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On the geometry of wave solutions of a delayed reaction–diffusion equation

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The aim of this paper is to study the existence and the geometry of positive bounded wave solutions to a non-local delayed reaction–diffusion equation of the monostable type.

1. Introduction and main results

This paper concerns positive wave solutions of the non-local delayed reaction–diffusion equation

$$u_t(t,x) = u_{xx}(t,x) - u(t,x) + \int_{\mathbb{R}} K(x-y)g\big(u(t-h,y)\big)\,dy, \quad u \geqslant 0, \tag{1}$$

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which is widely used in applications, see e.g. [4,7,10–12,16,24,30,33]. In part, our research was inspired by several problems raised in [12,17,21,27]. We suppose that Eq. (1) has exactly two equilibria $u_1 \equiv 0$, $u_2 \equiv \kappa > 0$ and

$$K(\cdot) \geqslant 0, \quad \int_{\mathbb{R}} K(s)e^{\lambda s}\,ds \text{ is finite for all } \lambda \in \mathbb{R}, \quad \text{and} \quad \int_{\mathbb{R}} K(s)\,ds = 1. \tag{2}$$

Note that the usual restriction $K(s) = K(-s)$, $s \in \mathbb{R}$, is not required here. In a biological context, u is the size of an adult population, so we will consider only non-negative solutions of Eq. (1). The nonlinear g is called *the birth function*, it is often assumed to satisfy the following hypothesis

(H) $g \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\mathbb{R}_+ := [0, +\infty)$, has only one local extremum at $s = s_M$ (maximum) and $g(s) > 0$ if $s > 0$. Next, 0 and $\kappa > 0$ are the only two solutions of $g(s) = s$, and g is differentiable at $s = 0$, with $g'(0) > 1$.

For example, this is the case in the Nicholson's blowflies model [7,12,16,21,24] where $g(s) = pse^{-s}$. See also Section 1.6 below.

Let us fix some terminology. Following [9], we call bounded positive classical solutions $u(x,t) = \phi(x+ct)$ satisfying $\phi(-\infty) = 0$ semi-wavefronts. We say that the semi-wavefront $u(x,t) = \phi(x+ct)$ is a wavefront [is a pulse], if the profile function ϕ satisfies $\phi(+\infty) = \kappa$ [respectively, satisfies $\phi(+\infty) = 0$]. Wavefronts constitute the most studied subclass of semi-wavefronts. Asymptotically periodic semi-wavefronts represent another subclass, see [31]. Some of our results are proved for semi-wavefronts, and some of them, for wavefronts. For example, the setting of semi-wavefronts is more convenient to work with the problem of the minimal speed of propagation, cf. [30, Section 3].

In Sections 1.1–1.5 below, we present our main results. Their proofs and some additional comments can be found in Sections 2–9.

1.1. Two critical speeds and non-existence of pulse waves

In this subsection, we consider more general equation

$$u_t = u_{xx} - qu + F(u, \mathcal{K}_1 u, \ldots, \mathcal{K}_m u), \tag{3}$$

where $F : \mathbb{R}_+^{m+1} \to \mathbb{R}_+$ is a continuous function and

$$(\mathcal{K}_j u)(t,x) := \int_{\mathbb{R}} K_j(x-y) f_j\big(u(t-h,y)\big)\,dy.$$

This equation includes (1) as a particular case (as well as Eqs. (7), (8) considered below). We assume that each kernel K_j satisfies condition (2) and the continuous non-negative functions F, f_j are differentiable at the origin. Set

$$p := \sum_{j=1}^{m} F_{s_j}(0) f_j'(0), \qquad K(s) := p^{-1} \sum_{j=1}^{m} F_{s_j}(0) f_j'(0) K_j(s). \tag{4}$$

Everywhere in this subsection, we assume additionally that $p > q > 0$ and $F(0) = f_j(0) = F_{s_0}(0) = 0$; and $F_{s_j}(0), f_j'(0) \geqslant 0$ for all $j = 1, \ldots, m$. As a consequence, K satisfies (2). Next, consider

$$\psi(z,\epsilon) = \epsilon z^2 - z - q + p\exp(-zh) \int_{\mathbb{R}} K(s)\exp(-\sqrt{\epsilon} zs)\,ds, \tag{5}$$

and let $\epsilon_i = \epsilon_i(h, p, q) > 0, i = 0, 1$, be as in Lemma 20 of Appendix A. Set $c_* := 1/\sqrt{\epsilon_0}$ and $c_\# := 1/\sqrt{\epsilon_1}$. By Lemma 20, $c_* \geqslant c_\#$, and $c_* = c_\#$ if and only if $c_* = c_\# = 0$. As we show in Appendix A, $c_\# = 0$ if $\int_{\mathbb{R}} sK(s)\,ds \geqslant 0$ and $c_* > 0$ if $\int_{\mathbb{R}} sK(s)\,ds \leqslant 0$. Moreover, $c_\# > 0$ if the equation

$$z^2 - q + p \int_{\mathbb{R}} \exp(-zs)K(s)\,ds = 0$$

has negative roots. The main result of this subsection is the following

Theorem 1. *Let $u(t,x) = \phi(x+ct)$, $c > 0$, be a positive bounded solution of Eq.* (3)*. If $c < c_*$ then $\liminf_{s\to-\infty} \phi(s) > 0$ and therefore $\phi(x+ct)$ is not a semi-wavefront. Next, if $c > c_\#$, then $\phi(x+ct)$ is persistent*: $\liminf_{s\to+\infty} \phi(s) > 0$. *In consequence, Eq.* (3) *does not have non-stationary pulses.*

Observe that even when g is monotone on $[0,\kappa]$, it was not known whether every semi-wavefront to Eq. (1) is separated from zero as $x+ct \to +\infty$. The persistence of semi-wavefronts was established in [32] for a local version of model (1). The proof in [32] is based on the local estimations technique which does not apply to Eq. (1). To overcome this obstacle, we will use a Laplace transform approach developed in [22,23] and successfully applied in [2, Proposition 4], [25, Theorem 4.1], [32, Theorem 5.4].

We emphasize that c_* can be different from the minimal speed of propagation of semi-wavefronts even for the simpler case of Eq. (1), cf. [9]. However, as it was shown in [30,32,34], c_* coincides with the minimal speed for Eq. (1) if g satisfies (H) together with the additional condition

$$g(s) \leqslant g'(0)s \quad \text{for all } s \geqslant 0. \tag{6}$$

Remark 2. A lower bound for the admissible speeds of semi-wavefronts to the reaction–diffusion functional equation

$$u_t(t,x) = \Delta u(t,x) + g(u_t), \quad u(t,x) \geqslant 0, \ x \in \mathbb{R}^n,$$

was already calculated in the pioneering work of Schaaf, see Theorem 2.7(i) and Lemma 2.5 in [26]. Recent work [32] complements Schaaf's investigation in two aspects: (i) analyzing the case of non-hyperbolic trivial equilibrium and (ii) taking into account the problem of *small solutions*.

