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A Population Model with Nonlinear Diffusion

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A model is presented for a single species population moving in a limited one-dimensional environment. The birth-death process is specialized by assuming a constant death modulus and a birth modulus which is an exponential in the age. The diffusion mechanism is nonlinear and results in a problem for the space population density which has a degenerate parabolic form and is similarly to the porous media equation. It is shown that the effect of the nonlinearity in the diffusion is to produce an approach to steady state even when the process is birth dominant. The interaction of the birth-death and diffusion processes is studied and is shown to yield a modified birth-death mechanism which is both time and space dependent.

1. Introduction

The model to be discussed here derives from earlier work in [2–4]. It concerns a single species population moving in a one-dimensional environment. References [2, 3] describe a quite general birth-death process, including age effects, but with no diffusion. Reference [4], on the other hand, treats diffusion in an infinite environment but with the simplest possible birth-death process, one containing no age dependence. The principal result of [4] is that the diffusion mechanism should be nonlinear.

This paper is a first step toward a general model combining all the aspects of [2–4] and at the same time treating the effect of limitations on the environment. Let us begin by describing the general model even though we are able to treat only a very special case.

The independent variables are time t, age a and position x. The dependent variables are *age-space population density* $\rho(t, a, x)$ and *space population density* $P(t, x)$ with

$$P(t, x) = \int_0^\infty \rho(t, a, x)\, da. \tag{1.1}$$

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Following [2, 3] there would be a *birth-law* of the form

$$\rho(t, 0, x) = \int_0^\infty \beta(a, P(t, x))\, \rho(t, a, x)\, da. \tag{1.2}$$

β is called the *birth-modulus*. Again from [2, 3] there would be a death-law which gives σ, the time rate of loss of individuals of age a through death, as

$$\sigma = -\lambda(a, P(t, x))\, \rho(t, a, x), \tag{1.3}$$

where λ is the *death modulus*.

If the population is free to move Ref. [4] postulates an additional rate of change of individuals of age a through diffusion. This rate is

$$-\frac{\partial}{\partial x}(q(t, a, x)\, \rho(t, a, x)), \tag{1.4}$$

where q is the *diffusion velocity*. The idea of [4] is to assume that individuals move to avoid crowding and this is modelled by choosing

$$q(t, a, x) = -K(a)\, P_x(r, x), \tag{1.5}$$

where $K(a) > 0$. The basic equation of the model is then the *law of population balance* for individuals of age a. This is

$$\rho_t + \rho_a = \sigma - \frac{\partial}{\partial x}(q\rho) = -\lambda(a, P)\, \rho + \frac{\partial}{\partial x}(KP_x \rho). \tag{1.6}$$

The general model would involve solving (1.6) subject to (1.2). To limit the environment one requires $0 < x < L$ in which case boundary conditions are needed. The most interesting ones, and those used here, represent extremely inhospitable ends, which means

$$\rho(t, a, 0) \equiv \rho(t, a, L) \equiv 0. \tag{1.7}$$

The object then would be to start with an initial age-space distribution,

$$\rho(0, a, x) = \rho_0(a, x), \tag{1.8}$$

and determine how ρ evolves with time.

The general problem has so far proved too difficult. We have, however, succeeded in treating a very special case which we believe indicates the essential features. We take the death modulus to be a (positive) constant λ and we take the birth modulus to have the special form $\beta e^{-\alpha a}$, where $\beta > 0$ and $\alpha \geqslant 0$ are constants.[1] The case $\alpha = 0$ we will term *age-independent*. This

[1] This special case was considered in [2, 3].

was the case considered in [4]. We also assume $K(a)$ is a constant which we can normalize to one. These assumptions lead to the specific problem we study. Let D denote the set

$$D = \{(t, a, x): t > 0, a > 0, 0 < x < L\}.$$

PROBLEM. $(\mathscr{P}_\alpha)$ For $\alpha \geqslant 0$ find $\rho: D \to R^+$ such that:

(i) $\rho(t, \cdot, x) \in L_1(0, \infty)$, $\lim_{a\to\infty} \rho(t, a, x) = 0$ for each (t, x);

(ii) $\rho_t + \rho_a = -\lambda\rho + (\partial/\partial x)(\rho P_x)$; $P(t, x = \int_0^\infty \rho(t, a, x)\, da$;

(iii) $\rho(t, 0, x) = \beta B(t, x)$, $B(t, x) = \int_0^\infty e^{-\alpha a}\rho(t, a, x)\, da$;

(iv) $\rho(t, a, 0) \equiv \rho(t, a, L) \equiv 0$;

(v) $\rho(0, a, x) = \rho_0(a, x)$.

A precise statement for our results is given in the next section but there are two features which we wish to emphasize:

(I) *In a limited environment the nonlinear diffusion stabilizes the system in the following sense. Even if the birth-death process is unstable there will be a unique (positive) steady state distribution which is globally asymptotically stable.*

(II) *If there is age dependence ($\alpha > 0$ here) then even when the birth-death process is temporally and spatially homogeneous the interaction of the diffusion produces an effective birth-death process which is both temporally and spattially inhomogeneous.*

In the next section we will compare our model with the earlier linear diffusion model of Skellam [9].

2. Statement of Results

Since we want to illustrate feature (I) of the introduction we study here the case in which the birth-death process is unstable. We require that

$$\beta > \lambda + \alpha. \qquad (c_1)$$

Without diffusion this would imply that the populations would grow exponentially. We will impose two further conditions to simplify the analysis. These are

$$\rho_0(a, x) > 0 \qquad \text{for } 0 < x < L, \qquad (c_2)$$

$$\frac{\int_0^\infty \rho_0(a, x)\, da}{\int_0^\infty e^{-\alpha a}\rho_0(a, x)\, da} \leqslant \bar{m} < \frac{\beta}{\beta - \alpha} \qquad \text{for } 0 < x < L. \qquad (c_3)$$

We assume that (c_1)–(c_3) hold throughout the paper, except for Remark 2.2 below.

The basic idea of our procedure is easy to state. We view (ii) of $(\mathscr{P}_\alpha)$ as a linear, homogeneous first order partial differential equation for ρ with P given. This equation can be solved by integrating along characteristics.[2] Substitution in the other conditions then yields a problem for P and B. We can describe the latter easily. We perform two calculations. First integrate (ii) with respect to a from 0 to ∞ and second multiply (ii) by $e^{-\alpha a}$ and integrate. One obtains the following:

PROBLEM (P_α). For $\alpha \geqslant 0$ find (P, B): $(0, \infty) \times (0, L) \to R^+$ such that

(i) $P_t = \beta B - \lambda P + (\partial/\partial x)(PP_x)$;
$B_t = (\beta - \lambda - \alpha) B + (\partial/\partial x)(BP_x)$;

(ii) $P(t, 0) \equiv P(t, L) \equiv B(t, 0) \equiv B(t, L) \equiv 0$;

(iii) $P(0, x) = P_0(x) \equiv \int_0^\infty \rho_0(a, x)\, da$,
$B(0, x) = B_0(x) \equiv \int_0^\infty e^{-\alpha a} \rho_0(a, x)\, da$.

For the special case $\alpha = 0$ it is easy to see that $B = \beta P$ and the problem reduces to the single differential equation problem:

PROBLEM (P_0). Find P: $(0, \infty) \times (0, L) \to R^+$ such that

(i) $P_t = \delta P + (\partial/\partial x)(PP_x)$, $\delta = \beta - \lambda$,

(ii) $P(t, 0) \equiv P(t, L) \equiv 0$,

(iii) $P(0, x) = P_0(x)$.

