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Note

Ordered matroids and regular independence systems

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Abstract

We consider a class of matroids which we call ordered matroids. We show that these are the matroids of regular independence systems. (If E is a finite ordered set, a regular independence system on E is an independence system $(E,\mathscr{F})$ with the following property: if $A \in \mathscr{F}$ and $a \in A$, then $(A-\{a\})\cup\{e\} \in \mathscr{F}$ for all $e \in E-A$ such that $e \leqslant a$.) We give a necessary and sufficient condition for a regular independence system to be a matroid. This condition is checkable with a linear number of calls to an independence oracle. With this condition we rediscover some known results relating regular 0/1 polytopes and matroids.

Keywords: Matroids; Regular 0/1 polytopes; 0/1 solutions of linear inequalities

1. Introduction

There are some well-known classes of matroids [5].

We consider a class of matroids which we call ordered matroids. These matroids are defined from submodular functions on a given intersecting ring family, which is a chain of subsets.

We prove that ordered matroids are the matroids of regular independence systems. (If E is a finite ordered set, a regular independence system on E is an independence system $(E,\mathscr{F})$ with the following property: if $A \in \mathscr{F}$ and $a \in A$, then $(A-\{a\})\cup\{e\} \in \mathscr{F}$ for all $e \in E-A$ such that $e \leqslant a$.)

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We give a necessary and sufficient condition for a regular independence system to be a matroid. This condition is checkable with a linear number of calls to an independence oracle.

With that characterization we extend some results presented by Wolsey [6], relating regular 0/1 polytopes with matroids.

2. Ordered matroids

Let E be a finite set, t a positive integer, $0 = r_0 < r_1 < \cdots < r_t$ integers, and $\emptyset = E_0 \subset E_1 \subset \cdots \subset E_t = E$ a chain of subsets of E. Assume $|E_t - E_{t-1}| \geqslant r_t - r_{t-1}$, and $|E_i - E_{i-1}| > r_i - r_{i-1}$ for $i = 1, \ldots, t-1$.

Clearly, $\mathscr{R} = \{E_1, \ldots, E_t\}$ is an intersecting ring family, i.e., for any pair $i, j = 1, \ldots, t$, $E_i \cap E_j \in \mathscr{R}$ and $E_i \cup E_j \in \mathscr{R}$. Moreover, $r(E_i) = r_i$ is a submodular function defined on $\mathscr{R}$.

The following theorem is immediate in the context of matroids from submodular functions on intersecting ring families, essentially due to Edmonds (see [2]).

Let $\mathscr{F}$ be the collection of subsets of E defined as follows:

$$E \supseteq A \in \mathscr{F} \quad \text{iff } |A \cap E_i| \leqslant r_i, \text{ for } i = 1, \ldots, t,$$

Theorem 1. *$M = (E, \mathscr{F})$ is a matroid, whose polytope $\mathscr{P}$ is:*

$$\mathscr{P} = \{x \in \mathbb{R}^{|E|} : \textstyle\sum_{e \in E_i} x_e \leqslant r_i,\ i = 1, \ldots, t,\ 0 \leqslant x_e \leqslant 1\ \forall e \in E\}.$$

Note that, whenever $|E_t - E_{t-1}| = r_t - r_{t-1}$, the inequality $\sum_{e \in E_t} x_e \leqslant r_t$ is no longer necessary for the definition of $\mathscr{P}$, since in this case $E = E_t$ becomes a separable set of M.

Definition 2. If E; t; $r_1, \ldots, r_t$; $E_1, \ldots, E_t$; and $\mathscr{F}$ are as defined above, $M = (E, \mathscr{F})$ is called an *ordered* matroid.

3. Regular independence systems and ordered matroids

Let $E = \{e_1, \ldots, e_n\}$ be a finite ordered set, and assume that $e_1 \geqslant e_2 \geqslant \cdots \geqslant e_n$.

Definition 3. An independence system $(E, \mathscr{F})$ on E is *regular* if for all $A \in \mathscr{F}$ and $e_j \in A$, $(A - \{e_j\}) \cup \{e_{j+k}\} \in \mathscr{F}$ for $k = 1, \ldots, n-j$.

Examples of regular independence systems are regular 0/1 polytopes [4] which, in particular, include the 0/1 solutions of linear inequalities.