Note that very few theoretical studies are devoted to the minimal speed problem for the *non-local* equation (3). To the best of our knowledge, the first accurate proof of the non-existence of semi-wavefronts was provided by Thieme and Zhao in [30, Theorem 4.2 and Remark 4.1] for the equation

$$u_t = u_{xx} - f(u) + \int_{\mathbb{R}} K_\alpha(x-y)g\big(u(t-h,y)\big)\,dy, \qquad K_\alpha(x) = \frac{e^{-x^2/(4\alpha)}}{\sqrt{4\pi\alpha}}. \tag{7}$$

In order to prove this result, Thieme and Zhao have extended an integral-equations approach [5,29] to scalar non-local and delayed reaction–diffusion equations. Their proof makes use of the special form of the kernel K which is the fundamental solution of the heat equation.

Besides the above mentioned work [30], a non-existence result was proved for the equation

$$u_t(t,x) = u_{xx}(t,x) + g\left(u(t,x), \int_{\mathbb{R}} K(x-y)u(t-h,y)\,dy\right) \tag{8}$$

in the recent work [34] by Wang, Li and Ruan. Their method required C^2-smoothness of g and the fulfillment of several convexity conditions.

Our approach is different from those in [30] and [34] and it allows us to impose minimal restrictions on the right-hand side of Eq. (3). In any case, the problem of non-existence of semi-wavefronts to Eqs. (3), (7), (8) is non-trivial, and the corresponding proofs are not easy. In fact, some papers provide only a heuristic explanation for why non-local models similar to (3) do not have positive wavefronts propagating at velocity c which is less than some critical speed c_*; see, for instance, [3, 12,16,27,33]. In the mentioned works, c_* is defined as the unique positive number for which some associated characteristic function ψ_{c_*} (similar to (5)) has a positive multiple root while ψ_c does not have any positive root for all $c < c_*$, cf. Lemma 20. However, it seems that this argument is incomplete. Indeed, some linear autonomous functional differential equations of mixed type may have a nonoscillatory solution in spite of the non-existence of real roots of its characteristic equation. See remarkable examples proposed by Krisztin in [15].

1.2. Uniform persistence of waves

The second aspect of the problem we address is the uniform persistence of positive waves $u(t,x) = \phi(x+ct)$, $c > c_\#$, to Eq. (1). This property means that $\liminf_{s\to+\infty}\phi(s) \geqslant \zeta$, where $\zeta > 0$ depends only on g. The uniform persistence of positive bounded waves will be proved by assuming condition (2) and the following hypothesis

(B) $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies $g(s) > 0$ when $s > 0$ and, for some $0 < \zeta_1 < \zeta_2$,
1. $g([\zeta_1, \zeta_2]) \subseteq [\zeta_1, \zeta_2]$ and $g([0, \zeta_1]) \subseteq [0, \zeta_2]$;
2. $\min_{s\in[\zeta_1,\zeta_2]} g(s) = g(\zeta_1)$;
3. (i) $g(s) > s$ for $s \in (0, \zeta_1]$ and (ii) there exists $p = g'(0) \in (1, +\infty)$;
4. $\kappa > 0$ and 0 are the only two solutions of $g(s) = s$.

Remark that conditions in (B) are weaker than (H). Indeed, set $\zeta_2 = g(s_M)$ if g satisfies (H). It is easy to see that the map $g : [0, \zeta_2] \to [0, \zeta_2]$ is well defined. We can also consider the restrictions $g : [\zeta_1, \zeta_2] \to [\zeta_1, \zeta_2]$ for every positive $\zeta_1 \leqslant \min\{g^2(s_M),\ s_M\}$. Clearly, there exists ζ_1 satisfying (B-2).

Theorem 3. *Assume* (B) *and let $u = \phi(x + ct)$, $c > c_\#$, be a positive bounded solution to Eq.* (1)*. Then* $\zeta_1 \leqslant \liminf_{s\to+\infty}\phi(s) \leqslant \sup_{s\in\mathbb{R}}\phi(s) \leqslant \sup g([0, \sup_{s\in\mathbb{R}}\phi(s)])$.

1.3. Existence of semi-wavefronts

Set $\tilde{c}_* := (\epsilon_0(h, \sup_{s>0} g(s)/s, 1))^{-1/2}$. It is clear that $\tilde{c}_* \geqslant c_*$ and $\tilde{c}_* = c_*$ if conditions (6) and (B-3ii) hold. Our third result establishes the existence of semi-wavefronts for all $c \geqslant \tilde{c}_*$:

Theorem 4. *Assume* (B) *except the condition* (B-3ii)*. Then Eq.* (1) *has a positive semi-wavefront $u(t,x) = \phi(x + ct)$ for every positive $c \geqslant \tilde{c}_*$.*

The proof of Theorem 4 relies on the Ma–Wu–Zou method proposed in [35] and further developed in [20,21,27]. It uses the positivity and monotonicity properties of the integral operator

$$(A\phi)(t) = \frac{1}{\epsilon'}\left\{\int_{-\infty}^{t} e^{\lambda(t-s)}(G\phi)(s-h)\,ds + \int_{t}^{+\infty} e^{\mu(t-s)}(G\phi)(s-h)\,ds\right\},$$

$$(G\phi)(s) = \int_{\mathbb{R}} K(w)g\big(\phi(s-\sqrt{\epsilon}w)\big)\,dw, \quad \epsilon' := \epsilon(\mu-\lambda), \tag{9}$$

where $\lambda < 0 < \mu$ solve $\epsilon z^2 - z - 1 = 0$ and $\epsilon^{-1/2} = c > 0$ is the wave velocity. As it can be easily observed, the profiles $\phi \in C(\mathbb{R}, \mathbb{R}_+)$ of travelling waves are completely determined by the integral equation $A\phi = \phi$ and the Ma–Wu–Zou method consists in the use of an appropriate fixed point theorem to $A : \mathfrak{K} \to \mathfrak{K}$, where $\mathfrak{K} = \{\phi\colon\ 0 \leqslant \phi^-(t) \leqslant \phi(t) \leqslant \phi^+(t)\}$ is subset of an adequate Banach space $(C(\mathbb{R}, \mathbb{R}), |\cdot|)$. Hence, A-invariant set $\mathfrak{K}$ should be 'nice' enough to assure the compactness (or monotonicity) of A. These requirements are not easy to satisfy. Thus only relatively narrow subclasses of g (e.g. sufficiently smooth at the steady states and monotone or quasi-monotone in the sense of [35]) were considered within this approach. Our contribution to the above method is the very simple form of the bounds $\phi^{\pm}$ for $\mathfrak{K}$. For instance, due to the information provided by Theorem 3, we may take $\phi^-(t) = 0$ for all $t \geqslant 0$. Here, this finding allows to weaken the smoothness conditions imposed on $g(s)$ at $s = 0$. In particular, for Eq. (1), Theorem 4 improves Theorem 1.1 in [21]. Indeed, the method employed in [21] needs essentially that $K(s) = K(-s)$ and $\limsup_{u \to +0}(g'(0) - g(u)/u)u^{-\nu}$ is finite for some $\nu \in (0, 1]$.