We will be particularity interested in steady state solutions. For (P_0) and (P_α) these are:[3]

PROBLEM (P_0^s). Find p, with $p(x) > 0$ in $(0, L)$ such that

$$(pp')' + \delta p = 0, \qquad p(0) = p(L) = 0.$$

PROBLEM (P_α^s). For $\alpha > 0$ find (p, b); $p(x) > 0$, $b(x) > 0$ on $(0, L)$ such that

$$(pp') + \beta b - \lambda p = 0, \qquad (bp')' + (\beta - \lambda - \alpha)\, b = 0;$$
$$p(0) = p(L) = b(0) = b(L) = 0.$$

[2] As stated in [4] this procedure shows that our model guarantees the positivity of ρ given only that ρ_0 is positive.

[3] Note that the trivial solutions are ruled out here.

The steady state problem for $(\mathscr{P}_\alpha)$ is:

PROBLEM $(\mathscr{P}^s_\alpha)$. Find r, $r(a, x) > 0$ for $x \in (0, L)$ such that

(i) $r(\cdot, x) \in L_1(0, \infty)$, $\lim_{a \to \infty} r(a, x) = 0$ for each x in $(0, L)$;

(ii) $r_a = -\lambda r + (\partial/\partial x)(rp')$, $p(x) = \int_0^\infty r(a, x)\, da$;

(iii) $r(0, x) = \beta b(x)$, $b(x) = \int_0^\infty e^{-\alpha a} r(a, x)\, da$;

(iv) $r(a, 0) \equiv r(a, L) \equiv 0$.

We now state our results in the order in which they will be proved.

THEOREM I. (a) *There exists a unique solution of* (P^s_α) *for* $\alpha \geqslant 0$.

(b) *There exists a unique solution of* $(\mathscr{P}^s_\alpha)$ *for* $\alpha \geqslant 0$.

THEOREM II. (a) *There exists a unique solution of* (P_0).

(b) *There exists a unique solution of* $(\mathscr{P}_0)$.

(c) *As* $t \to \infty$ *the solution of* (P_0) $(\mathscr{P}_0)$) *tends to that of* (P^s_0) $((\mathscr{P}^s_0))$.

THEOREM III. (a) *There exists a solution of* (P_α) *for* $\alpha > 0$.

(b) *There exists a solution of* $(\mathscr{P}_\alpha)$ *for* $\alpha > 0$.

(c) *As* $t \to \infty$ *the solutions of parts* (a) *and* (b) *tend to slutions of* (P^s_α) *and* $(\mathscr{P}^s_\alpha)$, *respectively*.

Remark 2.1. Notice that uniqueness is missing in Theorem III. We believe the solution is unique but have not been able to prove it.

Remark 2.2. If the birth-death process is stable, $\beta < \lambda + \alpha$, Theorem II(a) and (b) and Theorem III(a) and (b) remain true. In this case the only steady-state solutions are the trivial ones and the populations all tend to zero.

Remark 2.3. Theorems II(c) and III(c) confirm feature (I) of the Introduction.

In Section 5 we will show that there exists a function $M(t, x)$ such that

$$M(t, x) > 1, \qquad M(t, x) \to (\beta - \alpha)/\beta \qquad \text{as} \quad t \to \infty,$$

and having the property that if (P, B) is the solution of (P_α) for $\alpha > 0$ then,

$$P_t = \frac{\beta}{M(t, x)} P - \lambda P + \frac{\partial}{\partial x}(P P_x). \tag{2.1}$$

As $\alpha \to 0$ the quantity M becomes identically equal to one. This is the technical statement of feature (II) of the Introduction. The quantity M is actually a functional of P and this demonstrates the essentially hereditary aspect of the age-dependent model, $\alpha > 0$.

In the Introduction we mentioned the work of ref. [9]. That paper dealt only with the space population density and thus its results can be compared only with (P_0). The difference is in the diffusion mechanism. In [9] it was assumed that P satisfies $P_t = \delta P + P_{xx}$. The corresponding steady state equation is then $\delta p + p'' = 0$. Observe that the qualitative behavior is very different. There will be steady state solutions only for one value of L. For any other value of L the solution would grow exponentially or decay exponentially to zero.

3. Steady State: Proof of Theorem I

We begin with the observation that the proof of Theorem I can be reduced to the case $\alpha = 0$. Suppose we have existence for $\alpha = 0$. Then given $\alpha > 0$ we can solve the problem

$$(pp')' + \delta p = 0, \qquad p(0) = p(L) = 0, \tag{3.1}$$

for $\delta = \beta - \lambda - \alpha$, $\delta > 0$ by (c_1). Then one can check that p and $b = [(\beta - a)/\beta]\, p$ satisfies (P^s_α).

Next we consider the uniqueness of solutions of (P^s_α). Suppose (p, b) is a solution. We multiply the first equation by $(\beta - \alpha)/\beta$ and subtract from the second. The result is

$$(Zp')' = \lambda Z, \qquad Z = b - \frac{\beta - \alpha}{\beta}\, p. \tag{3.2}$$

We assert that $Z \equiv 0$. If we can show this then the substitution of $b = [(\beta - \alpha)/\beta]\, p$ into the first equation in (P^s_α) yields $(pp')' - (\beta - a - \lambda)\, P = 0$. Thus p is the (unique) solution of (3.1) for $\delta = \beta - a - \lambda$.

The argument that $Z = 0$ is complicated a little by the fact, to be seen later, that p' becomes infinite at $x = 0$ and $x = L$. Thus we cannot conclude from (3.2) that $z \equiv 0$ because it is zero at $x = 0$. We argue as follows. Since $p(0) = p(L) = 0$ there is an $\bar{x} \in (0, L)$ such that $p'(\bar{x}) = 0$ but $p'(x) > 0$ on $(0, \bar{x})$. Now if Z were zero anywhere in $0, \bar{x})$ then it would be zero identically by uniqueness and we would be done. Suppose then that $Z > 0$. Then we integrate (3.2) from ε to $\bar{x}$ and find

$$\limsup_{\epsilon \downarrow 0} \lambda \int_\epsilon^{x} Z(x)\, dx = \limsup_{\epsilon \downarrow 0} (-Z(\varepsilon)\, p'(\varepsilon)) \leqslant 0,$$

a contradiction. Similarly we cannot have $Z < 0$ on $(0, \bar{x})$.

Theorem I is thus reduced to showing that (3.1) has a unique solution which is positive on $(0, L)$. Put $u = p^2$. Then (3.1) becomes

$$u'' + 2\delta\sqrt{u} = 0, \qquad u(0) = u(L) = 0. \tag{3.3}$$

We first establish the uniqueness. Let u and v be solutions which are positive on $(0, L)$. Suppose there is an interval (x_1, x_2) such that $v(x_1) = u(x_1)$, $v(x_2) = u(x_2)$, $v(x) > u(x)$ on (x_1, x_2). We cannot have $u'(x_1) = v'(x_1)$ or $u'(x_2) = v'(x_2)$ (by uniqueness); hence $v'(x_1) > u'(x_1)$ and $v'(x_2) < u'(x_2)$. But then we have

$$[v'(x_2)\,u(x_2) - v(x_2)\,u'(x_2)] - [v'(x_1)\,u(x_1) - v(x_1)\,u'(x_1)]$$
$$+ \delta\int_{x_1}^{x_2} \sqrt{uv}(\sqrt{u} - \sqrt{v})\,dx = 0,$$

which is a contradiction since every term on the left is negative.

