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The sorting order on a Coxeter group

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ABSTRACT

Let (W, S) be an arbitrary Coxeter system. For each word ω in the generators we define a partial order—called the ω -sorting order—on the set of group elements $W_\omega \subseteq W$ that occur as subwords of ω . We show that the ω -sorting order is a supersolvable join-distributive lattice and that it is strictly between the weak and Bruhat orders on the group. Moreover, the ω -sorting order is a “maximal lattice” in the sense that the addition of any collection of Bruhat covers results in a nonlattice.

Along the way we define a class of structures called supersolvable antimatroids and we show that these are equivalent to the class of supersolvable join-distributive lattices.

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1. Introduction

In this paper we will describe a very general phenomenon regarding reduced words in Coxeter groups. Let (W, S) be an arbitrary Coxeter system and let

$$S^* := S^0 \cup S^1 \cup S^2 \cup \dots \cup S^\infty$$

denote the collection of finite and semi-infinite words in the generators. Given an arbitrary word $\omega \in S^*$ —called the sorting word—let $W_\omega \subseteq W$ denote the set of group elements that occur as subwords of ω . We identify every subword $\alpha \subseteq \omega$ with the index set $I(\alpha) \subseteq I(\omega) = \{1, 2, 3, \dots\}$ describing the positions of its letters.

For each element $u \in W_\omega$ let $\text{sort}_\omega(u)$ denote the reduced word for u that is lexicographically first among subwords of ω —we call this the ω -sorted word of u . In this way, ω induces a canonical reduced word for each element of W_ω . We define the ω -sorting order on W_ω as the inclusion order on index sets of sorted words. It turns out that there always exists a *reduced* subword $\omega' \subseteq \omega$ such that the

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ω' - and ω -sorting orders coincide. Furthermore, the sorting order is not affected by the exchange of adjacent commuting generators in the sorting word. In the case of a finite Coxeter group W , we may summarize some of our results as follows:

For each commutation class of reduced words for the longest element $w_\circ \in W$ we obtain a supersolvable join-distributive lattice on the elements of the group. This lattice is graded by the usual Coxeter length $\ell : W \rightarrow \mathbb{Z}$ and it is strictly between the weak and Bruhat orders. Furthermore, the poset is a “maximal lattice” in the sense that the addition of any collection of Bruhat covers results in a nonlattice.

More generally, the collection of ω -sorted subwords of ω has a remarkable structure, related to the study of abstract convexity. Given a ground set E and a collection of finite subsets $\mathcal{F} \subseteq 2^E$ —called feasible sets—we say that the pair (E, \mathcal{F}) is an antimatroid if it satisfies:

The empty set \emptyset is in \mathcal{F} , and

Given $A, B \in \mathcal{F}$, $B \not\subseteq A$, there exists $x \in B \setminus A$ such that $A \cup \{x\} \in \mathcal{F}$.

Edelman showed that a lattice P is join-distributive (see Section 2.1.3) if and only if it arises as the lattice of feasible sets of an antimatroid.

We will prove an extension of Edelman’s theorem based on the following concept. Let (E, \leq_E) be a totally ordered ground set and let $\mathcal{F} \subseteq 2^E$ be a collection of feasible finite subsets. We say that (E, \mathcal{F}, \leq_E) is a supersolvable antimatroid if it satisfies:

The empty set \emptyset is in \mathcal{F} , and

Given $A, B \in \mathcal{F}$, $B \not\subseteq A$, let $x = \min_{\leq_E}(B \setminus A)$. Then $A \cup \{x\} \in \mathcal{F}$.

It is clear that a supersolvable antimatroid is, in particular, an antimatroid. We prove that a lattice is join-distributive and supersolvable if and only if it arises as the lattice of feasible sets of a supersolvable antimatroid. Finally, our main result states that

The collection of ω -sorted subwords of a given sorting word $\omega \in S^*$ is a supersolvable antimatroid.

The paper is organized as follows.

In Section 2 we review the concepts of antimatroid, convex geometry, and join-distributive lattice. After this we define supersolvable antimatroids and prove that they are equivalent to supersolvable join-distributive lattices.

Section 3 contains the definitions of ω -sorted words and the ω -sorting order. We give an algorithmic characterization of ω -sorted words and discuss how this algorithm is a generalization of classical sorting algorithms.

We prove our main results in Section 4. Namely, we show that the ω -sorting order is strictly between the weak and Bruhat orders; we prove that the collection of ω -sorted words forms a supersolvable antimatroid and hence that the ω -sorting order is a supersolvable join-distributive lattice; and we show that the ω -sorting order is constant on commutation classes of reduced words. Finally, we prove that the sorting orders are “maximal lattices” and discuss how this is related to the weak and Bruhat orders.

After this we discuss two important special cases. In Section 5 we consider the case when infinitely many group elements occur as subwords of the sorting word ω . In this case most of our results still hold. In particular, the ω -sorting order is still a lattice in which every interval is join-distributive and supersolvable. This is remarkable because the weak order on an infinite Coxeter group is *not* a lattice.

Finally, in Section 6 we discuss our motivation for the current paper. This is the work of Reading on “Coxeter-sortable elements.”

2. Abstract convexity

In this section we review the theory of “abstract convexity” by discussing three equivalent structures: antimatroids, convex geometries and join-distributive lattices. All of the structures here are finite but in Section 5 we will relax this condition. Except where stated otherwise, this material can be found in *Greedoids* [11] by Korte, Lovász and Schrader.

At the end of the section we will add the criterion of “supersolvability” and prove a characterization of supersolvable join-distributive lattices.

2.1. Equivalent structures

2.1.1. Antimatroids

A set system is a pair (E, \mathcal{F}) , where E is a finite ground set and $\mathcal{F} \subseteq 2^E$ is a collection of subsets—called feasible sets. The system is accessible if it satisfies:

For each nonempty $A \in \mathcal{F}$, there exists $x \in A$ such that $A \setminus \{x\} \in \mathcal{F}$.

More specifically, an accessible set system is called an antimatroid if it satisfies any of the following equivalent conditions.

Lemma 2.1. (See [11, Lemma III.1.2].) *Given an accessible set system (E, \mathcal{F}) , the following statements are equivalent:*

For all feasible sets $A, B \in \mathcal{F}$ with $B \not\subseteq A$, there exists $x \in B \setminus A$ such that $A \cup \{x\} \in \mathcal{F}$. (2.2)

\mathcal{F} is closed under taking unions. (2.3)

Given $A, A \cup \{x\}$ and $A \cup \{y\}$ in \mathcal{F} , it follows that $A \cup \{x, y\} \in \mathcal{F}$. (2.4)

2.1.2. Convex geometries

The way in which antimatroids encode the idea of “convexity” is expressed by the equivalent concept of an “abstract convex geometry,” introduced by Edelman and Jamison [6].

Let (E, \mathcal{C}) be a set system in which \mathcal{C} is closed under intersections and $\emptyset \in \mathcal{C}$. This gives rise to a closure operator $\tau : 2^E \rightarrow 2^E$,

$$\tau(X) := \bigcap \{A \in \mathcal{C}, A \supseteq X\},$$

which satisfies the following properties:

$$\tau(\emptyset) = \emptyset,$$

$$A \subseteq \tau(A) \text{ for all } A \in 2^E,$$

$$A \subseteq B \text{ implies } \tau(A) \subseteq \tau(B),$$

$$\tau(\tau(A)) = \tau(A).$$

Conversely, every closure operator arises in this way. That is, if we are given a map τ satisfying the above properties, then \mathcal{C} is the collection of sets satisfying $\tau(A) = A$ —called the τ -closed sets. The triple (E, \mathcal{C}, τ) (where either \mathcal{C} or τ is redundant) is called a closure space.

Closure operators are ubiquitous in combinatorics. If a closure space (E, \mathcal{C}, τ) satisfies the exchange axiom:

$$\text{If } x, y \notin \tau(A) \text{ then } x \in \tau(A \cup \{y\}) \text{ implies } y \in \tau(A \cup \{x\}),$$

then it is called a matroid. In this case τ models the notion of “linear span.” If instead we wish to model the notion of “convex hull,” we will require the following anti-exchange axiom:

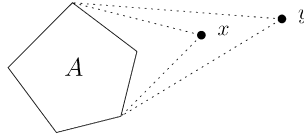


Fig. 1. The anti-exchange axiom.

If $x, y \notin \tau(A)$ then $x \in \tau(A \cup \{y\})$ implies $y \notin \tau(A \cup \{x\})$,

which is illustrated in Fig. 1.

In this case we say that τ is a convex closure and that (E, \mathcal{C}, τ) is an abstract convex geometry—or just a convex geometry. Convex geometries and antimatroids are complementary structures.

Lemma 2.5. (See [11, Theorem III.1.3].) *Given an accessible set system (E, \mathcal{F}) on finite ground set E , let $\mathcal{F}^c = \{E \setminus A : A \in \mathcal{F}\}$ denote the collection of complements of feasible sets. Then (E, \mathcal{F}) is an antimatroid if and only if (E, \mathcal{F}^c) is a convex geometry. In this case \mathcal{F} consists of the open sets and \mathcal{F}^c consists of the closed sets of a convex closure.*

The contrast between the exchange and anti-exchange properties is one reason for the term *anti*-matroid.

