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A conjecture on the number of SDRs of a (t, n)-family☆

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ABSTRACT

A system of distinct representatives (SDR) of a family $F = (A_1, \ldots, A_n)$ is a sequence $(x_1, \ldots, x_n)$ of n distinct elements with $x_i \in A_i$ for $1 \leq i \leq n$. Let $N(F)$ denote the number of SDRs of a family F; two SDRs are considered distinct if they are different in at least one component. For a nonnegative integer t, a family $F = (A_1, \ldots, A_n)$ is called a (t, n)-family if the union of any $k \geq 1$ sets in the family contains at least $k + t$ elements. The famous Hall's theorem says that $N(F) \geq 1$ if and only if F is a $(0, n)$-family. Denote by $M(t, n)$ the minimum number of SDRs in a (t, n)-family. The problem of determining $M(t, n)$ and those families containing exactly $M(t, n)$ SDRs was first raised by Chang [G.J. Chang, On the number of SDR of a (t, n)-family, European J. Combin. 10 (1989) 231–234]. He solved the cases when $0 \leq t \leq 2$ and gave a conjecture for $t \geq 3$. In this paper, we solve the conjecture.

1. Introduction

A system of distinct representatives (SDR) of a family $F = (A_1, \ldots, A_n)$ is a sequence $(x_1, \ldots, x_n)$ of n distinct elements with $x_i \in A_i$ for $1 \leq i \leq n$. The famous Hall's theorem tells us that a family has an SDR if and only if the union of any $k \geq 1$ sets of this family contains at least k elements. Several quantitative refinements of Hall's theorem were given in [3,5,6]. Their results are all under the assumption of Hall's condition plus some extra conditions on the cardinalities of A_i's.

Chang [1] extends Hall's theorem as follows: let t be a nonnegative integer. A family $F = (A_1, \ldots, A_n)$ is called a (t, n)-*family* if $|\bigcup_{i \in I} A_i| \geq |I| + t$ holds for any non-empty subset $I \subseteq \{1, \ldots, n\}$. Denote by $N(F)$ the number of SDRs of a family F. Let $M(t, n) = \min\{N(F) \mid F \text{ be a } (t, n) - \text{family}\}$. Hall's theorem says that $M(0, n) \geq 1$. In fact, it is easy to see that $M(0, n) = 1$. By the

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result in [3], a $(0, n)$-family in which every set has two or more elements has at least two SDRs. Using this fact and with a straightforward induction, Chang showed that F is a $(0, n)$-family with $N(F) = 1$ iff F can be permuted into $H = (B_1, \dots, B_n)$ such that there exist n distinct elements $b_1, \dots, b_n$ with $b_i \in B_i \subseteq \{b_1, \dots, b_i\}$ for $1 \le i \le n$. He also showed that $M(1, n) = n + 1$ and F is a $(1, n)$-family with $N(F) = n + 1$ iff $|A_i| = 2$ for $1 \le i \le n$ and $G(F)$ is a tree, where $G(F)$ is a graph with vertex set $V(G) = A_1 \cup \cdots \cup A_n$ and edge set $E(G) = \{A_1, \dots, A_n\}$.

Consider the (t, n)-family $F^* = (A_1^*, \dots, A_n^*)$, where $A_i^* = \{i, n+1, \dots, n+t\}$ for $1 \le i \le n$. Then,

$$N(F^*) = U(t, n) = \sum_{j=0}^{t} \binom{t}{j} \binom{n}{j} j!.$$

Chang [1] has shown that F^* as above is the only $(2, n)$-family F with $N(F) = M(2, n) = n^2 + n + 1$, and he raised the following conjecture.

Conjecture 1 *([1]). $M(t, n) = U(t, n)$ and F^* is the only (t, n)-family F with $N(F) = M(t, n)$ for all $t \ge 3$.*

Leung and Wei [4] claimed that they proved the conjecture by means of a comparison theorem. But their proof has a fatal mistake (see [2]). Hence, the conjecture is still open. The main purpose of this paper is to solve the conjecture. In fact, we will show a more general result, which settles the above conjecture.