This uniqueness proof is adapted from one in [5]. One can also use ideas from that reference to give a variational proof of existence as we indicate in Section 4 (Remark 4.1). We present here, however, a more direct proof of existence. Clearly if we solve

$$v'' + 2\delta\sqrt{v} = 0, \qquad v(0) = v'(L/2) = 0, \tag{3.4}$$

then the function $u(x) = v(x)$ on $(0, L/2)$, $u(x) = v(L - x)$ on $(L/2, L)$ solves (3.3).

We define the function $\Gamma(\xi)$ for $0 \leqslant \xi \leqslant 1$ by

$$\Gamma(\xi) = \int_0^{\xi} \frac{d\eta}{\sqrt{1 - \eta^{3/2}}}, \tag{3.5}$$

and define $\bar{c}$ by

$$\left(\frac{3}{8\delta}\right)^{2/3} \bar{c}^{1/3}\Gamma(1) = \frac{L}{2}. \tag{3.6}$$

Then we define $v(x)$ by the formula

$$\left(\frac{3}{8\delta}\right)^{2/3} \bar{c}^{1/3}\Gamma\left(\left(\frac{8\delta}{3\bar{c}^2}\right)^{2/3} v(x)\right) \equiv x. \tag{3.7}$$

We see that this equation uniquely determines $v(x)$ on $[0, L/2]$, since $(3/8\delta)^{2/3}\,\bar{c}^{1/3}\Gamma(\xi)$ is a monotone increasing function of ξ for $\xi \in [0, 1]$. It is zero at $\xi = 0$ and, by (3.6), is $L/2$ at $\xi = 1$. We have $v(0) = 0$. Moreover differentiating (3.7) we obtain,

$$v' = \sqrt{\bar{c}^2 - \frac{8\delta}{3}(v(x))^{3/2}}. \tag{3.8}$$

Squaring, differentiating again and cancelling $v'(x)$ we obtain $v'' + 2\delta\sqrt{v} = 0$. We have $v(0) = 0$ and $v(L/2)(8\delta/3\bar{c}^2)^{3/2} = 1$ and hence, by (3.8), $v'(L/2) = 0$. Thus we have a solution of (3.4). Note that $\bar{c} = v'(0)$.

Remark 3.1. Since $v(x) \sim \bar{c}x$ as $x \to 0$ we see that $p(x) = \sqrt{v(x)}$ will satisfy $p'(x) \sim \sqrt{c}/2\sqrt{x}$ as $x \to 0$, that is, $p'(x) \to +\infty$ as $x \to 0$. Similarly $p'(x) \to -\infty$ as $x \to L$.

The proof of Theorem I(a) is complete and we turn to that of Theorem I(b). Let us proceed formally at first. Suppose r is a solution of $(\mathscr{P}_\alpha)$. Then we have $r_a - p'r_x + (\lambda - p'')\,r = 0$. We integrate this (first order) equation along characteristic curves defined by $x = X(a; \bar{a}, \bar{x})$,

$$\frac{\partial X}{\partial a} = -p'(X), \qquad X(\bar{a}; \bar{a}, \bar{x}) = \bar{x}. \tag{3.9}$$

If we integrate from $a = 0$ to $\bar{a}$ and use $r(0, x) = \beta b(x)$ we find

$$r(\bar{a}, \bar{x}) = \text{ewp}\left\{-\lambda\bar{a} - \int_0^{\bar{a}} p''(X(a; \bar{a}, \bar{x}))\, da\right\} b(X(0; \bar{a}, \bar{x})). \tag{3.10}$$

We can simplify formula (3.10). From (3.9) we have

$$\frac{\partial}{\partial a}\left(\frac{\partial X}{\partial \bar{x}}\right) = -p''(X)\, X_{\bar{x}}, \tag{3.11}$$

so that $p'' = -(\partial/\partial a) \ln X_{\bar{x}}$. If we substitute this into the integral in (3.10) we obtain

$$r(\bar{a}, \bar{x}) = \beta e^{-\lambda\bar{a}} X_{\bar{x}}(0; \bar{a}, \bar{x})\, b(X(0, \bar{a}, \bar{x})). \tag{3.12}$$

So far our work is formal. We prove Theorem I(b) by verifying that r, as defined by (3.9) and (3.12), with $(p, b) = (p, [(\beta - \alpha)/\beta]\, p)$ the solution of (P^s_α), yields a solution of $(\mathscr{P}^s_\alpha)$. That the solution is unique follows from the fact that if we do have a solution of $(\mathscr{P}^s_\alpha)$ then the above calculations show that it must be given by (3.12) with (p, b) a solution of (P^s_α). We start by establishing some properties of X.

LEMMA 3.1. (a) *For any* $(\bar{a}, \bar{x})$ *problem* (3.9) *has a unique solution for* $a \in [0, \bar{a}]$. *This solution is differentiable in* $\bar{a}$, $\bar{x}$.

(b) $0 \leqslant X_{\bar{x}}(a; \bar{a}, \bar{x}) \leqslant 1$.

(c) $X_{\bar{a}}(a; \bar{a}, \bar{x}) - p'(\bar{x}) X_{\bar{x}}(a; \bar{a}, \bar{x}) \equiv 0$.

Proof. Since p' is locally Lipschitz (3.9) will have a local solution. This solution can fail to be global only if X tends to 0 or L as a tends to some $a_1 > 0$. But Remark 3.1 shows that this cannot happen. The differentiability with respect to $\bar{a}$ and $\bar{x}$ follow in a standard way. To prove (b) we note that $X_{\bar{x}}$ satisfies the linear equation (3.11) with $X_{\bar{x}}(\bar{a}; \bar{a}, \bar{x}) = 1$. We see by (3.3) that $p'' < 0$ in $(0, L)$ and hence (b) follows. Part (c) is proved in a similar way. Let χ be the quantity in question. Then, from (3.9), $\partial\chi/\partial a - p''(X)\chi$ and $\chi(\bar{a}; \bar{a}, \bar{x}) = X_{\bar{a}}(\bar{a}; \bar{a}, \bar{x}) - p'(\bar{x}) X_{\bar{x}}(\bar{a}; \bar{a}, \bar{x}) = -X_a(\bar{a}; \bar{a}, \bar{x}) - p'(\bar{x}) = p'(\bar{x}) - p'(\bar{x}) = 0$.

