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Determination of mechanical properties of non-linear coatings from measurements with coated beams

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ABSTRACT

Coatings are applied to structural components for several various reasons, such to protect against erosion or corrosion, as thermal barrier coatings, or to increase the energy dissipation. As determining the material properties of such coatings from homogeneous specimens is often difficult, it is sometimes necessary to conduct testing on coated specimens, with the properties of the coating then to be extracted from the results of testing. A methodology for doing this is given here. While applicable to other materials, the properties of such coatings as ceramics, metallics, or compounds to be applied to rotating and static components of gas turbines are of special interest. Such materials present a special challenge as the mechanical properties have generally been found to display a strong dependence on the amplitude of cyclic strain. Application of the methodology requires careful measurement of specimen dimensions, weights, natural frequencies, and system loss factors before and after coating. From these, the storage (Young's) modulus, the loss modulus, and the loss factor can be extracted. The methodology is demonstrated through the use of data taken on flat specimens of titanium with plasma-sprayed coatings of NiCrAlY and a titania–alumina blend ceramic, vibrating in a cantilever mode.

1. Introduction

In the ideal test for the determination of the mechanical properties of a coating material a specimen of a macroscopically homogeneous material would be subjected to a spatially uniform stress or strain and temperature. In this case, the energies dissipated and stored per unit volume would be simply the total energy dissipated per cycle of a fully reversed oscillation and the peak energy stored during that cycle, each divided by the volume. In many cases, however, the testing of homogeneous specimens is not possible. The material may flow (creep), or it may be of such brittleness that fractures at support points preclude valid tests. The alternative is to apply the coating to a substrate specimen of known properties and then extract the properties of the coating from the measured response of the system.

An ASTM standard (ASTM, 2005) has been adopted for determining the damping properties of coatings. It provides for a coating applied to one or both sides of a cantilever beam, or for the coating material applied as the core of a sandwich beam. However, the test methodology and procedures for extracting properties from the system response are applicable only if the coating material is linear, i.e. if the mechanical properties (modulus and loss factor) are independent of vibratory amplitude.

The extraction of the *material* properties of the coating from *system* measurements is more challenging when the observed response is that of a non-linear system, i.e., when the natural frequencies and system loss factors show a dependence on the amplitude of the cyclic strain. An extensive treatment of the problem of extracting material damping properties of non-linear materials from macroscopically uniform specimens subjected to non-uniform distributions of cyclically applied stress has been given by Lazan (1968).

Material properties to be considered here are the storage modulus and loss modulus of the coating material, defined respectively on the basis of the energy stored and dissipated in a unit volume of material subjected to a uniform state of strain during a complete cycle of fully reversed oscillation. Knowledge of the material storage modulus is necessary if natural frequencies and response amplitudes of objects to be coated are to be evaluated during design; knowledge of a material damping measure is necessary if the amplitudes of responses at resonance are to be predicted. Although the loss factor is often used as the measure of the inherent damping ability of a material, the loss modulus (the product of loss factor and storage modulus) is the more relevant measure as the response of structures with thin free-layer coatings is driven by the value of this parameter.

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We will consider here procedures for determining the amplitude-dependent material properties of a macroscopically uniform coating applied at uniform thickness over both sides of a beam from comparisons of the system response (resonant frequencies and system loss factors) before and after application of the coating.

2. Methodology: storage modulus

2.1. Formulation

The change in resonant frequency of a beam resulting from application of a coating provides a means of estimating the storage modulus of the coating. We first write the Rayleigh quotients, $\omega^2 = U/K$, for the frequencies of the coated ($\omega = 2\pi f$) and uncoated ($\omega_0 = 2\pi f_o$) systems in terms of the maximum strain energies, U, and maximum kinetic energies, K. The ratio of the Rayleigh quotients then gives

$$\frac{f^2}{f_o^2} = \frac{\omega^2}{\omega_o^2} = \frac{U_{\text{Coated}}/K_{\text{Coated}}}{U_B/K_B} = \frac{1 + U_C/U_B}{1 + K_C/K_B} = \frac{1 + R_{SE}}{1 + 2t\rho_C/(h\rho_B)} \tag{1}$$

where C and B denote properties of the coating alone and of the bare beam, respectively. The masses per unit area (the products of thickness and density) can be determined by weighing and measuring before and after application of the coating. The use of Eq. (1) with observed frequencies at common values of test temperature and reference strains enables the extraction of a strain energy ratio, $R_{SE} = U_C/U_B$, at each value of cyclic strain amplitude and temperature

$$R_{SE} = \frac{f^2}{f_0^2}\left(1 + \frac{2t\rho_C}{h\rho_B}\right) - 1 \tag{2}$$

If the substrate beam is linear (i.e. amplitude independent) but the modulus of the coating is amplitude-dependent, the ratio of energies stored in coating and substrate will vary with the value of maximum strain at which the frequencies are observed. Such changes in the strain energy ratio with strain provide a means of evaluating the storage modulus of the coating material.

The need for high quality measurements of frequencies, densities, and thicknesses is evident. In particular, the thickness ratio, t/h must be such that errors in measurements and uncertainties in the determination of the factors in the right hand side of Eq. (2) do not negate the validity of the determined strain energy ratios.

If the coating were linear (i.e. an amplitude-independent storage modulus), the mode shape for a beam of thickness h, fully covered with a uniform thickness (t) of a linear coating would be the same as that of the uncoated beam. With the assumption that the coating is quite thin, or nearly linear, the mode shape of the coated beam may be approximated by that of the uncoated beam. The strain energies of the coating and the substrate beam can then be evaluated from the mode shape for the nth mode, expressed in terms of the local amplitude of strain at the beam-coating interface, $\varepsilon_n(x)$. For a coating of uniform thickness, fully covering both sides of a beam of width W, modulus, E_B, and uniform thickness, the strain energy ratio is

$$\begin{aligned} R_{SE} &= \frac{U_C}{U_B} = \frac{2W\int_{\text{Coating}} \frac{E_{1C}(\varepsilon)}{2}\varepsilon^2(x,z)dxdz}{W\int_{\text{Beam}} \frac{E_B}{2}\varepsilon^2(x,z)dxdz} \\ &= \frac{E_E(\varepsilon_{\max})\int_{h/2}^{h/2+t}(\frac{2z}{h})^2 dz \int_0^L \varepsilon_n^2(x)dx}{\frac{E_B}{2}\int_{-h/2}^{h/2}(\frac{2z}{h})^2 dz \int_0^L \varepsilon_n^2(x)dx} \end{aligned} \tag{3}$$

The last integrals in the numerator and denominator, being identical, cancel in consequence of the assumption of a common mode shape for the coated and uncoated beams. Note that the energy stored in each unit volume of the coating is taken as $E_{1C}\varepsilon^2/2$, where the strain is the local amplitude and the modulus is presumed to be an amplitude-dependent secant modulus.

For convenience, the true, or amplitude-dependent value of coating storage modulus, $E_{1C}(\varepsilon)$ has been expressed in terms of an effective value, $E_E(\varepsilon_{\max})$, defined by

$$E_E(\varepsilon_{\max})\int_{\text{Vol}} \varepsilon^2 dv \equiv \int_{\text{Vol}} E_{1C}(\varepsilon)\varepsilon^2 dv \tag{4}$$

This is not a local (material) modulus, but is rather a global or mean value, an average weighted for the range of strains in the coating with a particular strain distribution and maximum amplitude of the interface strain, $\varepsilon_{\max}$. Amplitude-dependent values of this effective modulus may then be evaluated from Eq. (3), with values of the strain energy ratio found from the observed frequencies and Eq. (2):

$$E_E(\varepsilon_{\max}) = R_{SE}(\varepsilon_{\max})\frac{hE_B}{6tT(2,t/h)} \tag{5}$$

The variation in strain through the coating thickness in Eq. (3) has been accounted for by the evaluation for $N = 2$ of the quantity

$$T(N,t/h) \equiv \frac{1}{t}\int_{h/2}^{h/2+t}\left(\frac{2z}{h}\right)^N dz = \frac{h}{2t}\frac{1}{N+1}\{(1+2t/h)^{N+1} - 1\} \tag{6}$$

The relative simplicity of the process for obtaining this measure of coating stiffness is a consequence of the use of a beam fully coated on both sides. Tests conducted for the purpose of determining material properties have also been performed with beams coated only over a short segment in a region of high strain (Patsias et al., 2004a; Tassini et al., 2006; Reed, 2007) In such cases, the mode shape changes markedly when the coating is applied, and the ratio of kinetic energies does not take the simple form as in Eq. (1), nor does the strain energy ratio take the simple form of Eq. (3).