1.4. Delay-depending conditions of the existence of wavefronts

Similarly to the case of semi-wavefronts, the existence of the wavefronts depends not only on the local behavior of g at the equilibria but also on the entire shape of g. In fact, we have to analyze here some one-dimensional dynamical systems associated to g. The property of the negative Schwarzian $(Sg)(s) = g'''(s)/g'(s) - (3/2)(g''(s)/g'(s))^2$ is instrumental in simplifying the analysis of these systems in some cases, see Proposition 24 and further comments in Appendix A. The next result presents delay-depending conditions of the existence of the wavefronts, it follows from more general Theorem 15.

Theorem 5. *Assume* (H) *and let* $\zeta_1, \zeta_2 = g(s_M)$ *be as in* (B). *Suppose further that* $g \in C^3(\mathbb{R}_+, \mathbb{R}_+)$, $(Sg)(s) < 0$, $s \in [\zeta_1, \zeta_2] \setminus \{s_M\}$, *and* $g(g(\zeta_2)) \geqslant \kappa$. *If, for some* $\epsilon > 0$,

$$\left(1 - \min\left\{e^{-h}, \int_{-h/\sqrt{\epsilon}}^{0} K(u)\, du\right\}\right) g'(\kappa) \geqslant -1,$$

then Eq. (1) *has a wavefront* $u(t, x) = \phi(x + ct)$ *for every* $c \geqslant \max\{\tilde{c}_*, 1/\sqrt{\epsilon}\}$. *Moreover, for these values of* c, *each semi-wavefront is in fact a wavefront.*

1.5. Non-monotonicity of wavefronts

The problem of non-monotonicity of wavefronts to Eq. (1) was widely discussed in the literature. The state of the art is surveyed in [12, Section 4.3]. As far as we know, the paper [3] by Ashwin et al. contains the first heuristic explanation of this phenomenon. Recent works [8,31] have provided rigorous analysis of non-monotonicity in the local case. Here, we follow the approach of [8] to indicate conditions inducing the loss of monotonicity of wavefronts in the simpler case when the kernel K has compact support.

Theorem 6. *Suppose that* $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ *is differentiable at* κ, *with* $g'(\kappa) < 0$. *Assume further that* $g(\kappa) = \kappa$, $\operatorname{supp} K \subseteq [-\eta, \eta]$, *and the equation*

$$(z/c)^2 - z - 1 + g'(\kappa)\exp(-zh) \int_{-\eta}^{\eta} K(s)\exp(-zs/c)\, ds = 0 \tag{10}$$

does not have any root in $(-\infty, 0)$ *for some fixed* $c = \bar{c}$. *If* $\phi(+\infty) = \kappa$ *for a non-constant solution* $\phi(x + \bar{c}t)$ *of Eq.* (1), *then* $\phi(s)$ *oscillates about* κ.

1.6. An example

We apply our results to the reaction–diffusion–advection equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D_m \frac{\partial u}{\partial x} + Bu\right) - u + \int_{\mathbb{R}} K_1(x+Bh-y) g\big(u(t-h,y)\big)\,dy, \tag{11}$$

where $g(s) = pse^{-s}$, $K_\alpha(s) = (4\pi\alpha)^{-1/2}e^{-s^2/(4\alpha)}$. This equation was studied numerically in [17] for various values of parameters p, B, h, D_m. Plugging the traveling wave ansatz $u(x,t) = \phi(x+ct)$ into (11), we obtain that

$$D_m\phi''(t) - (c-B)\phi'(t) - \phi(t) + \int_{\mathbb{R}} K_1(s) g\big(\phi\big(t-(c-B)h-s\big)\big)\,ds = 0.$$

Next, setting $\gamma = 1/D_m$, $\phi(s) = z(s/(c-B))$, $\epsilon = D_m/(c-B)^2$, we find that

$$\epsilon z''(t) - z'(t) - z(t) + \int_{\mathbb{R}} K_\gamma(s) g\big(z(t-h-\sqrt{\epsilon}s)\big)\,ds = 0.$$

For this equation, g satisfies condition (6) and the function $\psi(z,\epsilon)$ from (5) can be found explicitly: $\psi(z,\epsilon) = \epsilon z^2 - z - 1 + p\exp(\epsilon\gamma z^2 - zh)$. An easy calculation shows that $\kappa = \ln p$, $\zeta_2 = p/e$, $g'(\kappa) = \ln(e/p)$. As in Section 4.1 of [17], we select $D_m = 5, h = 1, p = 9$. Then we find that $g(g(\zeta_2)) = 3.29\ldots > \kappa = 2.19\ldots$. Analyzing $\psi(z,\epsilon)$, we obtain that $\epsilon_0 = 0.37\ldots$. In consequence, Eq. (11) has semi-wavefronts if and only if $c - B \geqslant \sqrt{5/0.37\ldots} = 3.66\ldots$. This can explain (see also Remark 19) the emergence of unsteady multihump waves in the numerical experiments realized in [17]: indeed, the value $c - B = 3$ taken in [17] is less than the minimal speed of semi-wavefronts. Finally, an application of Theorem 5 shows that Eq. (11) with $D_m = 5$, $h = 1$, $p = 9$ has *wavefronts* if and only if $c - B \geqslant 3.66\ldots$.

2. Proof of Theorem 1 for $c < c_*$

Let $u(t,x) = \phi(x+ct)$ be a positive bounded solution of (3) and suppose that $\epsilon := c^{-2} > \epsilon_0(h,p,q)$. Then $\xi(t) = \phi(-ct)$ satisfies

$$\epsilon\xi''(t) + \xi'(t) - q\xi(t) + (\mathcal{F}\xi)(t) = 0, \quad t \in \mathbb{R}, \tag{12}$$

where $(\mathcal{F}\xi)(t) = F((\mathbf{I}\xi)(t))$ with $(\mathbf{I}\xi)(t) \in \mathbb{R}_+^{m+1}$ denoting

$$\left(\xi(t), \int_{\mathbb{R}} K_1(s) f_1\big(\xi(t+\sqrt{\epsilon}s+h)\big)\,ds, \ldots, \int_{\mathbb{R}} K_m(s) f_m\big(\xi(t+\sqrt{\epsilon}s+h)\big)\,ds\right).$$

Since $\xi(t)$ is a bounded solution of Eq. (12), it must satisfy

$$\xi(t) = \frac{1}{\epsilon(\tilde{\mu}-\tilde{\lambda})}\left\{\int_{-\infty}^{t} e^{\tilde{\lambda}(t-s)}\big(\mathcal{F}(\xi)\big)(s)\,ds + \int_{t}^{+\infty} e^{\tilde{\mu}(t-s)}\big(\mathcal{F}(\xi)\big)(s)\,ds\right\}, \tag{13}$$

where $\tilde{\lambda} < 0 < \tilde{\mu}$ are roots of $\epsilon z^2 + z - q = 0$.