We now define r by (3.12). It follows immediately from Lemma 3.1(b) that (i) of $(\mathscr{P}^s_\alpha)$ is satisfied. Next we observe that

$$\begin{aligned} r_{\bar{a}}(\bar{a}, \bar{x}) &+ \lambda r(\bar{a}, \bar{x}) - (\partial/\partial\bar{x})(r(\bar{a}, \bar{x})\, p'(\bar{x})) \\ &= e^{-\lambda\bar{a}}(\beta - \alpha)\{ p(X(0; \bar{a}, \bar{x})) \\ &\quad \times [X_{\bar{x}\bar{a}}(0; \bar{a}, \bar{x}) - p'(\bar{x}) X_{\bar{x}\bar{x}}(0; \bar{a}, \bar{x}) - p''(\bar{x}) X_{\bar{x}}(0; a, x)] \\ &\quad + X_{\bar{x}}(0; \bar{a}, \bar{x})\, p'(X(0; \bar{a}, \bar{x}))[X_{\bar{a}}(0; \bar{a}, \bar{x}) - p'(\bar{x}) X_{\bar{x}}(0; \bar{a}, \bar{x})] \}. \end{aligned}$$

Lemma 3.1(c) shows that the two quantities in square brackets are zero and hence we have

$$r_{\bar{a}}(\bar{a}, \bar{x}) + \lambda r(\bar{a}, \bar{x}) = \frac{\partial}{\partial\bar{x}} (r(\bar{a}, \bar{x})\, p'(\bar{x})). \tag{3.13}$$

From the definition of X we have $X_{\bar{x}}(0; 0, \bar{x}) \equiv 1$; hence (3.12) also yields

$$r(0, \bar{x}) = \beta b(\bar{x}) = (\beta - \alpha)\, p(x). \tag{3.14}$$

What remains to be shown is that if we define $\tilde{p}(x)$ and $\tilde{b}(\bar{x})$ by

$$\tilde{p}(\bar{x}) = \int_0^\infty r(\bar{a}, \bar{x})\, d\bar{a}, \qquad \tilde{b}(\bar{x}) = \int_0^\infty e^{-\alpha\bar{a}} r(\bar{a}, \bar{x})\, d\bar{a}, \tag{3.15}$$

then $\tilde{p}(\bar{x}) = p(\bar{x})$ and $\tilde{b}(\bar{x}) = b(\bar{x})$. Then (3.13) and (3.14) will imply that (ii) and (iii) of $(\mathscr{P}^s_\alpha)$ are satisfied. Since $r > 0$ it will follow ffrom $p(0) = p(L) = b(0) = b(L) = 0$ that (iv) is also satisfied.

We have from (3.15), (3.14) that

$$\begin{aligned} (p'(\bar{x})\,\tilde{p}(\bar{x}))' &= \int_0^\infty (p'(\bar{x})\, r(\bar{a}, \bar{x}))_{\bar{x}}\, d\bar{a} = \int_0^\infty (r_{\bar{a}}(\bar{a}, \bar{x}) + \lambda r(\bar{a}, \bar{x}))\, d\bar{a} \\ &= -(\beta - \alpha)\, p(\bar{x}) + \lambda\tilde{p}(\bar{x}). \end{aligned}$$

If we subtract the first of the equations in (P_α^s) we have

$$(p'(x)\,z)' = \lambda z, \qquad z = \tilde{p} - p.$$

Now we argue as in the proof of uniqueness for (P_α^s) and conclude that $z \equiv 0$. A similar calculation gives

$$(p'(\bar{x})\,\tilde{b}(\bar{x}))' = -\beta b(\bar{x}) + (\lambda + \alpha)\,\tilde{b}$$

and $(p'\zeta)' = (\lambda + \alpha)\,\zeta$, $\zeta = \tilde{b} - b$ so that $\zeta = 0$. This concludes the proof of Theorem I.

4. Age Independence: Proof of Theorem II

The treatment of Problem (P_0) is essentially that of the porous media equation (that is $\delta = 0$) in [7, 8]. We note, in fact, that the equation in (P_0) can be reduced to the porous media equation. This was done in [4]. First put $P(t, x) = e^{\delta t} v(t, x)$ and obtain $v_t = e^{\delta t}(vv_x)_x$. Now change the independent variable putting $\tau = (1/\delta)(e^{\delta t} - 1)$ and $u(\tau, x) = v((1/\delta)\log(1 + \delta\tau), x)$. Problem (P_0) is then converted into

$$u_\tau = \frac{\partial}{\partial x}(uu_x); \qquad u(\tau, 0) = u(\tau, L) = 0; \qquad u(0, x) = P_0(x). \tag{4.1}$$

The results of [8] yield the uniqueness of solutions of (4.1). They also yield the existence. We will describe the existence portion of the proof since we will use some information gained in the rest of the proof of Theorem II. For the outline of existence we return to the original problem. We consider the approximate problems,

$$\begin{aligned} P_t^n &= \frac{\partial}{\partial x}(P^n P_x^n) + \delta P^n + c^n, \\ p^n(t, 0) &= P^n(t, L) = c^n, \\ P^n(0, X) &= P_0(x) + c^n. \end{aligned} \tag{4.2}$$

Here $\{c_n\}$ is a monotone decreasing sequence of constants with $c_n \downarrow 0$.

We establish a priori bounds for solutions of (4.2).

LEMMA 4.1. *There exists a constant γ, independent of n, such that any solution of* (4.2) *satisfies* $0 < c^n \leqslant P^n(t, x) \leqslant \gamma$ *for all* (x, t).

Proof. The lower bound is an immediate consequence of the maximum principle. The upper bound is established as follows. Let $u^n = (P^n)^2$ as in Section 3. We then have

$$u_t^n/\sqrt{u^n} = u_{xx}^n + 2\delta\sqrt{u^n} + 2c^n; \qquad u_t^n(t, o) \equiv u_t^n(t, L) \equiv 0. \tag{4.3}$$

Now define $Q^n[v]$ for $v \in H_1(0, L)$ by

$$Q^n[v] = \int_0^L [\tfrac{1}{2}(v'(x))2 - \tfrac{4}{3}\delta(v(x))^{3/2} - 2c^n v(x)]\, dx. \tag{4.4}$$

Then for u^n satisfying (4.3) we have

$$\begin{aligned} \frac{d}{dt} Q^n[u^n(t, \cdot)] &= \int_0^L [u_x^n u_{xt}^n - 2\delta(u^n)^{1/2} u_t^n - 2c^n u_t^n]\, dx \\ &= -\int_0^L (u_{xx}^n + 2\delta\sqrt{u^n} + 2c^n)\, u_t^n\, dx \\ &= -\int_0^L \frac{(u_t^n)^2}{\sqrt{u^n}}\, dx. \end{aligned}$$

Thus $Q^n[u^n]$ is a decreasing function of t.

We verify next that $Q^n[u^n(t, \cdot)]$ is bounded below. For $v \in H_1(0, L)$ put $\| v \|_1 = (\int_0^L v'(x)^2\, dx)^{1/2}$. We have then, since $u^n(t, 0) - c^n = 0$ and $c^n \leqslant c^1$,

$$\begin{gathered} |u^n(t, x)| \leqslant c^1 + \sqrt{L}\, \|u^n(t, \cdot)\|_1, \\ \int_0^L (u^n(t, x))^{3/2}\, dx \leqslant M(1 + \|u^n(t, \cdot)\|_1^{3/2}), \\ \int_0^L u^n(t, x)\, dx \leqslant M'(1 + \|u^n(t, \cdot)\|_1), \end{gathered} \tag{4.5}$$

where M and M' are independent of n. Hence (4.4) yields

$$\begin{aligned} Q^n[u^n(t, \cdot)] \geqslant \tfrac{1}{2} \|u^n(t, \cdot)\|_1^2 - \tfrac{4}{3}M\delta \|u^n(t, \cdot)\|_1^{3/2} - 2c_1 M' \|u^n(t, \cdot)\|_1 \\ - \tfrac{4}{3}M\delta - 2c_1 M' \geqslant M'', \end{aligned} \tag{4.6}$$

where again M'' is independent of n. It follows that

$$M'' \leqslant Q^n[u^n(t, \cdot)] \leqslant Q^n[u^n(0, \cdot)] \leqslant M''' \tag{4.7}$$

where M''' is independent of n. Equations (4.6) and (4.7) imply that $\|u^n(\cdot, t)\|_1$ is bounded independently of n and thus $(4.5)_1$ yields the second inequality of the lemma.

