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Spectral Theory of Stationary $\mathscr{H}$-Valued Processes

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For weakly stationary stochastic processes taking values in a Hilbert space, spectral representation and Cramér decomposition are studied. Using these ideas and the moving average representation for such processes established earlier by the authors, some necessary and sufficient spectral conditions for such stochastic processes to be purely nondeterministic are given in both discrete and continuous parameter cases.

1. INTRODUCTION

This note is devoted to the study of the spectral theory of weakly stationary stochastic processes taking values in an infinite-dimensional, separable Hilbert-space $\mathscr{H}$ ($\mathscr{H}$-valued processes). The attempt to extend to infinite-dimensional processes the theory for finite vector-valued processes developed by Wiener and Masani [25] has given rise to several papers in recent years [9, 18, 5, 7]. The main effort in these papers has been to extend the factorability criterion of [25] to certain spectral density operators. However, as has been shown by P. Lax [16], in the infinite-dimensional processes of interest, i.e., in the "genuinely" infinite-dimensional case, the conditions of Wiener and Masani are not extendable. (see also [9, p. 909]). It is this case that is considered here.

In two earlier papers we gave a time domain analysis of infinite-dimensional processes in which the central role was played by the notion of multiplicity,

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first introduced by H. Cramér [2] and T. Hida [11] and discussed by the former in a series of papers [3, 4]. In the present paper we continue in the same spirit and show that the multiplicity of an $\mathscr{H}$-valued process equals the rank of the spectral density operator of its purely nondeterministic part. Our definition of the rank follows Zasuhin [26] and uses the Wold decomposition given in [14] to show that it coincides with the rank of the purely nondeterministic part. All this is done in Section 2. Section 3 contains the Bochner theorem and the Cramér decomposition of the spectral distribution operator.

Our main results are given in Section 5. Theorem 5.1, which is the extension of Rozanov's Theorem 11 of [23] to the infinite-dimensional case, gives necessary and sufficient conditions (in terms of the spectral distribution operator) for an $\mathscr{H}$-valued process $\{X_n\}$ to be a purely nondeterministic process of rank M. The factorability criterion for the spectral density operator is given in Theorem 5.1. Using a theorem of Payen [20], we give a necessary and sufficient analytic condition for an $\mathscr{H}$-valued process to be purely nondeterministic of rank M. In Section 6 we give, without proof, analogous results for continuous-parameter $\mathscr{H}$-valued processes.

Most of the results of this paper were announced in [13].

In the following section, we give some preliminaries which will be used throughout the paper.

2. Preliminaries

Let $(\Omega, \mathfrak{F}, P)$ be a probability space and $\mathscr{H}$ be a separable Hilbert space with inner-product $\langle\ ,\ \rangle$ and norm $\|\ \|$. We say that a random variable X on Ω to $\mathscr{H}$ is $\mathscr{H}$-valued if for each $\omega \in \Omega$, $X(\omega) \in \mathscr{H}$, and $E\|X\|^2 = \int_\Omega \|X\|^2\, dP$ is finite.

Definition 2.1. A sequence $\{X_n; n = 0, \pm 1, \pm 2,...\}$ of $\mathscr{H}$-valued random variables is called an $\mathscr{H}$-valued stationary stochastic process (SSP) if for each $h_1, h_2 \in \mathscr{H}$, and $n \leqslant m$,

$$E\langle X_m, h_1\rangle\overline{\langle X_n, h_2\rangle} = \gamma((m-n); h_1, h_2) = \langle \Gamma(m-n)h_1, h_2\rangle,$$

where $\Gamma(m-n)$ is the *gramian* of X_m, X_n (see Gangolli [9]). Since for each n, $E\|X_n\|^2 < \infty$, we get for $h \in \mathscr{H}$, $\langle X_n, h\rangle \in L_2(\Omega, P)$, the space of complex-valued square integrable functions on $(\Omega, \mathfrak{F}, P)$, we can associate with $\{X_n\}$ the following (closed) subspaces of $L_2(\Omega, P)$.

(a) The subspace $L_2(X)$ of $L_2(\Omega, P)$ generated by the family of random variables $\{\langle X_n, h\rangle$ for $n = 0, \pm 1, \pm 2,...$ and $h \in \mathscr{H}\}$.

(b) The subspace $L_2(X : n)$ of $L_2(\Omega, P)$ generated by $\{\langle X_m, h\rangle, m \leqslant n, h \in \mathscr{H}\}$.

We now consider $P_{L_2(X:n-1)}\langle X_n, h\rangle$, projection of $\langle X_n, h\rangle$ onto $L_2(X : n-1)$. By Lemma 7.1 of [14] we have

$$P_{L_2(X:n-1)}\langle X_n, h\rangle = \langle Y_{n,n-1}, h\rangle \qquad \text{for each } h \in \mathscr{H}, \tag{2.1}$$

where $Y_{n,n-1}$ is an $\mathscr{H}$-valued random variable. Clearly, $Z_n = X_n - Y_{n,n-1}$ is $\mathscr{H}$-valued. This is the *innovation* in X_n. Because of stationary, we get, for $h, h' \in \mathscr{H}$,

$$E\langle Z_n, h\rangle\overline{\langle Z_n, h'\rangle} = E\langle Z_0, h\rangle\langle Z_0, h'\rangle. \tag{2.2}$$

Since Z_0 is an $\mathscr{H}$-valued random variable, we get (see [13])

$$E\langle Z_0, h\rangle\overline{\langle Z_0, h'\rangle} = \langle Sh, h'\rangle, \tag{2.3}$$

where $S \in T(\mathscr{H}, \mathscr{H})$, the class of self-adjoint nonnegative definite, compact operators of finite trace.