Once values of the effective modulus $E_E(\varepsilon_{\max})$ have been determined for a range of amplitudes of maximum strain, the true value of the local or material modulus $E_{1C}(\varepsilon)$ can then be extracted by solving Eq. (4), an integral equation. One approach to this is to assume an appropriate functional form for $E_{1C}(\varepsilon)$, with the necessary parameters for that form to be found by using in Eq. (4) the actual strain distribution together with values of the effective modulus, $E_E(\varepsilon_{\max})$ determined from the measured values of frequency.

2.2. A polynomial representation of the storage modulus

Let it be assumed that the strain dependence of the true material storage modulus can be adequately represented by a few terms of a polynomial

$$E_{1C}(\varepsilon) = \sum_{m=0}^{M} E_m \varepsilon^m \tag{7}$$

and that the effective or mean value, as extracted from test data with Eqs. (2) and (5) can be represented by a polynomial of similar form

$$E_E(\varepsilon_{\max}) = \sum_{p=0}^{P} A_p \varepsilon_{\max}^p \tag{8}$$

With the assumption that the presence of the coating influences the maximum *amplitude* of the cyclic strain but not the *distribution*, the strain throughout the substrate beam and the coating as found from the curvature is

$$\varepsilon(x,z) = Cz\partial^2 X_n(x)/\partial x^2 \tag{9}$$

where $X_n(x)$ is the distribution of transverse displacements in one of the normal modes of a Bernoullli–Euler beam (Timoshenko,

1955). If both surfaces are coated at the same uniform thickness, the neutral axis ($z = 0$) is at the geometric center of the beam. Substitution of Eqs. (7) and (8) with these strains into both sides of Eq. (4) gives that

$$\sum_{p=0}^{P} A_p \varepsilon_{\max}^{p} T(2,t/h)\varepsilon_{\max}^{2} \int_0^L \left(\frac{\varepsilon_n(x)}{\varepsilon_{\max}}\right)^2 dx = \sum_{m=0}^{M} E_m T(2+m,t/h)\varepsilon_{\max}^{2+m} \int_0^L \left(\frac{\varepsilon_n(x)}{\varepsilon_{\max}}\right)^{2+m} dx \tag{10}$$

It is convenient to define an integral to be evaluated numerically from the mode shapes as

$$I(N,n) \equiv \frac{1}{L}\int_0^L \left(\frac{\varepsilon_n(x)}{\varepsilon_{\max}}\right)^N dx = \int_0^1 \left[\frac{X_n\prime\prime(x/L)}{X_n''(\max)}\right]^N d(x/L) \tag{11}$$

By taking the same number of terms in the expansions for the effective and true modulus (M=P) and using the integrals defined in Eqs. (6) and (11), the coefficients for Eq. (7) are found to be related those from Eq. (8) by

$$E_m = A_m \frac{T(2,t/h)I(2,n)}{T(2+m,t/h)I(2+m,n)} \tag{12}$$

Values of the thickness integral, Eq. (6), are given in Fig. 1a for several values of t/h.

As cantilever beams are most frequently used in such tests, it is of particular interest to consider values of the function $I(N, n)$ for this boundary condition. The integral $I(2,n)$ has value of 1/4 for any mode of a cantilever beam; values for other choices of $N = 2 + m$ as found by numerical integration of Eq. (11) with Simpson's rule and 101 integration points are shown in Fig. 1b. Values for the second and higher modes are very similar, but notably less than those for the first mode.

Values for the ratio of coefficients, E_m/A_m, as computed from Eq. (12) for the first several terms of the expansion for each of several modes with several ratios of coating to substrate thicknesses, are all found to be greater than one. This indicates that the strain at which the true modulus has some particular value E_{1C} is always less than the maximum strain $\varepsilon_{\max}$ at which the effective modulus, E_E, has that same value.

A quick estimate of the true value of the material storage modulus may be obtained. The most representative maximum coating strain is the mid-plane value

$$\varepsilon(x) = (1 + t/h)\varepsilon_n(x) \tag{13}$$

Substitution of Eqs. (12) and (13) into Eq. (7) then gives the true modulus, as a function of the representative strain, of

$$E_{1C}(\varepsilon) = \sum_{m=0}^{M} A_m \frac{I(2,n)T(2,t/h)}{I(2+m,n)T(2+m,t/h)}[\varepsilon_{\max}(1+t/h)]^m = \sum_{m=0}^{M} A_m(\alpha_m\varepsilon_{\max})^m \tag{14}$$

This suggests that the material modulus at local strain ε can be constructed from the effective modulus by replacing the maximum strain in each term of the expansion by a scaled strain, $\alpha_m\varepsilon_{\max}$, where

$$\alpha_m = (1 + t/h)(E_m/A_m)^{1/m} \tag{15}$$

Values by which the maximum strains must be multiplied to create the relationship between true modulus and the strain ε_0 (i.e. $1/\alpha_m$) are given in Table 1 for the first several coefficients for each of several modes (n) and thickness ratios (t/h). Average values of coefficients for terms $m = 1 - 4$ are also given. It is of interest that individual values of the coefficients $1/\alpha_m$, vary only slightly from the average values, and that the average values for modes 2–4 differ only slightly from a common value of 0.665. Values for higher modes (not shown) are nearly identical to the values given for mode 4. Thus, a quick estimate of the storage modulus of the coating can be obtained from the effective modulus determined from tests in cantilever mode 2 or higher with

$$E_{1C}(\varepsilon) \cong E_E\{1.50\varepsilon_{\max}\} \tag{16}$$

2.2.1. An alternative evaluation

An alternative means of determining the coefficients for the expansion of the true material storage modulus from the mean or effective values is to use an adaptation of the concept of the volume-strain integral as employed by Lazan (1968). After substituting the expansions for the true and effective modulus into Eq. (4) with the volume element written as dzdS, substituting the strain distribution for the nth mode from Eq. (9) and then using Eq. (6), the result is

$$\sum_{p=0}^{P} A_p \varepsilon_{\max}^{p} tT(2,t/h)\int_0^S \varepsilon_n^2 dS = \sum_{m=0}^{M} E_m tT(2+m,t/h)\int_0^S \varepsilon_n^{2+m} dS \tag{17}$$

After making a change in variable of integration with S now being the area at which the surface strain is less than ε_n

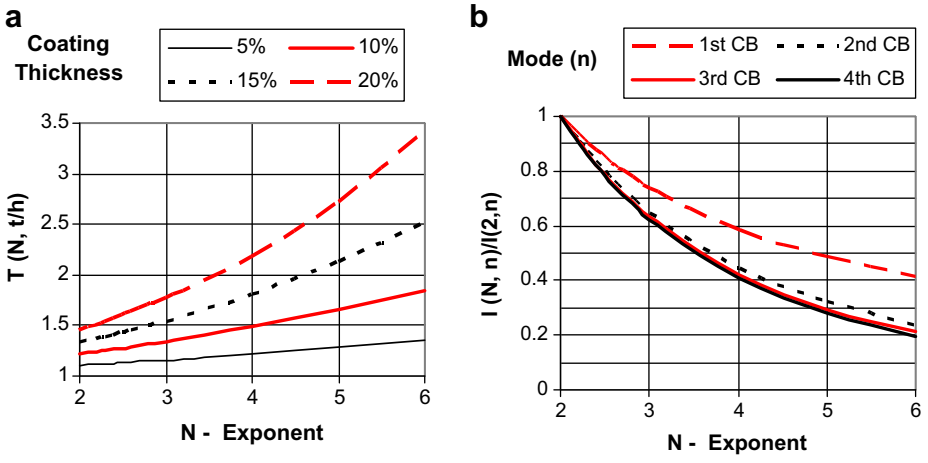

Fig. 1. Geometric factors for coated beams. (a) Dimensionless thickness integral. (b) Dimensionless area integral.