The following inequality is crucial in the coming discussion.

Lemma 7. *If* $\xi : \mathbb{R} \to (0, +\infty)$ *is a bounded solution of Eq.* (12)*, then*

$$\xi(t) \geqslant e^{\tilde{\lambda}(t-s)}\xi(s), \quad t \geqslant s. \tag{14}$$

Proof. Since $(\mathcal{F}\xi)(t)$ is non-negative, after differentiating (13), we obtain

$$\xi'(t) - \tilde{\lambda}\xi(t) = \frac{1}{\epsilon}\int_t^{+\infty} e^{\mu(t-s)}(\mathcal{F}\xi)(s)\,ds \geqslant 0. \tag{15}$$

Therefore $(\xi(t)e^{-\tilde{\lambda}t})' \geqslant 0$, which implies (14). □

Arguing by contradiction, we suppose that $\liminf_{t\to+\infty} \xi(t) = 0$ for some $\epsilon > \epsilon_0$.

Case I. $\limsup_{t\to+\infty} \xi(t) = \lim_{t\to+\infty} \xi(t) = 0$.

In virtue of (14) and Lemma 21 from Appendix A, we can find a real number $D > 1$ and a sequence $t_n \to +\infty$ such that $\xi(t_n) = \max_{s\geqslant t_n} \xi(s)$ and

$$\max_{s\in[t_n-8\sqrt{\epsilon},t_n]} \xi(s) \leqslant D\xi(t_n).$$

It is easy to see that, for every fixed n, $\xi'(t)$ is either negative on $[t_n - 4\sqrt{\epsilon}, t_n]$ or there is $t_n' \in [t_n - 4\sqrt{\epsilon}, t_n]$ such that $\xi'(t_n') = 0$, and $\xi(t) \leqslant \xi(t_n')$ for all $t \geqslant t_n'$ (thus $\xi(t) \leqslant D\xi(t_n')$ if $t \in [t_n' - 4\sqrt{\epsilon}, t_n']$). Now, if $\xi'(t)$ is negative then

$$\left|\xi'\left(t_n''\right)\right| = \left(\xi(t_n - 4\sqrt{\epsilon}\,) - \xi(t_n)\right)/(4\sqrt{\epsilon}\,) \leqslant (D-1)\xi(t_n)/(4\sqrt{\epsilon}\,) := D_1\xi(t_n)$$

for some $t_n'' \in [t_n - 4\sqrt{\epsilon}, t_n]$. Since $\xi(t_n'') \geqslant \xi(t_n)$, we obtain that

$$\left|\xi'\left(t_n''\right)\right| \leqslant D_1\xi(t_n) \leqslant D_1\xi\left(t_n''\right) \quad \text{and} \quad \xi(t) \leqslant D\xi\left(t_n''\right) \quad \text{for all } t \in \left[t_n'' - 4\sqrt{\epsilon}, t_n''\right].$$

Hence, by the above reasoning, we may assume that D and $\{t_n\}$ are such that

$$\left|\xi'(t_n)\right| \leqslant D\xi(t_n), \qquad \max_{s\in[t_n-4\sqrt{\epsilon},t_n]} \xi(s) \leqslant D\xi(t_n) \quad \text{and} \quad \xi(t) \leqslant \xi(t_n), \quad t \geqslant t_n.$$

Next, since continuous F is differentiable at 0 and $F(0) = 0$, we obtain that

$$F(s_0, s_1, \dots, s_m) = \sum_{j=0}^{m} A_j(s_0, s_1, \dots, s_m)s_j, \quad s_j \geqslant 0,$$

for some continuous A_j satisfying $A_j(0) = F_{s_j}(0)$, $j = 0, \dots, m$. In consequence, $y_n(t) = \xi(t+t_n)/\xi(t_n)$, $t \in \mathbb{R}$, should satisfy the equation

$$\epsilon y''(t) + y'(t) - a_{0,n}(t)y(t) + \int_{\mathbb{R}} \mathcal{K}_n(t, s, \epsilon)y(t + \sqrt{\epsilon}s + h)\,ds = 0, \tag{16}$$

where $\mathscr{K}_n(t,s,\epsilon) := \sum_{j=1}^m K_j(s) a_{j,n}(t, t+\sqrt{\epsilon}s+h)$ and

$$a_{0,n}(t) := q - A_0\big((\mathbf{I}\xi)(t+t_n)\big), \qquad a_{j,n}(t,u) := A_j\big((\mathbf{I}\xi)(t+t_n)\big)\frac{f_j(\xi(u+t_n))}{\xi(u+t_n)}.$$

From (14), it is clear that $e^{\tilde{\lambda}t} \leqslant y_n(t) \leqslant 1$ for all $t \geqslant 0$ and $y_n(t) \leqslant e^{\tilde{\lambda}t}$, $t \leqslant 0$. In particular, $y_n(0) = 1$. Note also that there is $C_\xi > 0$ such that $|a_{j,n}(t,u)| \leqslant C_\xi$ for all $j = 0, \dots, m$; $n \in \mathbb{N}$; $t, u \in \mathbb{R}$. Moreover, $\lim_{n\to\infty} a_{0,n}(t) = q$, $\lim_{n\to\infty} a_{j,n}(t,u) = F_{s_j}(0) f_j'(0)$ pointwise. Set

$$\mathscr{G}_n(t) := \int_{\mathbb{R}} \mathscr{K}_n(t,s,\epsilon) y_n(t+\sqrt{\epsilon}s+h)\, ds.$$

We claim that for arbitrary fixed $\sigma, \tau > 0$ there exists $c_{\sigma,\tau} > 0$ such that $|\mathscr{G}_n(t)| \leqslant c_{\sigma,\tau}$ for all $t \in [-\sigma, \tau]$ and for all $n \in \mathbb{N}$. Indeed,

$$\begin{aligned}
\big|\mathscr{G}_n(t)\big| &\leqslant \int_{\mathbb{R}} \big|\mathscr{K}_n(t,s,\epsilon)\big| y_n(t+\sqrt{\epsilon}s+h)\, ds \leqslant C_\xi \int_{\mathbb{R}} \sum_{j=1}^m K_j(s) y_n(t+\sqrt{\epsilon}s+h)\, ds \\
&\leqslant C_\xi \int_{-\frac{t+h}{\sqrt{\epsilon}}}^{+\infty} \sum_{j=1}^m K_j(s)\, ds + C_\xi \int_{-\infty}^{-\frac{t+h}{\sqrt{\epsilon}}} \sum_{j=1}^m K_j(s) e^{\tilde{\lambda}(t+\sqrt{\epsilon}s+h)}\, ds \\
&\leqslant C_\xi \left(m + e^{\tilde{\lambda}(h-\sigma)} \int_{\mathbb{R}} \sum_{j=1}^m K_j(s) e^{\tilde{\lambda}\sqrt{\epsilon}s}\, ds \right) =: c_{\sigma,\tau}, \quad t \in [-\sigma, \tau].
\end{aligned}$$