We extend the definition of a (t, n)-family as follows: let $a_1, \dots, a_n$ be positive integers, a family $F = (A_1, \dots, A_n)$ is called a *$(t, n; a_1, \dots, a_n)$-family* if $|\bigcup_{i\in I} A_i| \ge \sum_{i\in I} a_i + t$ for any non-empty subset $I \subseteq \{1, \dots, n\}$. Hence, a (t, n)-family is a $(t, n; 1, \dots, 1)$-family.

Let $\tilde{F}$ be a $(t, n; a_1, \dots, a_n)$-family such that each A_i has $a_i + t$ elements and $|\bigcap_{i\in I} A_i| = t$ for any $|I| \ge 2$. Hence, F^* is $\tilde{F}$ with $a_1 = \cdots = a_n = 1$. Define $M(t, n; a_1, \dots, a_n) = \min\{N(F) \mid F$ be a $(t, n; a_1, \dots, a_n)$-family $\}$, and let

$$U(t, n; a_1, \dots, a_n) = N(\tilde{F}) = \sum_{j=0}^{t} \left[\binom{t}{j} j! \sum_{1\le i_1 < \cdots < i_{n-j} \le n} a_{i_1} \cdots a_{i_{n-j}} \right].$$

In this paper, we will prove that $M(t, n; a_1, \dots, a_n) = U(t, n; a_1, \dots, a_n)$ and $\tilde{F}$ is the only $(t, n; a_1, \dots, a_n)$-family F satisfying $N(F) = M(t, n; a_1, \dots, a_n)$ for $t \ge 2$. Conjecture 1 is a direct corollary of the conclusion.

Some notations are needed. Suppose F is a $(t, n; a_1, \dots, a_n)$-family. Let $N = \{1, 2, \dots, n\}$ and $\mathcal{B} = \bigcup_{i\in N} A_i$, and let $I_x = \{i \in N \mid x \in A_i\}$ and $I_x^c = N - I_x$ for $x \in \mathcal{B}$. A pair of elements $\{x, y\} \subseteq \mathcal{B}$ is *exclusive* if $I_x \cap I_y^c \ne \emptyset$ and $I_y \cap I_x^c \ne \emptyset$. A subset I of N is *full* if $|\bigcup_{i\in I} A_i| = \sum_{i\in I} a_i + t$. An exclusive pair $\{x, y\}$ is *saturated* if there exists a full subset $I \subseteq N$ satisfying $I \cap I_x \ne \emptyset$, $I \cap I_y \ne \emptyset$, $I \cap I_x \cap I_y = \emptyset$; otherwise, the exclusive pair $\{x, y\}$ is *unsaturated*.

2. Necessary conditions for $(t, n; a_1, \dots, a_n)$-family F with $N(F) = M(t, n; a_1, \dots, a_n)$

We call a $(t, n; a_1, \dots, a_n)$-family $F = (A_1, \dots, A_n)$ *strict* if $|A_i| = a_i + t$ for $1 \le i \le n$.

Theorem 2. *If $t \ge 1$ and $F = (A_1, \dots, A_n)$ is a $(t, n; a_1, \dots, a_n)$-family with $N(F) = M(t, n; a_1, \dots, a_n)$, then F is strict and so all A_i's are distinct.*

Proof. Let $F = (A_1, \dots, A_n)$ be a $(t, n; a_1, \dots, a_n)$-family with $N(F) = M(t, n; a_1, \dots, a_n)$. We first claim that the deletion of any element from A_i $(1 \le i \le n)$ results in a family that is not a $(t, n; a_1, \dots, a_n)$-family.