Table 1
Strain-scaling factors for determining the coating modulus.

n	t/h	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	Ave
2	0.05	1	0.6481	0.6685	0.6857	0.7012	0.676
3	0.05	1	0.6330	0.6518	0.6677	0.6818	0.659
4	0.05	1	0.6237	0.6408	0.6552	0.6680	0.647
2	0.1	1	0.6507	0.6718	0.6898	0.7061	0.680
3	0.1	1	0.6355	0.6550	0.6716	0.6865	0.662
4	0.1	1	0.6261	0.6440	0.6591	0.6726	0.650
2	0.15	1	0.6544	0.6766	0.6956	0.7129	0.685
3	0.15	1	0.6391	0.6597	0.6773	0.6932	0.667
4	0.15	1	0.6297	0.6486	0.6646	0.6791	0.656

$$\sum_{p=0}^{P} A_p \varepsilon_{\max}^{p} T(2,.t/h) \int_0^{\varepsilon_{\max}} \varepsilon_n^2 \frac{dS}{d\varepsilon_n} d\varepsilon_n = \sum_{m=0}^{M} E_m T(2+m,t/h) \int_0^{\varepsilon_{\max}} \varepsilon_n^{2+m} \frac{dS}{d\varepsilon_n} d\varepsilon_n \tag{18}$$

An area-strain function, $S/S_0 = f(\varepsilon_n/\varepsilon_{\max})$, analogous to Lazan's (1968) volume-strain function can then be introduced to represent the faction of the total surface area at which the strain is less than some fraction $\varepsilon_n/\varepsilon_{\max}$ of the maximum surface strain for the nth mode

$$\sum_{p=0}^{P} A_p \varepsilon_{\max}^{p+2} T(2,t/h) S_0 \int_0^1 \left(\frac{\varepsilon_n}{\varepsilon_{\max}}\right)^2 \frac{dS/S_0}{d\varepsilon_n/d\varepsilon_{\max}} d\left(\frac{\varepsilon_n}{\varepsilon_{\max}}\right) = \sum_{m=0}^{M} E_m T(2+m,t/h) \varepsilon_{\max}^{2+m} S_0 \int_0^1 \left(\frac{\varepsilon_n}{\varepsilon_{\max}}\right)^{2+m} \frac{dS/S_0}{d\varepsilon_n/d\varepsilon_{\max}} d\left(\frac{\varepsilon_n}{\varepsilon_{\max}}\right) \tag{19}$$

Values of the area-strain function, S/S_0 are shown in Fig. 2 for the first several modes of a cantilever beam, as obtained by evaluating the surface strain at 201 equally spaced points on the surface. The histogram for higher modes is nearly identical to that of mode 4. A linear approximation for modes 2–4 is also shown as the dashed line.

In principle, values of the function S/S_0 could be evaluated numerically and applied in Eq. (19) so as to relate the material modulus, E_{1C} to the effective modulus, E_E. However, the resulting requirement to evaluate the derivatives from such numerical values would likely introduce considerable uncertainty. In the case of the first cantilever mode and lower modes of a free-free beam, a low order polynomial approximation might suffice. But in the case of the cantilever beam, the abrupt variations seen in Fig. 2 would necessitate the use of a large number of terms in order to obtain an adequate representation. However, in the case of cantilever modes 2 and higher, the area strain histogram of Fig. 2 is quite well approximated by the simple relationship

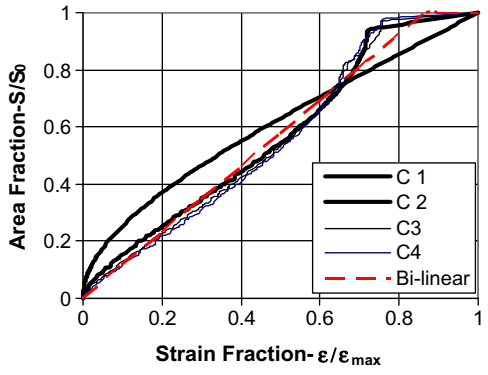

Fig. 2. Histograms of the strain distribution, cantilever modes.

$$S/S_0 = c(\varepsilon/\varepsilon_{\max}) \quad \text{for } \varepsilon/\varepsilon_{\max} < \beta = 1/c \tag{20a}$$

$$S/S_0 = \beta c = 1 \quad \text{for } \varepsilon/\varepsilon_{\max} > \beta = 1/c \tag{20b}$$

The average slope $c = 1/\beta$ for the histograms of modes 2 and higher is found to be 1.150, with differences between modes being only 0.1%. Upon substitution into Eq. (19), integrating, and then taking the same number of terms in both expansions ($M = P$), the coefficients of the expansion for the true modulus, Eq. (7), are found to be related to those for the effective modulus by

$$E_m = A_m \frac{T(2.t/h)}{T(m+2.t/h)} \frac{3+m}{3\beta^m} \tag{21}$$

The resulting values of the ratios of the coefficients, E_m/A_m, while not identical, are similar to those computed from Eq. (12). The same process as used previously may again be used to deduce strain scaling factors for obtaining an approximate value of the material modulus directly from the effective modulus. In this case, the factors, as introduced with Eq. (15) become

$$\alpha_m = \frac{(1+t/h)}{\beta} \left[\frac{T(2,t/h)}{T(m+2,t/h)} \frac{3+m}{3}\right]^{1/m} \tag{22}$$

Values of $1/\alpha_m$, analogous to those of Table 1, are shown in Table 2. The average value for the first four terms and all thicknesses is 0.685, in quite good agreement with that obtained (0.665) by integration of the strain distributions.

The ratio $T(2,t/h)/T(m+2,t/h)$ is quite well approximated by $1/(1+t/h)^m$ for lower values of m. Using this in Eq. (22) gives an average scaling factor of 0.679 for all modes and thickness ratios, suggesting that the material storage modulus at strain ε be estimated as the effective or average modulus at a strain of $\varepsilon_{\max} = 1.47\varepsilon$.

2.3. Summary

Values of the storage modulus for a non-linear coating (amplitude dependent values) can be extracted by comparing the frequencies of beams with and without coatings of uniform thickness applied to both sides of the beams. Two methods were given for extracting the true storage modulus from an average or effective value; and it was found that a suitable estimate of the coating modulus at a strain ε can be found from the average or effective modulus at a strain of about 150% of the strain of interest.

3. Methodology: loss modulus

3.1. Formulation

Amplitude-dependent coating dissipation gives rise to system loss factors varying with amplitude. If the loss factors are to determined from the decay of free vibrations, this may be satisfactorily accounted for by determining the decrement over a small number of cycles (Patsias et al., 2004b). But if the loss factors are to be determined from the frequency response function at constant amplitude of excitation, compensation for the influence of the amplitude dependence on the observed values must be made. A methodology for doing this is given in an appendix.

Given that credible data for the system loss factors of a beam with and without coating can be obtained, the total energy

Table 2
Approximate strain-scaling factors for determining the coating modulus.

n	t/h	$m=0$	$m=1$	$m=2$	$m=3$	$m=4$	Average
$n \geqslant 2$	0.05	1	0.653	0.674	0.691	0.705	0.681
$n \geqslant 2$	0.1	1	0.655	0.678	0.695	0.710	0.684
$n \geqslant 2$	0.15	1	0.659	0.682	0.701	0.717	0.690

dissipated in the coating, D_C, can be determined. The system loss factor, η_S, is related to the energies dissipated and the energies stored in the coated and uncoated beams (U_C and U_B, respectively) by the definition:

$$\eta_S(\varepsilon_{\max}) \equiv \frac{D_C + D_B}{2\pi(U_C + U_B)} = \frac{D_C(\varepsilon_{\max})}{2\pi U_B(1 + R_{SE})} + \frac{D_B(\varepsilon_{\max})}{2\pi U_B(1 + R_{SE})} \tag{23}$$

R_{SE} is the strain energy ratio of Eq. (3) and the reference strain is the maximum value at the beam-coating interface. The energy dissipated by all means other than the coating, denoted D_B, includes air damping, beam damping, grip losses, etc. Assuming once again that the presence of the coating does not affect the mode shape, this dissipated energy can be evaluated from an observed loss factor for the uncoated beam vibrating in the same mode at the same maximum amplitude of surface strain

$$\eta_B(\varepsilon_{\max}) = \frac{D_B(\varepsilon_{\max})}{2\pi U_B} \tag{24}$$

After substitution of Eq. (24) into Eq. (23)

$$D_C(\varepsilon_{\max}) = 2\pi[(1 + R_{SE})\eta_S(\varepsilon_{\max}) - \eta_B(\varepsilon_{\max})]U_B \tag{25}$$

If the substrate beam is assumed to be linear with energy, U_B, stored is as in the denominator of Eq. (3)

$$U_B = WhLE_B I(2,n)\varepsilon_{\max}^2/6 \tag{26}$$

Let $D(\varepsilon)$ denote the unit damping capacity, i.e., the energy dissipated per cycle by a unit volume of material subjected to a fully reversed strain of amplitude ε. The total energy dissipated, D_C, in the coating volume V_0 with non-uniform distribution of strain amplitudes is, for a maximum strain $\varepsilon_{\max}$ at the beam-coating interface

$$D_C(\varepsilon_{\max}) = \int_{V_0} D(\varepsilon) d\text{vol} \tag{27}$$

Extracting the unit damping function for the material requires the numerical solution of this integral equation, given numerical values of the total energy dissipated at numerous values of the maximum strain. But if an appropriate functional form for the dependence of the unit damping on the local amplitude of cyclic strain can be assumed, such parameters as are required for the assumed form may be evaluated from the system data.