Now, since $z_n(t) = y_n'(t)$ solves the initial value problem $z_n(0) = \xi'(t_n)/\xi(t_n) \in [-D, D]$ for

$$\epsilon z'(t) + z(t) - a_{0,n}(t) y_n(t) + \mathscr{G}_n(t) = 0,$$

we deduce the existence of $k_{\sigma,\tau} > 0$ such that, for all $t \in [-\sigma, \tau]$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\big|y_n'(t)\big| &= \left| e^{-t/\epsilon} z_n(0) + \frac{1}{\epsilon} \int_0^t e^{(s-t)/\epsilon} \big(a_{0,n}(s) y_n(s) - \mathscr{G}_n(s)\big)\, ds \right| \\
&\leqslant D e^{\sigma/\epsilon} + \frac{1}{\epsilon} \left| \int_0^t e^{(s-t)/\epsilon} \big(C_\xi \max\{1, e^{-\tilde{\lambda}s}\} + c_{\sigma,\tau}\big)\, ds \right| \leqslant k_{\sigma,\tau}.
\end{aligned} \tag{17}$$

Therefore, we may apply the Ascoli–Arzelà compactness criterion together with a diagonal argument on each of the intervals $[-i, i]$ to find a subsequence $\{y_{n_j}(t)\}$ converging, in the compact open topology, to a non-negative function $y_* : \mathbb{R} \to \mathbb{R}_+$. It is evident that $y_*(0) = 1$ and $e^{\tilde{\lambda}t} \leqslant y_*(t) \leqslant 1$ for all $t \geqslant 0$ and $y_*(t) \leqslant e^{\tilde{\lambda}t}$, $t \leqslant 0$. By the Lebesgue's dominated convergence theorem, we have that, for every $t \in \mathbb{R}$,

$$\mathscr{G}_n(t) \to \mathscr{G}_*(t) := p \int_{\mathbb{R}} K(s) y_*(t+\sqrt{\epsilon}s+h)\, ds,$$

where $K(s)$, p are defined in (4). In consequence, integrating (17) without $|\cdot|$ between 0 and t and then taking the limit as $n_j \to \infty$ in the obtained expression, we establish that $y_*(t)$ satisfies

$$\epsilon y''(t) + y'(t) - qy(t) + p \int_{\mathbb{R}} K(s) y(t + \sqrt{\epsilon} s + h)\, ds = 0. \tag{18}$$

Then Lemma 22 implies that $y_*(t) = w(t) + O(\exp(2\tilde{\lambda} t))$, $t \to +\infty$, where w is a non-empty finite sum of eigensolutions of (18) associated to the eigenvalues $\nu_j \in F = \{2\tilde{\lambda} < \Re \nu_j \leqslant 0\}$. Observe now that ν is an eigenvalue of (18) if and only if $-\nu$ is a root of (5). In this way, F does not contain any real eigenvalue for $\epsilon > \epsilon_0$ (by Lemma 20), and therefore $y_*(t)$ should be oscillating on $\mathbb{R}_+$, a contradiction.

Case II. $\liminf_{t \to +\infty} \xi(t) = 0$ and $S = \limsup_{t \to +\infty} \xi(t) > 0$.

Owing to (15), we conclude that $\sup_{t \in \mathbb{R}} (|\xi(t)| + |\xi'(t)|) < +\infty$. This guarantees the pre-compactness of the one-parametric family $\{\xi(t+s),\ s \in \mathbb{R}\}$ in the compact open topology of $C(\mathbb{R}, \mathbb{R})$. We use this fact repeatedly in what follows. Next, for every fixed $j > S^{-1}$ there exists a sequence of intervals $[p_i', q_i']$, $\lim p_i' = +\infty$ such that $\xi(p_i') = 1/j$, $\lim \xi(q_i') = 0$, $\xi'(q_i') = 0$ and $\xi(t) \leqslant 1/j$, $t \in [p_i', q_i']$. Note that $\limsup (q_i' - p_i') = +\infty$ since otherwise we get a contradiction: the sequence $\xi(t + p_i')$ of solutions to Eq. (13) contains a subsequence converging to a non-negative bounded solution $\xi_1(t)$ such that $\xi_1(0) = 1/j$, $\xi_1(\sigma) = 0$ for some finite $\sigma > 0$. In consequence, $w_i(t) = \xi(t + p_i')$, $t \in \mathbb{R}$, has a subsequence converging to some bounded non-negative solution $w_*(t)$ of (13) satisfying $0 < w_*(t) \leqslant 1/j$ for all $t \geqslant 0$. Since the case $w_*(+\infty) = 0$ is impossible due to the first part of the proof, we conclude that $0 < S^* = \limsup_{t \to +\infty} w_*(t) \leqslant 1/j$. Let $r_i \to +\infty$ be such that $w_*(r_i) \to S^*$, then $w_*(t + r_i)$ has a subsequence converging to a positive solution $\zeta_j : \mathbb{R} \to [0, 1/j]$ of (13) such that $\max_{t \in \mathbb{R}} \zeta_j(t) = \zeta_j(0) = S^* \leqslant 1/j$. Next, arguing as in Case I after formula (16), we can use sequence $\{y_j(t) := \zeta_j(t)/\zeta_j(0)\}$ to obtain a bounded positive solution $y_* : \mathbb{R} \to (0, 1]$ of linear equation (18). For the same reason as given in Lemma 7, *bounded* y_* decays at most exponentially. Now, invoking Lemma 22 and the oscillation argument as in Case I, we get a contradiction.

3. Proof of Theorem 1 for $c > c_\#$

The case $c > c_\#$ is similar to case considered in Section 2. Below we give some details. Let $u(t, x) = \phi(x + ct)$ be a positive bounded solution of (3) and suppose that $\epsilon := c^{-2} < \epsilon_1(h, p, q)$. Set $\varphi(s) = \phi(cs)$ and $\tilde{K}_j(s) = K_j(-s - 2h/\sqrt{\epsilon})$. Then each $\tilde{K}_j(s)$ satisfies (2) and $\varphi(t)$ verifies

$$\epsilon \varphi''(t) - \varphi'(t) - q\varphi(t) + (\mathcal{H}\xi)(t) = 0, \quad t \in \mathbb{R},$$

where $(\mathcal{H}\varphi)(t) = F((\mathbf{J}\varphi)(t))$ with $(\mathbf{J}\varphi)(t) \in \mathbb{R}_+^{m+1}$ denoting

$$\left(\varphi(t), \int_{\mathbb{R}} \tilde{K}_1(s) f_1\big(\varphi(t + \sqrt{\epsilon} s + h)\big)\, ds, \ldots, \int_{\mathbb{R}} \tilde{K}_m(s) f_m\big(\varphi(t + \sqrt{\epsilon} s + h)\big)\, ds\right).$$

Since $(\mathcal{H}\xi)(t)$ is non-negative, the same argument as used to prove Lemma 7 shows that $\varphi(t) \geqslant e^{\lambda(t-s)} \varphi(s)$, $t \geqslant s$, where $\lambda = -\tilde{\mu} < 0 < \mu = -\tilde{\lambda}$ are the roots of $\epsilon z^2 - z - q = 0$. All this allows to repeat the proof given in Section 2, with a few obvious changes, to establish the persistence of $\varphi(t)$. For example, the paragraph below (18) should be modified in the following way:

"... we establish that $y_*(t)$ satisfies

$$\epsilon y''(t) - y'(t) - qy(t) + p \int_{\mathbb{R}} K(s) y(t - \sqrt{\epsilon} s - h)\, ds = 0. \tag{19}$$

Then Lemma 22 implies that $y_*(t) = w(t) + O(\exp(2\lambda t))$, $t \to +\infty$, where w is a *non-empty* finite sum of eigensolutions of (19) associated to the eigenvalues $\lambda_j \in F = \{2\lambda < \Re\lambda_j \leqslant 0\}$. Now, since the set F does not contain any real eigenvalue for $\epsilon \in (0, \epsilon_1)$ (see Lemma 20), we conclude that $y_*(t)$ should be oscillating on $\mathbb{R}_+$, a contradiction."