The motivating example of a convex geometry is a pair (E, \mathcal{C}) where $E \subseteq \mathbb{R}^n$ is a finite subset of Euclidean space and \mathcal{C} is the collection of intersections of E with convex subsets of \mathbb{R}^n .

2.1.3. Join-distributive lattices

Closure spaces, in turn, give rise to lattices. A partially-ordered set (poset) (P, \leq) is a finite set together with a reflexive, antisymmetric and transitive relation. If each pair of elements $x, y \in P$ possesses a least upper bound $x \vee y$ (their join) and a greatest lower bound $x \wedge y$ (their meet), then we call (P, \leq) a lattice.

Given a finite closure space (E, \mathcal{C}, τ) let $(P_{\mathcal{C}}, \leq)$ denote the collection of τ -closed sets, partially ordered by inclusion. Since \mathcal{C} is closed under intersections, every pair $X, Y \in P_{\mathcal{C}}$ has a meet $X \wedge Y = X \cap Y$. Furthermore, since the ground set E is τ -closed, the collection of upper bounds $U(X, Y) = \{Z \in P_{\mathcal{C}} : X \subseteq Z \text{ and } Y \subseteq Z\}$ is nonempty and

$$X \vee Y = \bigwedge_{Z \in U(X, Y)} Z$$

defines a join operation. Hence $(P_{\mathcal{C}}, \leq)$ is a lattice.

The relationship between closure spaces and lattices has been fruitful and the two most important types of closure spaces—matroids and antimatroids—have been classified in terms of their lattice structure. To express this we need some poset notation.

Given elements x, y in a poset (P, \leq) , we say that y covers x (and write $x \prec y$) when $x \leq y$ and there does not exist $z \in P$ such that $x < z < y$. If P has a minimum element $\hat{0}$ then the elements covering $\hat{0}$ are called atoms. More generally, the atoms of an interval $[x, y] = \{z \in P : x \leq z \leq y\}$ are the elements in $[x, y]$ that cover x . We call a lattice P upper-semimodular if whenever x and y both cover $x \wedge y$ it follows that $x \vee y$ covers both x and y . A lattice P is called geometric if it is upper-semimodular and if every element in P can be written as a join of atoms. The following characterization was known to Birkhoff, Whitney and MacLane in the 1930s (see [7, p. 179]).

Theorem 2.6. (See [11, Theorem I.1.7].) *A finite lattice is geometric if and only if it occurs as the lattice of closed sets of an exchange closure.*

The characterization of anti-exchange closures is more recent and is due to Edelman. We say that a lattice is boolean if it is isomorphic to the collection of all subsets of a finite set under inclusion. Given an element x in a lattice and a set Y of elements covering x , we call the interval

$$\left[x, \bigvee_{y \in Y} y \right]$$

an atomic interval. If $\hat{1}$ is the maximum element in a lattice, then an element $x \neq \hat{1}$ is called meet-irreducible if $x = y \wedge z$ implies either $x = y$ or $x = z$. A lattice that satisfies any of the following equivalent conditions is called join-distributive.

Theorem 2.7. (See [1, Theorems 1.7 and 1.9].) *Given a finite lattice (P, \leq) , the following statements are equivalent:*

1. Every atomic interval in P is boolean.
2. Every element of P has a unique irredundant decomposition as a meet of meet-irreducible elements.
3. P is upper-semimodular and it satisfies the meet-semidistributive property: for all $x, y, z \in P$, we have

$$x \wedge y = x \wedge z \implies x \wedge y = x \wedge (y \vee z).$$

These lattices were first considered by Dilworth [4], for whom condition 2 was the defining property. Edelman’s theorem is the following.

Theorem 2.8. (See [5, Theorem 3.3].) *A finite lattice is join-distributive if and only if it occurs as the lattice of open sets of an anti-exchange closure.*²

That is, a finite join-distributive lattice is precisely the lattice of feasible sets of some antimatroid. In this sense, geometric lattices and join-distributive lattices are opposite. In fact, since a geometric lattice is itself an atomic interval, the only lattices that are both geometric and join-distributive are the boolean lattices.

2.2. Supersolvable join-distributive lattices

In [17] Stanley introduced another generalization of distributive lattices—the supersolvable lattices. (See also [16, Example 3.13.4].) The motivating example of a supersolvable lattice is the lattice of subgroups of a supersolvable group. In the same paper Stanley also discussed join-distributive lattices—under the name 1-lattices—and he used the code 1-SS to refer to lattices that are both supersolvable and join distributive. Hawrylycz and Reiner [8] later found an important class of examples: the lattice of closure relations on any finite poset is 1-SS.

In this paper we will explore a new source of examples. Hence we find it convenient to give a characterization of 1-SS lattices in the spirit of Edelman’s theorem (Theorem 2.8). First let us review the notion of “supersolvability.”

A finite lattice (P, \leq) is called distributive if either of the equivalent “distributive laws,”

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$

holds for all $x, y, z \in P$. Given a subset $X \subseteq P$, the smallest sublattice of P containing X is called the sublattice generated by X . A maximal chain m in P is called an M-chain if together with any other maximal chain in P it generates a distributive sublattice. If such an M-chain exists we say that P is a supersolvable lattice.

A characterizing property of supersolvable lattices is the fact that they have a certain kind of “edge-labelling.” Given a poset P , its Hasse diagram is a directed graph with an edge from x to y for each cover relation $x < y$ in P . (The edges are directed “up” in the diagram.) An edge-labelling of P is

² Edelman [5] used the term meet-distributive for the lattice of closed sets.

just a labelling of the edges of its Hasse diagram by the elements of some ordered set—typically the integers. Now suppose that P is supersolvable with M-chain

$$\hat{0} = a_0 < a_1 < \dots < a_n = \hat{1},$$

and define an edge-labelling λ by setting

$$\lambda(x, y) := \min\{i: y \leq a_i \vee x\} \tag{2.9}$$

for each cover $x < y$ in P . Stanley [17] showed that this labelling has the following properties:

The labels on any maximal chain in P form a permutation of the index set, which we may assume is $\{1, \dots, n\}$. It follows that the labels on the maximal chains in an interval $[x, y] \subseteq P$ are permutations of some fixed subset $I_{[x,y]} \subseteq \{1, \dots, n\}$. In any interval $[x, y] \subseteq P$ there exists a unique maximal chain labelled by the increasing permutation of $I_{[x,y]}$.

Such a labelling is called an S_n EL-labelling of P (where EL stands for “edge-lexicographic”). McNamara proved that the existence of an S_n EL-labelling characterizes a supersolvable lattice.

Lemma 2.10. (See [12, Theorem 1].) *A finite lattice is supersolvable if and only if it possesses an S_n EL-labelling. In particular, the unique increasing maximal chain in this labelling is an M-chain.*

Now we can give our characterization of supersolvable join-distributive lattices, based on the following definition. We say that \leq_E is a total order on E if either $x \leq_E y$ or $y \leq_E x$ for all $x, y \in E$.

Definition 2.11. A set system (E, \mathcal{F}, \leq_E) , with a total order \leq_E on the ground set E , is called a supersolvable antimatroid if it satisfies $\emptyset \in \mathcal{F}$ and the following condition:

Given feasible sets $A, B \in \mathcal{F}$ with $B \not\subseteq A$ and $x = \min_{\leq_E}(B \setminus A)$, it follows that

$$A \cup \{x\} \in \mathcal{F}. \tag{2.12}$$

In particular, a supersolvable antimatroid is an antimatroid. Indeed, taking $A = \emptyset$, condition (2.12) implies that every prefix of a feasible set—with respect to the order on E —is feasible. Thus the maximum element of any feasible set may be deleted, and we conclude that (E, \mathcal{F}) is an accessible system. Finally, note that condition (2.12) strengthens condition (2.2).

Theorem 2.13. *Let (E, \mathcal{F}) be an antimatroid with join-distributive lattice P of feasible sets. The following statements are equivalent:*

1. *There exists a total order on E with respect to which (E, \mathcal{F}) is a supersolvable antimatroid.*
2. *P is a supersolvable lattice.*
3. *There exists an order on E with respect to which the natural edge-labelling of P by E is an S_n EL-labelling.*

Proof. We will show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

First, suppose that (E, \mathcal{F}) is supersolvable with respect to a certain total order (E, \leq_E) and consider the join-distributive lattice (P, \leq) of feasible sets. Note that a cover relation $X < Y$ in P is naturally labelled by the element $x \in E$ where $\{x\} = Y \setminus X$. We claim that this is an S_n EL-labelling with respect to the order \leq_E . Indeed, consider an interval $[A, B]$ in P . By property (2.2) of antimatroids we know that every maximal chain in $[A, B]$ is labelled by some permutation of the set $B \setminus A \subseteq E$, and condition (2.12) guarantees that the unique increasing permutation occurs. Thus P possesses an S_n EL-labelling and by Lemma 2.10 it is supersolvable.

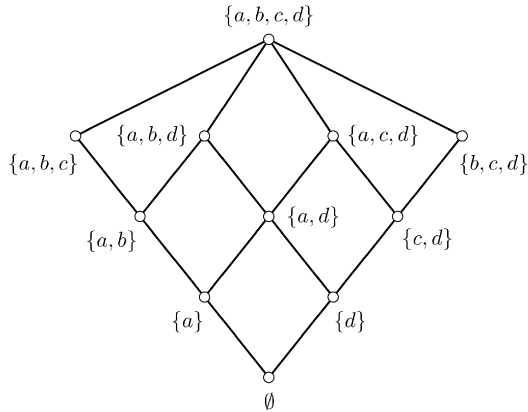


Fig. 2. A join-distributive lattice that is not supersolvable.