Suppose that the claim is not true. Without loss of generality we can assume that $F' = (A_1 - \{x\}, A_2, \dots, A_n)$ is a $(t, n; a_1, \dots, a_n)$-family for some $x \in A_1$. Then $N(F') \ge M(t, n; a_1, \dots, a_n)$. On the other hand, $F'' = (A_2 - \{x\}, A_3 - \{x\}, \dots, A_n - \{x\})$ is a $(t-1, n-1; a_2, \dots, a_n)$-family. As $t \ge 1$, by Hall's theorem, F'' has an SDR $(x_2, \ldots, x_n)$. Hence, $(x, x_2, \ldots, x_n)$ is an SDR of F but not F'. Then $M(t, n; a_1, \ldots, a_n) = N(F) > N(F') \geq M(t, n; a_1, \ldots, a_n)$, which is impossible.

Now we show that $|A_i| = a_i + t$ for $1 \leq i \leq n$. Suppose to the contrary that there is some $|A_i| \geq a_i + t + 1$, say A_1. For each $x \in A_1$, by the above claim, $F_x = (A_1 - \{x\}, A_2, \ldots, A_n)$ is not a $(t, n; a_1, \ldots, a_n)$-family. Hence there exists a non-empty subset $J_x \subseteq \{2, \ldots, n\}$ such that $|(A_1 - \{x\}) \cup (\bigcup_{i\in J_x} A_i)| \leq a_1 + \sum_{i\in J_x} a_i + t - 1$, which implies that $|A_1 \cup (\bigcup_{i\in J_x} A_i)| = a_1 + \sum_{i\in J_x} a_i + t$ and $x \notin \bigcup_{i\in J_x} A_i$. Now we select such J_x with a minimum size.

For any element $y \in A_1 \setminus \{x\}$, let $S = A_1 \cup (\bigcup_{i\in J_x} A_i)$ and $T = A_1 \cup (\bigcup_{i\in J_y} A_i)$. Then

$$\begin{aligned}
\sum_{i\in J_x} a_i + \sum_{i\in J_y} a_i + 2a_1 + 2t &= |S| + |T| = |S \cup T| + |S \cap T| \\
&\geq \left|\left(\bigcup_{i\in J_x\cup J_y} A_i\right) \cup A_1\right| + \left|\left(\bigcup_{i\in J_x\cap J_y} A_i\right) \cup A_1\right| \\
&\geq \begin{cases} \displaystyle\sum_{i\in J_x\cup J_y} a_i + a_1 + t + \sum_{i\in J_x\cap J_y} a_i + a_1 + t, & \text{if } J_x \cap J_y \neq \emptyset; \\ \displaystyle\sum_{i\in J_x\cup J_y} a_i + a_1 + t + a_1 + t + 1, & \text{if } J_x \cap J_y = \emptyset. \end{cases} \\
&= \begin{cases} \displaystyle\sum_{i\in J_x} a_i + \sum_{i\in J_y} a_i + 2a_1 + 2t, & \text{if } J_x \cap J_y \neq \emptyset; \\ \displaystyle\sum_{i\in J_x} a_i + \sum_{i\in J_y} a_i + 2a_1 + 2t + 1, & \text{if } J_x \cap J_y = \emptyset. \end{cases}
\end{aligned}$$

Hence, $J_x \cap J_y \neq \emptyset$ and $|(\bigcup_{i\in J_x\cap J_y} A_i) \cup A_1| = \sum_{i\in J_x\cap J_y} a_i + a_1 + t$. By the minimality of J_x, we have $J_x = J_y$. Therefore, $y \notin \bigcup_{i\in J_x} A_i$. This implies that $A_1 \cap (\bigcup_{i\in J_x} A_i) = \emptyset$. Hence,

$$\begin{aligned}
\sum_{i\in J_x} a_i + a_1 + t &= \left|A_1 \cup \left(\bigcup_{i\in J_x} A_i\right)\right| \\
&= |A_1| + \left|\bigcup_{i\in J_x} A_i\right| \\
&\geq \sum_{i\in J_x} a_i + t + a_1 + t + 1.
\end{aligned}$$

This is a contradiction. Hence $|A_i| = a_i + t$ for $1 \leq i \leq n$. If $A_i = A_j$ for two distinct i and j, then $a_i + t = |A_i| = |A_i \cup A_j| \geq a_i + a_j + t$ is a contradiction. So all A_i's are distinct. □