DEFINITION 2.3. (a) The operator S will be called the Prediction Error Operator of the process $\{X_n, n = 0, \pm 1, ...\}$, and the dimension of the closure of the range of S is called the rank of the process X_n.

(b) An $\mathscr{H}$-valued process $\{X_n, 0, \pm 1, \pm 2, ...\}$ is said to be of nearly full rank if $\mathscr{n}(S)$ the null space of S is trivial (i.e., $\{0\}$).

As in [13], we call an $\mathscr{H}$-valued stationary process purely nondeterministic if

$$\bigcap_{n=-\infty}^{+\infty} L_2(X : n) = \{0\}, \qquad \text{where 0 is the zero of } L_2(X) \tag{2.4}$$

and deterministic if

$$L_2(X : n) = L_2(X) \qquad \text{for some (and hence for all)} \quad n. \tag{2.5}$$

We now recall the following Wold-decomposition [14, Proposition 7.1]:

LEMMA 2.1. *Let $\{X_n, n = 0, \pm 1, ...\}$ be an $\mathscr{H}$-valued stationary process; then*

$$X_n = X_n^{(1)} + X_n^{(2)}, \qquad n = 0, \pm 1, ...,$$

where $\{X_n^{(1)}\}$ and $\{X_n^{(2)}\}$ are $\mathscr{H}$-valued stationary processes such that $\{X_n^{(1)}\}$ is purely nondeterministic, $\{X_n^{(2)}\}$ is deterministic, and $L_2(X^{(1)}) \perp L_2(X^{(2)})$.

From the above proposition it follows that $Z_n = Z_n^{(1)}$, and hence

$$\langle Sh, h' \rangle = E \langle Z_0^{(1)}, h \rangle \overline{\langle Z_0^{(1)}, h' \rangle} = \langle S^{(1)}h, h' \rangle. \tag{2.6}$$

Thus we observe that the rank of the process is equal to the rank of its purely nondeterministic part.

LEMMA 2.2. *Let $\{X_n, n = 0, \pm 1, \pm 2,...\}$ be a purely nondeterministic process, then the subspace $R^\perp$ of $L_2(X)$ generated by $\langle Z_0, h \rangle$, $h \in \mathscr{H}$, is isomorphic to the closure of the range $\mathscr{R}(S^{1/2})$ of $S^{1/2}$, where $S^{1/2}$ denotes the square root of the operator S* [22, p. 265].

Proof. In view of (2.3), the operator T, given by

$$T\langle Z_0, h \rangle = S^{1/2}h \qquad \text{for} \quad h \in \mathscr{H},$$

extends to an isometry on $R^\perp$ onto $\mathscr{R}(S^{1/2})$.

It can be easily observed that [22, p. 263] $\eta(S) = \eta(S^{1/2})$. Hence we get the following theorem from (2.6), Lemma 2.2, and [14, Theorem 4.1(i)].

THEOREM 2.1. *The rank of an $\mathscr{H}$-valued stationary stochastic process is equal to the multiplicity of its purely nondeterministic part.*

This shows that our definition of rank is a natural generalization of the one in the finite-dimensional case.

3. BOCHNER THEOREM AND CRAMÉR DECOMPOSITION

This section establishes the Bochner theorem for an $\mathscr{H}$-valued stationary process. This allows us to define the operator-valued *spectral* measure of the stationary process.

Let $\{X_n, n = 0, \pm 1,...\}$ be an $\mathscr{H}$-valued stationary process, then we can introduce on $L_2(X)$ the shift-operator U as

$$U\langle X_n, h \rangle = \langle X_{n+1}, h \rangle \quad \text{for each} \quad h \quad \text{and} \quad n. \tag{3.1}$$

We get from the stationarity of $\{X_n\}$ that U is a unitary operator. Hence by Stone's Theorem [22, p. 281],

$$U = \int_{(0,2\pi]} e^{-i\lambda} E(d\lambda), \tag{3.2}$$

where E is a projection-valued measure on the Borel subsets $\mathscr{B}$ of $(0, 2\pi)$.

Consider now $E(\Delta)\langle X_0, h\rangle$. It equals by [14, Proposition 7.1] $\langle X(\Delta), h\rangle$, where for each Δ, $X(\Delta)$ is an $\mathscr{H}$-valued random variable. Let us define the operator $F(\Delta)$ as

$$\langle F(\Delta)h, h'\rangle = E\langle X(\Delta), h\rangle\overline{\langle X(\Delta), h'\rangle}.$$

We now have the following Bochner Theorem.

THEOREM 3.1. *Let* $\{X_n\}_{-\infty}^{+\infty}$ *be an* $\mathscr{H}$*-valued stationary stochastic process; then for each* n, *the* gramian

$$\Gamma(n) = \int_0^{2\pi} e^{-in\lambda} IF(d\lambda),$$

where (i) $F(\Delta)$ *for each* $\Delta \in \mathscr{B}$ *belongs to* $T(\mathscr{H}, \mathscr{H})$;

(ii) F *is a measure in the sense that if* Δ_i *are disjoint members in* $\mathscr{B}$, *then* $\tau\{F(\bigcup_{i=1}^{\infty} \Delta_i) - \sum_1^k F(\Delta_i)\} \to 0$ as $k \to \infty$, *where* τ *denotes the trace*[1];

(iii) $\Gamma(n) = \int_0^{2\pi} e^{-in\lambda} IF(d\lambda) = \int_0^{2\pi} e^{-in\lambda} IG(d\lambda)$ *implies* $F(\Delta) = G(\Delta)$.