Remark 4.1. If one sets $c^n = 0$ in (4.4) then one can show that the minimum of the resulting functional over $H_0^{(1)}(0, L)$ is the solution u of (3.3).

With the bounds of Lemma (4.1) one can show (this is done in [8]) that problems (4.2) have solutions for all t. Another use of the maximum principle shows that $u^{n+1}(t, x) \leqslant u^n(t, x)$ and thus for each (t, x) the $u^n(t, x)$ converge. The idea of [8] is to show that the limit function $u(t, x) = P^2(t, x)$, where P is a solution of (P_0). This proof is carried out in [8] (for the case $\delta = 0$) by showing first that the limit is a generalized solution. We will not carry out the details of this argument but we will give the modification to our problem of an estimate which is a part of it.

LEMMA 4.2. *For any $T > 0$ there exists a constant γ, independent of n such that $u_x^n(t, x)| \leqslant \gamma'$ for $0 \leqslant t \leqslant T$, $0 \leqslant x \leqslant L$.*

Proof. The idea is that u_x^n satisfies an equation to which we can apply the maximum principle. Indeed from (4.3) we have

$$(u_x^n)_t = (u_x^n)_{xx} \sqrt{u^n} + \left(2\delta + \frac{c^n}{\sqrt{u^n}}\right) u_x^n + \frac{u_{xx}^n}{2\sqrt{u^n}} u_x^n. \tag{4.8}$$

At an interior maximum or minimum of u_x^n we will have $u_{xx}^n = 0$. Moreover we have $2\delta + c^n/\sqrt{u^n} \leqslant 2\delta + 1$. We conclude that if $\lambda > 2\delta + 1$ and $v^n = e^{-\lambda t} u^n$ then $|v_x^n|$ will have its maximum on the boundary. We estimate u_x^n on the boundary. Put $w^n = u^n + \mu e^{-x}$. Then from (4.3) and Lemma (4.1) we have

$$\frac{w_t^n}{\sqrt{u^n}} - w_{xx}^n = \frac{u_t^n}{\sqrt{u^n}} - u_{xx}^n - \mu e^{-x} = 2\delta \sqrt{u^n} + c^n - \mu e^{-x} < 0, \tag{4.9}$$

for μ sufficiently large, independently of n. Hence w has its maximum on the boundary. We have

$$w^n(t, 0) = c^n + \mu > c^n + \mu e^{-L} = w^n(t, L).$$

Also,

$$\begin{aligned} w_x^n(x, 0) &= u_x^n(x, 0) - \mu e^{-x} \\ &= P_0^1(x) - \mu e^{-x} < \max |P_0^1(x)| - \mu e^{-L} < 0, \end{aligned}$$

for μ sufficiently large. Hence if μ is large enough w has its maximum on the boundary on the line $x = 0$. This is the maximum for the whole region (again for μ large enough); hence $w_x^n(t, 0) \leqslant 0$ or $u_x^n(t, 0) \leqslant \mu$. Similar arguments show that $u_x^n(t, L)$ is bounded independently of n and we obtain the conclusion of the lemma.

From Lemma (4.2) one can infer that on any rectangle $Q_T = [0, T] \times [0, L]$ a subsequence of the u_x^n converges weak star in $L_\infty(Q_T)$ and then, as in [8], u can be shown to be a generalized solution. Further, by using regularity results for parabolic problems as in [1, 5], it is shown that this generalized solution is a classical solution wherever it is positive. The next result shows that this is true for all $x \in (0, L)$.

LEMMA 4.3. *Given any* $\varepsilon > 0$ *there is a constant* $\delta_\epsilon > 0$ *such that* $P(t, x) \geqslant \delta_\epsilon$ *for* $t \geqslant 0$, $\varepsilon \leqslant x \leqslant L - \varepsilon$.

Proof. With ε fixed define the function $q_\epsilon(x)$, $\varepsilon/2 \leqslant x \leqslant L - \varepsilon/2$ as the solution of the problem

$$(q_\epsilon q_\epsilon')' + \delta q_\epsilon = 0, \qquad q_\epsilon\left(\frac{\varepsilon}{2}\right) = q_\epsilon\left(L - \frac{\varepsilon}{2}\right) = 0,$$

as in Section 3. Now put[4]

$$p^\epsilon(t, x) = \frac{ce^{\delta t}}{ce^{\delta t + 1}}\, q_\epsilon(x).$$

Then

$$p_t^\epsilon = (p^\epsilon p_x^\epsilon)_x + \delta p^\epsilon, \qquad p^\epsilon(t, 0) = p^\epsilon(t, L) = 0.$$

We can use p as a comparison function for all the P^n. We have $P^n(t, \varepsilon/2) > p^\epsilon(t, \varepsilon/2) = 0$ and $P^n(t, L - \varepsilon/2) > p^\epsilon(t, L - \varepsilon/2) = 0$. Moreover, $P^n(0, x) = P^0(x) + c^n > P^0(x) \geqslant \mu_\epsilon > 0$ on $[\varepsilon, L - \varepsilon/2]$ while $p^\epsilon(0, x) = [c/(1 + c)]\, q_\epsilon(x)$. Hence $p^\epsilon(0, x) < P^n(0, x)$ on $[\varepsilon/2, L - \varepsilon/2]$ for c sufficiently small and Lemma 4.3 follows. This completes our outline of the proof of Theorem II(a).

Proof of Theorem II(b). The proof of this theorem is very close to the one given in Section 3 and we simply outline it pointing out the differences. Once again one takes P as the solution of (P_0) and writes the equation for ρ as

$$\rho_a + \rho_t - P_x \rho + (\lambda - P_{xx})\, \rho = 0. \tag{4.10}$$

The integration along characteristics this time is somewhat more complicated in that one must distinguish between the cases $\bar{a} \geqslant \bar{t}$ and $\bar{a} \leqslant \bar{t}$. In the first case backward characteristics from $(\bar{t}, \bar{a}, \bar{x})$ will reach $t = 0$ first and will use the initial data $\rho_0(a, x)$. In the second case they reach $a = 0$ first and will use as data $\rho(t, 0, x)$, which one takes as $\beta_0 P(t, x)$.

[4] The use of this function was suggested to the author by D. G. Aronson.

Both of the integrations can be carried out by means of the function $X(t; \bar{t}, \bar{x})$ defined by

$$\partial X/\partial t = -P_x(t, X), \qquad X(\bar{t}; \bar{t}, \bar{x}) = \bar{x}. \tag{4.11}$$

LEMMA 4.4. (a) *For any* $(\bar{t}, \bar{x}) \in (0, \infty) \times (0, L)$ *Eq.* (4.11) *has a solution for all* $t \in [0, t]$. *This solution is differentiable in* $\bar{t}$ *and* $\bar{x}$.

(b) $X_{\bar{t}}(t; \bar{t}, \bar{x}) - P_x(\bar{t}, \bar{x})\, X(t; \bar{t}, \bar{x}) \equiv 0$.

Proof. The proof of (a) is just like that of Lemma 3.1. The trajectories can never reach $x = 0$ or $x = L$ since P has a minimum on $x = 0$ and $x = L$; hence $P_x > 0$ on $x = 0$, $P_x < 0$ on $x = L$. (Actually P_x will be $\pm\infty$ on $x = 0, L$.) Relation (b) is proved exactly as in Lemma 3.1.