The process is analogous to that used in the extraction of the amplitude dependent storage modulus from values of the effective modulus determined from test data, Eq. (4).

3.2. A polynomial representation of the loss modulus

We assume that the experimentally determined loss factors (after correction for the influence of the non-linearity on the measurement) can be adequately represented in terms of the maximum strain at the beam-coating interface by a polynomial of modest order. Noting that the strain energy ratio as determined from Eq. (2) is also generally a function of the maximum strain, we let

$$[(1 + R_{SE})\eta_S(\varepsilon_{\max}) - \eta_B(\varepsilon_{\max})] = \sum_{r=0}^{R} C_r \varepsilon_{\max}^r \tag{28}$$

We also assume that the unit damping of the material can be written in a similar form

$$D(\varepsilon) = \sum_{q=0}^{Q} D_q \varepsilon^q \tag{29}$$

After substitution of Eqs. 25, 28 and 29; the strain distribution for the nth mode from Eq. (9); and the integrals defined in Eqs. (6) and (11) into Eq. (27)

$$2\pi WhLE_B \frac{I(2,n)}{6} \sum_{r=0}^{R} C_r \varepsilon_{\max}^{r+2} = 2WtL \sum_{q=0}^{Q} D_q \varepsilon_{\max}^q T(q, t/h) I(q,n) \tag{30}$$

The coefficients D_1 and D_2 are seen to be zero. After renumbering the index in the right hand sum ($q = r + 2, Q = R + 2$), the coefficients for the expansion of the unit damping are

$$D_{r+2} = \frac{\pi h}{6t} E_B \frac{I(2,n)}{T(r+2, t/h) I(r+2,n)} C_r \quad (r \geqslant 0) \tag{31}$$

The customary definition of the loss modulus as the imaginary part of a complex modulus is not strictly applicable to non-linear materials as the concept of a complex modulus is based on the presumption of harmonic values of both stress and strain, an attribute of a linear system. But an energy-based definition is applicable to both linear and non-linear materials, and in the latter case is consistent with the customary definition as the imaginary part of a complex modulus. Thus, we define the loss modulus of a material in terms of the energy dissipated in a unit volume at uniform strain ε, or

$$E_{2C}(\varepsilon) \equiv \frac{D(\varepsilon)}{\pi \varepsilon^2} \tag{32}$$

from which

$$E_{2C}(\varepsilon) = \frac{1}{\pi} \sum_{q=2}^{Q} D_q \varepsilon^{q-2} = \frac{1}{\pi} \sum_{r=0}^{R} D_{r+2} \varepsilon^r = E_B \frac{h}{6t} \sum_{r=0}^{R} \frac{I(2,n)}{T(r+2, t/h) I(r+2,n)} C_r \varepsilon^r \tag{33}$$

where the coefficients C_r of Eq. (28) are found from experimentally determined values of the loss factor with and without coating, Eq. (28), using the amplitude-dependent strain energy ratios determined from measured frequencies.

3.2.1. An alternative evaluation

As was done in the extraction of the material storage modulus from the average values, a change in variable of integration in Eq. (27) may be used to relate the coefficients for the expansion of the unit damping (and then the loss modulus) to those obtained from an expansion of the observed loss factors. Eq. (27) becomes:

$$\begin{aligned} D_C(\varepsilon_{\max}) &= \int_{V_0} D(\varepsilon) d\text{vol} \\ &= 2 \int_{h/2}^{6} \int_{S_0} \sum_{q=0}^{Q} D_q \left[\varepsilon_{\max} \left(\frac{2z}{h} \right) \left(\frac{\varepsilon_n}{\varepsilon_{\max}} \right) \right]^q dz dS \\ &= 2 \sum_{q=0}^{Q} D_q \varepsilon_{\max}^q t T(q, t/h) S_0 \int_0^1 \left(\frac{\varepsilon_n}{\varepsilon_{\max}} \right)^q \frac{dS/S_0}{d\varepsilon_n/\varepsilon_{\max}} d\left(\frac{\varepsilon_n}{\varepsilon_{\max}} \right) \end{aligned} \tag{34}$$

After substitution of the same approximation to the area strain integral (Eq. (20)) as was used previously into the right-hand side of Eq. (34), integrating, and then substituting Eqs. 25, 26, and 28 into the left-hand side

$$\frac{2\pi}{6} WhLE_B I(2,n) \varepsilon_{\max}^2 \sum_{r=0}^{R} C_r \varepsilon_{\max}^r = 2 \sum_{q=0}^{Q} D_q \varepsilon_{\max}^q t T(q, t/h) S_0 \frac{\beta^q}{q+1} \tag{35}$$

After setting $S_0 = WL$, $q = r + 2$, and $Q = R + 2$; there results that $D_0 = D_1 = 0$ and

$$D_{r+2} = \frac{\pi}{6} \frac{h}{t} E_B \frac{I(2,n)}{T(r+2, t/h)} \frac{r+3}{\beta^{r+2}} C_r \quad (r \geqslant 0) \tag{36}$$

Coefficients for the expansion obtained with this approximation, again using an average slope $c = 1.15 = 1/\beta$, differ slightly than those

obtained from Eq. (35), but by less by about 1%, 2%, 9%, 11% and 13% for the first five values of r.

3.3. Discussion

The total energy dissipated in a coating during a fully reversed cycle of strain amplitude ε_{max}, can be determined from system loss factors measured either from the decay of a free vibration or from the frequency response function in forced vibration. If the beam is fully covered, it is necessary in either case that the range of strain amplitudes be somewhat higher than the strains at which material properties are desired. As may be inferred Fig. 2, about 80% of the coating volume on a beam vibrating in the second or higher cantilever beam never experiences strains above about 2/3 of the maximum value. As the *system* damping at ε_{max} is only slightly influenced by the *material* damping at strains above (about) 2/3 of that value, material properties at high strains are not well represented in the data obtained and therefore can not be extracted with high resolution. As it is the highest order terms in the polynomial of Eq. (28), and consequently of Eq. (33), that are (relatively) the most sensitive to measured values at higher strains, it is critical that the lowest order polynomial adequately capturing the results be used.

As the first cantilever and free-free modes have more uniform strain distributions than do the higher cantilever modes, the use of such data might enable the reliable extraction of material properties at slightly higher fractions of the maximum test strain. However, the use of these modes introduces such other challenges as accounting for high air damping and providing a low-loss suspension, respectively.