DEFINITION 3.1. We say that F is absolutely continuous if the (possibly complex) measure $\langle F(\cdot)h, h'\rangle$ is absolutely continuous with respect to ($\ll$), the Lebesgue measure l on $(0, 2\pi]$, and F is called singular if for each $h, h' \in \mathscr{H}$, $\langle F(\cdot)h, h'\rangle$ is singular with respect to l.

We denote $\mathscr{m}_{ac} = \{x \mid x \in L_2(X), \langle E(\Delta)x, x\rangle \ll l\}$ and $\mathscr{m}_S = \{x \mid x \in L_2(X), \langle E(\Delta)x, x\rangle$ is singular to $l\}$. It is known [10, p. 104–105] that $\mathscr{m}_{ac}$ and $\mathscr{m}_S$ are mutually orthogonal subspaces of $L_2(X)$ and $L_2(X) = \mathscr{m}_{ac} \oplus \mathscr{m}_S$. Clearly, $\mathscr{m}_{ac}$ and $\mathscr{m}_S$ are U-invariant. Also by Lemma 7.1 of [13],

$$P_{\mathscr{m}_{ac}}\langle X_0, h\rangle = \langle X_0^{ac}, h\rangle \qquad \text{and} \qquad P_{\mathscr{m}_S}\langle X_0, h\rangle = \langle X_0{}^S, h\rangle,$$

where X_0^{ac} and $X_0{}^S$ are $\mathscr{H}$-valued random variables. We thus have

THEOREM 3.2 (Cramér decomposition). *Let* $\{X_n, n = 0, \pm 1, \ldots\}$ *be a stationary* $\mathscr{H}$*-valued process. Then*

$$X_n = X_n^{(ac)} + X_n^{(S)} \qquad n = 0, \pm 1, \ldots,$$

where the processes $\{X_n^{(ac)}\}$ *and* $\{X_n^{(S)}\}$ *are mutually orthogonal* $\mathscr{H}$*-valued stationary processes with shift-operator* U, *as for* $\{X_n\}$, *the spectral distribution operator of* $X_n^{(ac)}$ *is absolutely continuous, and that of* $X_n^{(S)}$ *is singular.*

[1] For a $T(\mathscr{H}, \mathscr{H})$-valued measure, this condition is precisely equivalent to the fact that $\langle F(\Delta)h, h'\rangle$ is countably additive for each $h, h' \in \mathscr{H}$.

Corollary 3.1 (Cramér decomposition). *Let F be the spectral distribution operator of a stationary $\mathscr{H}$-valued process, then*

$$F = F_{ac} + F_S ,$$

where F_{ac} and F_S are $T(\mathscr{H}, \mathscr{H})$-valued measures on $\mathscr{B}$ such that $F_{ac} \ll l$ and F_S is singular with respect to l.

Applying Theorem 3.2 to the deterministic part in Theorem 2.1, we get

Corollary 3.2. *Every stationary $\mathscr{H}$-valued process X_n can be decomposed into three pairwise orthogonal stationary $\mathscr{H}$-valued processes in the following manner:*

$$X_n = X_n^{(1)} + X_n^{(2)} + X_n^{(3)},$$

where $\{X_n^{(1)}\}$ is a purely nondeterministic stationary $\mathscr{H}$-valued process, $\{X_n^{(2)}\}$ is a deterministic $\mathscr{H}$-valued stationary process with absolutely continuous spectral distribution operator, and $\{X_n^{(3)}\}$ is a deterministic stationary $\mathscr{H}$-valued process with singular spectral distribution operator.

For an example of the $\mathscr{H}$-valued process, where the second part of the above decomposition is present, see M. Nadkarni [19].

We now obtain a Radon–Nikodym theorem for the operator-valued measure F_{ac}. Let us recall that a $T(\mathscr{H}, \mathscr{H})$-valued measure G is said to be absolutely continuous if for each $h, h' \in \mathscr{H}$, the (possibly complex-valued) measure $\langle G(\cdot)h, h'\rangle$ is absolutely continuous with respect to l.

Let $\{e_n\}$ be a CONS in $\mathscr{H}$; we then define by $g_{ij}(\lambda) = (d/dl) < G(\lambda)\, e_i , e_j\rangle$ the Radon–Nikodym derivative of $\langle G(\Delta)\, e_i , e_j\rangle$. Let us define by $g = \{g_{ij}(\cdot)\}$ the infinite-dimensional matrix. Then g can be considered as a spectral density of G. It should be noticed that the spectral density operator is defined *uniquely* only as a matrix, i.e., when the CONS is fixed. In the infinite-dimensional case, this point has been emphasized by Gangolli [9, p. 903]. This is also the case in the finite-dimensional case although Rozanov [23] seems to have overlooked this fact.

Obviously the matrix g has the following properties:

(i) $g = g^*$; (ii) g is nonnegative definite; (iii) $\sum_i g_{ii}(\lambda)$ converges a.e. $[l]$. (3.11)

From (i) and (ii) we get

$$(g_{ij})^2 \leqslant g_{ii}\, g_{jj} \quad \text{a.e.} \quad [l]. \tag{3.12}$$

Let us now define, for each $h \in \mathscr{H}$ and $\lambda \in (0, 2\pi]$, the spectral density operator

$$G'(\lambda)h = \sum_{i,j} (h, e_i)\, g_{ij}(\lambda) e_j\,. \tag{3.13}$$

Then (3.11), (3.12), and (3.13) give the following theorem by standard procedure;

THEOREM 3.3. *Let $G \ll l$ be a $T(\mathscr{H}, \mathscr{H})$-valued measure, then* (i) *the spectral density operator G' can be defined* uniquely *for each* CONS *by* (3.13), (ii) $G'(\lambda) \in T(\mathscr{H}, \mathscr{H})$ e.a. [l], *and* (iii) *For each $\Delta \in \mathscr{B}$, $G(\Delta) = \int_\Delta G'(\lambda)\, d\lambda$, where the integral is defined as in* [12, p. 79].

