The bounds of the smallest and largest eigenvalues for rank-one modification of the Hermitian eigenvalue problem

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The bounds of the smallest and largest eigenvalues for rank-one modification of the Hermitian matrices are studied in this paper. The sharper bounds are obtained. Numerical examples illustrate that our bounds give accurate estimates.

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1. Introduction

Eigenvalue problems have lots of applications in science and engineering such as structure calculation, calculation of the propagation modes in optical fiber, and absorptive photonic crystals. Recently, rank-one modification of the symmetric eigenvalue problems has been discussed by Huang et al. in [1]. Ding and Zhou studied a spectral perturbation theorem for rank-one updated matrices of special structure in [2], and considered two applications of the result. Eigenvalue bounds for perturbations of Hermitian matrices have been considered by Ipsen and Nadler in [3]. In this paper, we consider the bounds of the smallest and the largest eigenvalues for rank-one modification of the Hermitian matrices. The ideas of this paper were motivated by ones of [2,3]. The results of this paper extend ones of [2], are sharper than ones of [3] under some assumptions.

We study the following form

\[ [A \pm yy^*]x = \lambda x, \]  

(1.1)

where \( A \) is a Hermitian matrix, \( y \) is a complex column vector.

The paper is organized as follows. In Section 2 we give some notations and present the bounds of the smallest eigenvalue and the largest eigenvalue for the Hermitian rank-one modification. In Section 3 we consider some examples for illustrating our bounds.

2. The bounds of the smallest eigenvalue and the largest eigenvalue

For convenience, \( \| \cdot \| \) denotes the 2-norm of matrix \( \cdot \), and \( e_i \) denotes the \( i \)th column of the identity matrix. \( A^* \) and \( \bar{A} \) denote the conjugate transpose and the complex conjugation of a matrix \( A \), respectively. Let \( A \) be a Hermitian matrix, then \( A \) can be factorized as

\[ A = V \Lambda V^*, \]

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where $V = (v_1, \ldots, v_n)$ is a unitary matrix consisting of eigenvectors of $A$ associated with eigenvalues $\lambda_1, \ldots, \lambda_n$, and $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$. The eigenvalues of a Hermitian matrix $A \in \mathbb{C}^{n \times n}$ are numbered in an increasing order of magnitude

$$
\lambda_{\min}(A) \equiv \lambda_n \leq \cdots \leq \lambda_1 \equiv \lambda_{\max}(A).
$$

For an $n$ dimensional vector $y$ let a $j - i + 1$ dimensional vector $y_{ij}$ be defined by

$$
y_{ij} \equiv (v_i, \ldots, v_j)^* y, \quad 1 \leq i \leq j \leq n.
$$

In the following, we recall the well-known theorem which shows the interlacing properties between the eigenvalues of a Hermitian matrix $A$ and its constant rank-one modified matrix $A \pm yy^*$.

**Theorem 2.1** ([4–6]). Suppose $B = A + \tau yy^*$ where $A \in \mathbb{C}^{n \times n}$ is Hermitian, $y \in \mathbb{C}^n$ and $\tau \in \mathbb{R}$. If $\tau = 1$, then

$$
\lambda_n(A) \leq \lambda_n(B) \leq \lambda_{n-1}(A) \leq \lambda_{n-1}(B) \leq \cdots \leq \lambda_1(B) \leq \lambda_1(A) + \|y\|^2
$$

while if $\tau = -1$ then

$$
\lambda_n(A) - \|y\|^2 \leq \lambda_n(B) \leq \lambda_{n-1}(A) \leq \lambda_{n-1}(B) \leq \cdots \leq \lambda_1(B) \leq \lambda_1(A).
$$

In order to obtain our results, the following three lemmas are given.

**Lemma 2.1.** Let $A = \text{diag}(\lambda_1, \ldots, \lambda_n)$ and $(ae_i^\tau \ b)$ where $e_i$ denotes the $i$th column of the identity matrix of order $n - 1$, and $a \geq 0$ and $b$ are two scalars. Then the eigenvalues of $A + (ae_i^\tau \ b)$ are the eigenvalues of $(\lambda_i \ 0 \ 
\lambda_n) + (a \ b)$, and $\lambda_i$, $j \neq i, j \neq n$.