Assume that $F = (A_1, \ldots, A_n)$ is a $(t, n; a_1, \ldots, a_n)$-family and a pair of elements $\{x, y\}$ is exclusive for F. Let

$$A_i(x, y) = \begin{cases} A_i - \{x\} \cup \{y\}, & \text{if } i \in I_x \cap I_y^c; \\ A_i, & \text{otherwise.} \end{cases}$$

Then we get a new family $F_y^x = (A_1(x, y), \ldots, A_n(x, y))$, but it is possible that F_y^x is not a $(t, n; a_1, \ldots, a_n)$-family. For any $I \subseteq N$, by calculating $|\bigcup_{i\in I} A_i|$ and $|\bigcup_{i\in I} A_i(x, y)|$, we get the relationship between the two values as follows:

$$\left|\bigcup_{i\in I} A_i(x, y)\right| = \begin{cases} \left|\bigcup_{i\in I} A_i\right| - 1, & \text{if } I \cap I_x \neq \emptyset, I \cap I_y \neq \emptyset, I \cap I_x \cap I_y = \emptyset; \\ \left|\bigcup_{i\in I} A_i\right|, & \text{otherwise.} \end{cases}$$

Hence, F_y^x is still a $(t, n; a_1, \ldots, a_n)$-family if and only if $\{x, y\}$ is unsaturated for F. Furthermore, we have the following theorem.

Theorem 3. *If $t \geq 2$, then any $(t, n; a_1, \ldots, a_n)$-family F with $N(F) = M(t, n; a_1, \ldots, a_n)$ does not contain any unsaturated pair $\{x, y\}$.*

Proof. Suppose to the contrary that $\{x, y\}$ is unsaturated for F. Then, F_y^x is also a $(t, n; a_1, \ldots, a_n)$-family. We will prove that $N(F_y^x) < N(F)$ which leads to a contradiction.

Without loss of generality, we can assume that $I_x \cap I_y^c = \{1, \ldots, k_1\} \neq \emptyset, I_y \cap I_x^c = \{k_1+1, \ldots, k_2\} \neq \emptyset$, $I_x \cap I_y = \{k_2 + 1, \ldots, k_3\}$ and $I_x^c \cap I_y^c = \{k_3 + 1, \ldots, n\}$. So $F_y^x = (A_1(x, y), \ldots, A_n(x, y)) = (A_1 - \{x\} \cup \{y\}, \ldots, A_{k_1} - \{x\} \cup \{y\}, A_{k_1+1}, \ldots, A_n)$. Let $(x_1, \ldots, x_n)$ be an SDR of F_y^x. Define a function f from the set of all SDRs of F_y^x to the set of all SDRs of F as follows:

(a) if $x_i = y$ for some $i \in \{1, \ldots, k_1\}$ and $x_j = x$ for some $j \in \{k_2 + 1, \ldots, k_3\}$, then

$$(x_1, \ldots, y, \ldots, x, \ldots, x_n) \to (x_1, \ldots, x, \ldots, y, \ldots, x_n).$$

(b) If $x_i = y$ for some $i \in \{1, \ldots, k_1\}$ and $x_j \neq x$ for all x_j, then

$$(x_1, \ldots, y, \ldots, x_n) \to (x_1, \ldots, x, \ldots, x_n).$$

(c) Otherwise,

$$(x_1, \ldots, x_n) \to (x_1, \ldots, x_n).$$

f is clearly one to one and so $N(F_y^x) \leq N(F)$. Define

$$F' = (A_2 - \{x, y\}, \ldots, A_{k_1} - \{x, y\}, A_{k_1+2} - \{x, y\}, \ldots, A_n - \{x, y\}).$$

Since $t \geq 2$, F' satisfies Hall's theorem and has an SDR $(x_2, \ldots, x_{k_1}, x_{k_1+2}, \ldots, x_n)$. Hence, F has an SDR such as

$$(x, x_2, \ldots, x_{k_1}, y, x_{k_1+2}, \ldots, x_n),$$

which is not an f-image of an SDR of F_y^x, so f is not subjective. Hence, $N(F_y^x) < N(F)$. □