In view of the fact that the spectral density operator is defined with respect to a CONS, it seems natural to consider the coordinatization of X_n with respect to $\{e_k\}$ from now on. That is, we can treat X_n as a "column" vector $(X_n^{(k)}) = (\langle X_n, e_k \rangle)$ where, of course, $\sum_{k=1}^{\infty} E \mid \langle X_n, e_k \rangle|^2 < \infty$.

Using an extension of the ideas of [25], we call vector $X_n = (X_n^{(k)})$ $(k = 1, 2,...,)$ an $L_2^{\infty}(X)$-valued random variable, and the process $\{X_n, n = 0, \pm 1,...\}$ of these vectors (when co-ordinatized) an $L_2^{\infty}(X)$-valued stationary process. If U is the shift-operator given before, we can write $X_n^{(k)} = \int_0^{2\pi} e^{-in\lambda} E(d\lambda)\, X_0^{(k)}$. Let us consider the vector $\xi(\Delta) = (E(\Delta)\, X_0^{(k)})$. Then $\xi(\Delta)$ is an $L_2^{\infty}(X)$-valued countably additive orthogonally scattered (c.a.o.s.) measure in the sense of Mandrekar and Salehi [17, Section 6] and the matrix $F(\Delta) = [\xi(\Delta), \xi(\Delta)]$ is a $T(l_2, l_2)$-valued operator measure in the sense of [17, Section 2]. The stochastic integrals with respect to ξ are defined in [17, Section 6]. We now quote the following Isomorphism Theorem [17, Section 6] which will be used in Section 5.

THEOREM 3.4. *The space $L_2^{\infty}(X) = \{\int_0^{2\pi} A(\lambda)\, \xi(d\lambda), A \in L_{2,F}\}$, where $L_{2,F}$ is the space of all operator-valued functions square-integrable with respect to $T(l_2, l_2)$-valued measure F* [17, (4.10)] *and*

$$\left[\int_0^{2\pi} A(\lambda)\, \xi(d\lambda), \int_0^{2\pi} A(\lambda)\, \xi(d\lambda)\right] = \int_0^{2\pi} A(\lambda) F(d\lambda)\, A^*(\lambda) \qquad [17, (4.9)].$$

In the next section we obtain a Riesz–Fischer Theorem following Wiener and Masani [25, 3.9(b)], which will be used to obtain the factorization theorem.

4. FACTORIZATION OF THE DENSITY

Let $\mathscr{H}$, $\mathscr{K}$ be two separable Hilbert spaces and $HS(\mathscr{H}, \mathscr{K})$ be the class of Hilbert–Schmidt operators on $\mathscr{H}$ into $\mathscr{K}$ [8, 1010]. We denote $|\ |_E$ the Hilbert–Schmidt norm. Let us denote by $\mathscr{L}_2(0, 2\pi)$ the space of $HS(\mathscr{H}, \mathscr{K})$-valued functions $A(\cdot)$ on $(0, 2\pi]$, where $\int_{(0,2\pi]} | A(\lambda)|_E^2 \, dl$ is finite. If we identify in $\mathscr{L}_2(0, 2\pi]$ the functions A, B such that $| A(\cdot) - B(\cdot)|_E = 0$ a.e. $[l]$, then $\mathscr{L}_2(0, 2\pi]$ becomes a Hilbert space with the inner product $((A, B)) = 1/2\pi \int_0^{2\pi} \tau\{A, B^*\} \, dl$ and norm $\| A \|_E = ((A, A))^{1/2}$.

The proof of the above statement is a consequence of the fact that $HS(\mathscr{H}, \mathscr{K})$ is a Hilbert space with norm $| \; |_E$ given by the inner product $(A, B) = \tau AB^*$ (e.g., Schatten [24, p.32]) and the classical arguments [12, p. 46]. For a function A in $\mathscr{L}_2(0, 2\pi]$ we define the n-th Fourier coefficient

$$A_n = \frac{1}{2\pi} \int_0^{2\pi} A(\theta) e^{-in\theta} \, d\theta,$$

where the integral is as in [12, p. 79]. The following is a straightforward extension of Wiener and Masani [25, 3.9 (b) & (c)] for $\mathscr{L}_2(0, 2\pi]$.

THEOREM 4.1. (a) *If A_n is the n-th Fourier coefficient of $A \in \mathscr{L}_2(0, 2\pi]$, then $\sum_{-\infty}^{+\infty} | A_n |_E^2 < \infty$; conversely, if the A_n are such that $\sum_{-\infty}^{+\infty} | A_n |_E^2 < \infty$, then there is a function $A \in \mathscr{L}_2(0, 2\pi]$ whose n-th Fourier coefficient is A_n (Riesz–Fischer Theorem).*

(b) *If $A, B \in \mathscr{L}_2$ and have n-th Fourier coefficients A_n, B_n, then the integral $1/2\pi \int_0^{2\pi} A(\theta) B^*(\theta) \, d\theta$ exists as a weak operator integral and equals $\sum_{-\infty}^{+\infty} A_n B_n^*$ (Parseval's Identity).*

(c) *With the hypothesis of* (b), *the n-th Fourier coefficient of AB is $\sum_{-\infty}^{+\infty} A_k B_{n-k}$ (convolution formula).*

As these results follow readily as in the complex-valued case[2], we omit the proofs. We shall denote by $\mathscr{L}_2^+$ the subspace of $\mathscr{L}_2(0, 2\pi]$ consisting of functions A such that $A(n) = 0$, $n < 0$.

