sqrt[3]{\tau})\frac{\mathrm{d}\tau}{\tau^{2/3}},\quad 0\le k\le l-1,$$
$$0=\int_{-b^3}^{-a^3}\tau^k\,\Psi_{6l+4}(\sqrt[3]{\tau})s_2(\sqrt[3]{\tau})\frac{\mathrm{d}\tau}{\tau^{2/3}},\quad 0\le k\le l.$$

As a consequence of Proposition 3.2, we obtain:

Corollary 3.3. *For each $j\in\{0,1,2,3,5\}$, the function Ψ_{6l+j} has at least l sign change knots in the interval $(-b,-a)$, and the function Ψ_{6l+4} has at least $l+1$ sign change knots in the interval $(-b,-a)$. Therefore the functions Ψ_{6l+j}, $j\in\{0,1,2,3,5\}$ have at least $3l$ zeros, counting multiplicities, in $\mathbb{C}\setminus S_0$, and Ψ_{6l+4} has at least $3l+3$ zeros, counting multiplicities, in $\mathbb{C}\setminus S_0$.*

Observe that the function Ψ_n satisfies the property

$$\Psi_n(\overline{z}) = -\overline{\Psi_n(z)}, \quad z \in \mathbb{C} \setminus S_0. \tag{47}$$

Let $j \in \{0, 1, 2, 3, 5\}$ and assume that $x_1, \ldots, x_l$ are l distinct zeros of Ψ_{6l+j} in $(-b, -a)$. It follows from (40) that the points

$$e^{\frac{2\pi i}{3}} x_1, \ldots, e^{\frac{2\pi i}{3}} x_l, \qquad e^{\frac{4\pi i}{3}} x_1, \ldots, e^{\frac{4\pi i}{3}} x_l,$$

are also zeros of Ψ_{6l+j}. Let

$$R_1(z) := \prod_{k=1}^{l} (z - x_k) \prod_{k=1}^{l} \left(z - e^{\frac{2\pi i}{3}} x_k\right) \prod_{k=1}^{l} \left(z - e^{\frac{4\pi i}{3}} x_k\right) = \prod_{k=1}^{l} (z^3 - x_k^3).$$

Assume further that Ψ_{6l+j} has more than $3l$ zeros in $\mathbb{C} \setminus S_0$, counting multiplicities. Then there exists a point $z_0 \in \mathbb{C} \setminus S_0$ such that the polynomial

$$R_2(z) := R_1(z)(z^3 - z_0^3)$$

satisfies $\Psi_{6l+j}/R_2 \in H(\overline{\mathbb{C}} \setminus S_0)$. If $z_0 \in \mathbb{R}$, then R_2 is a polynomial in z^3 with real coefficients. If $z_0 \notin \mathbb{R}$, then R_2 may not have real coefficients, but the polynomial

$$R_3(z) := R_1(z)(z^3 - z_0^3)(z^3 - \overline{z}_0^3)$$

is a polynomial in z^3 with real coefficients, and $\Psi_{6l+j}/R_3 \in H(\overline{\mathbb{C}} \setminus S_0)$ (here we use (47)).

In conclusion, we see that if Ψ_{6l+j}, $j \in \{0, 1, 2, 3, 5\}$, has more than $3l$ zeros in $\mathbb{C} \setminus S_0$, counting multiplicities, then we can find a polynomial R_{6l+j} with real coefficients and degree at least $3l + 3$ satisfying:

$$R_{6l+j}(z) = R_{6l+j}\left(e^{\frac{2\pi i}{3}} z\right), \quad z \in \mathbb{C}, \quad \text{and} \quad \frac{\Psi_{6l+j}}{R_{6l+j}} \in H(\overline{\mathbb{C}} \setminus S_0). \tag{48}$$

Similarly, if we assume that Ψ_{6l+4} has more than $3l + 3$ zeros in $\mathbb{C} \setminus S_0$, counting multiplicities, then there exists a polynomial R_{6l+4} with real coefficients and degree at least $3l + 6$ such that (48) holds for $j = 4$.

Proof of Proposition 1.3. Suppose that Ψ_{6l} has more than $3l$ zeros in $\mathbb{C} \setminus S_0$, counting multiplicities. Let R_{6l} be a polynomial with real coefficients and degree at least $3l + 3$ satisfying (48). By (46), $\Psi_{6l}(z)/R_{6l}(z) = O(1/z^{6l+4})$ as $z \to \infty$.

Let Γ be a Jordan curve surrounding S_0 such that the zeros of R_{6l} lie outside Γ. By Cauchy's theorem, Fubini's theorem, and Cauchy's integral formula, for $\nu = 0, \ldots, 6l + 2$,

$$\begin{aligned}
0 &= \int_{\Gamma} z^{\nu} \frac{\Psi_{6l}(z)}{R_{6l}(z)} dz \\
&= \int_{\Gamma} \frac{z^{\nu}}{R_{6l}(z)} \frac{1}{2\pi i} \int_0^{\alpha} \left(\frac{1}{t - z} + \frac{1}{e^{\frac{2\pi i}{3}} t - z} + \frac{1}{e^{\frac{4\pi i}{3}} t - z} \right) Q_{6l}(t) s_1(t) dt dz \\
&= \int_0^{\alpha} t^{\nu} \left[\frac{1}{R_{6l}(t)} + \frac{e^{2\pi i \nu/3}}{R_{6l}\left(e^{\frac{2\pi i}{3}} t\right)} + \frac{e^{4\pi i \nu/3}}{R_{6l}\left(e^{\frac{4\pi i}{3}} t\right)} \right] Q_{6l}(t) s_1(t) dt,
\end{aligned}$$

and applying (48), we obtain

$$0=\int_0^\alpha t^{3k}Q_{6l}(t)\frac{s_1(t)}{R_{6l}(t)}\mathrm{d}t,\quad 0\le k\le 2l.$$

Consequently, Q_{6l} has at least $2l+1$ sign change knots in $(0,\alpha)$, contradicting Proposition 1.1. This and Corollary 3.3 prove the claim for $n=6l$. In the remaining cases we use the same argument. Indeed, if Ψ_{6l+j}, $j\in\{1,2,3,5\}$, has more than $3l$ zeros in $\mathbb{C}\setminus S_0$ and Ψ_{6l+4} has more than $3l+3$ zeros in $\mathbb{C}\setminus S_0$, counting multiplicities, then we know (see the discussion after Corollary 3.3) that we can select polynomials R_{6l+j}, $1\le j\le 5$ satisfying (48) such that, as $z\to\infty$:

$$\frac{\Psi_{6l+1}(z)}{R_{6l+1}(z)}=O\left(\frac{1}{z^{6l+6}}\right),\qquad \frac{\Psi_{6l+2}(z)}{R_{6l+2}(z)}=O\left(\frac{1}{z^{6l+5}}\right),\qquad \frac{\Psi_{6l+3}(z)}{R_{6l+3}(z)}=O\left(\frac{1}{z^{6l+7}}\right),$$
$$\frac{\Psi_{6l+4}(z)}{R_{6l+4}(z)}=O\left(\frac{1}{z^{6l+9}}\right),\qquad \frac{\Psi_{6l+5}(z)}{R_{6l+5}(z)}=O\left(\frac{1}{z^{6l+8}}\right).$$

These estimates lead to the orthogonality conditions

$$\begin{aligned}0&=\int_0^\alpha t^{3k+2}Q_{6l+1}(t)\frac{s_1(t)}{R_{6l+1}(t)}\mathrm{d}t=\int_0^\alpha t^{3k+1}Q_{6l+2}(t)\frac{s_1(t)}{R_{6l+2}(t)}\mathrm{d}t,\quad 0\le k\le 2l,\\ 0&=\int_0^\alpha t^{3k}Q_{6l+3}(t)\frac{s_1(t)}{R_{6l+3}(t)}\mathrm{d}t=\int_0^\alpha t^{3k+2}Q_{6l+4}(t)\frac{s_1(t)}{R_{6l+4}(t)}\mathrm{d}t\\ &=\int_0^\alpha t^{3k+1}Q_{6l+5}(t)\frac{s_1(t)}{R_{6l+5}(t)}\mathrm{d}t,\quad 0\le k\le 2l+1,\end{aligned}$$

which contradict the number of zeros that the polynomials Q_{6l+j}, $1\le j\le 5$, have on $(0,\alpha)$ (see Proposition 1.1). □

Recall that $Q_{n,2}$ is defined as the monic polynomial whose zeros coincide with the finite zeros of Ψ_n outside S_0. The argument shown above proves the following:

Proposition 3.4. *For each $j\in\{0,1,2,3,5\}$, $\deg(Q_{6l+j,2})=3l$, and $\deg(Q_{6l+4,2})=3l+3$. Furthermore,*

$$0=\int_0^\alpha t^{3k}Q_{3l}(t)\frac{s_1(t)}{Q_{3l,2}(t)}\mathrm{d}t,\quad 0\le k\le l-1,\tag{49}$$

$$0=\int_0^\alpha t^{3k+2}Q_{3l+1}(t)\frac{s_1(t)}{Q_{3l+1,2}(t)}\mathrm{d}t,\quad 0\le k\le l-1,\tag{50}$$

$$0=\int_0^\alpha t^{3k+1}Q_{3l+2}(t)\frac{s_1(t)}{Q_{3l+2,2}(t)}\mathrm{d}t,\quad 0\le k\le l-1.\tag{51}$$

Proposition 3.5. *The following formulas are valid for $z\in\mathbb{C}\setminus S_0$. If q is a polynomial of degree at most $3k$, then*

$$\frac{q(z)\,\Psi_{3k}(z)}{Q_{3k,2}(z)}=\int_0^\alpha\frac{Q_{3k}(x)s_1(x)}{Q_{3k,2}(x)}\left(\frac{q(x)}{x-z}+\frac{q\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}x\right)}{\mathrm{e}^{\frac{2\pi\mathrm{i}}{3}}x-z}+\frac{q\left(\mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}x\right)}{\mathrm{e}^{\frac{4\pi\mathrm{i}}{3}}x-z}\right)\mathrm{d}x.\tag{52}$$

If $\deg q \leq 3k+2$, *then*

$$\frac{q(z)\,\Psi_{3k+1}(z)}{Q_{3k+1,2}(z)} = \int_0^\alpha \frac{Q_{3k+1}(x)s_1(x)}{Q_{3k+1,2}(x)} \left(\frac{q(x)}{x-z} + \frac{e^{\frac{2\pi i}{3}} q\left(e^{\frac{2\pi i}{3}} x\right)}{e^{\frac{2\pi i}{3}} x - z} + \frac{e^{\frac{4\pi i}{3}} q\left(e^{\frac{4\pi i}{3}} x\right)}{e^{\frac{4\pi i}{3}} x - z} \right) dx. \tag{53}$$

If $\deg q \leq 3k+1$, *then*

$$\frac{q(z)\,\Psi_{3k+2}(z)}{Q_{3k+2,2}(z)} = \int_0^\alpha \frac{Q_{3k+2}(x)s_1(x)}{Q_{3k+2,2}(x)} \left(\frac{q(x)}{x-z} + \frac{e^{\frac{4\pi i}{3}} q\left(e^{\frac{2\pi i}{3}} x\right)}{e^{\frac{2\pi i}{3}} x - z} + \frac{e^{\frac{2\pi i}{3}} q\left(e^{\frac{4\pi i}{3}} x\right)}{e^{\frac{4\pi i}{3}} x - z} \right) dx. \tag{54}$$

In particular, we have

$$\begin{aligned} \frac{Q_{3k}(z)\,\Psi_{3k}(z)}{Q_{3k,2}(z)} &= 3z^2 \int_0^\alpha \frac{Q_{3k}^2(x)}{Q_{3k,2}(x)} \frac{s_1(x)}{x^3 - z^3} dx, \\ \frac{Q_{3k+1}(z)\,\Psi_{3k+1}(z)}{Q_{3k+1,2}(z)} &= 3z \int_0^\alpha \frac{Q_{3k+1}^2(x)}{Q_{3k+1,2}(x)} \frac{x s_1(x)}{x^3 - z^3} dx, \\ \frac{Q_{3k+2}(z)\,\Psi_{3k+2}(z)}{Q_{3k+2,2}(z)} &= 3z^3 \int_0^\alpha \frac{Q_{3k+2}^2(x)}{Q_{3k+2,2}(x)} \frac{s_1(x)}{x(x^3 - z^3)} dx. \end{aligned} \tag{55}$$

Proof. By (46) and Proposition 3.4, we know that if q is a polynomial of degree at most $3k$, then

$$\frac{q(z)\,\Psi_{3k}(z)}{Q_{3k,2}(z)} = O(1/z), \quad z \to \infty. \tag{56}$$

For $z \in \mathbb{C} \setminus S_0$, let Γ be a Jordan curve surrounding S_0 and oriented clockwise, so that z and the zeros of $Q_{3k,2}$ lie outside Γ. From (56) and (43) it follows that

$$\begin{aligned} \frac{q(z)\,\Psi_{3k}(z)}{Q_{3k,2}(z)} &= \frac{1}{2\pi i} \int_\Gamma \frac{q(t)\,\Psi_{3k}(t)}{Q_{3k,2}(t)} \frac{dt}{t-z} \\ &= \int_0^\alpha Q_{3k}(x)s_1(x) \frac{1}{2\pi i} \int_\Gamma \frac{q(t)}{Q_{3k,2}(t)(t-z)} \\ &\quad \times \left[\frac{1}{x-t} + \frac{1}{e^{\frac{2\pi i}{3}} x - t} + \frac{1}{e^{\frac{4\pi i}{3}} x - t} \right] dt\,dx \\ &= \int_0^\alpha \frac{Q_{3k}(x)s_1(x)}{Q_{3k,2}(x)} \left(\frac{q(x)}{x-z} + \frac{q(e^{\frac{2\pi i}{3}} x)}{e^{\frac{2\pi i}{3}} x - z} + \frac{q(e^{\frac{4\pi i}{3}} x)}{e^{\frac{4\pi i}{3}} x - z} \right) dx, \end{aligned}$$

where in the last equality we used that $Q_{3k,2}(t) = Q_{3k,2}\left(e^{\frac{2\pi i}{3}} t\right) = Q_{3k,2}\left(e^{\frac{4\pi i}{3}} t\right)$. This proves (52). The proofs of (53)–(54) are identical. To obtain the first and second formulas in (55), we replace q in formulas (52) and (53) by Q_{3k} and Q_{3k+1}, respectively. The third formula in (55) follows from (54) by taking $q(z) = Q_{3k+2}(z)/z$. □