3. Saturated pairs of a strict $(t, n; a_1, \ldots, a_n)$-family

Theorem 4. *For a strict $(t, n; a_1, \ldots, a_n)$-family F, denote by $NSP(F)$ the number of saturated pairs of F, then $NSP(F) \leq \sum_{1 \leq i < j \leq n} a_i a_j$.*

Proof. We shall prove the theorem by induction on n. The theorem is clear for $n = 1$ or $NSP(F) = 0$. Now we assume that $n \geq 2$ and F has at least one saturated pair.

Claim 1. *If I and J are two full subsets of N with $I \cap J \neq \emptyset$, then $I \cup J$ and $I \cap J$ are also full.*

Since $I \cap J \neq \emptyset$,

$$\begin{aligned}\sum_{s \in I \cup J} a_s + t &\leq \left| \bigcup_{s \in I \cup J} A_s \right| = \left| \left(\bigcup_{s \in I} A_s \right) \cup \left(\bigcup_{s \in J} A_s \right) \right| \\ &\leq \left| \bigcup_{s \in I} A_s \right| + \left| \bigcup_{s \in J} A_s \right| - \left| \bigcup_{s \in I \cap J} A_s \right| \\ &\leq \sum_{s \in I} a_s + t + \sum_{s \in J} a_s + t - \left(\sum_{s \in I \cap J} a_s + t \right) \\ &= \sum_{s \in I \cup J} a_s + t.\end{aligned}$$

Hence, $|\bigcup_{s \in I \cup J} A_s| = \sum_{s \in I \cup J} a_s + t$ and $|\bigcup_{s \in I \cap J} A_s| = \sum_{s \in I \cap J} a_s + t$, i.e., $I \cup J$ and $I \cap J$ are full.

Since F has a saturated pair, N has a full subset of size at least two. Choose a minimal full subset I^* of N with size at least two, i.e., any proper full subset of I^* is of size one. Now consider two cases.

Case 1. $I^* \neq N$, say $I^* = \{k+1, k+2, \dots, n\}$ with $k \geq 1$. In this case, $F' = (A_1, \dots, A_k, \bigcup_{i \in I^*} A_i)$ is a strict $(t, k+1; a_1, \dots, a_k, \sum_{i \in I^*} a_i)$-family and $F'' = (A_{k+1}, \dots, A_n)$ is a strict $(t, n-k; a_{k+1}, \dots, a_n)$-family. We claim that any saturated pair of F is either a saturated pair of F' or a saturated pair of F''. From this and the induction hypothesis, we then have

$$\begin{aligned} NSP(F) &\leq NSP(F') + NSP(F'') \\ &\leq \sum_{1 \leq i < j \leq k} a_i a_j + \left(\sum_{1 \leq i \leq k} a_i \right) \left(\sum_{k+1 \leq j \leq n} a_j \right) + \sum_{k+1 \leq i < j \leq n} a_i a_j \\ &\leq \sum_{1 \leq i < j \leq n} a_i a_j. \end{aligned}$$

To see the above claim, suppose to the contrary that F has a saturated pair $\{x, y\}$ that is not a saturated pair of F' or F''. Choose a full subset I of N such that $I \cap I_x \neq \emptyset, I \cap I_y \neq \emptyset$ but $I \cap I_x \cap I_y = \emptyset$. Since $\{x, y\}$ is not a saturated pair of F' and so not a saturated pair of $(A_1, A_2, \dots, A_k)$. This gives that I is not a subset of $N - I^*$ and so $I \cap I^* \neq \emptyset$. By Claim 1, $I \cap I^*$ and $I \cup I^*$ are full sets. By the minimality of I^*, either $I \cap I^* = I^*$ or $|I \cap I^*| = 1$.