We now recall the following theorem as a consequence of [14, Theorem 9.2].

THEOREM 4.2. *Let $X_n = (X_n^{(k)})$ be an $\mathscr{H}$-valued weakly stationary, purely nondeterministic process, then*

$$X_n^{(k)} = \sum_{m \leqslant 0} \sum_{l=1}^{M} b_l(k; m) \, \xi_l(m + n), \tag{4.1}$$

where (i) $M = \dim R$,

[2] For (*c*) it should be noted that if $B \in \mathscr{L}_2(0, 2\pi]$, so is B^*.

(ii) $\{\xi_i(m)\}$ $(m = 0, \pm 1, \ldots)$ *are mutually orthogonal* [6] *stationary processes,*

(iii) $\sum_{k=1}^{\infty} \sum_{i=1}^{M} \sum_{m \leqslant 0} | b_i(k; m)|^2 E | \xi_i(m)|^2$ *is finite and*

(iv) $L_2(X; n) = \sum_1^M \oplus L_2(\xi_i ; n)$.

Let us denote by ξ_n the vector $\{\xi_n^{(k)}\}$, where for $k \leqslant M$, $\xi_0^{(k)} = \xi_k(0)$ and $\xi_0^{(k)} = 0$ a.e. $[P]$ for $k > M$. Also define the matrix $\tilde{B}(m)$ for $m \leqslant 0$ as $\tilde{b}_{ij}(m) = b_j(i; m)$ for $i = 1, 2, \ldots$, and $j \leqslant M$ and zero otherwise. Then (4.1) can be rewritten as

$$X_n = \sum_{m \leqslant 0} \tilde{B}_m \xi_{m+n} .$$

Let D be the diagonal matrix with entries $E | \xi_i |^2$ along the diagonal for $i \leqslant M$ and other entries zero, then $\tilde{B}_m D^{1/2} \in HS(l_2 , l_2)$, since $\sum_{m \leqslant 0} | \tilde{B}_m D^{1/2} |_E^2 < \infty$ by (iii) of the above theorem. We, therefore, have

$$X_n = \sum_{m \leqslant 0} \tilde{B}_m \xi_{m+n} . \tag{4.2}$$

From (4.2), by a simple transformation, we get

$$X_n = \sum_{m=0}^{\infty} B_m \xi_{n-m} , \qquad \text{where} \quad B_m = \tilde{B}_{-m} . \tag{4.3}$$

Furthermore,

$$\text{rank of } D = \text{multiplicity of } X_n . \tag{4.4}$$

The following is our factorization theorem.

THEOREM 4.3. *A stationary $\mathscr{H}$-valued process $\{X_n , n = 0, \pm 1, \ldots\}$ is purely nondeterministic* iff *the spectral distribution F is absolutely continuous, and $F'_{ac}(\lambda) = \Phi(\lambda)\, \Phi^*(\lambda)$, where $\Phi(\lambda) = \sum_{m=0}^{\infty} B_m D^{1/2} e^{im\lambda} \in \mathscr{L}_2^+$.*

Proof. We observe that the gramian

$$\Gamma(n) = [X_n , X_0] = \sum_{m=0}^{\infty} B_{m+n} D B_m^* .$$

Since $\sum_{m=-\infty}^{+\infty} | B_m D^{1/2} |_E^2$ is finite, we get by Theorem 4.1 (a) that there exists a function $\Phi \in \mathscr{L}_2$, such that $\Phi(\lambda) = \sum_{m=-\infty}^{+\infty} B_m D^{1/2} e^{im\lambda}$. But $B_m = 0$ for $m < 0$ and hence

$$\Phi(\lambda) = \sum_{m=0}^{\infty} B_m D^{1/2} e^{im\lambda}.$$

Using Theorem 4.1 (b) and (c), we get

$$\Gamma(n) = \int_0^{2\pi} e^{-in\lambda} \Phi(\lambda)\, \Phi^*(\lambda)\, d\lambda.$$

By uniqueness of the spectral density we now get the required result.

Conversely, consider a sequence of $\mathscr{H}$-valued random variables $\tilde{\xi}_n$ such that $[\tilde{\xi}_n, \tilde{\xi}_m] = D\delta_{mn}$ and consider $\tilde{X}_n = \sum_{m=0}^{\infty} B_m \tilde{\xi}_{n-m}$. Then

$$\tilde{\Gamma}(n) = \int_0^{2\pi} e^{-in\lambda} \Phi(\lambda)\, \Phi^*(\lambda)\, d\lambda = \Gamma(n).$$

Since $\tilde{X}_n$ is obviously purely nondeterministic, so is X_n [14, p. 616].

5. The Rank of the Spectral Density

In this section we obtain the main result of the paper and apply it to obtain analytic conditions on the spectral density in order that the process is purely nondeterministic of rank M. We now state the main theorem.

Theorem 5.1. *The necessary and sufficient conditions for an $\mathscr{H}$-valued stationary process $\{X_n, n = 0, \pm 1, ...\}$ to be purely nondeterministic of rank M are the following*:

(i) *The spectral distribution operator is absolutely continuous*;

(ii) *The rank of its spectral density operator f is M* a.e. [l];

(iii) $f(\lambda) = \Phi(\lambda)\, \Phi^*(\lambda)$, *where* $\Phi(\lambda) = \sum_{k=0}^{\infty} B_k D^{1/2} e^{ik\lambda}$ *is in* $\mathscr{L}_2^+(0, 2\pi]$, *where D is a diagonal matrix.*

Before we proceed to the proof of the theorem we observe that in Theorem 3.4 one can take $A(\lambda) = A_0(\lambda)[f(\lambda)]^{-1/2}$, where $A_0(\cdot)$ is $HS(l_2, l_2)$-valued [17, (4.12)]. With this notation, we have

$$\int_0^{2\pi} A(\lambda) F(d\lambda)\, A^*(\lambda) = \int_0^{2\pi} A_0(\lambda)\, P_r(\overline{R(f^{1/2}(\lambda))})\, A_0^*(\lambda)\, d\lambda, \tag{5.0}$$

where $P_r(\overline{R(f^{1/2}(\lambda))})$ denotes the orthogonal projection onto $\overline{R(f^{1/2}(\lambda))}$.