**Proof.** According to the definition of matrix eigenvalues, we have the characteristic equation of $[A + (ae_i^\tau \ b)]$ as follows

$$
\det \left[ \lambda I - A - \begin{pmatrix} ae_i^\tau \ b \end{pmatrix} \right] = \prod_{j \neq i, j \neq n} (\lambda - \lambda_j) \times \det \begin{pmatrix} \lambda - \lambda_1 & a \ b \\ a & \lambda_n - |b|^2 \end{pmatrix} = 0.
$$

Since

$$
\left( \begin{array}{ll}
\lambda_i + a^2 \ b \\
\lambda_n + |b|^2
\end{array} \right) = \left( \begin{array}{ll}
\lambda_i \ a \\
\lambda_n \ b
\end{array} \right) = \left( \begin{array}{ll}
\lambda_i \ 0 \\
0 \ 
\lambda_n
\end{array} \right) + (a \ b),
$$

we know that the eigenvalues of $[A + (ae_i^\tau \ b)]$ are the eigenvalues of $A + (ae_i^\tau \ b)$. \hspace{1cm} \square

**Lemma 2.2.** Let $A = \begin{pmatrix} 0 & x \ 0 & 0 \end{pmatrix} + (a \ b)$, $a > 0$, $b \neq 0$. Then $\lambda_{\min}(A)$ is a monotone increasing function in $x$.

**Proof.** According to the definition of the matrix eigenvalues, we have the characteristic equation of $A$ as follows

$$
\det(\lambda I - A) = \lambda^2 - (a^2 + \|b\|^2 + c)\lambda + (x + a^2)(\|b\|^2 + c) - a^2\|b\|^2.
$$

So,

$$
\lambda_{\min}(A) = \frac{1}{2} \left[ (x + a^2 + \|b\|^2 + c) - \sqrt{(x + a^2 - \|b\|^2 - c)^2 + 4a^2\|b\|^2} \right].
$$

Furthermore, we can easily obtain that $\frac{d\lambda_{\min}(A)}{dx} \geq 0$, i.e., $\lambda_{\min}(A)$ is a monotone increasing function in $x$. \hspace{1cm} \square

**Lemma 2.3.** Let $A = \begin{pmatrix} 0 & x \ 0 & 0 \end{pmatrix} - (a \ b)$, $a > 0$, $b \neq 0$. Then $\lambda_{\min}(A)$ is a monotone increasing function in $x$.

**Theorem 2.2** (Smallest Eigenvalue). Let $A \in \mathbb{C}^{n \times n}$ be Hermitian, $y \in \text{span}\{v_1, \ldots, v_i, v_n\}$, $i = 1, \ldots, n - 1$, and

$$
L_{i\pm} \equiv \left( \begin{array}{ll}
\lambda_i & 0 \\
0 & \lambda_n
\end{array} \right) \pm \left( \begin{array}{ll}
\|y_{1:2}^\perp \| & ||y_{1:2}^\perp & y_n^\perp \end{array} \right).
$$

Then $\ell_i \leq \lambda_n(A + yy^*)$ and $\lambda_2(L_{i-}) \leq \lambda_n(A - yy^*)$, where $\ell_i \equiv \min\{\lambda_{n-1}, \lambda_2(L_{i+})\}$.

$$
\lambda_n(A) \leq \ell_i \leq \lambda_{n-1}(A),
$$

and

$$
\lambda_n(A) - \|y\|^2 \leq \lambda_2(L_{i-}) \leq \lambda_n(A).
$$
Theorem 2.3
Let \( A \)
where \( V = (v_1, \ldots, v_n) \) is a unitary matrix, and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \).

**Proof.** For convenience, we abbreviate \( \lambda_i \equiv \lambda_i(A), \ 1 \leq i \leq n \), in this proof. Due to the fact that \( A \) is a Hermitian matrix, \( A \) can be factorized as

\[
A = V \Lambda V^*,
\]

where \( V = (v_1, \ldots, v_n) \) is a unitary matrix, and \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \).

**Lower bounds.** Let \( u \) be an eigenvector associated with the smallest eigenvalue of \((A+yy^*)\), i.e., \((A+yy^*)u = \lambda_n(A+yy^*)u\), where \( \|u\| = 1 \). Let \( z = V^*u \), then \( \|z\| = 1 \).