4. Justification for polynomial representations

While the representation of the loss and storage moduli of a coating in terms of polynomials is arbitrary, some justification for this choice can be found by noting that such relationships are consistent with a generalization of a well-known hypothetical stress–strain relationship for non-linear materials. Using a slightly different notation, Davidenkov (1938) related the instantaneous stress, σ_i, to the instantaneous strain by writing:

$$\sigma_i(\varepsilon_i) + \sigma = E[(\varepsilon_i + \varepsilon) - b(\varepsilon_i + \varepsilon)^P] \quad \text{for } d\sigma_i/dt > 0 \tag{37a}$$

$$\sigma - \sigma_i(\varepsilon_i) = E[(\varepsilon - \varepsilon_i) - b(\varepsilon - \varepsilon_i)^P] \quad \text{for } d\sigma_i/dt < 0 \tag{37b}$$

where ε is the maximum amplitude of strain and σ is the maximum amplitude of stress in a fully reversed cycle. The amplitude-dependent secant modulus is then obtained from the first of these evaluated at $\varepsilon_i = \varepsilon$ or the second at $\varepsilon_i = -\varepsilon$. Either gives that:

$$E(\varepsilon) = \frac{\sigma}{\varepsilon} = E[1 - b(2\varepsilon)^{P-1}] \tag{38}$$

As Eqs. (37) form a closed hysteresis loop, the energy dissipated per cycle may be obtained by subtracting the lower branch stress of Eq. (37b) from the upper branch stress of Eq. (37a) and integrating over $-\varepsilon < \varepsilon_i < \varepsilon$. The result, as computed by Pisarenko (1955) is that

$$D = Eb\left[\frac{P-1}{P+1}\right]2^{P+1}\varepsilon^{P+1} \tag{39}$$

A discussion and interpretation of this stress–strain relationship has been given (Lazan, 1968). Hysteresis loops formed with Eqs. (37) have the pointed ends characteristic of such dissipative mechanisms as plasticity and friction, rather than the (generally) elliptical loops associated with viscoelastic dissipation. The predicted dissipation, Eq. (39), is consistent with Lazan's finding that the energy dissipated per cycle for a wide variety of structural materials was proportional to the amplitude of cyclic stress raised to a power between 2 and 3 (corresponding to $1 < P < 2$) at low stresses and larger, history-dependent, exponents at high stresses.

While many materials used as coatings also appear to dissipate energy in proportion to a constant power of stress or strain at low amplitudes ($\varepsilon < 100$ ppm), at higher levels of strain the dissipation is typically found to vary at a lower power or to diminish with strain (Torvik, 2007). In consequence, a stress–strain law such as that of Eqs. (37) with a single exponent generally is not adequate for describing such materials at all levels of strain. However, a generalization is possible. Let

$$\sigma_i(\varepsilon_i) + \sigma = E[(\varepsilon_i + \varepsilon) - \sum_{p=1}^{P} b_p(\varepsilon_i + \varepsilon)^p] \quad \text{for } d\sigma_i/dt > 0 \tag{40a}$$

$$\sigma - \sigma_i(\varepsilon_i) = E[(\varepsilon - \varepsilon_i) - \sum_{p=1}^{P} b_p(\varepsilon - \varepsilon_i)^p] \quad \text{for } d\sigma_i/dt < 0 \tag{40b}$$

The exponents in Eq. (40) may be integers, p, or chosen as more general values, P_p. The Davidenkov relationships, Eqs. (37), are then the special case for a single term with a power that need not be an integer.

A secant modulus results from using the first of these to evaluate the ratio σ/ε. And, by choosing $E_0 = E(1 - b_1)$ and $E_m = -E(2)^m b_{m+1}$ for $m \geqslant 1$, the result with integer values of m is equivalent to the empirical form of Eq. (7)

$$E(\varepsilon) = E\left[1 - \sum_{p=1}^{P} b_p(2\varepsilon)^{p-1}\right] \equiv \sum_{m=0}^{P-1} E_m \varepsilon^m \tag{41}$$

The energy dissipation during a fully reversed cycle of maximum amplitude ε may again be found by subtracting the stress at the lower branch of Eq. (40) from that on the upper and integrating over $-\varepsilon \leqslant \varepsilon_i \leqslant \varepsilon$. The result is that

$$D = \int_{-\varepsilon}^{\varepsilon} [\sigma_U(\varepsilon_i) - \sigma_U(\varepsilon_i)] d\varepsilon_i = E \sum_{p=1}^{P} b_p \frac{p-1}{p+1} (2\varepsilon)^{p+1} \tag{42}$$

Thus, the assumed polynomial expansion for the energy dissipated per cycle, Eq. (29), is also consistent with a stress–strain relationship of the form given in Eq. (40). The loss modulus defined as in Eq. (32) is then

$$E_2(\varepsilon) = \frac{E}{\pi} \sum_{p=1}^{P} 4b_p \frac{p-1}{p+1} (2\varepsilon)^{p-1} = \frac{E}{\pi} \sum_{r=0}^{P-1} 4b_{r+1} \frac{r}{r+2} (2\varepsilon)^r \tag{43}$$

A polynomial representation of a true logarithmic decrement ($\delta \equiv \pi\eta \equiv \pi E_2/E_1$) was also been used (Ustinov et al., 2007) to extract an amplitude-dependent material damping property from observed decrements of a tapered titanium beam coated with vapor-deposited Co–20%Fe, vibrating in the first cantilever mode. In this work, however, the strain distribution was taken as that of a statically-loaded cantilever and the storage modulus of the coating was assumed to be independent of strain amplitude.

5. Example: a ceramic coating

5.1. Test procedures, parameters, and data

Flat strip specimens of 2.29 mm (0.090 in.) Ti–6Al–4V were first air plasma-sprayed on both sides with a bond coat of NiCrAlY, nominally 0.076 mm (0.003 in.) thick, and then with a coating, nominally 0.254 mm (0.010 in.) of a titania–alumina blend ceramic. Specimens were 19 mm (0.75 in.) wide and of a total length of 254 mm (10 in.). Coatings were applied over a 203 mm (8 in.)

test section, leaving an uncoated length for gripping. Specimens were weighed and measured before and after each application of coating to determine the masses per unit area, as are required for the application of Eq. (2).

Specimens were mounted on a large shaker and excited at various constant levels of base acceleration, with frequency slowly swept downwards through resonance from above. Testing was conducted in the second, third and fourth cantilever bending modes with decreasing levels of the constant base acceleration. Specimen responses (velocities at an interior maximum of displacement, appropriate to the mode) were observed with a laser vibrometer and converted to a common reference strain, the value of strain at the beam-coating interface at the root. The resonant frequency was taken as the frequency of maximum response and system loss factors were extracted from the frequency response function by vibrometer software. Additional details of the test apparatus and test procedures are given elsewhere (Torvik et al., 2007).

Resonant frequencies and system loss factors were measured for the bare beams, for the same beams after applying the bond coat, and again after applying the ceramic. As satisfactory consistency was found between results obtained with four specimens tested in the three modes (Hansel, 2008), only data obtained in the 3rd cantilever bending mode with two of the specimens (#214 and #216) are used here. Resonant frequencies measured for the beams with bond coat only, and after addition of the titania–alumina blend coating are shown in Fig. 3.

The moduli of the beams without bond coat as computed with the frequency equation for a Bernoulli–Euler beam using measured densities, thicknesses, lengths, and frequencies at low strain were found to be 111.4 and 111.1 GPa (16.15 and 16.12 Mpsi), in sufficient agreement with handbook values for the modulus of Ti 6Al–4V as to validate adequacy of the cantilever mounting.

As the mechanical properties of the ceramic alone were desired, the properties of the titanium beam with the NiCrAlY bond coat were used as the properties of the uncoated beam in Eqs. 2, 5, 28, and 33. Strains are given in units of microinches/inch or parts per million (ppm). A significant increase in stiffness after applying the ceramic coating is evident, as is a dramatic increase in the degree of strain softening.

Observed system loss factors (before correction) for the beams with bond coat only and with bond coat and the titania–alumina blend coatings are shown in Fig. 4.

As system loss factors for the beam with bond coat alone were found to be typically about 150% of that with the bare beams, the greater part of the losses seen in Fig. 4a may be attributed to such extraneous sources as air damping, grip losses, and the inherent damping of the titanium substrate. Losses observed after the addition of the ceramic topcoat (Fig. 4b) are sufficiently larger than those measured before (Fig. 4a) as to insure that the formation of the net addition due to the coating, Eq. (25) would enable a valid extraction of the energy dissipation due to the titania–alumina coating. However, the dissipation of plasma-sprayed ceramics is known (Patsias and Williams, 2003; Patsias et al., 2004a,b) to exhibit a significant degree of strain history. In consequence, the measurements at the highest levels of strain (not having been preceded by tests at higher strains) must be rejected as being unrepresentative. In this case, data taken at maximum strains above (about) 450 were not be used in the extraction of the damping properties of the material.