The above equation will be crucial to our proof.

Proof of the necessity. The process $\xi_m = (U^m \xi_0^{(k)}) \in L_2^\infty(X)$ by (iv) of Theorem 4.2, and hence by Theorem 3.4, gives us $\xi_0 = \int_{(0,2\pi]} A(\lambda)\, \zeta(d\lambda)$, where $A \in L_{2,F}$. This implies that $\xi_m = \int_{(0,2\pi]} e^{-im\lambda} A(\lambda)\, \zeta(d\lambda)$ since $\zeta(\Delta) = (E(\Delta) X_0^{(k)})$. Thus, by Theorem 3.4 and (5.0),

$$[\xi_m, \xi_n] = \int_{(0,2\pi]} e^{-im\lambda} A(\lambda) F(d\lambda)\, A^*(\lambda)$$

$$= \int_{(0,2\pi]} e^{-im\lambda} A_0(\lambda)\, P_r\overline{(R(f^{1/2}(\lambda)))}\, A_0^*(\lambda) d\lambda. \tag{5.1}$$

But $[\xi_m, \xi_0] = \delta_{m0} D$ by the definition of D, and, therefore,

$$[\xi_m, \xi_0] = \frac{1}{2\pi i} \int_{(0,2\pi]} e^{-im\lambda} D\, d\lambda. \tag{5.2}$$

Hence, by Theorem 3.1 (iii) and Theorem 3.3,

$$A_0(\lambda)\, P_r\overline{(R^{1/2}(f(\lambda)))}\, A_0^*(\lambda) = \frac{D}{2\pi i} \quad \text{a.e.} \quad [l]. \tag{5.3}$$

Since now ξ_m is a stationary $L_2^\infty(\xi)$-valued process with shift-operator U, we get, using Theorem 4.2 (iv) and Theorem 3.4 by similar arguments as above, that

$$f(\lambda) = \frac{1}{2\pi i} B_0(\lambda)\, P_r\overline{(R(D^{1/2}))}\, B_0^*(\lambda) \quad \text{a.e.} \quad [l], \tag{5.4}$$

where $B(\lambda)$ is given by the equation $X_0 = \int B(\lambda)\, \eta(d\lambda)$ and $\eta(\Delta) = (E(\Delta)\, \xi_0^{(k)})$. We get that $\operatorname{rank} f^{1/2}(\lambda) = \text{rank of } D^{1/2}$ which implies

$$\operatorname{rank} f(\lambda) = \operatorname{rank} D = M \quad \text{a.e.} \quad [l]. \tag{5.5}$$

The other two conditions follow from Theorem 4.3. Sufficiency follows by using Theorem 4.3 and the necessity part of the proof.

From the above theorem and Corollary 3.2 we get

COROLLARY 5.1. *The spectral distribution operator F can be written as the sum of three operators F_1, F_2, and F_3, where F_1 is absolutely continuous with factorizable density, F_2 is absolutely continuous, and F_3 is singular. This decomposition is unique.*

We now use the above theorem to give analytic conditions for an $\mathscr{H}$-valued stationary process to be purely nondeterministic of rank M, using Payen [20, Proposition 9, p. 379].

THEOREM 5.2. *Let $\{X_n, n = 0, \pm 1, \ldots\}$ be an $\mathscr{H}$-valued stationary process; then X_n is purely nondeterministic of rank M* iff (i) *The spectral distribution operator is absolutely continuous*; (ii) *The spectral density $f(\lambda) = \sum_{n=1}^{M} \rho_n(\lambda) Q_n(\lambda)$, where $Q_n(\lambda)$ is a projection on a subspace of dimension one such that its range function is conjugate analytic* [7, p. 124], and $\rho_n(\lambda)$ *is a strictly positive real function a.e.* [l] *such that* $\int_0^{2\pi} \log \rho_n(\lambda) d\lambda > -\infty$.

Proof. Sufficiency follows from Proposition 9 of [20, p. 379] and Theorem 5.1. To prove the necessity part we choose a complete orthonormal system $\{e_i\}_{i=1}^{M}$ consisting of the eigenelements of $D^{1/2}$. Consider then $\Phi(\lambda)\, e_n = g_n(\lambda)$. In view of Theorem 5.1 we get that $f_{i,j}(\lambda) = \sum_{n=1}^{M} (e_i, g_n(\lambda))(g_n(\lambda), e_j)$. Putting $\rho_n(\lambda) = \| g_n(\lambda)\|_{\mathscr{H}}$ $Q_n(\lambda) =$ Projection of $g_n(\lambda)$, we get that $f(\lambda) = \sum_{n=1}^{M} \rho_n(\lambda) Q_n(\lambda)$ and $\int_0^{2\pi} \log \rho_n(\lambda) > -\infty$ following Payen's arguments [20, p. 379].