4. Proof of Theorem 3

Let $\varphi(t)$ satisfy $0 < \varphi(t) \leqslant M_0$, $t \in \mathbb{R}$, and

$$\epsilon\varphi''(t) - \varphi'(t) - \varphi(t) + \int_{\mathbb{R}} K(s) g\big(\varphi(t - \sqrt{\epsilon}s - h)\big)\, ds = 0, \quad t \in \mathbb{R}. \tag{20}$$

Being bounded, φ must verify the integral equation

$$\varphi(t) = \frac{1}{\epsilon'}\left\{\int_{-\infty}^{t} e^{\lambda(t-s)}(G\varphi)(s-h)\, ds + \int_{t}^{+\infty} e^{\mu(t-s)}(G\varphi)(s-h)\, ds\right\}, \tag{21}$$

where ϵ', μ, λ and $G\varphi$ are as in (9). From (21), we obtain that $|\varphi'(t)| \leqslant \max_{s\in[0,M_0]} g(s)/\epsilon'$. This implies the pre-compactness of the one-parametric family $\mathcal{F} = \{\varphi(t+s), s \in \mathbb{R}\}$ in the compact open topology of $C(\mathbb{R}, \mathbb{R})$. It is an easy exercise to prove (by using (21)) that the closure of $\mathcal{F}$ consists from the positive bounded solutions of (20). Next, for φ as above, set

$$0 \leqslant m = \inf_{t\in\mathbb{R}} \varphi(t) \leqslant \sup_{t\in\mathbb{R}} \varphi(t) = M < +\infty.$$

Lemma 8. $[m, M] \subseteq g([m, M])$.

Proof. Indeed, if $M = \varphi(s') = \max_{s\in\mathbb{R}} \varphi(s)$, a straightforward estimation of the right-hand side of (21) at $t = s'$ generates $M \leqslant \max_{m\leqslant s\leqslant M} g(s)$. As long as the maximum M is not reached, using the pre-compactness of $\mathcal{F}$, we can find a solution $z(t)$ of (20) such that $z(0) = \max_{s\in\mathbb{R}} z(s) = M$ and $\inf_{s\in\mathbb{R}} z(s) \geqslant m$. Therefore, by the above argument, $M \leqslant \max_{m\leqslant s\leqslant M} g(s)$. The inequality $m \geqslant \min_{m\leqslant s\leqslant M} g(s)$ can be proved in a similar way. Thus we can conclude that $[m, M] \subseteq g([m, M])$. □

Note that Lemma 8 implies that $\sup \varphi(t) \leqslant \sup g([0, \sup \varphi(t)])$.
Analogously, we have

Lemma 9. *Let φ satisfy* (20) *and be such that* $0 \leqslant m' = \liminf_{t\to+\infty} \varphi(t) \leqslant \limsup_{t\to+\infty} \varphi(t) = M' < +\infty$. *Then* $[m', M'] \subseteq g([m', M'])$.

Theorem 10. *Assume* (B) *and consider a positive bounded solution φ of Eq.* (20) *for some fixed $\epsilon \in (0, \epsilon_1)$. If $m = \inf_{s\in\mathbb{R}} \varphi(s) < \zeta_1$ then, in fact, $\epsilon \in (0, \epsilon_0]$ and $\lim_{t\to-\infty} \varphi(t) = 0$.*

Proof. Set $M = \sup_{s\in\mathbb{R}} \varphi(s)$, then Lemma 8 and (B) imply that $[m, M] \subseteq g([m, M])$ and $M \leqslant \zeta_2$. Furthermore, the condition $m < \zeta_1$ makes impossible the inequality $m > 0$. In consequence, $m = 0$ and, due to Theorem 1, either $\varphi(-\infty) = 0$ or

$$0 = \liminf_{t\to-\infty} \varphi(t) < \limsup_{t\to-\infty} \varphi(t) = S.$$

However, as we will show it in the continuation, the second case cannot occur. Indeed, otherwise for every positive $\delta_1 < \min\{\zeta_1, S\}$, it would be possible to indicate two sequences of real numbers

$p_n < q_n$ converging to $-\infty$ such that $\varphi(p_n) = \max_{[p_n,q_n]} \varphi(u) = \delta_1$, and $\varphi(q_n) < \varphi(s) < \varphi(p_n)$ for all $s \in (p_n, q_n)$ with $\lim \varphi(q_n) = 0$. We notice that necessarily $\lim(q_n - p_n) = +\infty$, since in the opposite case an application of the "compactness argument" leads to the following contradiction: the sequence of solutions $\varphi(t + p_n)$ contains a subsequence converging to a solution $\psi \in C(\mathbb{R}, [0, M])$ of Eq. (21) verifying $\psi(0) = \delta_1$ and $\psi(t_0) = 0$, for some finite $t_0 > 0$. Hence, $\lim(q_n - p_n) = +\infty$ and the limit solution ψ is positive and such that $\psi(0) = \delta_1 = \max_{s \geqslant 0} \psi(s)$. Moreover, by Theorem 1, we have that $\delta_0 := \liminf_{t\to+\infty} \psi(t) > 0$. In consequence, using again the "compactness argument," we can construct a solution $\tilde{\psi}(t)$ of Eq. (21) such that $\delta_0 \leqslant \tilde{\psi}(t) \leqslant \delta_1 < \zeta_1$, for all $t \in \mathbb{R}$. But, in view of hypotheses (B), this contradicts to Lemma 8. □

Now we are ready to prove that $\liminf_{t\to+\infty} \varphi(t) \geqslant \zeta_1$. Indeed, otherwise, by the "compactness argument," we can construct a bounded solution $\tilde{\varphi}(t)$ such that $0 < \liminf_{t\to+\infty} \varphi(t) \leqslant \inf_{s\in\mathbb{R}} \tilde{\varphi}(s) < \zeta_1$, contradicting to Theorem 10.