For the case of $I \cap I^* = I^*$, by $I \cap I_x \cap I_y = \emptyset$, we have $I^* \cap I_x \cap I_y = \emptyset$. This, together with that $\{x, y\}$ is not a saturated pair of F'', implies that either $I^* \cap I_x = \emptyset$ or $I^* \cap I_y = \emptyset$. So, at most one of x and y is in $\bigcup_{i \in I^*} A_i$. This gives that $\{x, y\}$ is a saturated pair of F', which is impossible.

For the case of $|I \cap I^*| = 1$, assume $I \cap I^* = \{k+1\}$. Then, A_{k+1} contains at most one of x and y, say $y \notin A_{k+1}$. So, $\bigcup_{i \in I^* - I} A_i - \bigcup_{i \in I} A_i$ is a proper subset of $\bigcup_{i \in I^*} A_i - A_{k+1}$ since the latter contains y while the former does not. Hence

$$\begin{aligned} \left| \bigcup_{i \in I \cup I^*} A_i \right| &= \left| \bigcup_{i \in I} A_i \right| + \left| \bigcup_{i \in I^* - I} A_i - \bigcup_{i \in I} A_i \right| \\ &< \left| \bigcup_{i \in I} A_i \right| + \left| \bigcup_{i \in I^*} A_i - A_{k+1} \right| \\ &= \sum_{i \in I} a_i + t + \sum_{i \in I^*} a_i + t - (a_{k+1} + t) \\ &= \sum_{i \in I \cup I^*} a_i + t, \end{aligned}$$

contradicting to the fact that $I \cup I^*$ is full.

Case 2. $I^* = N$, an exclusive pair $\{x, y\}$ is saturated for F if and only if $I_x \cap I_y = \emptyset$. Let $C = \{\{x, y\} \mid I_x \cap I_y = \emptyset\}$. Then $NSP(F) = |C|$. Now we calculate $|C|$.

For an arbitrary element $z \in \mathcal{B}$, define $C(z) = \{\{x, z\} \mid I_x \cap I_z = \emptyset\}$. It is not difficult to see that $|C| = \frac{1}{2} \sum_{z \in \mathcal{B}} |C(z)|$ and $C(z) = \{\{x, z\} \mid I_x \cap I_z = \emptyset\} = \{\{x, z\} \mid x \notin \bigcup_{i \in I_z} A_i\}$. So,

$$|C(z)| = |\mathcal{B}| - \left| \bigcup_{i \in I_z} A_i \right| \leq \sum_{i \in I_z^c} a_i.$$

Therefore,

$$\begin{aligned} |C| &\leq \frac{\sum_{z \in \mathcal{B}} \sum_{i \in I_z^c} a_i}{2} = \frac{\sum_{z \in \mathcal{B}} \left(\sum_{i=1}^{n} a_i - \sum_{i \in I_z} a_i \right)}{2} \\ &= \frac{\left(\sum_{i=1}^{n} a_i + t \right) \left(\sum_{i=1}^{n} a_i \right) - \sum_{z \in \mathcal{B}} \sum_{i \in I_z} a_i}{2} \end{aligned}$$

$$= \frac{\left(\sum_{i=1}^{n} a_i + t\right)\left(\sum_{i=1}^{n} a_i\right) - \sum_{i=1}^{n}(a_i + t)a_i}{2}$$
$$= \sum_{1\le i<j\le n} a_i a_j. \quad \Box$$

4. Exclusive pairs of a strict $(t, n; a_1, \ldots, a_n)$-family

Theorem 5. *For a strict $(t, n; a_1, \ldots, a_n)$-family F, denote by $NEP(F)$ the number of exclusive pairs of F. If $t \ge 2$, then $NEP(F) \ge \sum_{1\le i<j\le n} a_i a_j$, and $\tilde{F}$ is the only strict $(t, n; a_1, \ldots, a_n)$-family F with $NEP(F) = \sum_{1\le i<j\le n} a_i a_j$.*