Proposition 3.6. *The recurrence coefficients* $\{a_n\}_{n\geq 2}^\infty$ *that appear in* (6) *are all positive.*

Proof. To prove that a_{2n} is positive it suffices to show that $\int_0^\alpha t^n Q_{2n}(t)s_1(t)\mathrm{d}t > 0$ for all $n \geq 0$. Let $n = 3l$. Since $\deg(t^{3l} Q_{6l,2}) = 6l$, by (49) we obtain

$$\int_0^\alpha t^{3l} Q_{6l}(t)s_1(t)\mathrm{d}t = \int_0^\alpha t^{3l} Q_{6l}(t)Q_{6l,2}(t)\frac{s_1(t)}{Q_{6l,2}(t)}\mathrm{d}t = \int_0^\alpha Q_{6l}^2(t)\frac{s_1(t)}{Q_{6l,2}(t)}\mathrm{d}t > 0.$$

For $n = 3l + 1$, using (51) and $\deg(t^{3l+2} Q_{6l+2,2}) = 6l + 2$, we get

$$\begin{aligned}\int_0^\alpha t^{3l+1} Q_{6l+2}(t)s_1(t)\mathrm{d}t &= \int_0^\alpha t^{3l+2} Q_{6l+2,2}(t)Q_{6l+2}(t)\frac{s_1(t)}{tQ_{6l+2,2}(t)}\mathrm{d}t\\ &= \int_0^\alpha Q_{6l+2}^2(t)\frac{s_1(t)}{tQ_{6l+2,2}(t)}\mathrm{d}t > 0.\end{aligned}$$

Finally, for $n = 3l + 2$, applying (50) and $\deg(t^{3l+1} Q_{6l+4,2}) = 6l + 4$, we obtain

$$\begin{aligned}\int_0^\alpha t^{3l+2} Q_{6l+4}(t)s_1(t)\mathrm{d}t &= \int_0^\alpha t^{3l+1} Q_{6l+4,2}(t)Q_{6l+4}(t)\frac{ts_1(t)}{Q_{6l+4,2}(t)}\mathrm{d}t\\ &= \int_0^\alpha Q_{6l+4}^2(t)\frac{ts_1(t)}{Q_{6l+4,2}(t)}\mathrm{d}t > 0.\end{aligned}$$

It is easy to see that the functions Ψ_n satisfy the same recurrence relation (6). In particular,

$$t\,\Psi_{2n+1}(t) = \Psi_{2n+2}(t) + a_{2n+1}\,\Psi_{2n-1}(t).$$

Using Proposition 3.2, if we multiply the above relation by an appropriate power of t and integrate, we obtain

$$\begin{aligned}\int_{-b}^{-a} t^{3l}\,\Psi_{6l+1}(t)s_2(t)\mathrm{d}t &= a_{6l+1}\int_{-b}^{-a} t^{3l-1}\,\Psi_{6l-1}(t)s_2(t)\mathrm{d}t,\\ \int_{-b}^{-a} t^{3l+1}\,\Psi_{6l+3}(t)s_2(t)\mathrm{d}t &= a_{6l+3}\int_{-b}^{-a} t^{3l}\,\Psi_{6l+1}(t)s_2(t)\mathrm{d}t,\\ \int_{-b}^{-a} t^{3l+2}\,\Psi_{6l+5}(t)s_2(t)\mathrm{d}t &= a_{6l+5}\int_{-b}^{-a} t^{3l+1}\,\Psi_{6l+3}(t)s_2(t)\mathrm{d}t.\end{aligned}$$

On the other hand, it is easy to deduce from (55) that if $t < 0$, then

$$\begin{aligned}&\operatorname{sign}\left(\frac{\Psi_{3k}(t)}{Q_{3k,2}(t)}\right) = (-1)^{3k}, \qquad \operatorname{sign}\left(\frac{\Psi_{3k+1}(t)}{Q_{3k+1,2}(t)}\right) = (-1)^{3k},\\ &\operatorname{sign}\left(\frac{\Psi_{3k+2}(t)}{Q_{3k+2,2}(t)}\right) = (-1)^{3k+1}.\end{aligned} \tag{57}$$

Observe that since $\deg Q_{6l-1,2} = 3l - 3$ and $\deg Q_{6l+1,2} = \deg Q_{6l+3,2} = 3l$, by the orthogonality conditions satisfied by Ψ_{2n+1} and (57), we obtain:

$$\begin{aligned}\int_{-b}^{-a} t^{3l-1}\,\Psi_{6l-1}(t)s_2(t)\mathrm{d}t &= \int_{-b}^{-a} Q_{6l-1,2}(t)\,\Psi_{6l-1}(t)t^2 s_2(t)\mathrm{d}t\\ &= \int_{-b}^{-a} Q_{6l-1,2}^2(t)\frac{\Psi_{6l-1}(t)}{Q_{6l-1,2}(t)}t^2 s_2(t)\mathrm{d}t > 0,\\ \int_{-b}^{-a} t^{3l}\,\Psi_{6l+1}(t)s_2(t)\mathrm{d}t &= \int_{-b}^{-a} Q_{6l+1,2}(t)\,\Psi_{6l+1}(t)s_2(t)\mathrm{d}t\\ &= \int_{-b}^{-a} Q_{6l+1,2}^2(t)\frac{\Psi_{6l+1}(t)}{Q_{6l+1,2}(t)}s_2(t)\mathrm{d}t > 0,\end{aligned}$$

$$\begin{aligned}\int_{-b}^{-a} t^{3l+1}\,\Psi_{6l+3}(t)s_2(t)\mathrm{d}t &= \int_{-b}^{-a} Q_{6l+3,2}(t)\,\Psi_{6l+3}(t)ts_2(t)\mathrm{d}t\\ &= \int_{-b}^{-a} Q_{6l+3,2}^2(t)\frac{\Psi_{6l+3}(t)}{Q_{6l+3,2}(t)}ts_2(t)\mathrm{d}t > 0.\end{aligned}$$

This shows that $a_{2n+1} > 0$ for all $n \geq 1$. □

4. Interlacing properties of the zeros of Q_n and Ψ_n

Proposition 4.1. *Let $A, B \in \mathbb{R}$ be two constants such that $|A| + |B| > 0$, and let*

$$Y_n(z) := Az\,\Psi_n(z) + B\,\Psi_{n+1}(z), \tag{58}$$

$$T_n(z) := AzQ_n(z) + BQ_{n+1}(z). \tag{59}$$

Then, for every $n \geq 0$, the function Y_n has only simple zeros on $(-\infty, 0)$. Similarly, for every $n \geq 0$, the polynomial T_n has only simple zeros on $(0, \alpha)$.

Proof. From Proposition 3.2 it follows that

$$0 = \int_{-b^3}^{-a^3} \tau^k Y_{6l+1}(\sqrt[3]{\tau})s_2(\sqrt[3]{\tau})\mathrm{d}\tau, \quad 0 \leq k \leq l-2,$$

$$0 = \int_{-b^3}^{-a^3} \tau^k Y_{6l+4}(\sqrt[3]{\tau})s_2(\sqrt[3]{\tau})\mathrm{d}\tau, \quad 0 \leq k \leq l-1,$$

$$0 = \int_{-b^3}^{-a^3} \tau^k Y_{6l+j}(\sqrt[3]{\tau})s_2(\sqrt[3]{\tau})\frac{\mathrm{d}\tau}{\tau^{2/3}}, \quad 0 \leq k \leq l-1,\ j = 0, 3,$$

$$0 = \int_{-b^3}^{-a^3} \tau^k Y_{6l+2+j}(\sqrt[3]{\tau})s_2(\sqrt[3]{\tau})\frac{\mathrm{d}\tau}{\tau^{1/3}}, \quad 0 \leq k \leq l-1,\ j = 0, 3.$$

Consequently, for each $j \in \{0, 2, 3, 4, 5\}$, the function Y_{6l+j} has at least l sign change knots in $(-b, -a)$, and Y_{6l+1} has at least $l-1$ sign change knots in $(-b, -a)$. From (40) it follows that for every n, $Y_n\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}z\right) = C_nY_n(z)$, where C_n denotes a constant. Therefore, the functions Y_{6l+j}, $j \in \{0, 2, 3, 4, 5\}$ have at least $3l$ zeros on S_1, and Y_{6l+1} has at least $3l-3$ zeros on S_1. For each $0 \leq j \leq 5$, let R_{6l+j} denote the monic polynomial whose zeros coincide with the zeros of Y_{6l+j} on $\cup_{k=0}^{2}\exp(2\pi \mathrm{i}k/3)(-\infty, 0] \setminus \{0\}$. Then R_{6l+j} satisfies (48), $Y_{6l+j}/R_{6l+j} \in H(\overline{\mathbb{C}} \setminus S_0)$, and using (46) we deduce that as $z \to \infty$:

$$\frac{Y_{6l}(z)}{R_{6l}(z)} = O\left(\frac{1}{z^{6l}}\right), \qquad \frac{Y_{6l+1}(z)}{R_{6l+1}(z)} = O\left(\frac{1}{z^{6l-1}}\right), \qquad \frac{Y_{6l+2}(z)}{R_{6l+2}(z)} = O\left(\frac{1}{z^{6l+1}}\right),$$
$$\frac{Y_{6l+3}(z)}{R_{6l+3}(z)} = O\left(\frac{1}{z^{6l+3}}\right), \qquad \frac{Y_{6l+4}(z)}{R_{6l+4}(z)} = O\left(\frac{1}{z^{6l+2}}\right), \qquad \frac{Y_{6l+5}(z)}{R_{6l+5}(z)} = O\left(\frac{1}{z^{6l+4}}\right).$$

Let Γ again denote a Jordan curve surrounding S_0, such that the zeros of the polynomials R_{6l+j} lie outside Γ. By (43),

$$\begin{aligned}0 &= \int_{\Gamma} z^{\nu}\frac{Y_{6l}(z)}{R_{6l}(z)}\mathrm{d}z\\ &= \int_0^{\alpha} x^{\nu}T_{6l}(x)(1 + \mathrm{e}^{2\pi\mathrm{i}(\nu+1)/3} + \mathrm{e}^{4\pi\mathrm{i}(\nu+1)/3})\frac{s_1(x)}{R_{6l}(x)}\mathrm{d}x, \quad \nu = 0, \ldots, 6l-2,\end{aligned}$$

which is equivalent to

$$0=\int_0^\alpha x^{3k+2}T_{6l}(x)\frac{s_1(x)}{R_{6l}(x)}\mathrm{d}x, \quad 0\le k\le 2l-2. \tag{60}$$

Similarly we obtain:

$$\begin{aligned}
0&=\int_0^\alpha x^{3k+1}T_{6l+1}(x)\frac{s_1(x)}{R_{6l+1}(x)}\mathrm{d}x, \quad 0\le k\le 2l-2,\\
0&=\int_0^\alpha x^{3k}T_{6l+5}(x)\frac{s_1(x)}{R_{6l+5}(x)}\mathrm{d}x, \quad 0\le k\le 2l,\\
0&=\int_0^\alpha x^{3k}T_{6l+2}(x)\frac{s_1(x)}{R_{6l+2}(x)}\mathrm{d}x=\int_0^\alpha x^{3k+2}T_{6l+3}(x)\frac{s_1(x)}{R_{6l+3}(x)}\mathrm{d}x\\
&=\int_0^\alpha x^{3k+1}T_{6l+4}(x)\frac{s_1(x)}{R_{6l+4}(x)}\mathrm{d}x, \quad 0\le k\le 2l-1.
\end{aligned} \tag{61}$$

From (60) it follows that T_{6l} has at least $2l-1$ sign change knots in $(0,\alpha)$. Since $T_{6l}\left(z e^{\frac{2\pi i}{3}}\right)=e^{\frac{2\pi i}{3}}T_{6l}(z)$, we see that any zero of T_{6l} in $(0,\infty)$ must be simple, otherwise T_{6l} would have at least $6l+3$ zeros, contradicting $\deg(T_{6l})\le 6l+1$. Similarly, using (61) we show that the polynomials T_{6l+j}, $1\le j\le 5$, have only simple zeros in $(0,\infty)$.

Now we prove that the functions Y_n have only simple zeros in $(-\infty,0)$. We know that Y_{6l} has at least l sign change knots in $(-\infty,0)$. If we assume that Y_{6l} has a zero of multiplicity ≥ 2, then $\deg R_{6l}\ge 3l+6$, and so we would have

$$Y_{6l}(z)/R_{6l}(z)=O(1/z^{6l+6}), \quad z\to\infty.$$

Reasoning as above, we arrive at the fact that $\deg T_{6l}\ge 6l+3$, which is impossible. Similarly we see that the zeros of Y_{6l+j}, $1\le j\le 5$, contained in $(-\infty,0)$, must be simple. □

Proof of Theorem 1.4. Let $x\in(0,\alpha)$ and assume that $Q_n(x)=Q_{n+1}(x)=0$. Take $A=1$, $B=-xQ_n'(x)/Q_{n+1}'(x)$. For this choice of A and B, the polynomial T_n defined by (59) satisfies $T_n(x)=T_n'(x)=0$, contradicting Proposition 4.1.