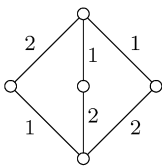
Next, suppose that P is supersolvable with M-chain

$$\emptyset = A_0 < A_1 < \dots < A_n = E,$$

and set $\{x_i\} = A_i \setminus A_{i-1}$ for all $1 \leq i \leq n$. This defines a total order $x_1 <_E x_2 <_E \dots <_E x_n$ on the set E . Now consider a cover relation $X < Y$ in P . Since the join operation in P is just union of feasible sets, we conclude that the edge-labelling (2.9) satisfies $\lambda(X, Y) = i$, where $Y \setminus X = \{x_i\}$. That is, the labelling λ coincides with the natural edge-labelling by E . Since λ is an S_n EL-labelling, so is the edge-labelling by E .

Finally, let (E, \leq_E) be a total order such that the edge-labelling of P by E is S_n EL. Now consider A and B in \mathcal{F} (hence also $A \cup B \in \mathcal{F}$) with $B \not\subseteq A$. By assumption there is a maximal chain in the interval $[A, A \cup B]$ that is labelled by the unique increasing permutation of $(A \cup B) \setminus A = B \setminus A \subseteq E$. If $x \in E$ is the first label on the chain, we conclude that $x = \min_{\leq_E} (B \setminus A)$ and $A \cup \{x\} \in \mathcal{F}$. Hence (E, \mathcal{F}) is a supersolvable antimatroid. \square

Both “supersolvable” and “join-distributive” are generalizations of the concept “distributive.” However, they are distinct concepts. For instance, the lattice



is supersolvable (it has an S_n EL-labelling) but it is not join-distributive since the lattice itself is an atomic interval that is not boolean. The next example illustrates that a join-distributive lattice need not be supersolvable.

Example 2.14. Consider the set $E = \{a < b < c < d\}$ of four distinct points in the line \mathbb{R} . Recall that the intersections of E with complements of convex sets in \mathbb{R} are the feasible sets of an antimatroid, and the inclusion order on these is a join-distributive lattice. We display this lattice in Fig. 2. It is easy to see, however, that this lattice is not supersolvable. For instance, the intervals $[\emptyset, \{a, b, c\}]$ and $[\emptyset, \{b, c, d\}]$ are chains, with edge-label sequences (a, b, c) and (d, c, b) . It is impossible to order the set E in such a way that both of these sequences are increasing.

At the moment we know relatively little about supersolvable antimatroids in general. This may be an interesting avenue for further study. In the next sections we will describe a natural class of supersolvable antimatroids arising from Coxeter groups.

3. Sorting in a Coxeter group

In the next two sections let (W, S) be an arbitrary Coxeter system and consider a word $\omega \in S^*$ in the generators S . At first we suppose that ω is an arbitrary finite word. However, we will see below (Corollary 4.6) that we lose nothing by taking ω to be reduced. In Section 5 we will allow ω to have infinite length.

We call ω the sorting word and we will use it to “sort” each element of W . This leads to a notion of “ ω -sorted” words and an associated partial order on the group.

3.1. Sorted words

First we set down some notation. Let $\omega = (\omega_1, \omega_2, \dots, \omega_m) \in S^*$ denote the sorting word. We will typically identify ω with the ground set $I(\omega) := \{1, 2, \dots, m\}$. We say that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in S^*$ is a subword of ω (and we write $\alpha \subseteq \omega$) if we have

$$(\alpha_1, \alpha_2, \dots, \alpha_k) = (\omega_{i_1}, \omega_{i_2}, \dots, \omega_{i_k})$$

for some $1 \leq i_1 < i_2 < \dots < i_k \leq m$. In this case, the index set of α is $I(\alpha) := \{i_1, i_2, \dots, i_k\} \subseteq I(\omega)$. Thus $I(\alpha) \subseteq I(\beta) \subseteq I(\omega)$ implies that α is a subword of β ; the converse is not true.

Warning. Because the sorting word ω may contain repeated letters, the index set $I(\alpha)$ of a subword $\alpha \subseteq \omega$ may not be uniquely recoverable from α . Thus we will always identify a subword with the pair $(\alpha, I(\alpha))$. (We may omit mention of the index set $I(\alpha)$ when no confusion will result.) Two subwords $\alpha, \beta \subseteq \omega$ are equal when $I(\alpha) = I(\beta)$. We will write $\alpha \cup \beta$ and $\alpha \cap \beta$ to denote the words corresponding to index sets $I(\alpha) \cup I(\beta)$ and $I(\alpha) \cap I(\beta)$, respectively.

To each word $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k) \in S^*$ we associate the group element

$$\langle \alpha \rangle := \alpha_1 \alpha_2 \dots \alpha_k \in W.$$

Note that the correspondence $\alpha \mapsto \langle \alpha \rangle$ is not injective. We say that α is a reduced word (for $\langle \alpha \rangle$) if there does not exist another word for $\langle \alpha \rangle$ of shorter length. In this case, the group element $\langle \alpha \rangle \in W$ has length k , and we write $\ell(\langle \alpha \rangle) = k$. Notice that $\ell(w) = \ell(w^{-1})$ for all $w \in W$ since we may reverse a reduced word for w to obtain one for w^{-1} .

In this paper we will consider the collection $2^{I(\omega)}$ of subsets of the ground set $I(\omega) = \{1, 2, \dots, m\}$ together with the total lexicographic order \leq_{lex} . Given A and B subsets of $I(\omega)$, we will say that $A \leq_{\text{lex}} B$ if either $A = B$ or the minimum element of $(A \cup B) \setminus (A \cap B)$ is in A .

We can now define sorted words.

Definition 3.1. A subword $\alpha \subseteq \omega$ of the sorting word is called ω -sorted if

1. α is a reduced word,
2. $I(\alpha) = \min_{\leq_{\text{lex}}} \{I(\beta) \subseteq I(\omega) : \beta \text{ is reduced and } \langle \beta \rangle = \langle \alpha \rangle\}$.

That is, α is ω -sorted if it is the lexicographically-least reduced word for $\langle \alpha \rangle$ among subwords of ω .

3.2. The sorting algorithm

Let $W_\omega \subseteq W$ denote the subset of group elements that occur as subwords of the sorting word ω . Each element of W_ω corresponds to a unique ω -sorted word, and we may think of this as a canonical form for the group element.

Table 1
An example of the ω -sorting algorithm.

Step	Reflection	$x \in W$	Descent?	Index set X
1	$s_1 = (12)$	41532	no	{
2	$s_2 = (23)$	41532	yes	{2
3	$s_3 = (34)$	41523	yes	{2, 3
4	$s_4 = (45)$	31524	yes	{2, 3, 4
5	$s_3 = (34)$	31425	no	{2, 3, 4
6	$s_2 = (23)$	31425	yes	{2, 3, 4, 6
7	$s_1 = (12)$	21435	yes	{2, 3, 4, 6, 7
8	$s_2 = (23)$	12435	no	{2, 3, 4, 6, 7
9	$s_3 = (34)$	12435	yes	{2, 3, 4, 6, 7, 9
10	$s_2 = (23)$	12345	no	{2, 3, 4, 6, 7, 9}

Recognizing canonical forms is a fundamental problem. In this section we give an algorithmic characterization of ω -sorted words.

The ω -sorting algorithm. Given a group element $u \in W_\omega$, let $\alpha \subseteq \omega = (\omega_1, \dots, \omega_m)$ be a subword of the sorting word such that $u = \langle \alpha \rangle$. We define another subword $\text{sort}_\omega(\alpha) \subseteq \omega$ as follows. Begin by setting $X := \emptyset$ and $x := u$. For i from 1 to m do

- If $\ell(\omega_i x) = \ell(x) - 1$ then put $X := X \cup \{i\}$ and $x := \omega_i x$.
- If $\ell(\omega_i x) = \ell(x) + 1$ do nothing.

Let $\text{sort}_\omega(\alpha) = \text{sort}_\omega(u)$ be the subword of ω with index set X . This is called the ω -sorted word for $\alpha \in S^*$ and $u \in W$.

That is, we proceed through the entries of ω , checking successively whether $\omega_i \in S$ is a left descent of our group element x . If it is, then we record the index i and replace our group element x by $\omega_i x$. Otherwise, we do nothing. We illustrate this algorithm with an example.

Example 3.2. Consider a Coxeter system (W, S) of type A_{n-1} . That is, let $W = \mathfrak{S}_n$ be the group of permutations of $\{1, 2, \dots, n\}$ with generating set S of adjacent transpositions,

$$S = \{s_i = (i, i + 1): 1 \leq i \leq n - 1\}.$$

We will express a permutation $\sigma \in \mathfrak{S}_n$ using the one-line notation

$$\sigma(1)\sigma(2) \cdots \sigma(n).$$

Notice that s_i is a left descent for $\sigma \in \mathfrak{S}_n$ (that is, $\ell(s_i \sigma) = \ell(\sigma) - 1$) precisely when the symbols i and $i + 1$ are out of order in the word for σ (that is, when $\sigma^{-1}(i + 1) < \sigma^{-1}(i)$).