The integration of (4.11) lead to the two formulas,

$$\rho(\bar{t}, \bar{a}, \bar{x}) = e^{-\lambda t} X_{\bar{x}}(0, \bar{t}, \bar{x})\, \rho_0(\bar{a} - \bar{t}, X(0; \bar{t}, \bar{x}), \qquad \bar{a} > \bar{t}, \tag{4.12}$$

$$\rho(\bar{t}, \bar{a}, \bar{x}) = \beta e^{-\lambda \bar{a}} X_{\bar{x}}(\bar{t} - \bar{a}; \bar{t}, \bar{x})\, P(\bar{t} - \bar{a}, X(\bar{t} - \bar{a}; \bar{t}, \bar{x}), \qquad \bar{t} > \bar{a}.^5 \tag{4.13}$$

The proof of Theorem II(b) amounts to verifying that (4.12) and (4.13) yield a solution of $(\mathscr{P}_0)$. The calculations are like those in Theorem I and we will not repeat them. It follows fairly easily from Lemma 4.4(b) that (4.10) is satisfied and the problem again is to show that if $\tilde{P}(\bar{t}, \bar{x}) = \int_0^\infty \rho(\bar{t}, \bar{a}, \bar{x})\, d\bar{a}$ then $\tilde{P}(\bar{t}, \bar{x}) \equiv P(\bar{t}, \bar{x})$. To do this one observes that $\tilde{P}$ satisfies $\tilde{P}_t = \beta P - \lambda \tilde{P} + (\partial/\partial x)(\tilde{P} P_x) = 0$. Subtracting the equation for P in (P_0) and putting $z = \tilde{P} - P$ one finds $z_t = (\partial/\partial x)(z P_x) - \lambda z$. For $t = 0$ one checks that $z = 0$, hence $z \equiv 0$. The uniqueness again follows from the facts that any solution must satisfy (4.12) and (4.13)) and that the solution of (P_0) is unique.

Proof of Theorem II(c). The proof consists of two parts. We establish first that the functions $u(t, x)$ have limits $\bar{u}^n(x)$ as t tends to infinity. Then we use this to show that $u(t, x)$ approaches the solution u of (3.3).

The first observation concerning the functions u^n is that we can obtain a priori estimates for their Hölder norms. For any rectangle Q_T, any ν, and any f on Q_T we define

$$|f|_\nu^T = \sup_{(t,x) \in Q_T} |f(t, x)| + \sup_{\substack{(t,x) \in Q_T \\ (t',x') \in Q_T}} \left\{ \frac{|f(t, x) - f(t', x')|}{[|x - x'|^2 + |t - t'|]^{\nu/2}} \right\}. \tag{4.14}$$

Now from Lemma 4.1 we know that the quantity $\delta P^n + c^n$ is uniformly bounded above and P^n is uniformly bounded below independently of T. We can then apply Theorem 13 of [6, p. 267], to conclude that $|P^n|_\nu^T$ is bounded

[5] The two formulas agree at $\bar{a} = \bar{t}$ if $\rho_0(0, x) = \int_0^\infty \rho_0(a, x)\, da$; that is, if the initial data satisfy (iii) of (P_0).

independently of T. (The proof immediately preceding Theorem 13 in [6] shows that the condition of vanishing at $x=0$ and L can be replaced by the vanishing of the t derivative. The earlier proofs in that reference show that the bound is independent of T.)

The preceding result shows that the P^n and hence the u^n are uniformly continuous on $[0, \infty) \times [0, L]$. Thus to show that $u^n(t, x) \to \bar{u}^n(x)$ as $t \to \infty$ it suffices to show that for any $\delta > 0$ there is a sequence $\{t_n\}$ with $0 \leqslant t_{k+1} - t_k < \delta$, $t_k \to \infty$ such that $u^n(t_k, x)$ converges to $n^n(x)$. This we do now.

We set $\varphi(t) = Q[u^n(t, \cdot)]$. The calculations in the proof of Lemma 4.1 show that φ is monotone decreasing and bounded below; hence $\varphi(t) \to \bar{\varphi}$. Moreover φ is differentiable and [6]

$$\dot{\varphi}(t) = -\int_0^L \frac{u_t^n(t, x)^2}{\sqrt{n^n(t, x)}}\, dx \leqslant 0. \tag{4.15}$$

We have $\dot{\varphi} \in L_1(0, \infty)$ with $\bar{\varphi} - \varphi(0) = \int_0^\infty \dot{\varphi}(\tau)\, d\tau$. Given $\delta > 0$ let $\{I_k\}$ denote the intervals $[k\delta, (k+1), \delta]$ and define $\{t_k\}$ by $\dot{\varphi}(t_k) = \sup_{I_k} \dot{\varphi}(t)$. Then $\dot{\varphi}(t_k) \leqslant 0$ and

$$-\infty < \int_0^\infty \dot{\varphi}(\tau)\, d\tau = \sum_0^\infty \int_{I_n} \dot{\varphi}(\tau)\, d\tau \leqslant \delta \sum_0^\infty \dot{\varphi}(t_k).$$

Hence $\dot{\varphi}(t_k) \to 0$ and, by (4.15),

$$\int_0^L \frac{u_t^n(t_k, x)^2}{\sqrt{u^n(t_k, x)}}\, dx \to 0 \qquad \text{as} \qquad k \to \infty. \tag{4.16}$$

Next we recall that the boundedness of $Q^n[u^n(t_k, \cdot)]$ implies that $\{u_x^n(t_k, \cdot)\}$ is a bounded sequence in $H_1(0, L)$. Hence there is a subsequence $\{t_{k_l}\}$ such $u_x^n(t_{k_l}, \cdot)$ converges weakly in $L_2(0, L)$ to $(\bar{u}^n)'$. Since $u^n(t_k, 0) - c_n = 0$ the boundedness of $\{u_x^n(t_k, \cdot)\}$ also implies (possibly with a further subsequence) that $\{u^n(t_k, \cdot)\}$ converges uniformly on $[0, L]$ to $\bar{u}^n$ with $(\bar{u}^n)'$ the weak derivative of $\bar{u}^n$. Note that $\bar{u}^n(0) = \bar{u}^n(L)$.

Now let $\eta \in c_0^\infty[0, L]$. Then by the above we have

$$-\int_0^L u_x^n(t_{k_l}, x)\, \eta'(x)\, dx + \int_0^L (2\delta\sqrt{u^n(t_{k_l}, x)} + 2c^n)\, \eta(x)\, dx$$

$$\to -\int_0^L (\bar{u}^n)'\,(x)\, \eta'(x)\, dx + \int_0^L (2\delta\sqrt{\bar{u}^n(t_{k_l}, x)} + 2c^n)\, \eta(x)\, dx.$$

[6] The u^n are smooth so that $\dot{\varphi}$ is continuous.