Let $x\in(0,\alpha)$ be arbitrary but fixed. Take now $A=Q_{n+1}(x)/x$ and $B=-Q_n(x)$. For this choice of A and B, we have $T_n(x)=0$, therefore $T_n'(x)\ne 0$, or equivalently

$$L_n(x):=\frac{Q_{n+1}(x)Q_n(x)}{x}+Q_{n+1}(x)Q_n'(x)-Q_n(x)Q_{n+1}'(x)\ne 0.$$

In particular, the sign of L_n is constant on $(0,\alpha)$. Evaluating L_n at two consecutive zeros of Q_n (Q_{n+1}) on $(0,\alpha)$, we see immediately that there must be an intermediate zero of Q_{n+1} (Q_n).

The same argument proves the interlacing property of the zeros of Ψ_n and Ψ_{n+1}.qed

Proposition 4.2. *Let the roots of the polynomials* Q_{3k+i}, $0\le i\le 2$, *in the interval* $(0,\alpha)$, *be defined as follows:*

$$x_1^{(3k+i)}<x_2^{(3k+i)}<x_3^{(3k+i)}<\cdots<x_{k-1}^{(3k+i)}<x_k^{(3k+i)}.$$

Then

$$x_1^{(3k)}<x_1^{(3k+1)}<x_2^{(3k)}<x_2^{(3k+1)}<\cdots<x_k^{(3k)}<x_k^{(3k+1)}, \tag{62}$$

$$x_1^{(3k+1)}<x_1^{(3k+2)}<x_2^{(3k+1)}<x_2^{(3k+2)}<\cdots<x_k^{(3k+1)}<x_k^{(3k+2)}, \tag{63}$$

$$x_1^{(3k+3)} < x_1^{(3k+2)} < x_2^{(3k+3)} < x_2^{(3k+2)} < \cdots < x_k^{(3k+2)} < x_{k+1}^{(3k+3)}. \tag{64}$$

Proof. If we write

$$Q_{3k-2}(z) = b_1^{(3k-2)} z + \cdots + z^{3k-2}, \qquad Q_{3k}(z) = b_0^{(3k)} + \cdots + z^{3k},$$
$$Q_{3k+1}(z) = b_1^{(3k+1)} z + \cdots + z^{3k+1},$$

from (6) we obtain the relation $b_0^{(3k)} - b_1^{(3k+1)} = a_{3k} b_1^{(3k-2)}$. Vieta formulas show that

$$b_0^{(3k)} = (-1)^{3k} (x_1^{(3k)} \cdots x_k^{(3k)})^3, \qquad b_1^{(3k+1)} = (-1)^{3k} (x_1^{(3k+1)} \cdots x_k^{(3k+1)})^3,$$

and similarly $b_1^{(3k-2)}$ equals $(-1)^{3k-1}$ times the product of all nonzero roots of Q_{3k-2}. Since $a_{3k} > 0$ and the product of all nonzero roots of Q_{3k-2} is also positive, we deduce that $(x_1^{(3k)} \cdots x_k^{(3k)})^3 < (x_1^{(3k+1)} \cdots x_k^{(3k+1)})^3$. This inequality and Theorem 1.4 imply (62). Similarly we show that $(x_1^{(3k+1)} \cdots x_k^{(3k+1)})^3 < (x_1^{(3k+2)} \cdots x_k^{(3k+2)})^3$, which implies (63). Finally, (64) follows directly from Theorem 1.4. □

5. Ratio asymptotics of the polynomials Q_n and $Q_{n,2}$

Let

$$H_n := \frac{Q_n \Psi_n}{Q_{n,2}}. \tag{65}$$

Notice that H_n is real valued on $(-\infty, 0)$ and has constant sign on this interval. Having in mind the definitions (10)–(11), we have:

Proposition 5.1. *Let $l \geq 0$ be an arbitrary integer. Then the following orthogonality conditions hold:*

$$0 = \int_{-b^3}^{-a^3} \tau^k P_{6l+j,2}(\tau) \frac{|H_{6l+j}(\sqrt[3]{\tau})| s_2(\sqrt[3]{\tau})}{|\sqrt[3]{\tau} P_{6l+j}(\tau)|} d\tau, \quad 0 \leq k \leq l-1, \ j = 0, 3,$$
$$0 = \int_{-b^3}^{-a^3} \tau^k P_{6l+2+j,2}(\tau) \frac{|H_{6l+2+j}(\sqrt[3]{\tau})| s_2(\sqrt[3]{\tau})}{|\tau^{2/3} P_{6l+2+j}(\sqrt[3]{\tau})|} d\tau, \quad 0 \leq k \leq l-1, \ j = 0, 3,$$
$$0 = \int_{-b^3}^{-a^3} \tau^k P_{6l+1,2}(\tau) \frac{|H_{6l+1}(\sqrt[3]{\tau})| s_2(\sqrt[3]{\tau})}{|\tau P_{6l+1}(\tau)|} d\tau, \quad 0 \leq k \leq l-1,$$
$$0 = \int_{-b^3}^{-a^3} \tau^k P_{6l+4,2}(\tau) \frac{|H_{6l+4}(\sqrt[3]{\tau})| s_2(\sqrt[3]{\tau})}{|\tau P_{6l+4}(\tau)|} d\tau, \quad 0 \leq k \leq l.$$

Proof. These orthogonality conditions follow immediately from Proposition 3.2. □

Proposition 5.2. *Let $k \geq 0$ be an arbitrary integer. Then the following orthogonality conditions hold:*

$$0 = \int_0^{\alpha^3} \tau^j P_{3k}(\tau) \frac{s_1(\sqrt[3]{\tau})}{P_{3k,2}(\tau)} \frac{d\tau}{\tau^{2/3}}, \quad 0 \leq j \leq k-1.$$
$$0 = \int_0^{\alpha^3} \tau^j P_{3k+1}(\tau) \frac{s_1(\sqrt[3]{\tau})}{P_{3k+1,2}(\tau)} \sqrt[3]{\tau} d\tau, \quad 0 \leq j \leq k-1.$$

$$0 = \int_0^{\alpha^3} \tau^j P_{3k+2}(\tau) \frac{s_1(\sqrt[3]{\tau})}{P_{3k+2,2}(\tau)} \sqrt[3]{\tau} \mathrm{d}\tau, \quad 0 \le j \le k-1.$$

Proof. These orthogonality conditions follow immediately from (49)–(51). ☐

Observe that by Proposition 1.3, for each $j \in \{0, 1, 2, 3, 5\}$, $P_{6l+j,2}$ is a polynomial of degree l, and $P_{6l+4,2}$ has degree $l+1$. By Proposition 1.1, for each $k \ge 0$ and $j \in \{0, 1, 2\}$, P_{3k+j} has degree k.

For each integer $j \ge 0$ we let

$$K_{3j} := \left(\int_0^{\alpha^3} P_{3j}^2(\tau) \frac{s_1(\sqrt[3]{\tau})}{P_{3j,2}(\tau)} \frac{\mathrm{d}\tau}{\tau^{2/3}} \right)^{-1/2},$$

$$K_{3j+1} := \left(\int_0^{\alpha^3} P_{3j+1}^2(\tau) \frac{s_1(\sqrt[3]{\tau})\sqrt[3]{\tau}}{P_{3j+1,2}(\tau)} \mathrm{d}\tau \right)^{-1/2},$$

$$K_{3j+2} := \left(\int_0^{\alpha^3} P_{3j+2}^2(\tau) \frac{s_1(\sqrt[3]{\tau})\sqrt[3]{\tau}}{P_{3j+2,2}(\tau)} \mathrm{d}\tau \right)^{-1/2}.$$

Similarly, we define for each integer $j \ge 0$ the following constants:

$$K_{3j,2} := \left(\int_{-b^3}^{-a^3} P_{3j,2}^2(\tau) \frac{|H_{3j}(\sqrt[3]{\tau})|}{|\sqrt[3]{\tau} P_{3j}(\tau)|} s_2(\sqrt[3]{\tau}) \mathrm{d}\tau \right)^{-1/2},$$

$$K_{3j+1,2} := \left(\int_{-b^3}^{-a^3} P_{3j+1,2}^2(\tau) \frac{|H_{3j+1}(\sqrt[3]{\tau})|}{|\tau P_{3j+1}(\tau)|} s_2(\sqrt[3]{\tau}) \mathrm{d}\tau \right)^{-1/2},$$

$$K_{3j+2,2} := \left(\int_{-b^3}^{-a^3} P_{3j+2,2}^2(\tau) \frac{|H_{3j+2}(\sqrt[3]{\tau})|}{|\tau^{2/3} P_{3j+2}(\tau)|} s_2(\sqrt[3]{\tau}) \mathrm{d}\tau \right)^{-1/2}.$$

We need to introduce more notations. Let

$$\kappa_n := K_n, \qquad \kappa_{n,2} := \frac{K_{n,2}}{K_n}, \tag{66}$$

consider the polynomials

$$p_n := \kappa_n P_n, \qquad p_{n,2} := \kappa_{n,2} P_{n,2}, \tag{67}$$

and the functions

$$h_n := K_n^2 H_n. \tag{68}$$

Finally, we introduce the following positive varying measures:

$$\begin{aligned}
&\mathrm{d}\nu_{3j}(\tau) := \frac{s_1(\sqrt[3]{\tau})}{P_{3j,2}(\tau)} \frac{\mathrm{d}\tau}{\tau^{2/3}}, \qquad \mathrm{d}\nu_{3j+1}(\tau) := \frac{s_1(\sqrt[3]{\tau})\sqrt[3]{\tau}}{P_{3j+1,2}(\tau)} \mathrm{d}\tau, \\
&\mathrm{d}\nu_{3j+2}(\tau) := \frac{s_1(\sqrt[3]{\tau})\sqrt[3]{\tau}}{P_{3j+2,2}(\tau)} \mathrm{d}\tau, \qquad \mathrm{d}\nu_{3j,2}(\tau) := \frac{|h_{3j}(\sqrt[3]{\tau})|}{|\sqrt[3]{\tau} P_{3j}(\tau)|} s_2(\sqrt[3]{\tau}) \mathrm{d}\tau, \\
&\mathrm{d}\nu_{3j+1,2}(\tau) := \frac{|h_{3j+1}(\sqrt[3]{\tau})|}{|\tau P_{3j+1}(\tau)|} s_2(\sqrt[3]{\tau}) \mathrm{d}\tau, \qquad \mathrm{d}\nu_{3j+2,2}(\tau) := \frac{|h_{3j+2}(\sqrt[3]{\tau})|}{|\tau^{2/3} P_{3j+2}(\tau)|} s_2(\sqrt[3]{\tau}) \mathrm{d}\tau.
\end{aligned} \tag{69}$$

Proposition 5.3. *The polynomials p_n and $p_{n,2}$ are orthonormal polynomials with respect to the measures $\mathrm{d}\nu_n$ and $\mathrm{d}\nu_{n,2}$, respectively. This is, for every $n \geq 0$, $\|p_n\|_{L^2(\mathrm{d}\nu_n)} = \|p_{n,2}\|_{L^2(\mathrm{d}\nu_{n,2})} = 1$, and*

$$\int_0^{\alpha^3} \tau^j p_n(\tau) \mathrm{d}\nu_n(\tau) = 0, \quad \textit{for all } j < \deg p_n,$$

$$\int_{-b^3}^{-a^3} \tau^j p_{n,2}(\tau) \mathrm{d}\nu_{n,2}(\tau) = 0, \quad \textit{for all } j < \deg p_{n,2}.$$

Proof. It follows immediately from Propositions 5.1 and 5.2. □

Using (55), it is easy to check that the functions h_n have the following representations:

$$h_{3k}(z) = z^2 \int_0^{\alpha^3} \frac{p_{3k}^2(\tau)}{\tau - z^3} \mathrm{d}\nu_{3k}(\tau), \qquad h_{3k+1}(z) = z \int_0^{\alpha^3} \frac{p_{3k+1}^2(\tau)}{\tau - z^3} \mathrm{d}\nu_{3k+1}(\tau),$$

$$h_{3k+2}(z) = z^3 \int_0^{\alpha^3} \frac{p_{3k+2}^2(\tau)}{\tau - z^3} \mathrm{d}\nu_{3k+2}(\tau). \tag{70}$$

Lemma 5.4. *Assume that $s_1 > 0$ a.e. on $[0, \alpha]$, and $s_2 > 0$ a.e. on $[-b, -a]$. Then*

$$p_n^2(\tau) \mathrm{d}\nu_n(\tau) \xrightarrow{*} \frac{1}{\pi} \frac{\mathrm{d}\tau}{\sqrt{(\alpha^3 - \tau)\tau}}, \quad \tau \in [0, \alpha^3], \tag{71}$$

$$p_{n,2}^2(\tau) \mathrm{d}\nu_{n,2}(\tau) \xrightarrow{*} \frac{1}{\pi} \frac{\mathrm{d}\tau}{\sqrt{(-a^3 - \tau)(\tau + b^3)}}, \quad \tau \in [-b^3, -a^3]. \tag{72}$$

Consequently, the following limits hold uniformly on closed subsets of $\overline{\mathbb{C}} \setminus S_0$:

$$\begin{aligned} \lim_{k\to\infty} h_{3k}(z) &= -\frac{z^2}{\sqrt{(z^3 - \alpha^3)z^3}}, \\ \lim_{k\to\infty} h_{3k+1}(z) &= -\frac{z}{\sqrt{(z^3 - \alpha^3)z^3}}, \\ \lim_{k\to\infty} h_{3k+2}(z) &= -\frac{z^3}{\sqrt{(z^3 - \alpha^3)z^3}}, \end{aligned} \tag{73}$$

where the branch of the square root is taken such that $\sqrt{x} > 0$ for $x > 0$.