Proof. We can assume that $n \ge 2$. For an arbitrary element $z \in \mathcal{B}$, $\{x, z\}$ is exclusive for F if and only if $x \in \bigcup_{i\in I_z^c} A_i$ and $x \notin \bigcap_{i\in I_z} A_i$. Define $D(z) = \{\{x, z\} \mid \{x, z\}$ is exclusive for $F\}$. Therefore,

$$D(z) = \left\{ \{x, z\} \mid x \in \bigcup_{i\in I_z^c} A_i - \bigcap_{i\in I_z} A_i \right\}.$$

Let $\mathcal{A} = \{z \mid |I_z| = n\}$ and $D = \{\{x, y\} \mid \{x, y\}$ is exclusive for $F\}$. Note that $D(z) = \emptyset$ if $z \in \mathcal{A}$. Then,

$$|D| = \frac{1}{2}\sum_{z\in\mathcal{B}} |D(z)| = \frac{1}{2}\sum_{z\in\mathcal{B}-\mathcal{A}} |D(z)|$$
$$= \frac{1}{2}\sum_{z\in\mathcal{B}-\mathcal{A}} \left(\left| \bigcup_{i\in I_z^c} A_i - \bigcap_{i\in I_z} A_i \right| \right).$$

We first assume that $|I_z| \ge 2$ and hence $|\bigcap_{i\in I_z} A_i| \le t$ for all $z \in \mathcal{B} - \mathcal{A}$. Hence,

$$|D| > \frac{1}{2}\sum_{z\in\mathcal{B}-\mathcal{A}} \left(\left| \bigcup_{i\in I_z^c} A_i \right| - \left| \bigcap_{i\in I_z} A_i \right| \right) \ge \frac{1}{2}\sum_{z\in\mathcal{B}-\mathcal{A}} \sum_{i\in I_z^c} a_i. \qquad (*)$$

We point out that the inequality strictly holds as $z \in \bigcap_{i\in I_z} A_i$ and $z \notin \bigcup_{i\in I_z^c} A_i$. To calculate $\sum_{z\in\mathcal{B}-\mathcal{A}} \sum_{i\in I_z^c} a_i$, we construct a weighted bipartite graph G as follows: $V(G) = V_1 \cup V_2$, where $V_1 = \mathcal{B} - \mathcal{A}$ and $V_2 = \{A_1, \ldots, A_n\}$; For $z \in V_1$, if $z \notin A_i$, then $zA_i \in E(G)$ and the weight of zA_i, denoted by $w(zA_i)$, is a_i. So,

$$\sum_{z\in\mathcal{B}-\mathcal{A}} \sum_{i\in I_z^c} a_i = \sum_{z\in V_1} \sum_{zA_i\in E(G)} w(zA_i) = \sum_{A_i\in V_2} \sum_{zA_i\in E(G)} w(zA_i). \qquad (**)$$

Let $|\mathcal{A}| = a$. Obviously, $a \le t$. Each set A_i contains $a_i + t - a$ elements in $\mathcal{B} - \mathcal{A}$ and there are at least $\sum_{j=1}^{n} a_j + t - a$ elements in $\mathcal{B} - \mathcal{A}$. By the construction of G, we know that the vertex A_i is incident to at least $\sum_{j=1}^{n} a_j - a_i$ edges in G and the weight of each edge incident to A_i is a_i. Therefore,

$$\sum_{A_i\in V_2} \sum_{zA_i\in E(G)} w(zA_i) \ge \sum_{i=1}^{n} a_i \left(\sum_{j=1}^{n} a_j - a_i \right) = \left(\sum_{i=1}^{n} a_i \right)^2 - \sum_{i=1}^{n} a_i^2. \qquad (* * *)$$

By above inequalities $(*)$, $(**)$ and $(* * *)$, we know that $|D| > \sum_{1\le i<j\le n} a_i a_j$ if $\deg z \ge 2$ for all $z \in \mathcal{B}$.

Now we assume that there exists an element x such that $\deg x = 1$, without loss of generality, we assume that $I_x = \{n\}$. Let $k = \sum_{i=1}^{n} a_i$. We use induction on k.

When $k = 2$, then $n = 2$ and $a_1 = a_2 = 1$, the conclusion is obvious. Assume that $k \geq 3$. As the conclusion is obvious when $n = 2$, we may assume that $n \geq 3$.