Specimen parameters required for the extraction of material properties from the system level response are given in Table 3. Values of the beam thickness, h, given here are the total thicknesses of

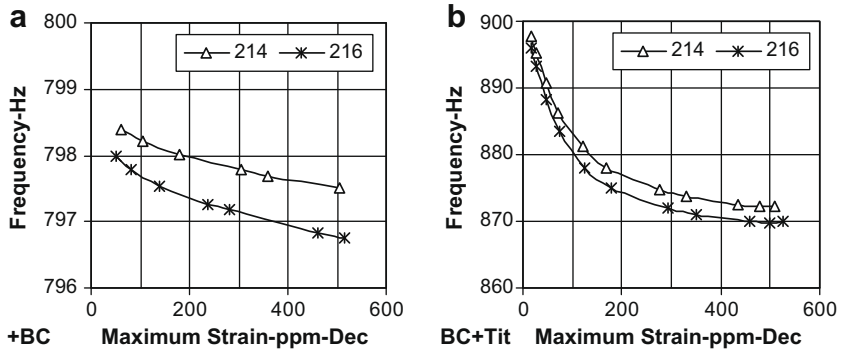

Fig. 3. Resonant frequencies of coated cantilever beams, mode 3. (a) NiCrAlY bond coat on Ti–6Al–4V. (b) Plus titania–alumina blend top coat.

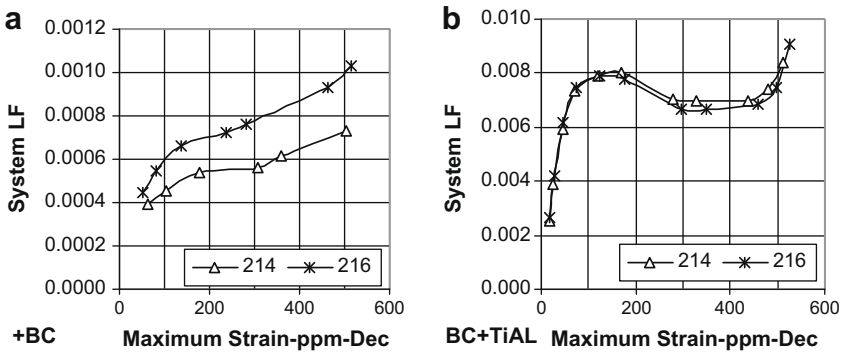

Fig. 4. System loss factors (uncorrected) of coated cantilever beams, mode 3. (a) NiCrAlY bond coat on Ti–6Al–4V. (b) Plus titania–alumina blend cop coat.

Table 3
Specimen parameters for extraction of material properties.

Specimen	t-Coat (mm)	t/h	Mass ratio	E_B (GPa)	$T(2,t/h)$
214	0.239	0.0966	0.1725	93.77	1.206
216	0.242	0.0977	0.1733	93.69	1.208

the titanium beam and bond coat; the modulus is that of an equivalent homogeneous beam having the frequency observed for the beam with bond coat at low amplitude of strain.

5.2. Determination of the storage modulus

Resonant frequencies of the beam with bond coat alone (Fig. 3a) were found to be well characterized by cubic polynomials. These were used as the bare beam frequencies in Eq. (2) with the mass ratios of Table 3 and the observed frequencies of the coated beam, Fig. 3b, to compute the strain energy ratio at each level of test strain. These are given in Fig. 5a and from these, values of the average or effective coating storage moduli shown in Fig. 5b were determined from Eq. (5).

In the development of Eq. (5) it was assumed that the modulus of the substrate beam was independent of amplitude. In the present case, where the substrate beam included the bond coat, a small decrease in frequency with strain (Fig. 3a) is indicative of a slight change in modulus. When significant, such changes can be accounted for by adjusting the bare beam modulus in proportion to changes in the square of the frequency. In the present case, however, such changes were neglected here as accounting for this softening would have influenced values for the storage and loss modulus of the coating by less than 1/2%, even at the highest strains.

The effective or average moduli for the two specimens as shown in Fig. 5b were found to be well characterized by fourth order polynomials. Using these coefficients, the coefficients for the expansion of the true or local value of the storage modulus were found from Eq. (12). These, when used in the expansion for the storage modulus, (Eq. (7), yield the solid lines of Fig. 6. Application of the approximation of Eq. (16) to each computed value of the average of effective modulus (Fig. 5b) produced the data points shown in Fig. 6. As these are in remarkably good agreement with values determined from the polynomial expansion (solid lines), the estimation of the true or material storage modulus from the effective or average modulus by expansion of the axis of strain appears to provide a rapid and satisfactory estimate of the storage modulus for coatings such as those considered here.

Estimates of the material storage modulus resulting from the use of the approximation to the area-strain histogram, Fig. 2 and Eq. (20), are not shown, as they differ from the data points given only by a 2% change in the values of strain.

5.3. Determination of the loss modulus

Corrections to loss factors determined from bandwidth measurements for systems with amplitude-dependent damping are necessary (Torvik, 2003) because the loss factor at the amplitude of the bandwidth frequencies is not the same as the loss factor at the maximum amplitude. The methodology of the appendix was used to apply the necessary corrections.

Observed system loss factors for the beam with bond coat alone (Fig. 4a) were first fit with linear relationships, and then adjusted so as to account for the damping non-linearity. Observed values for the beam with bond coat and top coat (Fig. 4b) were fit with fourth order polynomials and then adjusted for the damping non-linearity. The corrected values of the loss factor (Fig. A1 of the appendix) were then used with the strain energy ratios obtained by fitting fourth order polynomials to the values given in Fig. 5a to compute the net adjusted loss factors, Eq. (28). Results for the two specimens are shown in Fig. 7.

These net adjusted loss factors were also found to be adequately represented by fourth order polynomials. The coefficients are the values of C_r for Eq. (28) and, when used with values of the functions defined by Eqs. (6) and (11) and specimen parameters from Table 3, yield the coefficients for the expansion of Eq. (33). The resulting loss modulus is shown in Fig. 8a. A material loss factor for the coating can then be formed from the ratio of loss to storage modulus $\eta_C = E_{2C}/E_{1C}$), giving the values shown in Fig. 8b.

Fig. 8a and b was developed using only test data (Fig. 4b) for maximum strains below 450 ppm (corresponding to a local strain of about 300 ppm) as the responses at higher strains were believed to be influenced by strain history The slight increases seen in Fig. 8 above a local strain of 200 ppm may be a further consequence of the history effect, or of the emergence of a dissipative mechanism with different amplitude dependence. While the adequacy of the methodology can be expected to be influenced by the specific choice of the assumed functional form, and a choice other than the polynomials used here might give slightly different results, the presence of these same trends in the original test data (Figs. 3 and 4b) indicate that these are aspects of the material response, rather than artifacts introduced by the process used for the extraction of material properties.

5.4. An approximation for the loss modulus

A simplified procedure has been given elsewhere (Torvik et al., 2007) for the extraction of the material loss modulus from

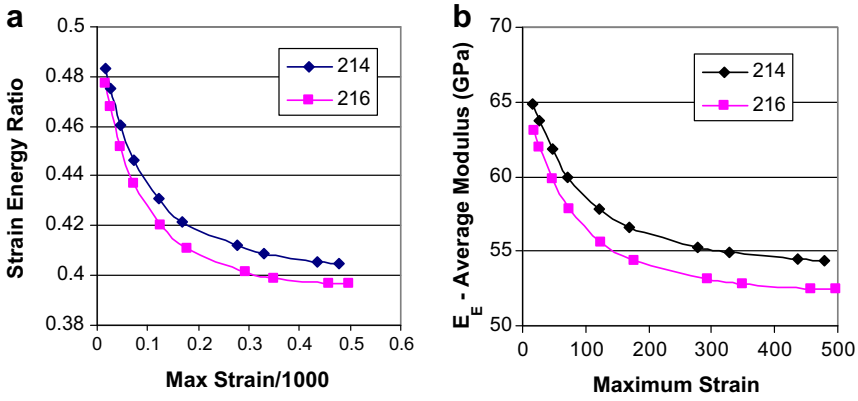

Fig. 5. Characteristics of titania–alumina blend coating (air plasma sprayed). (a) Strain energy ratios. (b) Average or effective modulus.