Proof. Let us define the measures

$$\mathrm{d}\mu_{3k}(\tau) = \frac{s_1(\sqrt[3]{\tau})}{\tau^{2/3}} \mathrm{d}\tau, \qquad \mathrm{d}\mu_{3k+1}(\tau) = \mathrm{d}\mu_{3k+2}(\tau) = s_1(\sqrt[3]{\tau})\sqrt[3]{\tau} \mathrm{d}\tau.$$

According to [5, Definition 2], for each $i \in \{0, 1, 2\}$ and $k \in \mathbb{Z}$, the system $(\{\mathrm{d}\mu_{3l+i}\}, \{P_{3l+i,2}\}, k)_{l\geq 1}$ is strongly admissible on $[0, \alpha^3]$. So by [5, Corollary 3],

$$\lim_{l\to\infty} \int_0^{\alpha^3} f(\tau) p_{3l+i}^2(\tau) \frac{\mathrm{d}\mu_{3l+i}(\tau)}{P_{3l+i,2}(\tau)} = \frac{1}{\pi} \int_0^{\alpha^3} f(\tau) \frac{\mathrm{d}\tau}{\sqrt{(\alpha^3 - \tau)\tau}},$$

for every f continuous on $[0, \alpha^3]$. Since $\mathrm{d}\nu_{3l+i}(\tau) = \mathrm{d}\mu_{3l+i}(\tau)/P_{3l+i,2}(\tau)$, (71) follows. The formulas (73) are a consequence of (71) and (70).

Similarly, if we define the measures

$$\mathrm{d}\lambda_{3k}(\tau) = \frac{|h_{3k}(\sqrt[3]{\tau})|}{|\sqrt[3]{\tau}|} s_2(\sqrt[3]{\tau})\mathrm{d}\tau, \qquad \mathrm{d}\lambda_{3k+1}(\tau) = \frac{|h_{3k+1}(\sqrt[3]{\tau})|}{|\tau|} s_2(\sqrt[3]{\tau})\mathrm{d}\tau,$$
$$\mathrm{d}\lambda_{3k+2}(\tau) = \frac{|h_{3k+2}(\sqrt[3]{\tau})|}{|\tau^{2/3}|} s_2(\sqrt[3]{\tau})\mathrm{d}\tau,$$

then for each $i \in \{0, 1, 2\}$ and each $k \in \mathbb{Z}$, the system $(\{\mathrm{d}\lambda_{3l+i}\}, \{|P_{3l+i}|\}, k)$ is strongly admissible on $[-b^3, -a^3]$, and (72) follows as before. □

For each $i \in \{0, \ldots, 5\}$, we consider the families of rational functions

$$\left\{ \frac{P_{6k+i+1}(z)}{P_{6k+i}(z)} \right\}_k, \qquad \left\{ \frac{P_{6k+i+1,2}(z)}{P_{6k+i,2}(z)} \right\}_k. \tag{74}$$

By Theorem 1.4, these families are uniformly bounded on compact subsets of $\mathbb{C} \setminus [0, \alpha^3]$ and $\mathbb{C} \setminus [-b^3, -a^3]$, respectively. Therefore, by Montel's theorem there exists a sequence of integers $\Lambda \subset \mathbb{N}$ so that for each $i \in \{0, \ldots, 5\}$,

$$\lim_{k\in\Lambda} \frac{P_{6k+i+1}(z)}{P_{6k+i}(z)} = \widetilde{F}_1^{(i)}(z), \quad z \in \mathbb{C} \setminus [0, \alpha^3], \tag{75}$$

$$\lim_{k\in\Lambda} \frac{P_{6k+i+1,2}(z)}{P_{6k+i,2}(z)} = \widetilde{F}_2^{(i)}(z), \quad z \in \mathbb{C} \setminus [-a^3, -b^3], \tag{76}$$

where the limits hold uniformly on compact subsets of the indicated regions. Our goal is to show that we obtain the same limiting functions $\widetilde{F}_j^{(i)}$, no matter which convergent subsequences we take.

Taking into account the degree of P_n and $P_{n,2}$, from (75)–(76) we deduce: $\widetilde{F}_1^{(i)}$ and $1/\widetilde{F}_1^{(i)}$ are analytic in $\mathbb{C} \setminus [0, \alpha^3]$, $\widetilde{F}_2^{(i)}$ and $1/\widetilde{F}_2^{(i)}$ are analytic in $\mathbb{C} \setminus [-b^3, -a^3]$, and as $z \to \infty$,

$$\begin{aligned}
&\widetilde{F}_1^{(i)}(z) = 1 + O(1/z), \quad i \in \{0, 1, 3, 4\},\\
&\widetilde{F}_1^{(i)}(z) = z + O(1), \quad i \in \{2, 5\},\\
&\widetilde{F}_2^{(i)}(z) = 1 + O(1/z), \quad i \in \{0, 1, 2\},\\
&\widetilde{F}_2^{(i)}(z) = z + O(1), \quad i \in \{3, 5\},\\
&\widetilde{F}_2^{(4)}(z) = 1/z + O(1/z^2).
\end{aligned} \tag{77}$$

Given a Borel measurable function $w \geq 0$ defined on the interval $[c, d]$ that satisfies the Szegő condition

$$\frac{\log w(t)}{\sqrt{(d-t)(t-c)}} \in L^1(\mathrm{d}t),$$

let

$$S(w; z) := \exp\left\{ \frac{d-c}{4\pi} \sqrt{\left(\frac{2z-c-d}{d-c}\right)^2 - 1} \int_c^d \frac{\log w(t)}{t-z} \frac{\mathrm{d}t}{\sqrt{(d-t)(t-c)}} \right\}$$

denote the Szegő function on $\overline{\mathbb{C}} \setminus [c, d]$ associated with w (see [16]). In particular, if w is continuous at $x \in [c, d]$ and $w(x) > 0$, then the limit

$$\lim_{z \to x} |S(w; z)|^2 = \frac{1}{w(x)} \tag{78}$$

holds. We will indicate this below by writing $|S(w; x)|^2 w(x) = 1$.

Throughout this section we are always assuming that $s_1 > 0$ a.e. on $[0, \alpha]$, and $s_2 > 0$ a.e. on $[-b, -a]$. If $f_n \in H(\Omega)$, $\Omega \subset \overline{\mathbb{C}}$, the notation

$$\lim_{n \in \widetilde{\Lambda}} f_n(z) = F(z), \quad z \in \Omega,\ \widetilde{\Lambda} \subset \mathbb{N},$$

stands for the uniform convergence of f_n to F on each compact subset of Ω.

By Proposition 5.2 we have:

$$0 = \int_0^{\alpha^3} \tau^j P_{6k}(\tau) \mathrm{d}\nu_{6k}(\tau), \quad 0 \le j \le 2k - 1,$$

$$0 = \int_0^{\alpha^3} \tau^j P_{6k+1}(\tau) g_{6k}(\tau) \mathrm{d}\nu_{6k}(\tau), \quad 0 \le j \le 2k - 1,$$

where $g_{6k}(\tau) := \tau P_{6k,2}(\tau)/P_{6k+1,2}(\tau)$. Using (76),

$$\lim_{k \in \Lambda} g_{6k}(\tau) = \frac{\tau}{\widetilde{F}_2^{(0)}(\tau)}, \quad \text{uniformly on } [0, \alpha^3].$$

Since $\deg(P_{6k}) = \deg(P_{6k+1})$, applying [5, Theorem 2] (result on relative asymptotics of polynomials orthogonal with respect to varying measures), we obtain

$$\lim_{k \in \Lambda} \frac{P_{6k+1}(z)}{P_{6k}(z)} = \frac{S_1^{(0)}(z)}{S_1^{(0)}(\infty)} = \widetilde{F}_1^{(0)}(z), \quad z \in \overline{\mathbb{C}} \setminus [0, \alpha^3], \tag{79}$$

where $S_1^{(0)}$ is the Szegő function on $\overline{\mathbb{C}} \setminus [0, \alpha^3]$ associated with the weight $\tau/\widetilde{F}_2^{(0)}(\tau)$, $\tau \in [0, \alpha^3]$.

By Proposition 5.2 we have:

$$0 = \int_0^{\alpha^3} \tau^j P_{6k+2}(\tau) \mathrm{d}\nu_{6k+2}(\tau), \quad 0 \le j \le 2k - 1,$$

$$0 = \int_0^{\alpha^3} \tau^j P_{6k+3}(\tau) g_{6k+2}(\tau) \mathrm{d}\nu_{6k+2}(\tau), \quad 0 \le j \le 2k,$$

where $g_{6k+2}(\tau) := P_{6k+2,2}(\tau)/(\tau P_{6k+3,2}(\tau))$. Let P^*_{6k+2} be the monic polynomial of degree $2k$ orthogonal with respect to the measure $\mathrm{d}\nu_{6k+3}(\tau) = g_{6k+2}(\tau)\mathrm{d}\nu_{6k+2}(\tau)$. Since $\deg(P^*_{6k+2}) = \deg(P_{6k+2})$, again by [5, Theorem 2] we obtain

$$\lim_{k \in \Lambda} \frac{P^*_{6k+2}(z)}{P_{6k+2}(z)} = \frac{S_1^{(2)}(z)}{S_1^{(2)}(\infty)}, \quad z \in \overline{\mathbb{C}} \setminus [0, \alpha^3],$$

where $S_1^{(2)}$ is the Szegő function on $\overline{\mathbb{C}} \setminus [0, \alpha^3]$ with respect to the weight $1/(\tau \widetilde{F}_2^{(2)}(\tau))$.

Let ϕ_1 denote the conformal mapping that maps $\overline{\mathbb{C}} \setminus [0, \alpha^3]$ onto the exterior of the unit circle and satisfies $\phi_1(\infty) = \infty$ and $\phi_1'(\infty) > 0$. Then, by [5, Theorem 1] (result on ratio asymptotics of polynomials orthogonal with respect to varying measures) we have

$$\lim_{k\in\Lambda}\frac{P_{6k+3}(z)}{P^*_{6k+2}(z)}=\frac{\phi_1(z)}{\phi_1'(\infty)},\quad z\in\mathbb{C}\setminus[0,\alpha^3].$$

Therefore, we conclude that

$$\lim_{k\in\Lambda}\frac{P_{3k+3}(z)}{P_{6k+2}(z)}=\frac{S_1^{(2)}(z)}{S_1^{(2)}(\infty)}\frac{\phi_1(z)}{\phi_1'(\infty)}=\widetilde{F}_1^{(2)}(z),\quad z\in\mathbb{C}\setminus[0,\alpha^3].\tag{80}$$

The same arguments used before show that

$$\lim_{k\in\Lambda}\frac{P_{6k+i+1}(z)}{P_{6k+i}(z)}=\frac{S_1^{(i)}(z)}{S_1^{(i)}(\infty)}=\widetilde{F}_1^{(i)}(z),\quad z\in\overline{\mathbb{C}}\setminus[0,\alpha^3],\ i\in\{1,3,4\},\tag{81}$$

$$\lim_{k\in\Lambda}\frac{P_{6k+6}(z)}{P_{6k+5}(z)}=\frac{S_1^{(5)}(z)}{S_1^{(5)}(\infty)}\frac{\phi_1(z)}{\phi_1'(\infty)}=\widetilde{F}_1^{(5)}(z),\quad z\in\mathbb{C}\setminus[0,\alpha^3],\tag{82}$$

where $S_1^{(1)}$, $S_1^{(3)}$, $S_1^{(4)}$, and $S_1^{(5)}$ are the Szegő functions on $\overline{\mathbb{C}}\setminus[0,\alpha^3]$ with respect to the weights $1/\widetilde{F}_2^{(1)}(\tau)$, $\tau/\widetilde{F}_2^{(3)}(\tau)$, $1/\widetilde{F}_2^{(4)}(\tau)$, and $1/(\tau\widetilde{F}_2^{(5)}(\tau))$, respectively.

Applying now the orthogonality conditions from Proposition 5.1 and (73), we deduce:

$$\lim_{k\in\Lambda}\frac{P_{6k+i+1,2}(z)}{P_{6k+i,2}(z)}=\frac{S_2^{(i)}(z)}{S_2^{(i)}(\infty)}=\widetilde{F}_2^{(i)}(z),\quad z\in\overline{\mathbb{C}}\setminus[-b^3,-a^3],\ i\in\{0,1,2\},\tag{83}$$

$$\lim_{k\in\Lambda}\frac{P_{6k+i+1,2}(z)}{P_{6k+i,2}(z)}=\frac{S_2^{(i)}(z)}{S_2^{(i)}(\infty)}\frac{\phi_2(z)}{\phi_2'(\infty)}=\widetilde{F}_2^{(i)}(z),\quad z\in\mathbb{C}\setminus[-b^3,-a^3],\ i\in\{3,5\},\tag{84}$$

$$\lim_{k\in\Lambda}\frac{P_{6k+5,2}(z)}{P_{6k+4,2}(z)}=\frac{S_2^{(4)}(\infty)}{S_2^{(4)}(z)}\frac{\phi_2'(\infty)}{\phi_2(z)}=\widetilde{F}_2^{(4)}(z),\quad z\in\mathbb{C}\setminus[-b^3,-a^3],\tag{85}$$

where $S_2^{(0)},\ldots,S_2^{(5)}$, are the Szegő functions on $\overline{\mathbb{C}}\setminus[-b^3,-a^3]$ associated with the weights

$$\frac{1}{|\tau\widetilde{F}_1^{(0)}(\tau)|},\qquad\frac{|\tau|}{|\widetilde{F}_1^{(1)}(\tau)|},\qquad\frac{1}{|\widetilde{F}_1^{(2)}(\tau)|},\qquad\frac{1}{|\tau\widetilde{F}_1^{(3)}(\tau)|},\qquad\frac{|\widetilde{F}_1^{(4)}(\tau)|}{|\tau|},\qquad\frac{1}{|\widetilde{F}_1^{(5)}(\tau)|},$$

respectively, and ϕ_2 is the conformal mapping that maps $\overline{\mathbb{C}}\setminus[-b^3,-a^3]$ onto the exterior of the unit circle that satisfies the conditions $\phi_2(\infty)=\infty$ and $\phi_2'(\infty)>0$.