In what follows, when refering to the first (last) element of a subset $A = \{e_{j_1}, \ldots, e_{j_k}\}$ $(e_{j_1} \geqslant \ldots \geqslant e_{j_k})$ of E we mean element e_{j_1} (e_{j_k}). For $l \leqslant k$ we will refer to $e_{j_1}, e_{j_2}, \ldots, e_{j_l}$

$(e_{j_k-l+1}, e_{j_k-l+2}, \ldots, e_{j_k})$ as the l leftmost (rightmost) elements in A. We will also refer to $\{e_1, \ldots, e_{j_1-1}\}$ ($\{e_{j_k+1}, \ldots, e_n\}$) as the set of elements on the left (right) of A.

Consider a regular independence system $M = (E, \mathscr{F})$. Without loss of generality we assume that all singletons are independent.

Let G denote the greedy set of M, i.e., the independent set obtained by applying the greedy algorithm to M. G consists of $t \geqslant 1$ pairwise disjoint blocks G_i of consecutive elements of E, i.e., $G = \bigcup_{i=1}^{t} G_i$, where each element of G_i is greater than the elements of G_{i+1}. We use $\bar{G}_i$ to denote the set of all elements which lay between G_i and G_{i+1}. Note that, for $i = 1, \ldots, t-1$, $\bar{G}_i \neq \emptyset$, and $\bar{G}_t = \emptyset$ iff $e_n \in G$.

Definition 4. Let t be defined as above (the number of blocks of G) and, for $i = 1, \ldots, t$, $E_i = \bigcup_{j \leqslant i}(G_j \cup \bar{G}_j)$ and $r_i = |G \cap E_i|$.

Clearly $\emptyset \neq E_1 \subset \cdots \subset E_t = E$, $0 < r_1 < \cdots < r_t = |G|$, $|E_i - E_{i-1}| = |G_i \cup \bar{G}_i| > r_i - r_{i-1}$ for $i = 1, \ldots, t-1$ and $|E_t - E_{t-1}| = |G_t \cup \bar{G}_t| \geqslant r_t - r_{t-1}$, with equality iff $e_n \in G$.

Lemma 5. *If $A \subseteq E$ satisfies $|A \cap E_i| \leqslant r_i$, $i = 1, \ldots, t$, either $A \subseteq G$, or $A \subseteq G'$, where G' is obtained from G interchanging some pairs of elements $g \in G$ and $a \notin G$, with $g \geqslant a$.*

Proof. We use induction on $|A - G|$. If $|A - G| = 0$, clearly $A \subseteq G$.

If $|A - G| > 0$, let a be the first element of $A - G$. Let j be such that $a \in E_j - E_{j-1}$, and denote by E_a the set of elements on the left of $\{a\}$. We thus have $G \cap E_a \supseteq A \cap E_a$. Notice that if $G \cap E_a = A \cap E_a$, then $r_j = |G \cap E_j| = |G \cap E_a| < |A \cap E_j|$, a contradiction.

Therefore there is some element $g \geqslant a$ in $G - A$. If $A' = (A - \{a\}) \cup \{g\}$, $|A' \cap E_i| \leqslant |G \cap E_i| = r_i$ for $i = 1, \ldots, j-1$ and $|A' \cap E_i| = |A \cap E_i| \leqslant r_i$ for $i = j, \ldots, t$. Now, as $|A' - G| < |A - G|$, it follows from the induction hypothesis that $A' \subseteq G'$ for some G' defined as in the statement of the lemma. If $G' \not\supseteq A$, then $(G' - \{g\}) \cup \{a\} \supseteq A$, and $g \geqslant a$. □

Lemma 5 directly implies

Corollary 6. *If $M = (E, \mathscr{F})$ is regular and $A \subseteq E$ satisfies $|A \cap E_i| \leqslant r_i$, $i = 1, \ldots, t$, then $A \in \mathscr{F}$.*

We now prove that whenever a regular independence system is a matroid, it is an ordered matroid.