5. An application of Ma–Wu–Zou reduction

Throughout this section, $\chi_{\mathbb{R}_-}(t)$ stands for the indicator of $\mathbb{R}_-$. Following the notations of Lemma 20 in Appendix A, for given $\epsilon \in (0, \epsilon_0)$ we will denote by $\lambda_1 = \lambda_1(\epsilon) < \lambda_2 = \lambda_2(\epsilon)$ the positive roots of $\psi(z, \epsilon) = 0$. Also we will require

(L) $g : (0, +\infty) \to (0, +\infty)$ is bounded and locally linear in some right δ-neighborhood of the origin: $g(s) = ps$, $s \in [0, \delta)$, with $p > 1$. Furthermore, $g(s) \leqslant ps$ for all $s \geqslant 0$.

Assuming this, for every $\epsilon \in (0, \epsilon_0)$, we will prove the existence of semi-wavefronts of Eq. (20). As it was shown by Ma, Wu and Zou [20,21,27,35], solving (20) can be successfully reduced to the determination of fixed points of the integral operator A from (9) which is considered in some closed, bounded, convex and A-invariant subset $\mathfrak{K}$ of an appropriate Banach space $(X, \|\cdot\|)$. In this section, the choice of $\mathfrak{K} \subset X$ is restricted by the following natural conditions: (i) constant functions cannot be elements of X; (ii) the convergence $\varphi_n \to \varphi$ in $\mathfrak{K}$ is equivalent to the uniform convergence $\varphi_n \rightrightarrows \varphi_0$ on compact subsets of $\mathbb{R}$. With this in mind, for some $\rho \in (\lambda_1, \mu)$ and δ as in (L), we set

$$X = \Big\{\varphi \in C(\mathbb{R}, \mathbb{R}):\ \|\varphi\| = \sup_{s\leqslant 0} e^{-\lambda_1 s/2}\big|\varphi(s)\big| + \sup_{s\geqslant 0} e^{-\rho s}\big|\varphi(s)\big| < \infty\Big\};$$

$$\mathfrak{K} = \big\{\varphi \in X:\ \phi^-(t) = \delta\big(e^{\lambda_1 t} - e^{\lambda_2 t}\big)\chi_{\mathbb{R}_-}(t) \leqslant \varphi(t) \leqslant \delta e^{\lambda_1 t} = \phi^+(t),\ t \in \mathbb{R}\big\}.$$

A formal linearization of A along the trivial steady state is given by

$$(L\varphi)(t) = \frac{p}{\epsilon'}\left\{\int_{-\infty}^{t} e^{\lambda(t-s)}(Q\varphi)(s-h)\,ds + \int_{t}^{+\infty} e^{\mu(t-s)}(Q\varphi)(s-h)\,ds\right\},$$

where

$$(Q\varphi)(s) = \int_{\mathbb{R}} K(w)\varphi(s - \sqrt{\epsilon}w)\,dw, \quad \epsilon' := \epsilon(\mu - \lambda).$$

Lemma 11. *We have $L\phi^+ = \phi^+$. Next, $(L\psi)(t) > \psi(t)$, $t \in \mathbb{R}$, where*

$$\psi(t) := \big(e^{\lambda_1 t} - e^{\nu t}\big)\chi_{\mathbb{R}_-}(t) \in \mathfrak{K}$$

is considered with $\nu \in (\lambda_1, \lambda_2]$.

Proof. It suffices to prove that $(L\psi)(t) > \psi(t)$ for $t \leqslant 0$. But we have

$$(L\psi)(t) > \frac{p}{\epsilon'}\left\{\int_{-\infty}^{t} e^{\lambda(t-s)}\big(\mathrm{Q}\big(e^{\lambda_1(\cdot)} - e^{\nu(\cdot)}\big)\big)(s-h)\,ds + \int_{t}^{+\infty} e^{\mu(t-s)}\big(\mathrm{Q}\big(e^{\lambda_1(\cdot)} - e^{\nu(\cdot)}\big)\big)(s-h)\,ds\right\} \geqslant \psi(t). \qquad \square$$

Lemma 12. *Let assumption* (L) *hold and* $\epsilon \in (0, \epsilon_0)$. *Then* $A(\mathfrak{K}) \subseteq \mathfrak{K}$.

Proof. We have $A\varphi \leqslant L\varphi \leqslant L\phi^+ = \phi^+$ for every $\varphi \leqslant \phi^+$. Now, if for some $u = s - \sqrt{\epsilon}w$ we have $0 < \phi^-(u) \leqslant \varphi(u)$, then $u < 0$, so that $\varphi(u) \leqslant \delta e^{\lambda_1 u} < \delta$ implying $g(\varphi(u)) = p\varphi(u) \geqslant p\phi^-(u)$. If $\phi^-(u_1) = 0$ then again $g(\varphi(u_1)) \geqslant p\phi^-(u_1) = 0$. Therefore $(G\varphi)(t) \geqslant p(\mathrm{Q}\phi^-)(t)$, $t \in \mathbb{R}$, so that $A\varphi \geqslant L\phi^- > \phi^-$ for every $\varphi \in \mathfrak{K}$. $\square$

Lemma 13. $\mathfrak{K}$ *is a closed, bounded, convex subset of* X *and* $A : \mathfrak{K} \to \mathfrak{K}$ *is completely continuous.*

Proof. Note that the convergence of a sequence in $\mathfrak{K}$ amounts to the uniform convergence on compact subsets of $\mathbb{R}$. Since g is bounded, we have $|(A\varphi)'(t)| \leqslant \max_{s\geqslant 0} g(s)/\epsilon'$ for every $\varphi \in \mathfrak{K}$. The lemma follows now from the Ascoli–Arzelà theorem combined with the Lebesgue's dominated convergence theorem. $\square$

Theorem 14. *Assume* (L) *and let* $\epsilon \in (0, \epsilon_0)$. *Then the integral equation* (21) *has a positive bounded solution in* $\mathfrak{K}$.

Proof. Due to the above lemmas, we can apply the Schauder's fixed point theorem to $A : \mathfrak{K} \to \mathfrak{K}$. $\square$

6. Proof of Theorem 4

Case I. $c > \tilde{c}_*$.

First, we assume that $\max_{s\geqslant 0} g(s) = \max_{s\in[\zeta_1,\zeta_2]} g(s) \leqslant \zeta_2$. Set $k = \sup_{s>0} g(s)/s$ (so that $ks \geqslant g(s)$ for all $s \geqslant 0$) and consider the sequence

$$\gamma_n(s) = \begin{cases} ks, & \text{for } s \in [0, 1/(nk)], \\ 1/n, & \text{when } s \in [1/(nk), \inf g^{-1}(1/n)], \\ g(s), & \text{if } s > \inf g^{-1}(1/n), \end{cases}$$

of continuous functions γ_n, all of them satisfying hypotheses (L), (B) (where ζ_1, ζ_2 do not depend on n). Obviously, γ_n converges uniformly to g on $\mathbb{R}_+$. Now, for each large n, Theorems 3, 14 guarantee the existence of a positive continuous function $\varphi_n(t)$ such that $\varphi_n(-\infty) = 0$, $\liminf_{t\to+\infty} \varphi_n(t) \geqslant \zeta_1$, and

$$\varphi_n(t) = \frac{1}{\epsilon'}\left\{\int_{-\infty}^{t} e^{\lambda(t-s)}\varGamma_n(s)\,ds + \int_{t}^{+\infty} e^{\mu(t-s)}\varGamma_n(s)\,ds\right\},$$

where

$$\Gamma_n(t) := \int_{\mathbb{R}} K(s)\gamma_n\big(\varphi_n(t-\sqrt{\epsilon}s-h)\big)\,ds.$$

Since the shifted functions $\varphi_n(s+a)$ satisfy the same integral equation, we can assume that $\varphi_n(0) = 0.5\zeta_1$.