On the other hand we have from Lemma (1) and (4.16),

$$-\int_0^L u_x^n(t_{k_l}, x)\, \eta'(x)\, dx + \int_0^L (2\delta\sqrt{u^n(t_{k_l}, x)} + 2c^n)\, \eta(x)\, dx$$

$$= \int_0^L (u_{xx}^n(t_{k_l}, x) + 2\delta\sqrt{u^n(t_k, x)} + 2c^n)\, \eta(x)\, dx$$

$$= \int_0^L \frac{u_t^n(t_{k_l}, x)}{\sqrt{u^n(t_{k_l}, x)}}\, \eta(x)\, dx \to 0.$$

Thus we conclude that

$$-\int_0^L (\bar{u}^n)'\ (x)\, \eta'(x)\, dx + \int_0^L (2\delta\sqrt{\bar{u}^n(x)} + 2c^n)\, \eta(x)\, dx = 0. \tag{4.17}$$

Equation (4.17) is the weak form of the problem

$$(\bar{u}^n)'' + 2\delta\sqrt{\bar{u}^n} + 2c^n = 0, \qquad \bar{u}^n(0) = \bar{u}n(L) = c^n. \tag{4.18}$$

The proof of Theorem I(a) can be modified to show that (4.18) has a unique solution $\bar{u}^n(x)$ on $[0, L]$. The fact that the limit function is a solution of (4.17), and hence (4.18), shows that it is unique and hence we can conclude that the restriction to a subsequence of the t_k's is not necessary. We recall that with the uniform continuity of the u^n this implies that $u^n(t, x) \to \bar{u}^n(x)$.

The proof of Theorem I(a), when applied to (4.18), shows that $\{\bar{u}^n\}$ is a non-increasing sequence, with $\bar{u}^n(x) \downarrow u(x)$, the unique solution of (3.3). Since, on the other hand, $u(t, x) \leqslant u^n(t, x)$ we have

$$\limsup_{t \to \infty} u(t, x) \leqslant \bar{u}(x). \tag{4.19}$$

Finally we return to the estimate in Lemma (4.3). Since $p^n(t, x) \geqslant p^\epsilon(t, x)$ on $[\varepsilon, L - \varepsilon]$, we have, for x in that interval,

$$\liminf_{t \to \infty} u^n(t, x) \geqslant q_\epsilon^2(x).$$

But as $\varepsilon \downarrow 0$, $q_\epsilon^2(x) \uparrow \bar{u}(x)$ and we conclude that $u(t, x) \to \bar{u}(x)$. This completes the proof of Theorem II(c).

The proof of the convergence of $\rho(t, a, x)$ as $t \to \infty$ is tedious and we will not give it. The main observation is the following. Since (3.9) is autonomous, one has for the solution of that equation $X(a; \bar{a}, \bar{x}) = \chi(\bar{a} - a, \bar{x})$. On the basis of the convergence of $P(t, x)$ to $\bar{p}(x)$, one can show that for the solution of (4.11) one has

$$\lim_{\bar{t}\to 0} X(\bar{t}-\bar{a}, \bar{t}, \bar{x}) = \chi(\bar{a}, \bar{x})$$

and the convergence of ρ can be deduced from (4.12).

5. Age Dependence: Proof of Theorem III

The essential idea in this section is contained in Eq. (2.1). Let us define $\mathscr{M}(x)$ by the formula

$$\mathscr{M}(x) = \frac{P_0(x)}{B_0(x)} \equiv \frac{\int_0^\infty \rho_0(a, x)\, da}{\int_0^\infty e^{-\alpha a}\rho_0(a, x)\, da}. \tag{5.1}$$

From (5.1) and conditions (c_2) and (c_3)

$$1 < \mathscr{M}(x) \leqslant \bar{m}. \tag{5.2}$$

Lemma 5.1. *Let* (P, B) *be a solution of* (P_α) *and let* $X(t; \bar{t}, \bar{x})$ *be defined in terms of* P *by* (4.18). *Then,*

$$P(t, x) = M(t, x)\, B(t, x), \tag{5.3}$$

where

$$M(t, x) = \frac{\beta}{\beta-\alpha} + \left\{\mathscr{M}(X(0; t, x)) - \frac{\beta}{\beta-\alpha}\right\} e^{-(\beta-\alpha)t}. \tag{5.4}$$

Proof. Multiply the first of equations (ii) of (P_α) by B and the second by P, then subtract and divide by B^2. The result is

$$(P/B_t - P_x(P/B)_x + (\beta-\alpha)(P/B) = \beta.$$

This is a linear first order equation for (P/B) which can be integrated along characteristics defined by (4.18). One obtains (5.3) and (5.4).

If we substitute (5.3) into the equation in (P_α) we obtain the new equation for P,

$$P_t + \left(\lambda - \frac{\beta}{M}\right) P = \frac{\partial}{\partial x}(PP_x). \tag{5.5}$$

We observe that (5.5) is not a parabolic equation since M is a functional of P as given by (5.4).

The above calculations are formal but they can be used to give a proof of Theorem III. In the remainder of this section we outline the steps of this proof, omitting many of the details.

Let us suppose first that we can obtain a solution of (5.5), with M given by (5.4), and such that $P(t,0) \equiv P(t,L) \equiv 0$ and $P(t,x) = P_0(x)$. Then we put $B(t,x) = P(t,x)/M(t,x)$. Frrom (5.4) we see that M satisfies $M_t - P_x M_x + (\beta - \alpha) M = \beta$. Then an easy calculation shows that B satisfies the second equation in (P_α) and (P, B) will then yield a solution of (P_α).

We will indicate a little later how to obtain the required solution of (5.5). First, however, we show how to obtain the asymptotic stability result, under the assumption that a solution exists. Define two sequences of constants $\{\underline{m}_k\}$ and $\{\bar{m}_k\}$ by,

$$\underline{m}_k = \sup_{\substack{t \geqslant k \\ 0 \leqslant x \leqslant L}} M(t,x); \qquad \bar{m}_k = \inf_{\substack{t \geqslant k \\ 0 \leqslant x \leqslant L}} M(t,x). \tag{5.6}$$

Then $\{m_k\}$ is a decreasing sequence while $\{\bar{m}_k\}$ is increasing. Moreover we have

$$1 < \bar{m}_k, \qquad \lim_{k \to \infty} \underline{m}_k = \lim_{k \to \infty} \bar{m}_k = \frac{\beta - \alpha}{\alpha}. \tag{5.7}$$

We now define two sequences $\{\underline{p}^k\}$ and $\{\bar{p}^k\}$ of comparison functions as solutions of the problems

$$\begin{aligned}
&\underline{p}_t^k = \left(\frac{\beta}{\underline{m}_k} - \lambda\right) \underline{p}^k + \frac{\partial}{\partial x}(\underline{p}^k \underline{p}_x^k), \\
&\bar{p}_t^k = \left(\frac{\beta}{\bar{m}_k} - \lambda\right) \bar{p}^k + \frac{\partial}{\partial x}(\bar{p}^k \bar{p}_x^k), \qquad t > k, \\
&\underline{p}^k(t,0) = \underline{p}^k(t,L) = \bar{p}^k(t,0) = \bar{p}^k(t,L) = 0, \\
&\underline{p}^k(k,x) = \bar{p}^k(k,x) = P(k,x).
\end{aligned}$$

The functions $\underline{p}^k$ and $\bar{p}^k$ exist by the methods of Section 4 and we have

$$\underline{p}^k(t,x) \leqslant P(t,x) \leqslant \bar{p}^k(t,x), \qquad t > k. \tag{5.8}$$