Theorem 7. *If $M = (E, \mathscr{F})$ is a matroid, $\mathscr{F} = \{A \subseteq E : |A \cap E_i| \leqslant r_i, i = 1, \ldots, t\}$.*

Proof. Since $G \cap E_i$ is a maximal independent set in E_i, for $A \subseteq E$ to be in $\mathscr{F}$ it is necessary $|A \cap E_i| \leqslant |G \cap E_i| = r_i$ for $i = 1, \ldots, t$. Sufficiency follows from Corollary 6. □

Conversely, we show that for every ordered matroid $M = (E, \mathscr{F})$ it is possible to define some order on the ground set E so that M becomes a regular independence system. This combined with Theorem 7 yields

Theorem 8. *A regular indepedence system is a matroid iff it is an ordered matroid.*

Proof. Impose on E the partial order defined by: $e \in E_i$, $e' \in E - E_i \Rightarrow e > e'$, $i = 1, \dots, t-1$, and denote by $\geqslant$ any linear extension of that partial order [3]. $(E, \geqslant)$ is a (totally) ordered set for which we now show that $M = (E, \mathscr{F})$ is regular. Let $A \in \mathscr{F}$, take any $a \in A$ and suppose $a \in E_j - E_{j-1}$. If b is such that $a \geqslant b$, then $b \in E - E_{j-1}$. Therefore, for $i = 1, \dots, j-1$, $|((A - \{a\}) \cup \{b\}) \cap E_i| = |A \cap E_i| \leqslant r_i$ and, for $i = j, \dots, t$, $|((A - \{a\}) \cup \{b\}) \cap E_i| \leqslant |A \cap E_j| \leqslant r_i$, showing that $(A - \{a\}) \cup \{b\}$ is still in $\mathscr{F}$, and that $M = (E, \mathscr{F})$ is regular. □

4. When is a regular independence system a matroid?

We now give a necessary and sufficient condition for deciding whether an arbitrary regular independence system is a matroid.

Consider a regular independence system $M = (E, \mathscr{F})$, $E = \{e_1, \dots, e_n\}$ with $e_1 \geqslant \cdots \geqslant e_n$. Let G be the greedy set and t, r_i and E_i as in Definition 4.

Definition 9. For $i = 1, \dots, t-1$ and, in case $G \neq E$, also for $i = t$, define A_i as the set of the $r_i + 1$ rightmost elements of E_i.

Theorem 10. *A regular independent system $M = (E, \mathscr{F})$ is a matroid iff all the above defined sets $A_i \notin \mathscr{F}$.*

Proof. Theorem 7 states that whenever M is a matroid, $\mathscr{F} = \{A \subseteq E : |A \cap E_i| \leqslant r_i, i = 1, \dots, t\}$.

If for some i we have $A_i \in \mathscr{F}$, clearly M is not a matroid given that $|A_i \cap E_i| = r_i + 1$.

Suppose now there is some $A \in \mathscr{F}$ with $|A \cap E_i| > r_i$. Since M is regular, the set A' consisting of the $|A \cap E_i|$ rightmost elements of E_i, together with the elements of $A \cap (E - E_i)$, is also in $\mathscr{F}$. But then a contradiction follows since $A_i \subseteq A'$ would be in $\mathscr{F}$ as well. □

Corollary 11. *Deciding whether a regular independence system $M = (E, \mathscr{F})$ is a matroid can be done in $O(|E| \log |E|)$ time, plus $O(|E|)$ calls to an oracle which checks the independence of subsets of E.*

Proof. According to Theorem 10, deciding whether a regular independence system $M = (E, \mathscr{F})$ is a matroid reduces to check the independence of the $t \leqslant \frac{1}{2}|E|$ sets A_i. Since those sets are defined from the greedy set, which can be obtained in $O(|E| \log |E|)$ time and $O(|E|)$ calls to the oracle, the result follows. □

5. Conclusions and final remarks

In this paper we considered a class of matroids which we called ordered matroids. The polytopes are described by t ($1 \leqslant t \leqslant \frac{1}{2}|E|$) linear inequalities, and the requirement of $x \in [0,1]^{|E|}$.

In Section 3 we considered independence systems defined on ordered sets. We called regular those systems for which independence is maintained whenever substituting any element e by some other element less than or equal to e. Regular independence systems generalize the definition of regular 0/1 polytopes [4].

The relation between ordered matroids and regular independence systems is established in Theorem 8. Whenever a regular independence system is a matroid, it is an ordered matroid. Moreover, it is possible to define some order on the ground set of any ordered matroid, so that the matroid turns out to be a regular independence system.