Now let $n = 5$ and fix the sorting word

$$\omega = (\omega_1, \dots, \omega_{10}) = (s_1, s_2, s_3, s_4, s_3, s_2, s_1, s_2, s_3, s_2).$$

(Incidentally, this is a reduced word for the longest element $w_\circ = 54321 \in \mathfrak{S}_5$ so that W_ω is the full group. This follows from the fact that w_\circ is the unique maximum element under Bruhat order—see Section 4.1.) We compute the ω -sorted word for $\sigma = 41532$ in Table 1. Since the resulting index set is $\{2, 3, 4, 6, 7, 9\}$, we obtain the ω -sorted word

$$\text{sort}_\omega(41532) = \text{sort}_\omega(\sigma) = (s_2, s_3, s_4, s_2, s_1, s_3).$$

One should imagine that the algorithm converts a group element $u \in W_\omega$ to the identity element and that the word $\text{sort}_\omega(u)$ records the steps in this process.

The sorting algorithm has the following properties.

Lemma 3.3. *Let $\alpha \subseteq \omega$ be a subword of the sorting word. We have*

1. $\text{sort}_\omega(\alpha)$ is a reduced word for $\langle \alpha \rangle \in W$.
2. α is ω -sorted if and only if $\text{sort}_\omega(\alpha) = \alpha$.

Proof. 1. Suppose that $\text{sort}_\omega(\alpha) = (\gamma_1, \dots, \gamma_k)$, so the algorithm will terminate with $x = \gamma_k \cdots \gamma_2 \gamma_1 \langle \alpha \rangle$. We also have

$$\ell(x) = \ell(\langle \alpha \rangle) - k,$$

since by construction each multiplication with a generator decreases the length of x by 1. Thus we will be done if we can show that the algorithm terminates with $x = 1 \in W$ —the identity element.

To show this we use induction and the Exchange Property (see Section 4.1 below)—hereafter invoked just as “Exchange.” Suppose that the following statement holds for some index i :

We have completed $i - 1$ steps of the algorithm and we currently have $x = \langle \alpha' \rangle$, where $\alpha' = (\alpha'_1, \dots, \alpha'_\ell)$ is a subword of $(\omega_i, \omega_{i+1}, \dots, \omega_m)$.

Note that this statement is true with $i = 1$ and $\alpha' = \alpha$. On the i th step, we perform the multiplication $\omega_i \langle \alpha' \rangle$. If $\ell(\omega_i \langle \alpha' \rangle) = \ell(\langle \alpha' \rangle) + 1$, then α' is in fact a subword of $(\omega_{i+1}, \dots, \omega_m)$. Thus we do nothing, and the statement remains true. On the other hand, suppose that $\ell(\omega_i \langle \alpha' \rangle) = \ell(\langle \alpha' \rangle) - 1$. In this case, by Exchange there exists $1 \leq j \leq \ell$ such that

$$\omega_i \langle \alpha' \rangle = \omega_i \alpha'_1 \cdots \alpha'_\ell = \alpha'_1 \cdots \hat{\alpha}'_j \cdots \alpha'_\ell.$$

Replacing α' by $(\alpha'_1, \dots, \hat{\alpha}'_j, \dots, \alpha'_\ell)$, the statement remains true. By induction, the statement is true for $i = m$ —thus the algorithm terminates with α' equal to the empty word as desired.

2. First suppose that $\alpha \subseteq \omega$ is not ω -sorted and let $\beta = \text{sort}_\omega(\alpha)$. If $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_\ell)$, let j be the minimum integer such that $\alpha_j \neq \beta_j$. Since $I(\beta) <_{\text{lex}} I(\alpha)$, this implies that $\beta_j = \omega_{j'}$ with $\omega_{j'} \notin \alpha$. Now let us apply the ω -sorting algorithm to α . At the $(j' - 1)$ th step we will have

$$x = \alpha_j \alpha_{j+1} \cdots \alpha_k = \beta_j \beta_{j+1} \cdots \beta_\ell.$$

Next, applying $\omega_{j'}$ on the left we get

$$\ell(\omega_{j'} x) = \ell(\beta_{j+1} \cdots \beta_\ell) = \ell(x) - 1,$$

which implies that $\omega_{j'} \in \text{sort}_\omega(\alpha)$. Since $\omega_{j'} \notin \alpha$, we conclude that $\text{sort}_\omega(\alpha)$ is not equal to α .

Conversely, suppose that $\text{sort}_\omega(\alpha) \neq \alpha$. Since α is a word for $\langle \alpha \rangle$ and $\text{sort}_\omega(\alpha)$ is a reduced word for $\langle \alpha \rangle$ (by part 1), there exists a minimum integer j such that $\omega_j \in \text{sort}_\omega(\alpha)$ and $\omega_j \notin \alpha$. In this case,

$$\text{sort}_\omega(\alpha) = (\alpha \cap (\omega_1, \dots, \omega_{j-1})) \cup (\text{sort}_\omega(\alpha) \cap (\omega_j, \dots, \omega_m))$$

is a reduced word for $\langle \alpha \rangle$ that is strictly lexicographically-less than α . Hence α is not ω -sorted. \square

This result gives us a convenient way to recognize ω -sorted words, which we will use in later proofs.

Corollary 3.4. *A word $\alpha \subseteq \omega = (\omega_1, \dots, \omega_m)$ is ω -sorted if and only if there does not exist $1 \leq j < m$ with $\omega_j \notin \alpha$ such that ω_j is a left descent of the group element $\langle \alpha \cap (\omega_{j+1}, \dots, \omega_m) \rangle \in W$.*

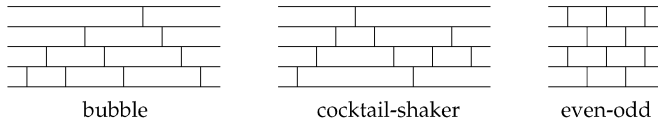


Fig. 3. Some classical sorting algorithms.

3.3. Remark—classical sorting algorithms

We have two justifications for our use of the term “sorting.”

First, reading [13] has defined the notion of “Coxeter-sorting” for elements in a Coxeter group (see Section 6 below). He notes that his Coxeter-sorting algorithm is related to the classical “stack-sorting” algorithm, described by Knuth [9, Exercise 2.2.1.4–5]. Reading’s work is the main motivation behind the current paper.

More generally, Knuth [10, Chapter 8] has described a framework for a wide variety of classical sorting algorithms. He defines a comparator $[i : j]$ as a map that operates on a sequence of numbers (x_1, \dots, x_n) , replacing x_i and x_j respectively by $\min(x_i, x_j)$ and $\max(x_i, x_j)$. A sorting network is a sequence of comparators that will sort any given sequence (x_1, \dots, x_n) , and a primitive sorting network consists entirely of comparators of the form $[i : i + 1]$. Thus, our sorting word ω and ω -sorting algorithm may be thought of as a generalized “primitive sorting network” for the set of “sequences” $W_\omega \subseteq W$.

Furthermore, Knuth notes that one may restrict attention to the *irredundant* primitive sorting networks, which correspond to commutation classes of reduced words for the longest permutation $w_\circ = n(n - 1) \cdots 321$. (In Section 4.2 we will show that the same reduction can be made in general.) Various reduced words for w_\circ then correspond to different classical sorting algorithms. Fig. 3 (adapted from Knuth [10, page 29], which contains an error) shows some reduced words for $w_\circ = 54321$ in \mathfrak{S}_5 corresponding to three classical sorting algorithms. Here, “bubblesort” corresponds to the lexicographically first reduced word. In our Example 3.2, we have performed the “cocktail-shaker sort.”

3.4. The sorting order

To end this section we define a natural partial order on the collection of ω -sorted words, or equivalently on the set of group elements $W_\omega \subseteq W$.

Definition 3.5. Given a Coxeter system (W, S) and a sorting word $\omega \in S^*$ let (P_ω, \leq_ω) denote the set of ω -sorted words together with the subword inclusion order,

$$\alpha \leq_\omega \beta \iff I(\alpha) \subseteq I(\beta).$$

We also define a partial order on W_ω by identifying a group element with its ω -sorted word,

$$u \leq_\omega w \iff \text{sort}_\omega(u) \leq_\omega \text{sort}_\omega(w).$$

In the next section we will see that the sorting order has many remarkable properties and it is closely related to other important orders on the group W .

4. Properties of the sorting order

4.1. Between weak and Bruhat

Let (W, S) be an arbitrary Coxeter system. Recall that $(s_1, \dots, s_k) \in S^*$ is called a reduced word for the group element $w \in W$ when $w = s_1 \cdots s_k$ and there does not exist a word for w of length less than k . In this case $\ell(w) = k$ is the length of the element $w \in W$. There are two classical and important partial orders on the group W , both based on the combinatorics of reduced words (this topic is covered thoroughly in [3]).

Weak order. Given $u, w \in W$, we write $u \leq_R w$ if u occurs as a “prefix” of w —that is, if there exists a reduced word $w = s_1 s_2 \cdots s_\ell$ for w and an integer $1 \leq k \leq \ell$ such that $u = s_1 s_2 \cdots s_k$ is a reduced word for u . This is called the weak order on W .