6. CONTINUOUS PARAMETER PROCESSES

In this section we describe a procedure for extending results concerning a discrete-parameter process to the continuous parameter case. The approach depends mainly on [14]. Let $\mathscr{H}$ be a separable Hilbert space. We say that for $t \in R$, the real line, X_t is a continuous parameter $\mathscr{H}$-valued weakly stationary process if, for each t, X_t is an $\mathscr{H}$-valued random variable and

$$E_p \langle X_t, h_1 \rangle \overline{\langle X_s, h_2 \rangle} = \langle \Gamma(| s - t |) h_1, h_2 \rangle,$$

where $\Gamma(t)$ is the *gramian*. We say that $\{X_t\}_{t \in R}$ is mean continuous if for each $h \in \mathscr{H}$, $E\, |\langle X_t, h\rangle - \langle X_s, h\rangle|^2 \to 0$ as $s \to t$.

For a mean continuous purely nondeterministic process, the authors have shown [14] that the idea of multiplicity coincides with that of rank given by Gladyshev in the finite-dimensional case.

DEFINITION 6.1. The rank of the process $\{X_t\}$ is defined as the multiplicity of its purely nondeterministic part.

Remark. The above definition is unambiguous in view of the Wold-decomposition given in [14, Proposition 7.1]. It is consistent with the finite-dimensional case and the Zasuhin definition in the discrete parameter case in view of Theorem 6.2 of [14].

Let us define $L_2(X)$ as the (closed) subspace of $L_2(\Omega)$ generated by $\{\langle X_t, h\rangle t \in R,$

$h \in \mathscr{H}\}$ and the *shift operator* U_t on $L_2(X)$ as $U_t\langle X_s, h\rangle = \langle X_{s+t}, h\rangle$. Then U_t is a strongly continuous group of unitary operators and hence

$$U_t = \int_{-\infty}^{+\infty} e^{-it\lambda} E(d\lambda),$$

where $E(\Delta)$ is a projection-valued measure on the Borel sets $\mathscr{B}(R)$ of R. Using now the arguments paralleling those in discrete parameter case, we get the following analogs of Theorems 3.1, 3.2, and 3.3, and Corollaries 3.1, and 3.2.

THEOREM 6.1. *Let $\{X_t\}_{t \in R}$ be an $\mathscr{H}$-valued stationary stochastic process; then, for each t,*

$$\Gamma(t) = \int_{-\infty}^{+\infty} e^{-itu} F(du),$$

where (i) *for each $\Delta \in \mathscr{B}(R)$, $F(\Delta) \in T(\mathscr{H}, \mathscr{H})$;*

(ii) *F is a measure in the sense that if Δ_i are disjoint members in $\mathscr{B}(R)$, then $\tau\{F(\bigcup_1^\infty \Delta_i) - \sum_1^k F(\Delta_i)\} \to 0$ as $k \to \infty$ where τ denotes the trace;*

(iii) *$\Gamma(t) = \int_{-\infty}^{+\infty} e^{-itu} F(du) = \int_{-\infty}^{+\infty} e^{-itu}\, G(du)$ implies $F(\Delta) = G(\Delta)$.*

THEOREM 6.2. *Let $\{X_t, t \in R\}$ be a mean continuous stationary $\mathscr{H}$-valued process; then*

$$X_t = X_t^{(ac)} + X_t^{(s)} \qquad t \in R,$$

where the processes $\{X_t^{(ac)}\}$ and $\{X_t^{(S)}\}$ are mean continuous $\mathscr{H}$-valued processes and the spectral distribution operator of $\{X_t^{(ac)}\}$ is absolutely continuous and that of $\{X_t^{(s)}\}$ is singular with respect to the Lebesgue measure λ on R.

COROLLARY 6.1 (Cramér decomposition). *Let F be the spectral distribution operator of a stationary $\mathscr{H}$-valued process; then*

$$F = F_{ac} + F_s,$$

where F_{ac} and F_s are $T(\mathscr{H}, \mathscr{H})$-valued measures on $\mathscr{B}(R)$ such that $F_{ac} \ll \lambda$ and F_s is singular with respect to λ.

COROLLARY 6.2. *Every mean continuous stationary $\mathscr{H}$-valued process $\{X_t\}_{t \in R}$ can be decomposed into three pairwise orthogonal stationary $\mathscr{H}$-valued processes as follows:*

$$X_t = X_t^{(1)} + X_t^{(2)} + X_t^{(3)},$$

where $\{X_t^{(1)}\}$, $\{X_t^{(2)}\}$ *and* $\{X_t^{(3)}\}$ *are mean continuous stationary* $\mathscr{H}$*-valued processes such that* $\{X_t^{(1)}\}$ *is purely nondeterministic and* $F^{(1)} \ll \lambda$, $\{X_t^{(2)}\}$ *is deterministic with* $F^{(2)} \ll \lambda$, *and* $\{X_t^{(3)}\}$ *is deterministic with* $F^{(3)}$ *singular with respect to* λ.

THEOREM 6.3. *Let G be an absolutely continuous* $T(\mathscr{H}, \mathscr{H})$*-valued measure; then* (i) *the spectral density operator* G' *can be defined* uniquely *for each* CONS *in* $\mathscr{H}$; (ii) $G'(u) \in T(\mathscr{H}, \mathscr{H})$ a.e. $[\lambda]$; *and* (iii) *for each* Δ *in* $\mathscr{B}$, $G(\Delta) = \int_\Delta G'(u)\lambda(du)$, *where the integral is as in* [12].

As in Section 3, we now coordinatize the process to study the factorization.