The results of Section 4 show that as $t \to \infty$ we have

$$\underline{p}^k(t,x) \to \underline{\pi}^k(x), \qquad \bar{p}^k(t,x) \to \bar{\pi}^k(x), \tag{5.9}$$

where $\underline{\pi}^k$ and $\bar{\pi}^k$ are the solutions of the steady state problems,

$$(\underline{\pi}^k(\underline{\pi}^k)')' + \left(\frac{\beta}{\bar{m}_k} - \lambda\right)\underline{\pi}^k = 0, \qquad \underline{\pi}^k(0) = \underline{\pi}^k(L) = 0$$

$$(\bar{\pi}^k(\bar{\pi}^k)')' + \left(\frac{\beta}{\underline{m}_k} - \lambda\right)\bar{\pi}^k = 0, \qquad \bar{\pi}^k(0) = \bar{\pi}^k(L) = 0.$$

Thus we have

$$\underline{\pi}^k(x) \leqslant \liminf_{t\to\infty} P(t,x) \leqslant \limsup_{t\to\infty} P(t,x) \leqslant \bar{\pi}^k(x),$$

for all k. But as $k \to \infty$ both $\underline{m}_k$ and $\bar{m}_k$ tend to $(\beta-\alpha)/\beta$ and $\underline{\pi}_k$ and $\bar{\pi}^k$ tend to the solution p of $(p,p')' + (\beta - \lambda - \alpha)p = 0$, $p(0) = p(L) = 0$. We conclude that $\lim_{t\to\infty} P(t,x) = p(x)$. Since $B = P/M$ and $M \to (\beta-\alpha)/\beta$ as $t \to \infty$ we conclude that the solution (P, B) of (P_α) tends to the steady state solution $(p, [(\beta-\alpha)/\beta]\, p)$ of Theorem 1.

We now outline the steps in showing that (5.5), with (5.4), has a solution. As in Section 4, we begin with a sequence of approximating problems. Once again we choose a monotone decreasing sequence of constants $\{c_n\}$ with $c_n \downarrow 0$ and consider the sequence of problems

$$\begin{aligned} P_t^n &= \left(\frac{\beta}{M^n} - \lambda\right) P^n + \frac{\partial}{\partial x}(P^n P_x^n) + c^n, \\ P^n(t,0) &\equiv P^n(t,L) \equiv c^n, \\ P^n(0,x) &= P_0(x) + c^n. \end{aligned} \tag{5.10}$$

Here the M^n are defined in terms of P^n by (5.4) and (4.18).

The first task is to establish that (5.10) has a solution P^n. This can be done by a lengthy fixed point argument. For a given p first compute $X(p)$ as the solution of (4.18) with p on the right side. Then determine $M[p]$ from [5.4] with $X[p]$. Finally solve (5.10) with M_n replaced by $M[p]$. Call the result $P[p]$. The problem is then to find a fixed point of $P[p]$.

One needs a space in which to work. For $T > 0$ let Q_T denote $[0,T] \times [0,L]$. Then for T and $\nu > 0$ fixed let $C_{2+\nu}$ denote the space of function u with the Hölder norms $|u|_\nu^T$, $|u_x|_\nu^T$, $|u_{xx}|_\nu^T$ and $|u_t|_\nu^T$, as defined by (4.14), all finite. We let Γ_R denote the convex subset,

$$\Gamma_R = \{u\colon u \in C_{2+\nu}, |u|_{2+\nu} \leqslant R, u \geqslant c^n, u_x(t,0) > 0, u_x(t,L) < 0\}. \tag{5.11}$$

The map $P[p]$ takes Γ_R into itself. The first observation is that $X[p]$ is well defined. For the smoothness of the function in $C_{2+\nu}$ guarantees that (4.18) can be solved locally and the last condition in (5.11) shows, as in earlier calculations, that the solution is global. Next, we see from (5.4) that $1 < M[p] < \bar{m}$. It follows from (5.10) and (c_3) that $P[p]$ is bounded below by the solution of (4.2) with $\delta = \beta - \lambda - \alpha$. Hence by Lemma (4.1) we have $P[p] \geqslant c_n$. Since $P[p] = c_n$ on $x = 0$ and $x = L$ we conclude by the maximum principle that $P[p]$ satisfies the last condition in (5.11). Now, just as earlier, we can use the a priori estimates of Refs. [6, 1] in succession to show that the norm of $P[p]$ in $C_{2+\nu'}$ is bounded by a quantity depending only on bounds for P_0 and its derivatives. Hence, if $\nu' > \nu$ and R is large enough we insure that Γ_R is mapped into itself.

With $\nu' > \nu$ bounded sets in $C_{2+\nu'}$ are compact in $C_{2+\nu}$. Hence the image of Γ_R is precompact. The next step is to establish that the map $P[p]$ is continuous. This fact can be established by forming the difference of $P[p]$ and $P[q]$. The fact that p and q have second x-derivatives which are Hölder continuous makes it possible to obtain estimates on the difference between $X[p]$ and $X[q]$ by subtracting the corresponding equations (4.18). This in turn yields estimates for the difference between $M[p]$ and $M[q]$. Then one can subtract Eqs. (5.10), for $P[p]$ and $P[q]$ and apply once again the a priori estimates from [6] and then [1] to estimate the $C_{2+\nu}$ norm of $P[p] - P[q]$. The map $P[p]$ can now be extended to the closure of Γ_R and the application of the Schauder–Leray theorem yields the existence of a fixed point.

The task now is to show that the functions P^n defined by (5.10) converges to a solution of (P_α). We have already observed that the functions P^n are bounded below by the solutions of (4.2) with $\delta = \beta - \lambda\alpha$. The work of Section 4 shows that these are, in turn, all greater than or equal to the solution of (P_0) with β replaced by $\beta - \alpha$. Thus Lemma 4.3 can be invoked to show that the P^n are bounded below by a positive constant δ_ε on any interval $\varepsilon \leqslant x \leqslant L - \varepsilon$. We can also obtain a uniform upper bound on the P^n. For this we observe that by (5.4) the M^n are bounded below by one. Hence the solutions of (5.10) are bounded above by the solutions of (4.2) with $\delta = \beta - \lambda$ and these are in turn bounded above by Lemma (4.1).

Collecting our information we conclude the following. Given $\varepsilon > 0$ and $T > 0$ let $Q_T^\varepsilon = [0, T] \times [\varepsilon, L - \varepsilon]$. On Q_T^ε we have $P^n \geqslant \delta_\varepsilon$ and $|(\beta/M^n - \lambda) P^n + c^n|$ bounded above by K independently of n, x and t. With these facts we may enter the interior estimate theorems of paragraph (5) of Ref. [6] and conclude that $|P^n|_\nu^{T,\varepsilon}$ and $|P_x^n|_\nu^{T,\varepsilon}$ are bounded independently of n. These are the norms (4.14) computed for Q_T^ε. In turn these estimates allow us to use the interior estimate of Theorem 5, page 64, of Ref. [1] and obtain uniform estimates for $|P_t^n|_\nu^{T,\varepsilon}$ and $|P_{xx}^n|_\nu^{T,\varepsilon}$.

The boundedness of the Hölder norms of P^n, P_x^n, P_{xx}^n and P_t^n implies that we can choose a subsequence n_k such that the corresponding derivatives all converge uniformly. By choosing a sequence of ε's tending to zero and a sequence of T's tending to infinity and then diagonalizing we can get a subsequence such that the derivatives converge uniformly on any compact subset of $(0, \infty) \times (0, L)$. For this subsequence the X's of (4.18) and hence the M's of (5.4) will also converge as a will the functions $B = MP$. The limits (P, B) will be a solution of (P_α).

The solutions of (P_α) can be obtained from teh solution of (P_α) just as in SSection 4. The only difference is that in (4.13) one replaces βP by βB. We omit this calculation as well as that of limit behavior of ρ.

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