Now consider

\[
\lambda_n(A+yy^*) = u^*(A+yy^*)u
= z^*V^*(A+yy^*)Vz
= z^*V^*AVz + z^*V^*yy^*Vz
= z^*Az + |y^*Vz|^2
= z^*Az + |y_1^e|^2
= z^*(A+y_1^eV^e_z)z.
\]

Since \( y \in \text{span}\{v_1, \ldots, v_i, v_{n-1}\} \), we know \( y_1^e \) is a zero vector whose order is \( n-i \). According to the Householder reflection, we can get a unitary matrix \( Q_i \) whose order is \( i \) so that \( Q_iy_{1,i} = \|y_{1,i}\|e_i \), and set \( w_i = \left( \frac{Q_i}{0_{n-i}} \right) z \). Then \( \|w_i\| = 1 \) and

\[
\lambda_n(A+yy^*) \geq w_i^* \begin{bmatrix}
\lambda_i & \\
\lambda_{i+1} & \ddots \\
& \ddots & \ddots
\end{bmatrix} \begin{bmatrix}
\|y_{1,i}\|e_i \\
0 \\
\ddots \\
0 \\
\|y_{1,n}\|e_n
\end{bmatrix} w_i
= \lambda_n \begin{bmatrix}
\lambda_i & \\
\lambda_{i+1} & \ddots \\
& \ddots & \ddots
\end{bmatrix} \begin{bmatrix}
\|y_{1,i}\|e_i \\
0 \\
\ddots \\
0 \\
\|y_{1,n}\|e_n
\end{bmatrix}
= \min\{\lambda_{n-1}, \lambda_n(L_{i+})\} \quad \text{(By Lemma 2.1)}.
\]

Now consider the negative semidefinite modification. As above one shows

\[
\lambda_n(A-yy^*) \geq \lambda_n \begin{bmatrix}
\lambda_i & \\
\lambda_{i+1} & \ddots \\
& \ddots & \ddots
\end{bmatrix} \begin{bmatrix}
\|y_{1,i}\|e_i \\
0 \\
\ddots \\
0 \\
\|y_{1,n}\|e_n
\end{bmatrix}
= \min\{\lambda_{n-1}, \lambda_n(L_{i-})\}
\geq \lambda_n(L_{i-}) \quad \text{(By Theorem 2.1).}
\]

**Remark 2.1.** According to Lemmas 2.2 and 2.3, the results of Theorem 2.2 are sharper than the lower bounds of Theorem 2.1 in Ref. [3] if \( i \leq n-2 \); the results of Theorem 2.2 are the same with the lower bounds of Theorem 2.1 in Ref. [3] if \( i = n-2 \).

**Lemma 2.4.** Let \( A = \begin{pmatrix} c & 0 \\ 0 & x \end{pmatrix} \pm \begin{pmatrix} a & b \\ b & a \end{pmatrix} (\bar{a} b) \), \( a \neq 0 \), \( b > 0 \). Then \( \lambda_{\text{max}}(A) \) is a monotone increasing function in \( x \).

**Theorem 2.3 (Largest Eigenvalue).** Let \( A \in \mathbb{C}^{n \times n} \) be Hermitian, \( y \in \text{span}\{v_1, v_i, \ldots, v_n\} \), \( i = 2, \ldots, n \), and

\[
U_{i\pm} \equiv \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i \end{pmatrix} \pm \begin{pmatrix} y_1 \ |y_{1,n}| \end{pmatrix} \begin{pmatrix} y_1^e \ |y_{1,n}| \end{pmatrix}, \quad i = 2, \ldots, n.
\]

Then \( \lambda_1(A + yy^*) \leq \lambda_1(U_{i+}) \), and \( \lambda_1(A - yy^*) \leq \delta_i \), where \( \delta_i \equiv \max\{\lambda_2, \lambda_1(U_{i-})\} \).

\[
\lambda_1(A) \leq \lambda_1(U_{i+}) \leq \lambda_1(A) + \|y\|^2,
\]

and

\[
\lambda_2(A) \leq \delta_i \leq \lambda_1(A).
\]
Proof. Let \( u \) be an eigenvector associated with the largest eigenvalue of \((A + yy^*)\), i.e., \((A + yy^*)u = \lambda_1(A + yy^*)u\), where \( \|u\| = 1 \). Let \( z = V^*u \), then \( \|z\| = 1 \).