Proposition 5.5. *There exist positive constants $c_k^{(l)}$ so that the functions $F_k^{(l)}:=c_k^{(l)}\widetilde{F}_k^{(l)}$ satisfy the following boundary value conditions:*

$$|F_1^{(l)}(\tau)|^2\frac{\tau}{F_2^{(l)}(\tau)}=1,\quad\tau\in(0,\alpha^3],\ l=0,3,\tag{86}$$

$$|F_1^{(l)}(\tau)|^2\frac{1}{F_2^{(l)}(\tau)}=1,\quad\tau\in[0,\alpha^3],\ l=1,4,\tag{87}$$

$$|F_1^{(l)}(\tau)|^2\frac{1}{\tau F_2^{(l)}(\tau)}=1,\quad\tau\in(0,\alpha^3],\ l=2,5,\tag{88}$$

$$|F_2^{(l)}(\tau)|^2 \frac{1}{|\tau F_1^{(l)}(\tau)|} = 1, \quad \tau \in [-b^3, -a^3],\ l = 0, 3, \tag{89}$$

$$|F_2^{(l)}(\tau)|^2 \frac{|\tau|}{|F_1^{(l)}(\tau)|} = 1, \quad \tau \in [-b^3, -a^3],\ l = 1, 4, \tag{90}$$

$$|F_2^{(l)}(\tau)|^2 \frac{1}{|F_1^{(l)}(\tau)|} = 1, \quad \tau \in [-b^3, -a^3],\ l = 2, 5. \tag{91}$$

Proof. It follows from the relations (79)–(85), the definition of the Szegő functions $S_j^{(i)}$ and (78), that there exist positive constants $\omega_1^{(l)}, \omega_2^{(l)}$, such that

$$|\widetilde{F}_1^{(l)}(\tau)|^2 \frac{\tau}{\widetilde{F}_2^{(l)}(\tau)} = \frac{1}{\omega_1^{(l)}}, \quad \tau \in (0, \alpha^3],\ l = 0, 3, \tag{92}$$

$$|\widetilde{F}_1^{(l)}(\tau)|^2 \frac{1}{\widetilde{F}_2^{(l)}(\tau)} = \frac{1}{\omega_1^{(l)}}, \quad \tau \in [0, \alpha^3],\ l = 1, 4, \tag{93}$$

$$|\widetilde{F}_1^{(l)}(\tau)|^2 \frac{1}{\tau \widetilde{F}_2^{(l)}(\tau)} = \frac{1}{\omega_1^{(l)}}, \quad \tau \in (0, \alpha^3],\ l = 2, 5, \tag{94}$$

$$|\widetilde{F}_2^{(l)}(\tau)|^2 \frac{1}{|\tau \widetilde{F}_1^{(l)}(\tau)|} = \frac{1}{\omega_2^{(l)}}, \quad \tau \in [-b^3, -a^3],\ l = 0, 3, \tag{95}$$

$$|\widetilde{F}_2^{(l)}(\tau)|^2 \frac{|\tau|}{|\widetilde{F}_1^{(l)}(\tau)|} = \frac{1}{\omega_2^{(l)}}, \quad \tau \in [-b^3, -a^3],\ l = 1, 4, \tag{96}$$

$$|\widetilde{F}_2^{(l)}(\tau)|^2 \frac{1}{|\widetilde{F}_1^{(l)}(\tau)|} = \frac{1}{\omega_2^{(l)}}, \quad \tau \in [-b^3, -a^3],\ l = 2, 5, \tag{97}$$

where

$$\begin{aligned}
\omega_1^{(l)} &= (S_1^{(l)}(\infty))^2, \quad \text{for } l = 0, 1, 3, 4,\\
\omega_1^{(l)} &= (S_1^{(l)}(\infty)\phi_1'(\infty))^2, \quad \text{for } l = 2, 5,\\
\omega_2^{(l)} &= (S_2^{(l)}(\infty))^2, \quad \text{for } l = 0, 1, 2,\\
\omega_2^{(l)} &= (S_2^{(l)}(\infty)\phi_2'(\infty))^2, \quad \text{for } l = 3, 5,\\
\omega_2^{(4)} &= 1/(S_2^{(4)}(\infty)\phi_2'(\infty))^2.
\end{aligned} \tag{98}$$

The positive constants $c_k^{(l)}$ that satisfy the requirements are $c_1^{(l)} = [(\omega_1^{(l)})^2 \omega_2^{(l)}]^{1/3}, c_2^{(l)} = [\omega_1^{(l)} (\omega_2^{(l)})^2]^{1/3}, l = 0, \ldots, 5.$ □

In order to prove the uniqueness of the limiting functions $\widetilde{F}_j^{(i)}$, we need to use Lemma 5.6. More general versions of this result can be found in [4] (see Lemma 4.1) and [1] (see Proposition 1.1), so we omit the proof.

Let us first introduce some notations. Assume that Δ_1, Δ_2 are disjoint compact intervals in $\mathbb{R}$, and let $C(\Delta_i)$ denote the space of real-valued continuous functions on Δ_i. We write $\mathbf{u} = (u_1, u_2)^t \in C$ if $u_1 \in C(\Delta_2), u_2 \in C(\Delta_1)$. Given $u_1 \in C(\Delta_2)$, let $T_{2,1}(u_1)$ be the harmonic function in $\overline{\mathbb{C}} \setminus \Delta_2$ that solves the Dirichlet problem with boundary condition

$$T_{2,1}(u_1)(x) = u_1(x), \quad x \in \Delta_2,$$

and given $u_2 \in C(\Delta_1)$, let $T_{1,2}(u_2)$ denote the harmonic function in $\overline{\mathbb{C}} \setminus \Delta_1$ that solves the Dirichlet problem with boundary condition

$$T_{1,2}(u_2)(x) = u_2(x), \quad x \in \Delta_1.$$

Consider the linear operator $T : C \longrightarrow C$ defined as follows:

$$T = \begin{bmatrix} 0 & T_{1,2} \\ T_{2,1} & 0 \end{bmatrix},$$

and $I : C \longrightarrow C$ the identity operator. The auxiliary result is the following

Lemma 5.6. *If* $\mathbf{u} \in C$ *and* $(2I - T)(\mathbf{u}) = \mathbf{0}$, *then* $\mathbf{u} = \mathbf{0}$.

Now we prove that the limiting functions do not depend on the sequence $\Lambda \subset \mathbb{N}$ for which (75)–(76) hold.

Proposition 5.7. *The limiting functions* $\widetilde{F}_j^{(i)}$ *are unique for every* $j \in \{1, 2\}$ *and* $i \in \{0, \ldots, 5\}$.

Proof. For each fixed $i \in \{0, \ldots, 5\}$, by Proposition 5.5 the functions $\log|F_1^{(i)}|$, $\log|F_2^{(i)}|$ satisfy

$$\begin{cases} 2\log|F_1^{(i)}(\tau)| - \log|F_2^{(i)}(\tau)| = \log|f_i(\tau)|, & \tau \in (0, \alpha^3], \\ -\log|F_1^{(i)}(\tau)| + 2\log|F_2^{(i)}(\tau)| = \log|g_i(\tau)|, & \tau \in [-b^3, -a^3], \end{cases} \tag{99}$$

where $f_i(\tau), g_i(\tau)$ equal $1/\tau$, 1, or τ, depending on the value of i. Assume that the functions $\widetilde{G}_1^{(i)}, \widetilde{G}_2^{(i)}$ satisfy

$$\lim_{k \in \Lambda'} \frac{P_{6k+i+1}(z)}{P_{6k+i}(z)} = \widetilde{G}_1^{(i)}(z), \quad z \in \mathbb{C} \setminus [0, \alpha^3],$$

$$\lim_{k \in \Lambda'} \frac{P_{6k+i+1,2}(z)}{P_{6k+i,2}(z)} = \widetilde{G}_2^{(i)}(z), \quad z \in \mathbb{C} \setminus [-a^3, -b^3],$$

for some other subsequence $\Lambda' \subset \mathbb{N}$. As before, we can find positive constants $d_1^{(i)}, d_2^{(i)}$ so that the functions $G_j^{(i)} := d_j^{(i)} \widetilde{G}_j^{(i)}$ satisfy the same system (99). If we define the functions

$$u_1 := \log|F_1^{(i)}| - \log|G_1^{(i)}|, \qquad u_2 := \log|F_2^{(i)}| - \log|G_2^{(i)}|, \qquad \mathbf{u} = (u_1, u_2)^t,$$

observe that u_1 is harmonic in $\overline{\mathbb{C}} \setminus [0, \alpha^3]$, u_2 is harmonic in $\overline{\mathbb{C}} \setminus [-b^3, -a^3]$, and they are also bounded in the corresponding regions. Moreover,

$$\begin{cases} 2u_1(\tau) - u_2(\tau) = 0, & \tau \in (0, \alpha^3], \\ -u_1(\tau) + 2u_2(\tau) = 0, & \tau \in [-b^3, -a^3]. \end{cases} \tag{100}$$

Let $\Delta_1 := [0, \alpha^3]$, $\Delta_2 := [-b^3, -a^3]$. From (100) and the (generalized) maximum–minimum principle for harmonic functions, we obtain that $2u_1 - T_{1,2}(u_2) \equiv 0$ on $\overline{\mathbb{C}} \setminus \Delta_1$ and $2u_2 - T_{2,1}(u_1) \equiv 0$ on $\overline{\mathbb{C}} \setminus \Delta_2$. In particular, $(2I - T)(\mathbf{u}) = \mathbf{0}$, so by Lemma 5.6 we get $u_1 = 0$ on Δ_2 and $u_2 = 0$ on Δ_1. Therefore $T_{1,2}(u_2) \equiv 0$ on $\overline{\mathbb{C}} \setminus \Delta_1$ and $T_{2,1}(u_1) \equiv 0$ on $\overline{\mathbb{C}} \setminus \Delta_2$, implying that $u_1 \equiv 0$ and $u_2 \equiv 0$. From $|F_j^{(i)}| = |G_j^{(i)}|$ it easily follows that $c_j^i = d_j^i$ and $\widetilde{F}_j^{(i)} = \widetilde{G}_j^{(i)}$. □

Proof of Theorem 1.5. The existence of the limits (12)–(13) follows from the normality of the families (74) and Proposition 5.7. The polynomials P_n satisfy:

$$\begin{aligned}
P_{3k}(z) &= P_{3k+1}(z) + a_{3k}P_{3k-2}(z),\\
P_{3k+1}(z) &= P_{3k+2}(z) + a_{3k+1}P_{3k-1}(z),\\
zP_{3k+2}(z) &= P_{3k+3}(z) + a_{3k+2}P_{3k}(z),
\end{aligned}$$

and so (12) implies that the following limits hold:

$$\lim_{k\to\infty} a_{6k+i} = \widetilde{F}_1^{(i-2)}(z)\widetilde{F}_1^{(i-1)}(z)(1-\widetilde{F}_1^{(i)}(z)), \quad i \in \{0,1,3,4\}, \tag{101}$$

$$\lim_{k\to\infty} a_{6k+i} = \widetilde{F}_1^{(i-2)}(z)\widetilde{F}_1^{(i-1)}(z)(z-\widetilde{F}_1^{(i)}(z)), \quad i \in \{2,5\}, \tag{102}$$

where these relations are valid for every $z \in \mathbb{C}\setminus[0,\alpha^3](\widetilde{F}_1^{(-2)} = \widetilde{F}_1^{(4)}, \widetilde{F}_1^{(-1)} = \widetilde{F}_1^{(5)})$.