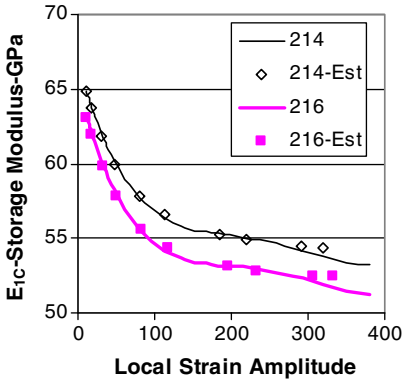

Fig. 6. Storage modulus for a titania–alumina blend coating.

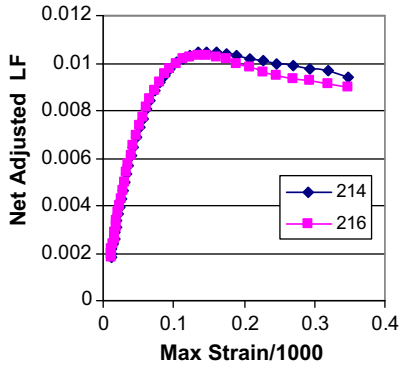

Fig. 7. Net adjusted loss factor.

observed system loss factors. In that case, the assumed functional form for the damping was that of Eqs. (37), corresponding to a Davidenkov stress–strain law with a single, non-integer exponent. Eq. (33) then can be rewritten, after the substitution of Eq. (28), as

$$E_{C2}(\varepsilon) = E_B \frac{h}{6tT(2,t/h)}[(1+R_{SE})\eta_S(\varepsilon_{\max}) - \eta_B(\varepsilon_{\max})] \times \left[\frac{T(2,t/h)I(2,n)}{T(r+2,t/h)I(r+2,n)}\right] \quad (43)$$

The last factor on the right represents the influence of the non-uniform distribution of strain and, for cantilever beams with typical coating/beam thickness ratios (~0.1), is about 1.1–1.3 at low strains and approximately 1 at high values. As the factor for correcting the observed system loss factors (η_{True}/η_{App} of the appendix) typically has values between 0.8 and 0.9 at low strains and nearly 1 at high, the influence of these two factors are largely offsetting. As a result, an estimate of a loss modulus from Eq. (43) using uncorrected system loss factors and taking the last factor as unity does not give significant error. Examples of the underestimates of the true loss modulus resulting from this approximation are given in Table 4 for various values of the thickness ratio and r, the exponent when a single term is used in Eq. (28) to approximate uncorrected system loss factors.

Application of this simplified method to the data of Figs. 3 and 4 gave values of the loss modulus within 10% of values shown in Fig. 8a.

5.5. Discussion

The material properties of a plasma-sprayed titania–alumina blend ceramic at room temperature were extracted from the natural frequencies and system loss factors before and after applying the ceramic to Ti–6Al–4V beams with a NiCrAlY bond coat. The storage modulus, loss modulus and loss factors (Figs. 6, 8a and b) obtained from two nominally identical specimens were found to be in satisfactory agreement. The storage modulus diminished with increasing amplitude of a fully reversed cyclic strain, whereas the loss modulus and loss factor both showed a strong increase with strain up to a critical value, and then diminished or became (relatively) constant. These dependencies of material properties on strain amplitude are characteristic of those seen in other plasma-sprayed coatings (Torvik, 2007).

Testing was conducted on cantilever beams fully covered on both sides and vibrating in the third cantilever bending mode. In consequence, significant variations in strain occurred within the coating material. At any given level of maximum strain about 80% of the coating volume is at strains below about 2/3 of this value. As observed frequencies and loss factors reflect the integrated influence of all strains from zero to the maximum value, measured values are only weakly influenced by the strains at the highest level. Moreover, the response of materials such as these is known to be influenced by the prior loading history. Accordingly, even after rejecting results from the highest test amplitudes, a lower level of confidence must be assigned to values obtained at the higher levels of strain. In particular, the results suggest an increase in loss modulus and loss factor at the higher strains of Fig. 8 and an increased rate of diminution of the storage modulus, Fig. 6. It can not be said with certainty if these are the consequences of a history effect, of

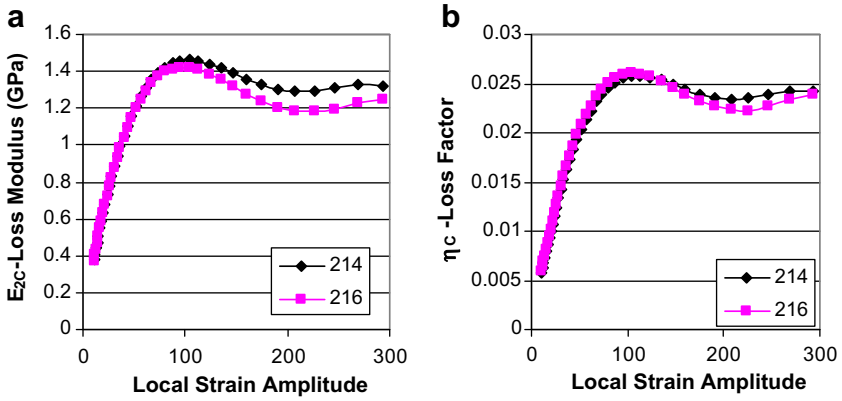

Fig. 8. Material damping properties of a titania–alumina blend ceramic. (a) Loss modulus. (b) Loss factor.

Table 4
Potential underestimates of loss modulus with approximate method.

r	0	0.2	0.4	0.6	0.8	1
$t/h = 0.05$	0	3%	7%	11%	16%	22%
$t/h = 0.1$	0	2%	4%	8%	12%	17%
$t/h = 0.15$	0	1%	2%	5%	8%	11%
$t/h = 0.2$	0	0	0	2%	4%	5%

the emergence and dominance of a different non-linear mechanism with characteristically different amplitude dependence.

The assumed form used here (polynomials) appears to capture well the amplitude dependence of the material properties of a plasma-sprayed ceramic. However, for other non-linear materials with characteristically different amplitude dependencies, other forms may prove to be more appropriate.

The applicability of a method for obtaining an approximate value of the local or material storage (Young's) modulus from the average or effective modulus was also considered. Taking the material modulus at strain ε to be the same as the effective modulus at stain $\varepsilon_{max} = 1.5\varepsilon$ was found to provide a very satisfactory estimate. Thus, once the effective or average modulus is found, the storage modulus of the material can be estimated by simply multiplying the abscissa by a factor of 2/3.

An abbreviated (single term) approximation for determining the loss modulus was described. In using this, the influences of the damping non-linearity on the system measure of damping and on the impact of a non-uniform strain distribution in the coating were seen to be largely offsetting. For coatings of similar properties to that considered here, applied at thickness ratios of about 0.1, this approximation yields results within 10% of those obtained by using the results of the more thorough analysis.

The nature of the amplitude dependence of the storage and loss modulus for the plasma-sprayed titania–alumina blend ceramic were found to be similar to that characteristic of other plasma-sprayed ceramics. Values of both were found to be greater than previously reported values for yttria-stabilized zirconia (Patsias et al., 2004a,b) and alumina (Torvik et al., 2007). The storage modulus appears to be higher, but the loss modulus lower than values reported for Rokide® (Patsias et al., 2004a,b) and magnesium aluminate spinel (Reed, 2007).

6. Summary and conclusions

A methodology suitable for extracting the material properties of coatings applied on substrate beams has been presented. Two intermediate quantifications of the observed response, an average or effective modulus and a net adjusted loss factor, were introduced and procedures for evaluating these from test system level test data were given. As these represent global or integrated measures of the coating stiffness and damping, processes were also given for extracting local or inherent material properties as functions of local strain from these global measures. The methodology requires the arbitrary choice of assumed functional forms for the amplitude dependence of material stiffness and damping. Integer-order polynomials were chosen, and terms up to the fourth power of the amplitude of strain were found to be adequate. Some justification for this choice was found by noting that such forms result from the generalization of an accepted stress–strain relationship for non-linear materials.

The methodology was applied to test data taken with a titania–alumina blend ceramic, air plasma-sprayed on a titanium beam with NiCrAlY bond coat. The nature of the amplitude dependencies of both storage modulus and loss modulus were found to be similar to that obtained with different methodologies and different materials. Abbreviated versions of the methodology for extracting properties from test data were also developed and found, for the test data used here, to provide satisfactory approximations.

It should be noted that the application of this methodology is predicated on the availability of high quality measurements of resonant or natural frequencies and system loss factors for the beams with and without the coating, and that these be sufficiently different. While the necessary thickness depends on coating properties, for coatings such as that used here a thickness of about 5% of the substrate thickness, fully covering both sides of the substrate beam, is sufficient.