If $a_n = 1$, let $F_1 = (A_1, \dots, A_{n-1})$, by induction hypothesis, $NEP(F_1) \geq \sum_{1 \leq i < j \leq n-1} a_i a_j$ and $NEP(F_1) = \sum_{1 \leq i < j \leq n-1} a_i a_j$ implies that F_1 is a strict $(t, n-1; a_1, \dots, a_{n-1})$-family such that $|\bigcap_{i \in I} A_i| = t$ for any $|I| \geq 2$. It is obvious that the exclusive pairs of F_1 are also exclusive for F. Since $(\bigcup_{i=1}^{n-1} A_i) - A_n = (\bigcup_{i=1}^{n} A_i) - A_n$, we know that $|(\bigcup_{i=1}^{n-1} A_i) - A_n| \geq \sum_{i=1}^{n-1} a_i$. Obviously, each element y in $(\bigcup_{i=1}^{n-1} A_i) - A_n$ is exclusive with x for F and $\{x, y\}$ is different from any exclusive pair of $(A_1, \dots, A_{n-1})$. Therefore,

$$NEP(F) \geq \sum_{1 \leq i < j \leq n-1} a_i a_j + \sum_{k=1}^{n-1} a_k = \sum_{1 \leq i < j \leq n} a_i a_j.$$

When $NEP(F) = \sum_{1 \leq i < j \leq n} a_i a_j$, it implies that $A_n \cap (\bigcup_{i=1}^{n-1} A_i) = t$ and $NEP(F) - NEP(F_1) = \sum_{k=1}^{n-1} a_k$. This requires that F is $\tilde{F}$.

If $a_n \geq 2$, let $F_2 = (A_1, \dots, A_{n-1}, A_n - \{x\})$, which is a $(t, n; a_1, \dots, a_{n-1}, a_n - 1)$-family, by induction hypothesis, $NEP(F_2) \geq \sum_{1 \leq i < j \leq n-1} a_i a_j + \sum_{k=1}^{n-1} a_k(a_n - 1)$ and $NEP(F_2) = \sum_{1 \leq i < j \leq n-1} a_i a_j + \sum_{k=1}^{n-1} a_k(a_n - 1)$ implies that F_2 is a strict $(t, n; a_1, \dots, a_{n-1}, a_n - 1)$-family such that $|\bigcap_{i \in I} A_i| = t$ for any $|I| \geq 2$. Similarly, the exclusive pairs of F_2 are also exclusive for F, $|\bigcup_{i=1}^{n-1} A_i - A_n| \geq \sum_{i=1}^{n-1} a_i$, and each element y in $\bigcup_{i=1}^{n-1} A_i - A_n$ is exclusive with x for F and $\{x, y\}$ is different from any exclusive pair of F_2. Therefore,

$$NEP(F) \geq \sum_{1 \leq i < j \leq n-1} a_i a_j + \sum_{k=1}^{n-1} a_k(a_n - 1) + \sum_{k=1}^{n-1} a_k = \sum_{1 \leq i < j \leq n} a_i a_j.$$

Similarly, $NEP(F) = \sum_{1 \leq i < j \leq n} a_i a_j$ implies that F_2 must be a strict $(t, n; a_1, \dots, a_{n-1}, a_n - 1)$-family such that $|\bigcap_{i \in I} A_i| = t$ for any $|I| \geq 2$. Since $I_x = \{n\}$, it is obvious that F is $\tilde{F}$. $\square$

5. The conclusion about $N(F)$

By Theorems 2–5, we can easily arrive at the following conclusion.

Theorem 6. *$M(t, n; a_1, \dots, a_n) = U(t, n; a_1, \dots, a_n)$ and $\tilde{F}$ is the only $(t, n; a_1, \dots, a_n)$-family F with $N(F) = M(t, n; a_1, \dots, a_n)$ for $t \geq 2$.*

Applying Theorem 6 to (t, n)-family, we immediately prove Conjecture 1.

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