We have actually defined the “right” weak order. There is a corresponding “left” weak order and the two are isomorphic via the map $w \mapsto w^{-1}$ which exchanges prefixes and suffixes. The Hasse diagram of the weak order is just the right Cayley graph of W with respect to the generating set S . In general the weak order is graded by the length function $\ell : W \rightarrow \mathbb{Z}$ and it is a meet-semilattice—it possesses meets, but not joins. However, when W is finite there exists a unique element $w_o \in W$ of maximum length—called the “longest element”—which satisfies $w \leq_R w_o$ for all $w \in W$. Hence the weak order on a finite Coxeter group is a lattice.

Bruhat order. Given $u, w \in W$, we write $u \leq_B w$ if u occurs as a “subword” of w —that is, if there exists a reduced word $w = s_1 s_2 \cdots s_\ell$ for w and integers $1 \leq i_1 < \cdots < i_k \leq \ell$ such that $u = s_{i_1} \cdots s_{i_k}$ is a reduced word for u . This is called the Bruhat order on W .

(Moreover, if $u \leq_B w$, it turns out that u occurs as a subword of **any** reduced word for w .) It is easy to see from the definition that the Bruhat order is also a graded poset, ranked by the length function. Since a “prefix” is a “subword” we note that $u \leq_R w$ implies $u \leq_B w$ for all $u, w \in W$ —that is, Bruhat order is a poset extension of the weak order. However Bruhat order is neither a meet- nor a join-semilattice. The Bruhat order arises in applications as the inclusion order on closures of Schubert cells in the generalized flag variety corresponding to W .

Since the Bruhat order is an extension of the weak order with the same rank function, one may obtain the Hasse diagram of Bruhat order from the hasse diagram of weak order by adding some extra edges (cover relations). Let $T = \{wsw^{-1} : w \in W, s \in S\}$ denote the generating set of reflections. By definition, the cover relations in weak order have the form $u < w$ where $w = us$ for some $s \in S$. It is also true—but not obvious—that the covers in Bruhat order have the form $u < w$ where $w = ut$ for some reflection $t \in T$ such that $\ell(w) = \ell(u) + 1$. We will find that the sorting orders are intermediate between weak and Bruhat order since they include some but not all of these extra covers.

A good first example is the dihedral group.

Example 4.1. Let $W = I_2(m)$ be the dihedral group of order $2m$ with Coxeter generators $S = \{s_1, s_2\}$. In this case the longest element $w_o \in W$ is of length m , and it has exactly two reduced words: $\omega_1 := (s_1, s_2, s_1, s_2, \dots)$ and $\omega_2 := (s_2, s_1, s_2, s_1, \dots)$. The following is a convenient notation for ω_1 - and ω_2 -sorted words: For example, when $m = 4$ and $\omega_1 = (s_1, s_2, s_1, s_2)$, the subword $\alpha = (s_2, s_1, s_2)$ is ω_1 -sorted with index set $I(\alpha) = \{2, 3, 4\} \subseteq I(\omega_1) = \{1, 2, 3, 4\}$. We encode both the word and the index set simultaneously with the string 0212, where the zeroes are placeholders. Fig. 4 displays the Hasse diagrams of the weak order, Bruhat order and both sorting orders on the group $I_2(4)$ (a.k.a. B_2).

Notice that the sorting orders on the full group $I_2(m)$ occur “between” the weak and Bruhat orders. To prove that this property holds in general, we will need the following well-known lemma regarding the combinatorics of reduced words (see [3, Theorem 1.4.3] for a proof).

The Exchange Property. Let $(s_1, s_2, \dots, s_k) \in S^*$ be a reduced word for $w \in W$ and suppose that $\ell(tw) < \ell(w)$ for some $t \in T$. Then there exists $1 \leq i \leq k$ such that

$$tw = s_1 \cdots \hat{s}_i \cdots s_k.$$

Here the notation \hat{s}_i indicates that the symbol s_i has been deleted from the word. Note that this new word for tw is not necessarily reduced since we might have $\ell(tw) < \ell(w) - 1$. A version of Exchange also holds when we multiply on the right by a reflection $t \in T$ —we see this by applying Exchange to w^{-1} and noting that $\ell(tw^{-1}) = \ell((wt)^{-1}) = \ell(wt)$.

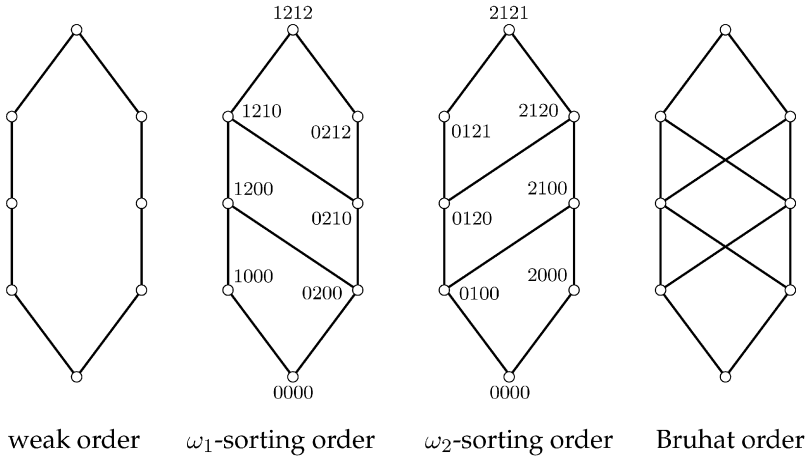


Fig. 4. Comparison of weak, Bruhat and sorting orders.

Theorem 4.2. Given a Coxeter system (W, S) and a sorting word $\omega \in S^*$, the ω -sorting order extends the weak order on W_ω and is extended by the Bruhat order on W_ω . That is, for all $u, w \in W_\omega$ we have

$$u \leq_R w \implies u \leq_\omega w \implies u \leq_B w.$$

Proof. The fact that Bruhat order extends ω -sorting order is immediate since the ω -sorting order is a special case of subword inclusion.

Now suppose that $u \leq_R w$ —that is, there exists a reduced word $\beta = (s_1, \dots, s_\ell)$ for w and an integer $1 \leq k \leq \ell$ such that $\alpha = (s_1, \dots, s_k)$ is a reduced word for u . To demonstrate that $u \leq_\omega w$ we must show that $\text{sort}_\omega(\alpha)$ is a subword of $\text{sort}_\omega(\beta)$. We do this by induction.

Let the sorting word be $\omega = (\omega_1, \omega_2, \dots, \omega_m) \in S^*$ and perform the ω -sorting algorithm on α and β . In the first step we multiply on the left by $\omega_1 \in S$. There are three cases:

1. If ω_1 is a left descent of u (that is, if $\ell(\omega_1 u) = \ell(u) - 1$), then, by the Exchange Property, there exists an integer $1 \leq i \leq k$ such that $\alpha' := (s_1, \dots, \hat{s}_i, \dots, s_k)$ is a reduced word for su . In this case we see that s is a left descent of w and $\beta' := (s_1, \dots, \hat{s}_i, \dots, s_\ell)$ is a reduced word for sw . We find that ω_1 is in both $\text{sort}_\omega(\alpha)$ and $\text{sort}_\omega(\beta)$.
2. If ω_1 is a left descent of w , but not of u , then by Exchange there exists an integer $1 \leq i \leq \ell$ such that $\beta' := (s_1, \dots, \hat{s}_i, \dots, s_\ell)$ is a reduced word for sw . Since ω_1 is not a left descent of u we must have $k < i$, hence we set $\alpha' := \alpha$. In this case ω_1 is in $\text{sort}_\omega(\beta)$ but not in $\text{sort}_\omega(\alpha)$.
3. Finally, if ω_1 is not a left descent of w then we set $\alpha' := \alpha$ and $\beta' := \beta$. This time ω_1 is in neither $\text{sort}_\omega(\alpha)$ nor $\text{sort}_\omega(\beta)$.

In any case we find that $\omega_1 \in \text{sort}_\omega(\beta) \implies \omega_1 \in \text{sort}_\omega(\alpha)$. Then, in the next step of the algorithm, we apply ω_2 on the left to α' and β' . Since α' is a prefix of β' , we may use the same reasoning as above to find that $\omega_2 \in \text{sort}_\omega(\beta) \implies \omega_2 \in \text{sort}_\omega(\alpha)$. Continuing in this way we conclude that $\text{sort}_\omega(\alpha)$ is a subword of $\text{sort}_\omega(\beta)$. \square

Note that the ω -sorting algorithm is defined in terms of *left* multiplication by generators; whereas the ω -sorting order is comparable to the *right* weak order.