The primary limitation to the methodology given appears to be the challenge of extracting material properties at strains much above about 50% of the highest strain levels present in the system level data. This is largely a consequence of the relatively low volume of the coating at high strain levels when applied to a cantilever beam vibrating in a higher mode.

Acknowledgements

The author is indebted to Mr. John Justice, Universal Technology Corporation (UTC) for the original suggestion that the local modulus might be related to the average value by an axis scaling; to APS Materials, Inc. for providing specimens and support for testing; and to Mr. Jason Hansel (UTC), who obtained the test data and measurements used here.

Appendix. Influence of non-linearity on measures of system damping

System damping is most typically determined either from the rate of decay of amplitude in a free vibration or from the variation of the amplitude of the response to an excitation at constant amplitude with frequency slowly swept through a resonance. In the first case (transient response), if the system stiffness is dependent on amplitude the frequency of the response will vary slowly throughout the decay, necessitating careful observations if the system stiffness is to be evaluated from the changes in frequency. If the measure of damping, e.g. the logarithmic decrement, is also dependent on amplitude, values must be extracted from narrow time windows if the amplitude dependence is to be captured accurately. Signal filtering may be necessary and the use of the Hilbert transform in the processing of the amplitude–time response has been suggested (Patsias et al., 2004a,b) as a means of obtaining valid measures of the amplitude dependence of system damping and stiffness.

In the second case (frequency response function), if the stiffness is amplitude-dependent a sine-sweep at constant excitation may reveal a system instability at some levels of maximum response. In such cases, observation of the complete frequency response function is not possible, and the system damping can not be evaluated from frequencies at the desired fraction of the maximum response. At excitation levels near the onset of the instability, rapid changes in response with changes in frequency make an accurate observation of the frequency response function difficult. Moreover, if the damping of the system is amplitude-dependent, then the system loss factor as determined from the half-power (or any other ratio) amplitude is not the true system loss factor (Torvik, 2003) as the damping at the amplitude corresponding to the bandwidth frequencies is not the same as the system damping at maximum amplitude. If the system loss factor increases with strain, bandwidth measurements lead to overestimates of true values. Conversely, with loss factors diminishing with strain (as occurs in ceramics coatings above a critical value of strain), a bandwidth determination underestimates the true system loss factor.

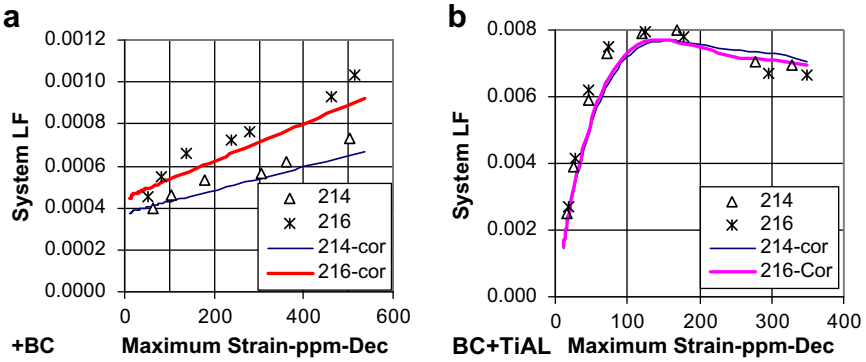

Fig. A1. Apparent and true values of the system loss factor. (a) NiCrAlY bond coat on Ti–6Al–4V. (b) NiCrAlY bond coat and titania blend top coat.

Let the response of a mildly non-linear system, such as an elastic beam coated with a non-linear material having amplitude-dependent stiffness and damping, be represented by a complex stiffness:

$$k* = k_0[1 - \kappa(A)] + jk_0\eta(A) \tag{A1}$$

where A is a measure of the maximum (resonant) response, typically the maximum strain. Amplitude-dependent softening is represented by positive values of $\kappa(A)$. $\eta(A)$ is an amplitude dependent loss factor. Although the response of a non-linear system to a harmonic excitation can not be truly harmonic, we assume here that the dominant portion of the response may be estimated from the complex amplitude of response of an equivalent linear system to a harmonic excitation $F(t) = F_0\cos\Omega t$ at any frequency ratio $\phi = \Omega/\omega_0$

$$\frac{X}{x_{ST}} = \frac{1}{(1 - \kappa(A) - \varphi^2) + j\eta(A)} \tag{A2}$$

where ω_0 is the undamped frequency at zero amplitude and $x_{xt} = F_0/k_0$. The dimensionless amplitude at resonance is simply $1/\eta(A_{\text{res}})$. The two frequencies away from resonance at which the amplitude is some fraction r of the value at resonance are the roots of

$$\frac{|rX|}{x_{ST}} = \frac{r}{\eta(A_{\text{res}})} = \frac{1}{\sqrt{(1 - \kappa(rA_{\text{res}}) - \varphi^2)^2 + \eta^2(rA_{\text{res}})}} \tag{A3}$$

The difference between the values of ϕ^2 is independent of the presumed form of the stiffness non-linearity so that, if the two frequencies are observable

$$\varphi_2^2 - \varphi_1^2 = (\varphi_2 - \varphi_1)(\varphi_2 + \varphi_1) = 2\sqrt{\left[\frac{\eta(A_{\text{res}})}{r}\right]^2 - \eta^2(rA_{\text{res}})} \tag{A4}$$

If bandwidth measurements are taken at the half-power point ($r = 1/\sqrt{2}$), the frequency difference, $f_2 - f_1$, is the apparent bandwidth, η_{APP}, which may be related to the true value by Eq. (A4). If the damping is relatively light, and softening parameter modest, $\kappa(A_{\text{res}}) \ll 1$, the sum of the two normalized frequencies is very nearly 2. The true loss factor at amplitude A_{res} is then related to the apparent (observed) value at that amplitude and the true value at the lower (half-power) amplitude by:

$$\eta(A_{\text{res}}) = \frac{1}{2}\sqrt{[\eta_{\text{App}}(A_{\text{res}})]^2 + [\eta(A_{\text{res}}/\sqrt{2})]^2} \tag{A5}$$

If the functional form of the true loss factor as a function of amplitude is known, but with unknown parameters, it may be substituted into Eq. (A5) and the necessary parameters extracted form the data. This is quite simply done if the dependence of the loss factor on amplitude is of a simple form, such as $\eta(A) = \eta_0 A^m$. For $m = 1$, the true loss factor is found to 0.8165 of the half-power bandwidth determination at all strains (Torvik, 2003).

If the true system loss factor is assumed to have a more complex form, such as a polynomial representation, coefficients for the expansion may be obtained by the following process:

1. Obtain a polynomial fit to observed system loss factors. First and fourth order polynomials, respectively, are found to be adequate for data such as those of Fig. A1a and b.
2. Using coefficients for a polynomial fit to the apparent (observed) system loss factor; evaluate the apparent loss factor at a number of values of strain. The choice of strains $\varepsilon_n = 10(2)^{n/8}$ is useful, as the half-power amplitude for any amplitude ε_n is then simply the amplitude at the strain corresponding to $n - 4$.
3. Estimate the loss factors for the first four values of strain as the intercept value for measurements such as Fig. A1a or by $\eta_{\text{True}} = 0.81645\eta_{\text{App}}$ when the amplitude dependence is nearly linear at low strain, as in Fig. A1b.
4. For $n \geqslant 4$, use Eq. (A5) sequentially

$$\eta_{\text{True}}(\varepsilon_n) = \sqrt{\frac{1}{2}[\{\eta_{\text{App}}(\varepsilon_n)\}^2 + \{\eta_{\text{True}}(\varepsilon_{n-4})\}^2]} \tag{A6}$$

This process was applied to the test data as given above in Fig. 4, using only data taken for the ceramic at maximum strains trains below 350 ppm, with the results shown in Fig. A1. Original data points for both specimens are shown as points without lines; data after correction for the damping non-linearity are given as the solid lines. For the data from the beams with the ceramic top coat, the influence of the non-linearity is seen to be the strongest in the low-strain region where the loss factor increases with strain, and relatively minor at the higher strains. As expected, when the apparent loss factor increases with strain, it is an overestimate of the true value, and when it diminishes with increasing strain, it provides an underestimate.

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