We have:

$$\begin{aligned}
&\widetilde{F}_1^{(i-2)}(z)\widetilde{F}_1^{(i-1)}(z)(1-\widetilde{F}_1^{(i)}(z)) = -C_1^{(i)} + O(1/z), \quad z\to\infty,\ i\in\{0,1,3,4\},\\
&\widetilde{F}_1^{(i-2)}(z)\widetilde{F}_1^{(i-1)}(z)(z-\widetilde{F}_1^{(i)}(z)) = -C_0^{(i)} + O(1/z), \quad z\to\infty,\ i\in\{2,5\},
\end{aligned}$$

and so (14) follows from (101)–(102). The ratio asymptotics of Q_n and $Q_{n,2}$ is a direct consequence of (12)–(13). □

Proposition 5.8. *Assume that the hypotheses of Theorem 1.5 hold. Then the polynomials p_n, $p_{n,2}$ defined in (67) satisfy for each $i \in \{0,\dots,5\}$:*

$$\lim_{k\to\infty}\frac{p_{6k+i+1}(z)}{p_{6k+i}(z)} = \kappa_1^{(i)}\widetilde{F}_1^{(i)}(z), \quad z\in\mathbb{C}\setminus[0,\alpha^3], \tag{103}$$

$$\lim_{k\to\infty}\frac{p_{6k+i+1,2}(z)}{p_{6k+i,2}(z)} = \kappa_2^{(i)}\widetilde{F}_2^{(i)}(z), \quad z\in\mathbb{C}\setminus[-b^3,-a^3], \tag{104}$$

uniformly on compact subsets of the indicated regions, where

$$\kappa_j^{(i)} = \sqrt{\omega_j^{(i)}}, \quad j = 1,2,$$

and the constants $\omega_j^{(i)}$ are defined in (98). Consequently, for the leading coefficients $\kappa_n, \kappa_{n,2}$ defined in (66) we have:

$$\lim_{k\to\infty}\frac{\kappa_{6k+i+1}}{\kappa_{6k+i}} = \kappa_1^{(i)}, \tag{105}$$

$$\lim_{k\to\infty}\frac{\kappa_{6k+i+1,2}}{\kappa_{6k+i,2}} = \kappa_2^{(i)}. \tag{106}$$

In addition, the following limits hold uniformly on compact subsets of $\mathbb{C}\setminus(S_0\cup S_1)$:

$$\lim_{k\to\infty}\frac{\Psi_{6k+i+1}(z)}{\Psi_{6k+i}(z)} = \frac{1}{\omega_1^{(i)}}\frac{\widetilde{F}_2^{(i)}(z^3)}{z^2\widetilde{F}_1^{(i)}(z^3)}, \quad i = 0,3, \tag{107}$$

$$\lim_{k\to\infty}\frac{\Psi_{6k+i+1}(z)}{\Psi_{6k+i}(z)} = \frac{1}{\omega_1^{(i)}}\frac{z\widetilde{F}_2^{(i)}(z^3)}{\widetilde{F}_1^{(i)}(z^3)}, \quad i = 1,2,4,5. \tag{108}$$

Proof. Using the same argument employed before and Theorems 1 and 2 from [5], we obtain

$$\lim_{k\to\infty}\frac{p_{6k+i+1}(z)}{p_{6k+i}(z)}=S_1^{(i)}(z),\quad z\in\mathbb{C}\setminus[0,\alpha^3],\ i=0,1,3,4,$$

$$\lim_{k\to\infty}\frac{p_{6k+i+1}(z)}{p_{6k+i}(z)}=S_1^{(i)}(z)\phi_1(z),\quad z\in\mathbb{C}\setminus[0,\alpha^3],\ i=2,5,$$

$$\lim_{k\to\infty}\frac{p_{6k+i+1,2}(z)}{p_{6k+i,2}(z)}=S_2^{(i)}(z),\quad z\in\mathbb{C}\setminus[-b^3,-a^3],\ i=0,1,2,$$

$$\lim_{k\to\infty}\frac{p_{6k+i+1,2}(z)}{p_{6k+i,2}(z)}=S_2^{(i)}(z)\phi_2(z),\quad z\in\mathbb{C}\setminus[-b^3,-a^3],\ i=3,5,$$

$$\lim_{k\to\infty}\frac{p_{6k+5,2}(z)}{p_{6k+4,2}(z)}=(S_2^{(4)}(z)\phi_2(z))^{-1},\quad z\in\mathbb{C}\setminus[-b^3,-a^3],$$

so (103) and (104) follow. (105)–(106) are immediate consequences of (103)–(104).

Observe that by (65) we can write

$$\frac{\Psi_{n+1}}{\Psi_n}=\frac{\kappa_n^2}{\kappa_{n+1}^2}\frac{h_{n+1}}{h_n}\frac{Q_n}{Q_{n+1}}\frac{Q_{n+1,2}}{Q_{n,2}},$$

so (105) together with Lemma 5.4 and Theorem 1.5 imply (107)–(108). □

Proof of Proposition 1.6. We first show that $a^{(i)}>0$ for all i. If $a^{(0)}=0$, then (101) implies $\widetilde{F}_1^{(0)}\equiv 1$, and using (86) we obtain that $\widetilde{F}_2^{(0)}(z)=z$ on $\mathbb{C}\setminus[-b^3,-a^3]$, contradicting (77). If $a^{(1)}=0$, then again by (101) we get $\widetilde{F}_1^{(1)}\equiv 1$, and so by (87) we have $\widetilde{F}_2^{(1)}\equiv 1$, contradicting (90). If $a^{(2)}=0$, then from (102) it follows that $\widetilde{F}_1^{(2)}(z)=z$ on $\mathbb{C}\setminus[0,\alpha^3]$, and so (88) implies that $\widetilde{F}_2^{(1)}(z)=z$, which is impossible. Similar arguments show that $a^{(i)}>0$ for $i\in\{3,4,5\}$.

Now we prove simultaneously that $\widetilde{F}_1^{(2)}(z)=z\widetilde{F}_1^{(0)}(z)$ and $\widetilde{F}_2^{(0)}=\widetilde{F}_2^{(2)}$. Let

$$u_1(z):=\log|F_1^{(2)}(z)|-\log|zF_1^{(0)}(z)|,\qquad u_2(z):=\log|F_2^{(2)}(z)|-\log|F_2^{(0)}(z)|.$$

Then u_1 is harmonic in $\overline{\mathbb{C}}\setminus[0,\alpha^3]$ and u_2 is harmonic in $\overline{\mathbb{C}}\setminus[-b^3,-a^3]$. By (89) and (91) we see that u_2 is bounded on $\overline{\mathbb{C}}\setminus[-b^3,-a^3]$. Taking into account the definitions of the functions $S_1^{(0)}$ and $S_1^{(2)}$, the boundedness of u_1 is equivalent to the boundedness of the expression

$$\frac{1}{2\pi}\int_0^{2\pi}\Re\left[\frac{e^{i\theta}+1/\phi_1(z)}{e^{i\theta}-1/\phi_1(z)}\right]\log(1+\cos\theta)d\theta-\log|z|,\quad z\notin[0,\alpha^3],$$

which follows trivially from the identity

$$\frac{1}{2\pi}\int_0^{2\pi}\Re\left[\frac{e^{i\theta}+w}{e^{i\theta}-w}\right]\log|1+e^{i\theta}|d\theta=\log|1+w|,\quad |w|<1.$$

Now Proposition 5.5 implies that $2u_1(\tau)-u_2(\tau)=0$ for $\tau\in(0,\alpha^3]$, and $-u_1(\tau)+2u_2(\tau)=0$ for $\tau\in[-b^3,-a^3]$. As in the proof of Proposition 5.7, this yields $u_1\equiv 0$, $u_2\equiv 0$. Similarly one proves the remaining relations in (16) and (19).

From (16), (14) and (15), it follows that $a^{(0)}=a^{(2)}$ and $a^{(3)}=a^{(5)}$. We have by (101)–(102) that

$$\widetilde{F}_1^{(0)}(z)\widetilde{F}_1^{(1)}(z)(z-\widetilde{F}_1^{(2)})=a^{(2)},\qquad \widetilde{F}_1^{(4)}(z)\widetilde{F}_1^{(5)}(z)(1-\widetilde{F}_1^{(0)})=a^{(0)}.$$

Since $a^{(0)} = a^{(2)}$ and $\widetilde{F}_1^{(2)}(z) = z\widetilde{F}_1^{(0)}(z)$, we deduce that $z\widetilde{F}_1^{(0)}\widetilde{F}_1^{(1)} = \widetilde{F}_1^{(4)}\widetilde{F}_1^{(5)}$, or equivalently $\widetilde{F}_1^{(1)}\widetilde{F}_1^{(2)} = \widetilde{F}_1^{(4)}\widetilde{F}_1^{(5)}$. The other two relations in (17) follow immediately using this equality and (16).

The relations in (20) are an easy consequence of (17) and (86)–(88). Now, (18) is obtained by dividing appropriate relations from (101)–(102), one by another, and taking into account (17). The equality $a^{(0)} + a^{(1)} = a^{(3)} + a^{(4)}$ follows immediately from $\widetilde{F}_1^{(0)}\widetilde{F}_1^{(1)} = \widetilde{F}_1^{(3)}\widetilde{F}_1^{(4)}$.

We next show that the functions $\widetilde{F}_1^{(i)}$, $i \in \{0, \dots, 5\}$, are all distinct. If $i \in \{0, 1, 3, 4\}$, then evidently $\widetilde{F}_1^{(i)} \neq \widetilde{F}_1^{(2)}$ and $\widetilde{F}_1^{(i)} \neq \widetilde{F}_1^{(5)}$. If $\widetilde{F}_1^{(0)} = \widetilde{F}_1^{(1)}$, then (92) and (93) imply that

$$\frac{\widetilde{F}_2^{(1)}(\tau)}{\widetilde{F}_2^{(0)}(\tau)} = \frac{\omega_1^{(1)}}{\omega_1^{(0)}}\frac{1}{\tau}, \qquad \tau \in (0, \alpha^3],$$

which is contradictory since $\widetilde{F}_2^{(1)}/\widetilde{F}_2^{(0)}$ is holomorphic outside $[-b^3, -a^3]$. The same argument proves that $\widetilde{F}_1^{(0)} \neq \widetilde{F}_1^{(4)}$, $\widetilde{F}_1^{(1)} \neq \widetilde{F}_1^{(3)}$, and $\widetilde{F}_1^{(3)} \neq \widetilde{F}_1^{(4)}$. If $\widetilde{F}_1^{(0)} = \widetilde{F}_1^{(3)}$, then (92) implies that $\widetilde{F}_2^{(0)} = \widetilde{F}_2^{(3)}$, which is impossible (cf. (77)). Similarly (using now (93) and (94)) we see that $\widetilde{F}_1^{(1)} \neq \widetilde{F}_1^{(4)}$ and $\widetilde{F}_1^{(2)} \neq \widetilde{F}_1^{(5)}$.

Now we show that the functions $\widetilde{F}_2^{(i)}$, $i \in \{0, 1, 3, 4\}$, are all different. If we assume that $\widetilde{F}_2^{(0)} = \widetilde{F}_2^{(1)}$, then (95)–(96) imply that

$$\frac{|\widetilde{F}_1^{(1)}(\tau)|}{|\widetilde{F}_1^{(0)}(\tau)|} = \frac{\omega_2^{(1)}}{\omega_2^{(0)}}\tau^2, \qquad \tau \in [-b^3, -a^3].$$

It follows that $\widetilde{F}_1^{(1)}(z) = z^2\widetilde{F}_1^{(0)}(z)$, which is impossible. The other cases are justified just by looking at the Laurent expansion at infinity.

By (18) we see that $a^{(0)} \neq a^{(3)}$ and $a^{(1)} \neq a^{(4)}$. Now we show that $a^{(1)} \neq a^{(3)}$. Applying (101) for $i = 0$ and the relation $\widetilde{F}_1^{(1)}\widetilde{F}_1^{(2)} = \widetilde{F}_1^{(4)}\widetilde{F}_1^{(5)}$, we get

$$\widetilde{F}_1^{(1)}\widetilde{F}_1^{(2)}(1 - \widetilde{F}_1^{(0)}) = a^{(0)}.$$

From this relation and (101) (for $i = 4$), we obtain

$$\widetilde{F}_1^{(1)}(1 - \widetilde{F}_1^{(0)}) = \frac{a^{(0)}}{a^{(4)}}\widetilde{F}_1^{(3)}(1 - \widetilde{F}_1^{(4)}).$$

Applying the first two equations from (18), we derive that

$$\widetilde{F}_1^{(1)}(1 - \widetilde{F}_1^{(0)}) = \frac{a^{(3)}}{a^{(1)}}(1 - \widetilde{F}_1^{(1)})(\widetilde{F}_1^{(0)} - 1) + \frac{a^{(0)}}{a^{(1)}}(1 - \widetilde{F}_1^{(1)}). \tag{109}$$

If we assume now that $a^{(1)} = a^{(3)}$, then (109) yields $(1 - \widetilde{F}_1^{(0)})/(1 - \widetilde{F}_1^{(1)}) = a^{(0)}/a^{(1)}$. But from (101) we know that

$$\frac{(1 - \widetilde{F}_1^{(0)})\widetilde{F}_1^{(4)}}{(1 - \widetilde{F}_1^{(1)})\widetilde{F}_1^{(0)}} = \frac{a^{(0)}}{a^{(1)}},$$

hence $\widetilde{F}_1^{(4)} = \widetilde{F}_1^{(0)}$, which is contradictory. Therefore $a^{(1)} \neq a^{(3)}$, and so by (21) we also obtain that $a^{(0)} \neq a^{(4)}$. $\square$

Corollary 5.9. *The following relations hold:*

$$\begin{aligned} &\omega_1^{(0)}\omega_1^{(1)} = \omega_1^{(3)}\omega_1^{(4)}, \qquad && \omega_1^{(0)} = \omega_1^{(2)}, \qquad && \omega_1^{(3)} = \omega_1^{(5)}, \\ &\omega_2^{(0)}\omega_2^{(1)} = \omega_2^{(3)}\omega_2^{(4)}, \qquad && \omega_2^{(0)} = \omega_2^{(2)}, \qquad && \omega_2^{(3)} = \omega_2^{(5)}. \end{aligned}$$

Proof. All these relations follow immediately from the relations established in Proposition 1.6 and the boundary value Eqs. (92)–(97) (multiply or divide appropriately these equations, one by another). □

6. The Riemann surface representation of the limiting functions $\widetilde{F}_j^{(i)}$

We will give now the proof of Theorem 1.7. Before doing so, we need some definitions and comments. Let

$$G_1^{(i,j)} := F_1^{(i)}/F_1^{(j)}, \qquad G_2^{(i,j)} := F_2^{(i)}/F_2^{(j)}, \quad 0 \le i, j \le 5.$$

Recall that the conformal representation ψ of $\mathcal{R}$ onto $\overline{\mathbb{C}}$ satisfies (23). As a consequence, we have $\psi(z) = \overline{\psi(\overline{z})}$. This property implies in particular that

$$\psi_k : \overline{\mathbb{R}} \setminus (\Delta_k \cup \Delta_{k+1}) \longrightarrow \overline{\mathbb{R}}, \quad k = 0, 1, 2, \ \Delta_0 = \Delta_3 = \emptyset,$$

and

$$\psi_k(x_\pm) = \overline{\psi_k(x_\mp)} = \overline{\psi_{k+1}(x_\pm)}, \quad x \in \Delta_{k+1}. \tag{110}$$

So all the coefficients in the Laurent expansion at infinity of the branches ψ_k are real. Given a function F that satisfies

$$F(z) = Cz^k + O(z^{k-1}), \qquad C \in \mathbb{R} \setminus \{0\}, \quad z \to \infty,$$

we use the symbol $\operatorname{sign}(F(\infty))$ to denote the sign of C (i.e., $\operatorname{sign}(F(\infty)) = 1$ if $C > 0$ and $\operatorname{sign}(F(\infty)) = -1$ if $C < 0$).