Now consider

\[
\lambda_1(A + yy^*) = u^*(A + yy^*)u \\
= z^*V^*(A + yy^*)Vz \\
= z^*V^*AVz + z^*V^*yy^*Vz \\
= z^*Az + |y^*Vz|^2 \\
= z^*Az + |y_{1:n}^*|^2 \\
= z^*(A + y_{1:n}y_{1:n}^*)z.
\]

Since \( y \in \text{span}\{v_1, v_2, \ldots, v_n\} \), we know \( y_{2:i-1} \) is a zero vector of order \( i - 2 \). According to the Householder reflection, we can get a unitary matrix \( Q_{n-i+1} \) whose order is \( n - i + 1 \) so that \( Q_{y_{i:n}} = \|y_{i:n}\|e_i \), and set \( w_i = \begin{pmatrix} \lambda_{i-1} & \lambda_{i-1} & \cdots & \lambda_{i-1} \\ \lambda_{i-1} & \cdots & \cdots & \lambda_{i-1} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{i-1} & \cdots & \cdots & \cdots \end{pmatrix} + \begin{pmatrix} y_i \\ \vdots \\ \vdots \\ \|y_{i:n}\|e_i \end{pmatrix} \begin{pmatrix} \bar{y}_i \\ \bar{y}_i \\ \|y_{i:n}\|e_i \end{pmatrix} \begin{pmatrix} y_{i:n}^* \\ y_{i:n}^* \\ y_{i:n}^* \end{pmatrix} = \lambda_1(U_{i+}) \). (By Theorem 2.1).

Now consider the negative semidefinite modification. As above one shows

\[
\lambda_1(A - yy^*) \leq \lambda_1 \begin{pmatrix} \lambda_1 & \lambda_1 & \cdots & \lambda_1 \\ \lambda_1 & \cdots & \cdots & \lambda_1 \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_1 & \cdots & \cdots & \cdots \end{pmatrix} + \begin{pmatrix} y_i \\ \vdots \\ \vdots \\ \|y_{i:n}\|e_i \end{pmatrix} \begin{pmatrix} \bar{y}_i \\ \bar{y}_i \\ \|y_{i:n}\|e_i \end{pmatrix} \begin{pmatrix} y_{i:n}^* \\ y_{i:n}^* \\ y_{i:n}^* \end{pmatrix} = \lambda_1(U_{i-}) \). (By Theorem 2.1).

\[
\lambda_1(A - yy^*) = \max\{\lambda_2, \lambda_1(U_{i-})\} = \delta_i. \quad \square
\]

Remark 2.2. According to Lemma 2.4, the results of Theorem 2.3 are sharper than the upper bounds of Theorem 2.4 in Ref. [3] if \( i \geq 2 \); the results of Theorem 2.3 are the same with the upper bounds of Theorem 2.4 in Ref. [3] if \( i = 2 \).

3. Numerical examples

In this section, we give some examples to illustrate our results. All the numerical experiments were performed with MATLAB 2009a. “Real” denotes the exact eigenvalues of matrix \( A \pm y \ast y^* \). “New” and “Old” denote the bounds which are obtained in this paper and Ref. [3], respectively. For illustrating the results, three types of the vector \( y \) are computed for the same \( i \). In order to repeat the results, the vectors \( y \) are obtained by the first three random vectors when MATLAB was opened.

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<tr>
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<td>Old</td>
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<tr>
<td>10.4640</td>
<td>10.4593</td>
<td>10.1000</td>
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<tr>
<td>10.7196</td>
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<td>10.1714</td>
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<th>( i = 80 )</th>
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<td>10.4932</td>
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<td>10.2965</td>
<td>10.2547</td>
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<td>10.4640</td>
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Table 1...
Example 3.1. Let \( A = 10 \cdot \text{diag}(n : -1 : 1) \) and \( n = 100, 400, y = 10 \cdot \text{randn}(n), \) and \( y(i + 1 : n - 1) = 0, i = 40, 80. \) Tables 1–4 are the comparison results of Theorem 2.2.

Example 3.2. Let \( A = \text{diag}(150 : -2 : -50) \) and \( n = \text{length}(150 : -2 : -50), y = \text{randn}(n), \) and \( y(i + 1 : n - 1) = 0, i = 40, 80. \) Tables 5 and 6 are the comparison results of Theorem 2.3.

According to two examples, we can conclude that our bounds can give accurate estimates for some vectors \( y. \) Furthermore, the bounds of the interior eigenvalues are also considered under some special the rank-one modifications.
Table 6
The lower bounds of the smallest eigenvalue are showed when \( B = A - y \ast y^* \).

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References