Example 4.3. For a more detailed example, we consider the symmetric group $W = \mathfrak{S}_4$ with the generating set of adjacent transpositions,

$$S = \{s_1 = (12), s_2 = (23), s_3 = (34)\}.$$

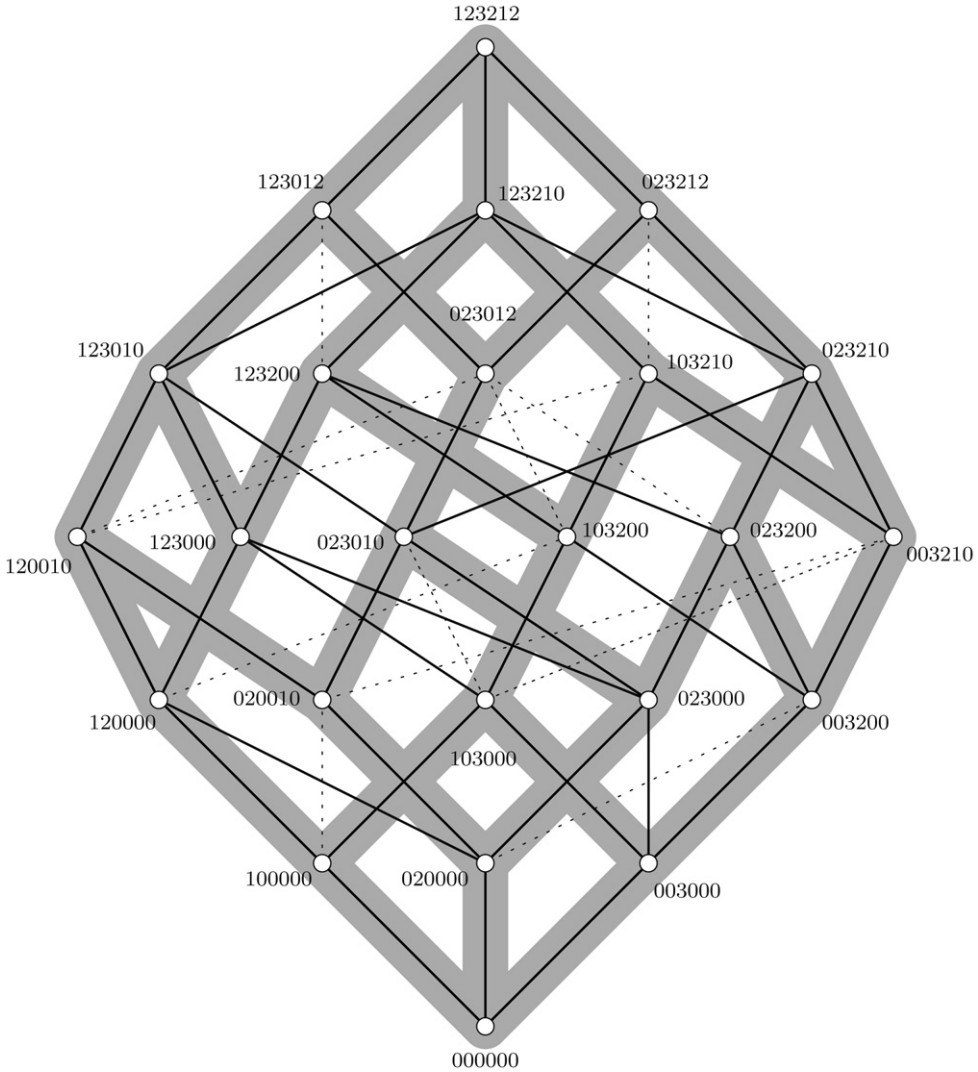


Fig. 5. The weak order, 123212-sorting order and Bruhat order on \mathfrak{S}_4 .

Let the sorting word be $\omega = (s_1, s_2, s_3, s_2, s_1, s_2)$ —another example of the “cocktail-shaker.” Since ω is a reduced word for the longest element $w_o \in W$, we again have $W_\omega = W$. We will use the same notation as in Example 4.1 to denote ω -sorted words. For example, the string 003210 denotes the ω -sorted word (s_3, s_2, s_1) with index set $\{3, 4, 5\} \subseteq I(\omega) = \{1, 2, 3, 4, 5, 6\}$. Fig. 5 shows the nested Hasse diagrams of weak order, ω -sorting order and Bruhat order. Weak order is shaded in grey, the solid black lines are the sorting order and the solid plus dotted lines together give Bruhat order.

4.2. A supersolvable antimatroid

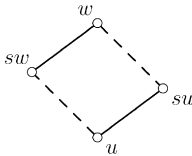
In this section we will show that the collection of ω -sorted words forms a supersolvable antimatroid, and hence that the ω -sorting order is a supersolvable join-distributive lattice. At the end of the section we will conclude that the sorting orders are parametrized by commutation classes of reduced words, instead of arbitrary single words.

In addition to the Exchange Property we will need the following standard properties of reduced words.

The Gluing Property. Let $s \in S$ be a generator and consider $\alpha, \beta \in S^*$. If the words $\alpha\beta, \alpha s$ and $s\beta$ are all reduced then so is $\alpha s\beta$.

Proof. In this case [3, Lemma 2.2.10] implies that $\ell(\langle\alpha\beta\rangle) < \ell(\langle\alpha s\beta\rangle)$. Since $\alpha\beta$ is reduced it follows that $\ell(\langle\alpha s\beta\rangle) = \ell(\langle\alpha\beta\rangle) + 1$ and that $\alpha s\beta$ is reduced. \square

The Lifting Property. Given $u \leq_B w$ and $s \in S$, suppose that $\ell(sw) < \ell(w)$ and $\ell(u) < \ell(su)$ —hence, by exchange we have $sw \leq_B w$ and $u \leq_B su$. It follows that $su \leq_B w$ and $u \leq_B sw$. See the diagram below.



Proof. See [3, Proposition 2.2.7] for a proof. \square

As with the Exchange Property, a version of the Lifting Property also holds when we multiply on the right by a simple reflection. To see this we replace u and w by u^{-1} and w^{-1} , respectively, and note that inversion $w \mapsto w^{-1}$ is an automorphism of Bruhat order.

Theorem 4.4. Given an arbitrary Coxeter system (W, S) and a finite sorting word $\omega = (\omega_1, \dots, \omega_m) \in S^*$, the collection of index sets of ω -sorted words,

$$\mathcal{F} := \{I(\alpha) \subseteq I(\omega) : \text{sort}_\omega(\alpha) = \alpha\},$$

is a supersolvable antimatroid with respect to the natural order on the ground set $E := I(\omega) = \{1, \dots, m\}$. (See Definition 2.11.)

Proof. In the proof we will abbreviate “lexicographic” as “lex.”

To show that (E, \mathcal{F}) is a supersolvable antimatroid, we must verify property (2.12). Consider ω -sorted words α and β with $\beta \not\leq \alpha$, such that $i = \min(I(\beta) \setminus I(\alpha))$. We must show that $\alpha \cup \omega_i$ is an ω -sorted word—that is, we must show that $\alpha \cup \omega_i$ is reduced and that it is the lex-least reduced word for $\langle\alpha \cup \omega_i\rangle \in W$ among subwords of ω .

In general, ω_i breaks the word α into

$$\alpha' := \alpha \cap (\omega_1, \dots, \omega_{i-1}) \quad \text{and} \quad \alpha'' := \alpha \cap (\omega_{i+1}, \dots, \omega_m),$$

either of which may be empty. Now observe that $\omega_i \alpha''$ is a reduced word. Otherwise, by Exchange we obtain a reduced word $\omega_i \hat{\alpha}''$ for $\langle\alpha''\rangle$ that is lex-less than α'' . Consequently, $\alpha' \omega_i \hat{\alpha}''$ is a reduced word for $\langle\alpha\rangle$ that is lex-less than α , a contradiction.

Next, we show that $\alpha' \omega_i$ is reduced. Set

$$\beta' := \beta \cap (\omega_1, \dots, \omega_{i-1}) \quad \text{and} \quad \beta'' := \beta \cap (\omega_{i+1}, \dots, \omega_m)$$

and note that β' is a subword of α' because i is the first place in which α and β differ. Furthermore, note that $\beta' \omega_i$ is a reduced word since it is a prefix of β . If $\alpha' \omega_i$ is not reduced then the facts $\langle\beta'\rangle \leq_B \langle\alpha'\rangle$, $\ell(\langle\alpha'\rangle \omega_i) < \ell(\langle\alpha'\rangle)$ and $\ell(\langle\beta'\rangle) < \ell(\langle\beta'\rangle \omega_i)$, together with the Lifting Property, imply that $\langle\beta'\rangle \omega_i \leq_B \langle\alpha'\rangle$. That is, there exists a reduced word for $\langle\beta'\rangle \omega_i = \langle\beta' \omega_i\rangle$ that is a subword of α' . Let φ be the lex-least such word. We claim that φ is lex-less than $\beta' \omega_i$. Indeed, suppose the opposite—let $i' < i$ be the first position in which φ and $\beta' \omega_i$ differ and suppose that $\omega_{i'} \in \beta' \setminus \varphi$. Set

$$\varphi' := \varphi \cap (\omega_1, \dots, \omega_{i-1}) \quad \text{and} \quad \varphi'' := \varphi \cap (\omega_{i+1}, \dots, \omega_{i-1}),$$

so that $\varphi = \varphi' \varphi''$. Note that the suffix φ'' cannot be empty since φ and $\beta' \omega_i$ have the same number of letters. However if φ'' is not empty then by Exchange there exists a subword $\hat{\varphi}'' \subseteq \varphi''$ such that $\omega_i \hat{\varphi}''$ is a reduced word for $\langle \varphi'' \rangle$ and hence $\varphi' \omega_i \hat{\varphi}'' \subseteq \alpha'$ is a reduced word for $\langle \varphi \rangle$ lex-less than φ , a contradiction. We conclude that φ is lex-less than $\beta' \omega_i$ and hence $\varphi \beta''$ is a reduced word for $\langle \beta \rangle$ lex-less than β . This contradiction proves that the word $\alpha' \omega_i$ is reduced.