The function $\psi_0\psi_1\psi_2$ is analytic and bounded on $\overline{\mathbb{C}}$, so this function is constant. Let us denote this constant by C (we will reserve in this section the letter C for this constant). So we have

$$(\psi_0\psi_1\psi_2)(z) \equiv C, \qquad (\widetilde{\psi}_0\widetilde{\psi}_1\widetilde{\psi}_2)(z) \equiv 1, \quad z \in \overline{\mathbb{C}}. \tag{111}$$

Proposition 6.1. *The following relations hold:*

$$G_1^{(0,3)}(z) = \frac{\operatorname{sign}((\psi_1\psi_2)(\infty))(\psi_1\psi_2)(z)}{|C|^{2/3}}, \qquad G_2^{(0,3)}(z) = \frac{\operatorname{sign}(\psi_2(\infty))\psi_2(z)}{|C|^{1/3}}. \tag{112}$$

Proof. By (86) and (89) we have

$$|G_1^{(0,3)}(\tau)|^2 \frac{1}{G_2^{(0,3)}(\tau)} = 1, \quad \tau \in (0, \alpha^3], \tag{113}$$

$$|G_2^{(0,3)}(\tau)|^2 \frac{1}{|G_1^{(0,3)}(\tau)|} = 1, \quad \tau \in [-b^3, -a^3]. \tag{114}$$

Observe also that $G_1^{(0,3)}$ and $G_2^{(0,3)}$ are bounded on $\overline{\mathbb{C}} \setminus \Delta_1$ and $\overline{\mathbb{C}} \setminus \Delta_2$, respectively. Let us call v_1 and v_2 the functions on the right-hand side of the relations (112), respectively. The function v_2 is positive on $\Delta_1 = [0, \alpha^3]$ since $\operatorname{sign}(v_2(\infty)) = 1$. Using (110)–(111), for any $x \in (0, \alpha^3)$,

$$\frac{|v_1(x_\pm)|^2}{v_2(x)} = \frac{|\psi_1(x_\pm)|^2 \psi_2(x)^2}{\operatorname{sign}(\psi_2(\infty))\psi_2(x)|C|} = \frac{|\psi_0(x_\mp)||\psi_1(x_\pm)||\psi_2(x)|}{|C|}$$
$$= \frac{|\overline{\psi_0(x_\pm)}||\psi_1(x_\pm)||\psi_2(x)|}{|C|} = 1,$$

i.e., v_1 and v_2 satisfy (113) on $(0, \alpha^3)$. On the other hand, for $x \in (-b^3, -a^3)$,

$$\frac{|v_2(x_\pm)|^2}{|v_1(x)|} = \frac{|\psi_2(x_\pm)|}{|\psi_1(x_\pm)|} = 1,$$

so v_1 and v_2 also satisfy (114) on $(-b^3, -a^3)$. Finally, the same argument used to prove Proposition 5.7 yields the validity of (112). □

Proof of Theorem 1.7. By Proposition 6.1 we have:

$$\widetilde{F}_1^{(4)} / \widetilde{F}_1^{(1)} = \widetilde{F}_1^{(0)} / \widetilde{F}_1^{(3)} = \widetilde{\psi}_1 \widetilde{\psi}_2 = 1/\widetilde{\psi}_0, \tag{115}$$

$$\widetilde{F}_2^{(0)} / \widetilde{F}_2^{(3)} = \widetilde{\psi}_2. \tag{116}$$

From the first relation in (18) and (115), simple algebraic manipulations show that

$$\widetilde{F}_1^{(0)} = \frac{a^{(0)} - a^{(3)}}{a^{(0)}\widetilde{\psi}_0 - a^{(3)}}, \qquad \widetilde{F}_1^{(3)} = \frac{(a^{(0)} - a^{(3)})\widetilde{\psi}_0}{a^{(0)}\widetilde{\psi}_0 - a^{(3)}}.$$

The representations of $\widetilde{F}_1^{(2)}$ and $\widetilde{F}_1^{(5)}$ follow immediately from the relations $\widetilde{F}_1^{(2)}(z) = z\widetilde{F}_1^{(0)}(z)$ and $\widetilde{F}_1^{(5)}(z) = z\widetilde{F}_1^{(3)}(z)$. The relation $\widetilde{F}_1^{(1)} / \widetilde{F}_1^{(4)} = \widetilde{\psi}_0$ and (18) prove the representations of $\widetilde{F}_1^{(1)}$ and $\widetilde{F}_1^{(4)}$.

Recall that

$$z\,\Psi_n(z) = \Psi_{n+1} + a_n \Psi_{n-2}, \quad n \geq 2. \tag{117}$$

Therefore, if we define the functions

$$U^{(i)}(z) := \lim_{k\to\infty} \frac{\Psi_{6k+i+1}(z)}{\Psi_{6k+i}(z)}, \quad z \in \mathbb{C} \setminus (S_0 \cup S_1),\ 0 \leq i \leq 5,$$

(by Proposition 5.8 we know that such limits exist) then we know by (117) that

$$a^{(i)} = U^{(i-2)}(z)U^{(i-1)}(z)(z - U^{(i)}(z)), \quad 0 \leq i \leq 5,$$

where we understand that $U^{(-2)} = U^{(4)}$, $U^{(-1)} = U^{(5)}$. In particular, applying (107) and (108) we obtain for $i = 0, 1, 4, 5$,

$$a^{(0)} = \frac{1}{\omega_1^{(4)}\omega_1^{(5)}} \frac{\widetilde{F}_2^{(5)}(z)}{\widetilde{F}_1^{(5)}(z)} \frac{\widetilde{F}_2^{(4)}(z)}{\widetilde{F}_1^{(4)}(z)} \left(z - \frac{\widetilde{F}_2^{(0)}(z)}{\omega_1^{(0)}\widetilde{F}_1^{(0)}(z)} \right), \tag{118}$$

$$a^{(1)} = \frac{1}{\omega_1^{(0)}\omega_1^{(5)}} \frac{\widetilde{F}_2^{(0)}(z)}{\widetilde{F}_1^{(0)}(z)} \frac{\widetilde{F}_2^{(5)}(z)}{\widetilde{F}_1^{(5)}(z)} \left(1 - \frac{\widetilde{F}_2^{(1)}(z)}{\omega_1^{(1)}\widetilde{F}_1^{(1)}(z)} \right), \tag{119}$$

$$a^{(4)} = \frac{1}{\omega_1^{(2)}\omega_1^{(3)}} \frac{\widetilde{F}_2^{(2)}(z)}{\widetilde{F}_1^{(2)}(z)} \frac{\widetilde{F}_2^{(3)}(z)}{\widetilde{F}_1^{(3)}(z)} \left(1 - \frac{\widetilde{F}_2^{(4)}(z)}{\omega_1^{(4)}\widetilde{F}_1^{(4)}(z)}\right), \tag{120}$$

$$a^{(5)} = \frac{1}{\omega_1^{(3)}\omega_1^{(4)}} \frac{\widetilde{F}_2^{(3)}(z)}{\widetilde{F}_1^{(3)}(z)} \frac{\widetilde{F}_2^{(4)}(z)}{\widetilde{F}_1^{(4)}(z)} \left(z - \frac{\widetilde{F}_2^{(5)}(z)}{\omega_1^{(5)}\widetilde{F}_1^{(5)}(z)}\right), \tag{121}$$

where these identities are valid for every $z \in \mathbb{C} \setminus ([-b^3, -a^3] \cup [0, \alpha^3])$. If we apply the relations $a^{(3)} = a^{(5)}$, $\widetilde{F}_1^{(5)} = z\widetilde{F}_1^{(3)}$, $\widetilde{F}_2^{(5)} = \widetilde{F}_2^{(3)}$, from (118) and (121) we obtain

$$z\frac{a^{(0)}}{a^{(3)}}\left(1 - \frac{1}{\omega_1^{(5)}} \frac{\widetilde{F}_2^{(3)}(z)}{\widetilde{F}_1^{(5)}(z)}\right) = \frac{\omega_1^{(3)}}{\omega_1^{(5)}}\left(z - \frac{\widetilde{F}_2^{(0)}(z)}{\omega_1^{(0)}\widetilde{F}_1^{(0)}(z)}\right).$$

Using (116) and substituting in this expression the functions $\widetilde{F}_1^{(0)}$ and $\widetilde{F}_1^{(5)}$ by their representations in terms of the branches $\widetilde{\psi}_k$, we get

$$z\left(\frac{a^{(0)}}{a^{(3)}} - \frac{\omega_1^{(3)}}{\omega_1^{(5)}}\right) = \frac{(a^{(0)}\widetilde{\psi}_0(z) - a^{(3)})}{(a^{(0)} - a^{(3)})}\left(\frac{a^{(0)}}{a^{(3)}\widetilde{\psi}_0(z)} - \frac{\omega_1^{(3)}\widetilde{\psi}_2(z)}{\omega_1^{(0)}}\right)\frac{\widetilde{F}_2^{(3)}(z)}{\omega_1^{(5)}}.$$

The factors on the right-hand side of this equation never vanish on $\mathbb{C} \setminus ([0, \alpha^3] \cup [-b^3, -a^3])$, and so we can write

$$\widetilde{F}_2^{(3)}(z) = \frac{z\left(\frac{a^{(0)}}{a^{(3)}} - \frac{\omega_1^{(3)}}{\omega_1^{(5)}}\right)\omega_1^{(5)}(a^{(0)} - a^{(3)})}{(a^{(0)}\widetilde{\psi}_0(z) - a^{(3)})\left(\frac{a^{(0)}}{a^{(3)}\widetilde{\psi}_0(z)} - \frac{\omega_1^{(3)}\widetilde{\psi}_2(z)}{\omega_1^{(0)}}\right)}.$$

If we move z to the left-hand side and evaluate at infinity we obtain

$$\omega_1^{(5)}\left(\frac{a^{(0)}}{a^{(3)}} - \frac{\omega_1^{(3)}}{\omega_1^{(5)}}\right) = \frac{a^{(0)}}{a^{(3)}}, \tag{122}$$

and so the Riemann surface representation for $\widetilde{F}_2^{(3)}$ follows. This also proves the representation for the functions $\widetilde{F}_2^{(5)}$, $\widetilde{F}_2^{(0)}$, and $\widetilde{F}_2^{(2)}$.

From (119) and (120) we derive the relation

$$\frac{a^{(1)}}{a^{(4)}}\left(1 - \frac{\widetilde{F}_2^{(4)}}{\omega_1^{(4)}\widetilde{F}_1^{(4)}}\right) = \frac{\omega_1^{(2)}\omega_1^{(3)}}{\omega_1^{(0)}\omega_1^{(5)}}\left(1 - \frac{\widetilde{F}_2^{(1)}}{\omega_1^{(1)}\widetilde{F}_1^{(1)}}\right).$$

From Corollary 5.9 we know that $\omega_1^{(2)}\omega_1^{(3)} = \omega_1^{(5)}\omega_1^{(0)}$. Since $\widetilde{F}_2^{(4)}/\widetilde{F}_2^{(1)} = \widetilde{F}_2^{(0)}/\widetilde{F}_2^{(3)} = \widetilde{\psi}_2$ and $\widetilde{F}_1^{(4)}/\widetilde{F}_1^{(1)} = 1/\widetilde{\psi}_0 = \widetilde{\psi}_1\widetilde{\psi}_2$, we get

$$\frac{a^{(1)}}{a^{(4)}} - 1 = \frac{\widetilde{F}_2^{(4)}}{\widetilde{F}_1^{(4)}}\left(\frac{a^{(1)}}{a^{(4)}\omega_1^{(4)}} - \frac{\widetilde{\psi}_1}{\omega_1^{(1)}}\right).$$

Evaluating at infinity we obtain the relation

$$\omega_1^{(1)} = \frac{a^{(4)}}{a^{(4)} - a^{(1)}}, \tag{123}$$

and so we can write

$$\widetilde{F}_2^{(4)} = \frac{\widetilde{F}_1^{(4)}}{(\widetilde{\psi}_1 - (\omega_1^{(1)} - 1)/\omega_1^{(4)})}.$$

Therefore, the Riemann surface representation of $\widetilde{F}_2^{(4)}$ follows from that of $\widetilde{F}_1^{(4)}$ and the representation of $\widetilde{F}_2^{(1)}$ follows from the relation $\widetilde{F}_2^{(4)} = \widetilde{\psi}_2 \widetilde{F}_2^{(1)}$.

Now from (122) and Corollary 5.9 we get $\omega_1^{(3)} = \omega_1^{(5)} = a^{(0)}/(a^{(0)} - a^{(3)})$. If we evaluate both sides of Eq. (121) at infinity we obtain $a^{(5)} = a^{(3)} = (1 - 1/\omega_1^{(3)})/(\omega_1^{(3)}\omega_1^{(4)})$, and so $\omega_1^{(4)} = (a^{(0)} - a^{(3)})/(a^{(0)})^2$. Finally, from Corollary 5.9 and the above computations we deduce that $\omega_1^{(0)} = \omega_1^{(2)} = (a^{(4)} - a^{(1)})/(a^{(0)}a^{(4)})$. □

Remark 6.2. Since $\omega_1^{(1)} > 0$, it follows from (123) that $a^{(4)} > a^{(1)}$.