Let $\mathscr{L}_2(R)$ be the space of $HS(\mathscr{H}, \mathscr{H})$-valued, operator-valued functions A on R with $\int_R | A(u)|_E^2 \lambda(du)$ finite. If we identify in $\mathscr{L}_2(R)$ the functions A, B such that $| A(\cdot) - B(\cdot)|_E = 0$ a.e. $[\lambda]$, then $\mathscr{L}_2(R)$ becomes a Hilbert space with inner product $((A, B)) = 1/2\pi \int_R \tau\{AB^*\}\, d\lambda$ and norm $\| A \|_E = ((A, A))^{1/2}$.

For a function $A \in \mathscr{L}_2(R)$ we define the Fourier–Plancherel (FP) transform following Paley and Wiener [21], as

$$\hat{A}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-itu} A(u)\, \lambda(du).$$

The following theorem can be proved by using essentially classical arguments.

THEOREM 6.4. *Let* $A, B \in \mathscr{L}_2(R)$ *with F-P-transform* $\hat{A}$, $\hat{B}$. *Then*

(i) $\int_R A(u)\, B^*(u)\, \lambda(du) = \int_R \hat{A}(t)\, \hat{B}^*(t)\, \lambda(dt)$ (*Parseval's Identity*);

(ii) $\int_R A(u)\, B(u)\, e^{-iut}\lambda(du) = \int \hat{A}(s)\, \hat{B}(t-s)\, ds$ (*convolution formula*).

We denote $\mathscr{L}_2^+(R) = \{A \mid A \in \mathscr{L}_2(R), \hat{A}(t) = 0, t < 0\}$. Now in order to obtain the factorization and analog of Theorem 5.1, we recall the analog of Theorem 4.2 [14, Theorem 9.3].

THEOREM 6.5. *Let* $X_t = \{X_t^{(k)}\}$ *be continuous parameter, weakly stationary, purely non-deterministic,* $\mathscr{H}$*-valued process. Then, for each t,*

$$X_t^{(k)} = \sum_1^M \int_{-\infty}^t b_n^{(k)}(u-t)\, \xi_n(du), \tag{6.1}$$

where (a) *the* ξ_n*'s are mutually orthogonal, and each* ξ_n *is a process with stationary orthogonal increments*; (b) $L_2(X; t) = \sum_{i=1}^M \oplus L_2(\xi_n; t)$ *for each t where* $L_2(X; t)$ *is the subspace* $L_2(X)$ *generated by* $\{\langle X_\tau, h\rangle, \tau \leqslant t, h \in \mathscr{H}\}$; (c) M *is the multiplicity of the process*; *and* (d) $\sum_1^M \sum_{k=1}^\infty \int_{-\infty}^0 a_n \mid b_n(u)|^2\, \lambda(du)$ *is finite. Here* $a_n > 0$ *denote the numbers such that* $E \mid \xi_n(\Delta)|^2 = a_n \lambda(\Delta)$.

Let us now define for $\Delta \in \mathscr{B}(R)$, $\xi(\Delta) = (\xi^{(i)}(\Delta))_{i=1}^{\infty}$, where $\xi^{(i)}(\Delta) = \xi_i$ for $i \leqslant M$ and $\xi^{(i)}(\Delta) = 0$ otherwise, and let the matrix $\tilde{B}(u) = \{b_{jk}(u)\}$, where $b_{jk}(u) = b_j^{(k)}(u)$, $k = 1, 2,\ldots$, $j = 1, 2,\ldots, M$ and equals zero otherwise. Let D be the matrix with diagonal entries $d_i = a_i$ for $i \leqslant M$, and $d_i = 0$ otherwise; then condition (d) of Theorem 6.5 shows that $\tilde{B}(u)\, D^{1/2} \in \mathscr{L}_2(R)$. Now by an argument similar to Theorem 4.3 and Theorem 6.4 we get

THEOREM 6.6. *The process* $\{X_t^{(k)}\}_{t \in R}$ *is purely nondeterministic* iff *the spectral distribution* F *is absolutely continuous and* $F'_{ac}(u) = \Phi(u)\, \Phi^*(u)$, *where* $\Phi \in \mathscr{L}_2^+(R)$ *with* $\hat{\Phi}(t) = B(t)$ *and* $B(u) = \tilde{B}(-u)$.

The analog of Theorem 5.1 can be proved by using the so-called discretized process. Given any continuous $\mathscr{H}$-valued process, one can construct a discretized process $\{\tilde{X}_n, n = 0, \pm 1, \pm 2,\ldots\}$ by choosing $\tilde{X}_n = \int_0^{2\pi} e^{in\lambda} E(\pi + 2 \tan^{-1}\lambda)$ [14, (6.1), p. 634]. Furthermore, we know that the multiplicity of $\tilde{X}_n$ is the same as that of X_t, and $\tilde{X}_n$ is purely nondeterministic iff X_t is purely nondeterministic [13, Section 6, Lemma G_1]. Let $F(\lambda)$ be the spectral distribution of the purely nondeterministic process X_t and $G(\pi + 2 \tan^{-1}\lambda)$ the spectral distribution of the corresponding discretized process; then $F'(\lambda) = 2/(1 + \lambda^2)\, G'(\pi + 2 \tan^{-1}\lambda)$. Hence we get that rank $F'(\lambda) =$ rank of $G'(\pi + 2 \tan^{-1}\lambda)$ a.e. $[\lambda]$ which equals M a.e., by Theorem 5.1. Now, therefore, we have the following analog of Theorem 5.1.

THEOREM 6.7. *The following is a necessary and sufficient condition for* $\{X_t\}$ *to be purely nondeterministic of rank* M:

(a) *The spectral distribution operator is absolutely continuous with density* $f(u)$;

(b) *The rank of* $f(u) = M$ a.e. $[\lambda]$;

(c) $f(u) = \Phi(u)\, \Phi^*(u)$, *where* $\Phi(u) \in \mathscr{L}_2^+(R)$.

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