Since the words $\alpha' \alpha'' = \alpha$, $\alpha' \omega_i$ and $\omega_i \alpha''$ are all reduced, the Gluing Property implies that $\alpha' \omega_i \alpha'' = \alpha \cup \omega_i$ is also reduced.

To complete the proof, we must show that $\alpha \cup \omega_i$ is lex-least among reduced words for $\langle \alpha \cup \omega_i \rangle$. Suppose not. Then by Corollary 3.4 there exists an integer $1 \leq j < m$ such that ω_j is not in $\alpha \cup \omega_i$ and such that $\gamma := (\alpha \cup \omega_i) \cap (\omega_{j+1}, \dots, \omega_m)$ has left descent ω_j —that is, $\ell(\omega_j \langle \gamma \rangle) < \ell(\langle \gamma \rangle)$. But $\delta := \alpha \cap (\omega_{j+1}, \dots, \omega_m)$ is a subword of γ and we have $\ell(\langle \delta \rangle) < \ell(\omega_j \langle \delta \rangle)$. Otherwise, by Exchange we obtain a reduced word δ' for $\langle \delta \rangle$ that is lex-less than δ , and hence $(\alpha \cap (\omega_1, \dots, \omega_{j-1})) \delta'$ is a reduced word for $\langle \alpha \rangle$ lex-less than α , a contradiction. Since we have $\langle \delta \rangle \leq_B \langle \gamma \rangle$, $\ell(\omega_j \langle \gamma \rangle) < \ell(\langle \gamma \rangle)$ and $\ell(\langle \delta \rangle) < \ell(\omega_i \langle \delta \rangle)$, the Lifting Property tells us that $\omega_j \langle \delta \rangle \leq_B \langle \gamma \rangle$. That is, there exists a subword $\gamma' \subseteq \gamma$ that is a reduced word for $\omega_j \langle \delta \rangle$. In this case, $(\alpha \cap (\omega_1, \dots, \omega_{j-1})) \omega_j \gamma'$ is a reduced word for $\langle \alpha \rangle$ that is lex-less than α . This final contradiction proves that $\alpha \cup \omega_i$ is lex-least among reduced words for $\langle \alpha \cup \omega_i \rangle$, as we wished to show. \square

Following this, Theorem 2.13 implies our main result. Recall that a join-distributive lattice is graded by the cardinality of feasible sets in the corresponding antimatroid.

Corollary 4.5. *The ω -sorting order on $W_\omega \subseteq W$ is a supersolvable join-distributive lattice, graded by the usual Coxeter length function $\ell : W \rightarrow \mathbb{Z}$.*

We also conclude that the sorting word might as well be reduced.

Corollary 4.6. *We lose nothing if we consider only reduced sorting words—that is, given a sorting word $\omega \in S^*$ there exists a reduced word $\omega' \subseteq \omega$ for which ω' -sorting is the same as ω -sorting.*

Proof. Given an arbitrary word $\omega \in S^*$, let $\omega' \subseteq \omega$ denote the union of all ω -sorted subwords of ω . Since the collection of feasible sets of an antimatroid is closed under taking unions (property (2.3)), we conclude that ω' is an ω -sorted word and hence it is reduced. Since the elements of $\omega \setminus \omega'$ will play no role in the antimatroid, nor in the sorting order, we may replace ω by the reduced word ω' without loss. \square

Furthermore, if $\omega' \subseteq \omega$ denotes the union of all ω -sorted words, we note that the collection of elements $W_\omega \subseteq W$ that occur as subwords of ω coincides with the lower interval $[1, \langle \omega' \rangle]_B$ in Bruhat order. Indeed, any element $w \in W_\omega$ has a reduced ω -sorted word, which by definition is a subword of the reduced word ω' . Thus we may think of the sorting orders as partial orders on a lower interval $[1, w]_B$ in Bruhat order, parametrized by reduced word for w .

Finally we note that the ω -sorting order depends only on the “commutation class” of the word ω . We say that two words are in the same commutation class if one can be obtained from the other by repeatedly exchanging adjacent commuting generators.

Lemma 4.7. *Let $\omega, \zeta \in S^*$ be two words that differ by the exchange of commuting generators in positions i and $i + 1$. Then we have*

$$u \leq_\omega w \iff u \leq_\zeta w$$

for all $u, w \in W_\omega = W_\zeta \subseteq W$.

Proof. Consider $\omega = (\omega_1, \dots, \omega_m)$ and $\zeta = (\zeta_1, \dots, \zeta_m)$, where $\omega_i = \zeta_{i+1}$, $\omega_{i+1} = \zeta_i$ and $\omega_i \omega_{i+1} = \omega_{i+1} \omega_i$. Note that the transposition $(i, i + 1)$ acts on subsets of $\{1, \dots, m\}$ by switching the indices i and $i + 1$. If α is a subword of ω with index set $I(\alpha) \subseteq I(\omega) = \{1, \dots, m\}$, let α' denote the subword of ζ with index set $(i, i + 1) \cdot I(\alpha)$. We claim that the involution $\alpha \mapsto \alpha'$ (which satisfies $\langle \alpha \rangle = \langle \alpha' \rangle$) is a bijection between ω -sorted and ζ -sorted words, from which the result follows.

Indeed, suppose α is ω -sorted. We wish to show that α' is ζ -sorted. (The proof that $\alpha' \mapsto \alpha$ preserves sortedness will be the same.) Since we have merely exchanged commuting generators, α' is reduced. We must show that α' is the lex-least reduced word for $\langle \alpha' \rangle$ among subwords of ζ . Suppose not. Then by Corollary 3.4 there exists $1 \leq j < m$ such that $\zeta_j \notin \alpha'$ and ζ_j is a left descent for $\alpha' \cap (\zeta_{j+1}, \dots, \zeta_m)$. If both or neither of ζ_i, ζ_{i+1} occur in α' , or if $j \notin \{i, i + 1\}$, we find that $\omega_j \notin \alpha$ and $\omega_j = \zeta_j$ is a left descent of

$$\langle \alpha \cap (\omega_{j+1}, \dots, \omega_m) \rangle = \langle \alpha' \cap (\zeta_{j+1}, \dots, \zeta_m) \rangle,$$

which implies that α is not lex-least, a contradiction. Otherwise, exactly one of ζ_i, ζ_{i+1} occurs in α —without loss of generality, say $\zeta_i \in \alpha'$ —and we have $j = i + 1$. In this case, $\omega_i \notin \alpha$ and $\omega_i = \zeta_{i+1}$ is a left descent of

$$\langle \alpha \cap (\omega_{i+2}, \dots, \omega_m) \rangle = \langle \alpha' \cap (\zeta_{i+2}, \dots, \zeta_m) \rangle.$$

Since ω_i and ω_{i+1} commute, ω_i is also a left descent of

$$\omega_{i+1} \langle \alpha \cap (\omega_{i+2}, \dots, \omega_m) \rangle = \langle \alpha \cap (\omega_{i+1}, \dots, \omega_m) \rangle,$$

which implies that α is not lex-least, again a contradiction. \square

4.3. Remark—a maximal lattice

Recall that Bruhat order is obtained from weak order by adding the extra cover relations of the form $u < w$ where $w = ut$ for some non-simple reflection $t \in T \setminus S$ such that $\ell(w) = \ell(u) + 1$. In general, let (P, \leq) be a graded poset. We say that another graded poset (P, \leq') is a graded extension of (P, \leq) if $x \leq y$ implies $x \leq' y$ for all $x, y \in P$ and if the rank function is the same for both posets. In this case we also say that (P, \leq) is a graded contraction of (P, \leq') . We are interested in the collection of graded extensions between the weak and Bruhat orders and the role that the sorting orders play among these.

Our main observation³ is the following.

Definition 4.8. Let (P, \leq) be a finite graded lattice with rank function $\text{rk} : P \rightarrow \mathbb{Z}$. We say that P is maximal if the addition of any finite collection of cover relations of the form $x < y$ with $\text{rk}(y) = \text{rk}(x) + 1$ yields a nonlattice.

It turns out that all supersolvable join-distributive lattices—hence, in particular, all distributive lattices—are maximal. The converse of this theorem is not true.

Theorem 4.9. Let (P, \leq) be a supersolvable join-distributive lattice. Then P is maximal.

Proof. By Theorem 2.13, P is the lattice of feasible sets of a supersolvable antimatroid (E, \mathcal{F}) . Consider a collection $\{(A_i, B_i) \in \mathcal{F}^2\}$ of pairs of feasible sets such that $|B_i| = |A_i| + 1$ and $A_i \not\subseteq B_i$ for all i , and choose j such that the cardinality of A_j —and hence B_j —is minimal.

Let P' denote the poset obtained from P by adding the cover relations $A_i < B_i$ for all i . We claim that P' is not a lattice. Indeed, let $C_j := A_j \cup \{x_j\}$ where $x_j = \min(B_j \setminus A_j)$. By property (2.12) we have

³ Thanks to Hugh Thomas for suggesting that our original observation for one cover could be generalized to any number of covers.