Proof of Proposition 1.8. It is straightforward to check that the function

$$\chi(z) = \psi\left(-\frac{a^3}{2}(1+z)\right) - \psi(\infty^{(0)}), \quad \infty^{(0)} \in \mathcal{R},$$

is a conformal representation of the Riemann surface $\mathcal{S}$ constructed as $\mathcal{R}$ (22) but formed by the sheets

$$\mathcal{S}_0 := \overline{\mathbb{C}} \setminus [-\mu, -1], \qquad \mathcal{S}_1 := \overline{\mathbb{C}} \setminus ([-\mu, -1] \cup [1, \lambda]), \qquad \mathcal{S}_2 := \overline{\mathbb{C}} \setminus [1, \lambda],$$

where λ and μ are defined in (24). χ also satisfies $\chi(z) = z + O(1)$ as $z \to \infty^{(1)}$, and has a simple zero at $\infty^{(0)} \in \mathcal{S}$. Observe that $\chi(\infty^{(2)}) = -\psi(\infty^{(0)})$ (the reader is cautioned that in this relation, $\infty^{(2)} \in \mathcal{S}$ and $\infty^{(0)} \in \mathcal{R}$).

χ and $\mathcal{S}$ are the types of conformal mappings and Riemann surfaces analyzed in [11]. It follows from [11, Theorem 3.1] that $\chi(\infty^{(2)}) = 2/H(\beta)$, where H and β are described in the statement of Proposition 1.8 (the uniqueness of β and γ is justified in [11]). So $\chi(z) = \psi(-a^3(1+z)/2) + 2/H(\beta)$. It also follows from [11, Theorem 3.1] that the function $w = H(\beta)\chi(z) - 1$ is the solution of the algebraic equation

$$w^3 - (H(\beta)z + \Theta_1 - \Theta_2 - h)w^2 - (1 + \Theta_1 + \Theta_2)w + H(\beta)z - h = 0,$$

where Θ_1, Θ_2, and h are the constants described in the statement of Proposition 1.8. Simple computations and a change of variable yield immediately that $w = \psi(z)$ is the solution of Eq. (25). □

7. The *n*th root asymptotics and zero asymptotic distribution of the polynomials Q_n and $Q_{n,2}$

It is well known (see [14]) that if $E \subset \mathbb{C}$ is a compact set that is regular with respect to the Dirichlet problem, and ϕ is a continuous real-valued function on E, then there exists a unique $\widetilde{\mu} \in \mathcal{M}_1(E)$ satisfying the variational conditions

$$V^{\widetilde{\mu}}(z) + \phi(z) \begin{cases} = w, & z \in \operatorname{supp}(\widetilde{\mu}), \\ \geq w, & z \in E, \end{cases}$$

for some constant w. The measure $\widetilde{\mu}$ is called the equilibrium measure in the presence of the external field ϕ on E, and w the equilibrium constant.

Recall that we defined λ_1 to be the positive, rotationally invariant measure on S_0 whose restriction to the interval $[0, \alpha]$ coincides with the measure $s_1(x)\mathrm{d}x$, and we defined λ_2 to be the positive, rotationally invariant measure on S_1 whose restriction to the interval $[-b, -a]$ coincides with the measure $s_2(x)\mathrm{d}x$.

Lemma 7.1. *Suppose that* $\lambda_1, \lambda_2 \in \mathbf{Reg}$. *Then the following measures are also regular:*

$$\frac{s_1(\sqrt[3]{\tau})}{\tau^{2/3}}\mathrm{d}\tau, \qquad s_1(\sqrt[3]{\tau})\sqrt[3]{\tau}\mathrm{d}\tau, \quad \tau \in [0, \alpha^3], \tag{124}$$

$$s_2(\sqrt[3]{\tau})\mathrm{d}\tau, \qquad \frac{s_2(\sqrt[3]{\tau})}{\sqrt[3]{\tau}}\mathrm{d}\tau, \qquad \frac{s_2(\sqrt[3]{\tau})}{\tau^{2/3}}\mathrm{d}\tau, \quad \tau \in [-b^3, -a^3]. \tag{125}$$

Proof. Let π_n be the nth monic orthogonal polynomial associated with λ_1, i.e., π_n is the monic polynomial of degree n that satisfies

$$\int_{S_0} \pi_n(t)\overline{t^k}\mathrm{d}\lambda_1(t) = 0, \quad 0 \le k \le n-1. \tag{126}$$

It is immediate to check that $\pi_n(e^{\frac{2\pi i}{3}}z) = e^{\frac{2\pi i n}{3}}\pi_n(z)$. We deduce from this property and (126) that the polynomials

$$\pi_{3k}(\sqrt[3]{\tau}), \qquad \frac{\pi_{3k+1}(\sqrt[3]{\tau})}{\sqrt[3]{\tau}}, \qquad \frac{\pi_{3k+2}(\sqrt[3]{\tau})}{\tau^{2/3}},$$

are precisely the monic orthogonal polynomials of degree k associated, respectively, with the measures

$$\frac{s_1(\sqrt[3]{\tau})}{\tau^{2/3}}\mathrm{d}\tau, \qquad s_1(\sqrt[3]{\tau})\mathrm{d}\tau, \qquad s_1(\sqrt[3]{\tau})\tau^{2/3}\mathrm{d}\tau. \tag{127}$$

We also have:

$$\begin{aligned}
\int_{S_0} |\pi_{3k}(t)|^2\mathrm{d}\lambda_1(t) &= \int_0^{\alpha^3} (\pi_{3k}(\sqrt[3]{\tau}))^2 \frac{s_1(\sqrt[3]{\tau})}{\tau^{2/3}}\mathrm{d}\tau,\\
\int_{S_0} |\pi_{3k+1}(t)|^2\mathrm{d}\lambda_1(t) &= \int_0^{\alpha^3} \left(\frac{\pi_{3k+1}(\sqrt[3]{\tau})}{\sqrt[3]{\tau}}\right)^2 s_1(\sqrt[3]{\tau})\mathrm{d}\tau,\\
\int_{S_0} |\pi_{3k+2}(t)|^2\mathrm{d}\lambda_1(t) &= \int_0^{\alpha^3} \left(\frac{\pi_{3k+2}(\sqrt[3]{\tau})}{\tau^{2/3}}\right)^2 s_1(\sqrt[3]{\tau})\tau^{2/3}\mathrm{d}\tau.
\end{aligned}$$

So taking into account (see [13, Theorem 5.2.5]) that

$$\mathrm{cap}(\mathrm{supp}(\lambda_1)) = \mathrm{cap}(\mathrm{supp}(\rho))^{1/3},$$

where $\mathrm{cap}(A)$ denotes the logarithmic capacity of A, and ρ is any of the three measures in (127), the regularity of λ_1 implies the regularity of the three measures in (127).

Let l_n denote the nth monic orthogonal polynomial associated with the measure $\mathrm{d}\rho_1(\tau) := s_1(\sqrt[3]{\tau})\sqrt[3]{\tau}\mathrm{d}\tau$, and let T_n be the nth Chebyshev polynomial (see [13], page 155) for the set $E := \mathrm{supp}(\rho_1)$. We have

$$\left(\int l_n^2(\tau)\mathrm{d}\rho_1(\tau)\right)^{1/2} \le \left(\int T_n^2(\tau)\mathrm{d}\rho_1(\tau)\right)^{1/2} \le \|T_n\|_E \rho_1(E)^{1/2},$$

where $\|T_n\|_E$ denotes the supremum norm of T_n on E, and so by [13, Corollary 5.5.5] we obtain

$$\limsup_{n\to\infty} \|l_n\|_2^{1/n} \le \lim_{n\to\infty} \|T_n\|_E^{1/n} = \operatorname{cap}(\operatorname{supp}(\rho_1)). \tag{128}$$

If we call $\widetilde{l}_n$ the nth monic orthogonal polynomial associated with the measure $d\rho_2(\tau) := s_1(\sqrt[3]{\tau})\tau^{2/3}d\tau$, we have

$$\left(\int \widetilde{l}_n^2(\tau)d\rho_2(\tau)\right)^{1/2} \le \alpha^{1/2}\left(\int l_n^2(\tau)d\rho_1(\tau)\right)^{1/2},$$

and so the regularity of ρ_2 and (128) imply the regularity of ρ_1. Similar arguments show that the measures in (125) are regular. □

Proof of Theorem 1.10. Recall that if P is a polynomial, we indicate by μ_P the associated normalized zero counting measure. Let $j \in \{0, \dots, 5\}$ be fixed, and assume that for some subsequence $\Lambda \subset \mathbb{N}$ we have:

$$\mu_{P_{6k+j}} \xrightarrow{*} \mu_1 \in \mathcal{M}_1(\Delta_1), \qquad \mu_{P_{6k+j,2}} \xrightarrow{*} \mu_2 \in \mathcal{M}_1(\Delta_2).$$

Consequently,

$$\lim_{k\in\Lambda} \frac{1}{2k}\log|P_{6k+j}(z)| = -V^{\mu_1}(z), \quad z \in \mathbb{C}\setminus\Delta_1, \tag{129}$$

$$\lim_{k\in\Lambda} \frac{1}{4k}\log|P_{6k+j,2}(z)| = -\frac{1}{4}V^{\mu_2}(z), \quad z \in \mathbb{C}\setminus\Delta_2, \tag{130}$$

uniformly on compact subsets of the indicated regions.

We know by Proposition 5.2 that there exists a fixed measure $d\rho$ supported on Δ_1 ($d\rho$ is one of the measures in (124)) such that

$$0 = \int_{\Delta_1} \tau^j P_{6k+j}(\tau)\frac{d\rho(\tau)}{P_{6k+j,2}(\tau)}, \quad 0 \le j < \deg(P_{6k+j}). \tag{131}$$

We know by Lemma 7.1 that the measure $d\rho$ is regular. If we apply [7, Lemma 4.2] (taking, in the notation of [7], $d\sigma = d\rho$, $\phi_{2k} = 1/P_{6k+j,2}$ and $\phi = -(1/4)V^{\mu_2}$), we obtain from (130) and (131) that μ_1 is the equilibrium measure in the presence of the external field $\phi = -(1/4)V^{\mu_2}$, hence

$$V^{\mu_1}(\tau) - \frac{1}{4}V^{\mu_2}(\tau)\begin{cases} = w_1, & \tau \in \operatorname{supp}(\mu_1),\\ \ge w_1, & \tau \in \Delta_1,\end{cases} \tag{132}$$

and

$$\lim_{k\in\Lambda}\left(\int_{\Delta_1} P_{6k+j}^2(\tau)d\nu_{6k+j}(\tau)\right)^{1/4k} = e^{-w_1}, \tag{133}$$

where the measure $d\nu_{6k+j}$ is defined in (69).

By Proposition 5.1, there exists a fixed measure $d\eta$ ($d\eta$ is one of the measures in (125)) supported on Δ_2 such that

$$0 = \int_{\Delta_2} \tau^j P_{6k+j,2}(\tau)\frac{|h_{6k+j}(\sqrt[3]{\tau})|}{|P_{6k+j}(\tau)|}d\eta(\tau), \quad 0 \le j < \deg(P_{6k+j,2}). \tag{134}$$

The function h_{6k+j} is defined in (68). We also know by Lemma 7.1 that $d\eta$ is regular. Taking into account the representations (70) and the fact that p_n is orthonormal with respect to $d\nu_n$ (see (67) and Proposition 5.3), it follows that there exist positive constants C_1, C_2 such that

$$C_1 \leq |h_{6k+j}(\sqrt[3]{\tau})| \leq C_2 \quad \text{for all } \tau \in \Delta_2.$$

So applying again [7, Lemma 4.2] (now take $d\sigma = d\eta$, $\phi_k(\tau) = |h_{6k+j}(\sqrt[3]{\tau})|/|P_{6k+j}(\tau)|$ and $\phi = -V^{\mu_1}$), we get from (134) and (129) that μ_2 is the equilibrium measure in the presence of the external field $\phi = -V^{\mu_1}$, and so

$$V^{\mu_2}(\tau) - V^{\mu_1}(\tau) \begin{cases} = w_2, & \tau \in \operatorname{supp}(\mu_2), \\ \geq w_2, & \tau \in \Delta_2, \end{cases} \tag{135}$$

and

$$\lim_{k \in \Lambda} \left(\int_{\Delta_2} P_{6k+j,2}^2(\tau) d\nu_{6k+j,2}(\tau) \right)^{1/2k} = e^{-w_2}, \tag{136}$$

where the measure $d\nu_{6k+j,2}$ is defined in (69).

By (132) and (135), the vector measure (μ_1, μ_2) solves the equilibrium problem determined by the interaction matrix (27) on the intervals Δ_1, Δ_2. Since the solution to this equilibrium problem must be unique, (26) follows. (133) and (136) imply (29). Finally, (28) is an immediate consequence of (26). □

Proof of Proposition 1.12. By Theorem 1.5 we know that the following limit holds:

$$\lim_{k \to \infty} \frac{Q_{6(k+1)}(z)}{Q_{6k}(z)} = \prod_{i=0}^{5} \widetilde{F}_1^{(i)}(z^3), \quad z \in \mathbb{C} \setminus S_0.$$

Therefore we obtain that

$$\lim_{k \to \infty} |Q_{6k}(z)|^{1/k} = \prod_{i=0}^{5} |\widetilde{F}_1^{(i)}(z^3)|, \quad z \in \mathbb{C} \setminus S_0,$$

and by Corollary 1.11 it follows that

$$e^{-\frac{1}{3} V^{\overline{\mu}_1}(z^3)} = \prod_{i=0}^{5} |\widetilde{F}_1^{(i)}(z^3)|^{1/6} \quad z \in \mathbb{C} \setminus S_0.$$

So the first relation in (30) is proved. The same argument justifies the other relation. □

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