Theorem 10 gives a necessary and sufficient condition for an arbitrary regular independence system to be a matroid. Testing this condition amounts to check the independence of a linear number of subsets defined from the greedy set. Therefore, as long as there is a polynomial time algorithm which checks the independence of arbitrary subsets, deciding whether a regular independence system is a matroid can be carried out in polynomial time.

We now generalize some of the results presented by Wolsey [6] for regular 0/1 polytopes and 0/1 solutions of linear inequalities to the case of regular independence systems.

The definition of ceiling [1], extended to regular independence systems $M = (E, \mathscr{F})$, follows.

Definition 12. A maximal independent set $C \subseteq E$ is a *ceiling* of M if for $e_j \in C$ and $e_{j-1} \notin C$, $(C - \{e_j\}) \cup \{e_{j-1}\} \notin \mathscr{F}$.

We will say that $(C - \{e_j\}).\cup \{e_{j-1}\}$ was obtained from C by *a ceiling interchange*.

Clearly the greedy set G is a ceiling of M. Moreover, Lemma 5 ensures that if $C \neq G$ is a ceiling of M, $|C \cap E_i| > r_i$ for some $i = 1, \ldots, t$.

Theorem 13. *A regular independence system $M = (E, \mathscr{F})$ is a matroid iff M has unique ceiling.*

Proof. Theorem 7 and the above remark ensures G to be the unique ceiling, whenever M is a matroid.

Suppose now M is not a matroid. Then there is some set $A \in \mathscr{F}$ for which $|A \cap E_i| > r_i$. Take A so to be maximal independent. If A is not a ceiling, let e_j be the first element of A for which a ceiling interchange preserves independence, and define $A' := (A - \{e_j\}) \cup \{e_{j-1}\}$. Given that M is regular, A' is maximal. Moreover, $|A' \cap E_i| \geqslant |A \cap E_i| > r_i$. While A' is not a ceiling, repeat the above procedure with $A := A'$.

This will end with $\{e_1, e_2, \ldots, e_{|A|}\}$ if no other ceiling, different from G, were found before. $\square$

Definition 14. If $M = (E, \mathscr{F})$ is a regular independence system, a minimal dependent set $S = \{e_{j_1}, \ldots, e_{j_r}\} \subseteq E$ is a *strong cover* of M if $(S - \{e_{j_1}\}) \cup \{e_k\} \in \mathscr{F}$ for every $e_k \in E - S$ on the right of $\{e_{j_1}\}$.

The next result identifies all the strong covers of M, whenever M is a matroid.

Lemma 15. *If a regular independence system $M = (E, \mathscr{F})$ is a matroid, the strong covers of M are A_i, $i = 1, \ldots, t-1$, and A_t iff $e_n \notin G$.*

Proof. Clearly A_i, with $i = 1, \ldots, t-1$ and A_t, when $e_n \notin G$, are all strong covers of M. When $e_n \in G$ (and $G \neq E$, which implies $t > 1$), as $|E_t - E_{t-1}| = r_t - r_{t-1}$, the set $A_t = (E_t - E_{t-1}) \cup A_{t-1}$ is not minimal.

Take any dependent set $S \subseteq E$, i.e., $|S \cap E_i| > r_i$ for some $i = 1, \ldots, t$, and supose $S \neq A_i$. If $S \cap (E - E_i) \neq \emptyset$, C is not minimal. If $S \cap (E - E_i) = \emptyset$, let e_j be the first element of S. If there is no element e_k on the right of e_j not in S, again $S \supset A_i$ is not minimal. Otherwise, $|((S - \{e_j\}) \cup \{e_k\}) \cap E_i| > r_i$, and S cannot be a strong cover. $\square$

For the set of 0/1 solutions of a linear inequality, Wolsey [6] proved a result which we now state in terms of regular independence systems.

Theorem 16. *If a regular independence system $M = (E, \mathscr{F})$ is a matroid, the polytope of M is*

$$\mathscr{X} = \Big\{ x \in \mathbb{R}^{|E|} : \textstyle\sum_{e \in E(S)} x_e \leqslant |S| - 1, \ \forall S \text{ strong cover of } M,$$
$$0 \leqslant x_e \leqslant 1 \ \forall e \in E \Big\},$$

where $E(S)$ is S together with all the elements on the left of S.

According to Lemma 15, each strong cover S is one of the sets A_i. Thus, $E(S) = E_i$, and the above inequalities are precisely the inequalities describing the polytope in Theorem 1.

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