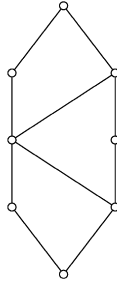


Fig. 6. A maximal lattice that is not a sorting order.

$C_j \in \mathcal{F}$. Note that the meet of C_j and B_j in P is given by the union of all feasible sets contained in $C_j \cap B_j$:

$$C_j \wedge B_j = \bigcup_{\substack{X \in \mathcal{F} \\ X \subseteq C_j \cap B_j}} X.$$

Since the prefix of every feasible set is feasible, so is the prefix of B_j ending in x_j . By construction this prefix is also contained in C_j , hence $x_j \in C_j \wedge B_j$. Thus A_j and $C_j \wedge B_j$ are two lower bounds for C_j and B_j in P' that are incomparable in P . Since the cardinality of A_j was chosen to be minimal, A_j and $C_j \wedge B_j$ are also incomparable in P' . We conclude that P' is not a lattice. \square

In particular, the ω -sorting order is a maximal lattice with respect to the addition of any collection of Bruhat cover relations of the form $u < ut$.

To extend this result, one might try to classify all maximal lattices among the graded extensions between weak and Bruhat order—or more generally among all graded contractions of Bruhat order. Note that the sorting orders do not provide the complete solution to this problem, since the lattice in Fig. 6 is maximal between the weak and Bruhat orders on the dihedral group $I_2(4)$ (a.k.a. B_2), but it is not a sorting order in our sense.

Our study of the general properties of ω -sorting order is now complete. In the remaining sections we consider two special cases—those of *infinite* sorting words and *cyclic* sorting words.

5. Infinite sorting orders

Thus far we have dealt exclusively with finite sorting words because some of the structures in Section 2 make sense only in the finite case. However, most of our results in this paper still hold when ω is a semi-infinite word,

$$\omega = (\omega_1, \omega_2, \omega_3, \dots).$$

For example, the characterization of ω -sorted words via the sorting algorithm and the definition of ω -sorting order remain the same.

If there are only finitely many group elements that occur as subwords of ω (which happens, for instance, when W is a finite group), then we may restrict ω to the (finite) union of all ω -sorted words and everything in the paper goes through as before. Hence, let us assume that W is infinite and that infinitely many group elements occur as subwords of the sorting word ω .

In this case, Theorem 4.2 still holds—the ω -sorting order extends the weak order and is extended by Bruhat order. Theorem 4.4 also goes through as before—the collection of ω -sorted words forms a supersolvable antimatroid (E, \mathcal{F}) on an infinite ordered ground set E . Although Korte, Lovász and

Schrader [11] did not consider antimatroids with infinite ground sets, the conclusions of Lemma 2.1 still hold in this case.⁴ Thus, we have the following.

Definition 5.1. Let (P, \leq) be a locally-finite lattice (all intervals are finite). We say that P is join-distributive if every atomic interval in P is boolean.⁵ We say that P is supersolvable if every interval in P is supersolvable in the usual sense.

Theorem 5.2. Let (W, S) be an infinite Coxeter system and let ω be a word in which infinitely many group elements occur as subwords. Then the ω -sorting order (P_ω, \leq_ω) is an infinite lattice that is supersolvable and join-distributive.

Proof. The poset P_ω contains joins because the collection of feasible sets \mathcal{F} is closed under taking unions (property (2.3)). Then since P_ω has a minimum element (the identity $1_W \in W$), it also contains meets. The other properties follow as before. \square

This conclusion is remarkable because the weak order on an infinite Coxeter group is not a lattice—while it possesses meets, it does not possess joins—and Bruhat order is not a lattice even in the finite case. It is thus interesting to have a new source of lattice structures on the elements of an infinite Coxeter group. Indeed, we know of no other source.

It also remains true that the sorting order only depends on the commutation class of the sorting word and we may still assume that the sorting word is reduced in the following sense.

Definition 5.3. We say that an infinite word ω in the generators S is reduced if every prefix of ω is reduced in the usual sense.

Indeed, we may restrict ω to the union $\omega' \subseteq \omega$ of all ω -sorted words. Any prefix α of ω' is then contained in the union of finitely many ω -sorted words, which is reduced. Since α is the prefix of a reduced word, it is reduced.

6. Cyclic sorting

Finally, we discuss an important special case of ω -sorting which has been the motivation for our work. Let (W, S) be an arbitrary Coxeter system with generators $S = \{s_1, s_2, \dots, s_n\}$. Any word of the form

$$c = (s_{\sigma(1)}, s_{\sigma(2)}, \dots, s_{\sigma(n)}),$$

where σ is a permutation of $\{1, \dots, n\}$, is called a Coxeter word; the corresponding group element $\langle c \rangle \in W$ is a Coxeter element. We say that a cyclic word is any semi-infinite word of the form

$$c^\infty := ccc \dots,$$

where c is a Coxeter word.⁶ In this case Reading was the first to consider the c^∞ -sorting algorithm—which he called “ c -sorting”—for elements of the group W (see [13,14]). However he did not consider the structure of the collection of sorted words nor the corresponding partial order.

His main interest was the collection of so-called “ c -sortable” elements. Because of the cyclic nature of c^∞ , each c^∞ -sorted word α naturally splits into a sequence of subwords $\alpha_{(1)}\alpha_{(2)}\alpha_{(3)} \dots$, where $\alpha_{(i)}$ is the intersection of α with the index set $\{(i-1)n+1, (i-1)n+2, \dots, in\}$. Given a word $\beta \in S^*$, let $\tilde{\beta} \subseteq S$ denote its underlying set of letters.

⁴ Infinite antimatroids and convex geometries are considered in [1].

⁵ This terminology is not entirely standard; see [1].

⁶ In the case that every irreducible component of W is infinite, Speyer [15] has recently shown that cyclic words are reduced in the sense of Definition 5.3.

Definition 6.1. Given a group element $w \in W$, let α denote its c^∞ -sorted word. We say that the element w is c -sortable if we have a descending chain

$$\tilde{\alpha}_{(1)} \supseteq \tilde{\alpha}_{(2)} \supseteq \tilde{\alpha}_{(3)} \supseteq \dots$$

of subsets of S .

Reading introduced this notion because the c -sortable elements are precisely the elements of his c -Cambrian lattice [14, Theorem 1.1]. Probably their most remarkable property is contained in the following theorem.

Theorem 6.2. (See [13, Theorem 9.1].) Let (W, S) be a finite Coxeter system with $|S| = n$. Given any Coxeter word c , the number of c -sortable elements in W is equal to

$$\text{Cat}(W) = \prod_{i=1}^n \frac{h + d_i}{d_i},$$

where h is the Coxeter number (the order of a Coxeter element) and $\{d_1, d_2, \dots, d_n\}$ is the multiset of degrees of fundamental invariants for (W, S) .

This “generalized Catalan number” $\text{Cat}(W)$ has played a central role in much recent work (for references see [2]), and the c -sortable elements provided Reading with a bridge between two important classes of “Catalan objects”: the clusters and the noncrossing partitions.

It is natural to ask what special properties the sorting order has in the case of a cyclic sorting word c^∞ . The following theorem provides a partial answer.

Theorem 6.3. Let c^∞ be a cyclic sorting word for an arbitrary Coxeter system (W, S) . The c^∞ -sorting order restricted to c -sortable elements is a join-distributive lattice.

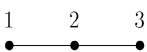
Proof. The proof follows from two observations. First note that the c -sortable elements form a join-sublattice of the full c^∞ -sorting order. This is because the defining property of c -sortability is preserved under taking unions of words.

Second, let u and w be c -sortable group elements with c^∞ -sorted words α and β , respectively, and suppose that $u < w$ is a cover in the sorting order on c -sortable elements. We claim that this is also a cover in the full c^∞ -sorting order. Suppose not, so that $\ell(w) > \ell(u) + 1$. If $x = \min(\beta \setminus \alpha)$, then $(\alpha \cup x)$ is another c -sortable group element strictly between u and w , contradicting the fact that $u < w$ is a cover.

Finally, it is easy to see that a join-sublattice of a join-distributive lattice that preserves covers is also a join-distributive lattice. \square

In a forthcoming paper we will show that the c^∞ -sorting and Bruhat orders coincide on c -sortable elements, and moreover that this order is supersolvable. For now we present an example.

Example 6.4. Consider the Coxeter system (W, S) of type A_3 with Coxeter diagram:



Among the six possible Coxeter words c , there are just two possibilities for the isomorphism type of the lattice of c -sortable elements under c^∞ -sorting order. These are displayed in Fig. 7 with the corresponding Coxeter words. Notice that one of these is the well-known lattice of order ideals of the root poset. This phenomenon, unfortunately, does not persist for all types.

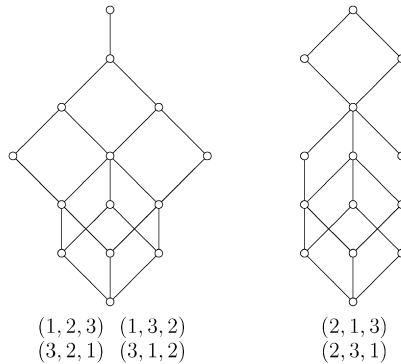


Fig. 7. The two possible lattices of sortable elements in type A_3 .

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