Now, taking into account the inequality $|\varphi_n(t)| + |\varphi_n'(t)| \leqslant \zeta_2 + \zeta_2/\epsilon'$, $t \in \mathbb{R}$, we find that the set $\{\varphi_n\}$ is pre-compact in the compact open topology of $C(\mathbb{R},\mathbb{R})$. Consequently we can indicate a subsequence $\varphi_{n_j}(t)$ which converges uniformly on compacts to some bounded element $\varphi \in C(\mathbb{R},\mathbb{R})$. Since

$$\lim_{j\to+\infty} \Gamma_{n_j}(t) = \int_{\mathbb{R}} K(s)g\big(\varphi(t-\sqrt{\epsilon}s-h)\big)\,ds$$

for every $t \in \mathbb{R}$, we can use the Lebesgue's dominated convergence theorem to conclude that φ satisfies integral equation (21). Finally, notice that $\varphi(0) = 0.5\zeta_1$ and thus $\varphi(-\infty) = 0$ (by Theorem 10) and $\liminf_{t\to+\infty}\varphi(t) \geqslant \zeta_1$ (by Theorem 3).

To complete the proof for Case I, we have to analyze the case when $\max_{s\geqslant 0} g(s) > \max_{s\in[\zeta_1,\zeta_2]} g(s)$. However, this case can be reduced to the previous one if we redefine $g(s)$ as $g(\zeta_2)$ for all $s \geqslant \zeta_2$, and then observe that $\sup_{t\in\mathbb{R}}\varphi(s) \leqslant \zeta_2$ for every solution obtained in the first part of this subsection.

Case II. $c = \tilde{c}_* > 0$.

Let $\epsilon_n = (n/(n+1)) \cdot 1/\tilde{c}_*^2$. The previous result (Case I) assures the existence of positive functions $\varphi_n(t)$ such that $\varphi_n(-\infty) = 0$, $|\varphi_n(t)| + |\varphi_n'(t)| \leqslant \zeta_2 + \zeta_2/\epsilon_1'$, $t \in \mathbb{R}$, $\liminf_{t\to+\infty}\varphi_n(t) \geqslant \zeta_1$, and

$$\varphi_n(t) = \frac{1}{\epsilon_n'}\left\{\int_{-\infty}^{t} e^{\lambda_n(t-s)}\Delta_n(s)\,ds + \int_{t}^{+\infty} e^{\mu_n(t-s)}\Delta_n(s)\,ds\right\},$$

where $\lambda_n < 0 < \mu_n$ satisfy $\epsilon_n z^2 - z - 1 = 0$, $\epsilon_n' := \epsilon_n(\mu_n - \lambda_n)$, and

$$\Delta_n(t) := \int_{\mathbb{R}} K(s)g\big(\varphi_n(t-\sqrt{\epsilon_n}s-h)\big)\,ds.$$

The rest of proof is exactly the same as in Case I and so is omitted.

7. Heteroclinic solutions of Eq. (20)

For $s \in (-\infty, 0)$ and $\lambda < 0 < \mu$ satisfying $\epsilon z^2 - z - 1 = 0$, set

$$\xi(s) = \frac{\mu-\lambda}{\mu e^{-\lambda s} - \lambda e^{-\mu s}}, \qquad \mathcal{D}(s) = \min\left\{\int_{-h/\sqrt{\epsilon}}^{-(s+h)/\sqrt{\epsilon}} K(u)\,du, \xi(-s)\right\}. \tag{22}$$

Everywhere in Section 7, g is C^3-smooth and we assume the hypothesis (H) so that all conditions of (B) are satisfied with $\zeta_2 = g(s_M)$ and some appropriate $\zeta_1 < \zeta_2$. Let $\varphi(t)$ be a semi-wavefront of Eq. (20). Set

$$m = \liminf_{t\to+\infty}\varphi(t) \leqslant \limsup_{t\to+\infty}\varphi(t) = M.$$

Our next result shows that $m = M = \kappa$ if, for some $s_* \in (-\infty, 0)$, it holds

$$\big(1 - \mathcal{D}(s_*)\big)g'(\kappa) \geqslant -1. \tag{23}$$

Theorem 15. *Assume* (H), $(Sg)(s) < 0$, $s \in [\zeta_1, \zeta_2] \setminus \{s_M\}$, *and* $g(g(\zeta_2)) \geqslant \kappa$. *If* (23) *holds for some fixed positive real number* $\epsilon \leqslant \tilde{c}_*^{-2}$, *then Eq.* (20) *with this* ϵ *has semi-wavefronts. Moreover, each of them is in fact a wavefront.*

Proof. Set $c := \epsilon^{-1/2} \in (0, \infty)$, then $c \geqslant \tilde{c}_*$. In light of Theorems 3, 4 and Lemma 9, the semi-wavefront $\varphi(t)$ exists and $\kappa, m, M \in [\zeta_1, \zeta_2]$ with $[m, M] \subseteq g([m, M])$. The latter inclusion and (H) imply that each of the following three relations $\kappa \leqslant s_M$, or $\kappa \leqslant m \leqslant M$, or $m \leqslant M \leqslant \kappa$ yields $m = M = \kappa$. Therefore, we will consider only the case when $m < \kappa < M$ so that $g'(\kappa) < 0$ and $m \geqslant g(\zeta_2)$ (due to Lemma 9). By the compactness argument, we can find a solution $y(t)$ of (20) such that $y(0) = \max_{s \in \mathbb{R}} y(s) = M$ and $\inf_{s \in \mathbb{R}} y(s) \geqslant m$. Fix some $s_* \in (-\infty, 0)$. Then either (I) $y(t) > \kappa$ for all $t \in [s_*, 0]$ or (II) there exists some $\hat{s} \in [s_*, 0]$ such that $y(\hat{s}) = \kappa$ and $y(t) > \kappa$ for $t \in (\hat{s}, 0]$.

In case (I), we have $y'(0) = 0$, $y''(0) \leqslant 0$ and thus, in view of Eq. (20),

$$\begin{aligned} M &\leqslant \int_{\mathbb{R}} K(w)g\big(y(-\sqrt{\epsilon}w - h)\big)\,dw \\ &= \int_{-h/\sqrt{\epsilon}}^{-(s_*+h)/\sqrt{\epsilon}} K(w)g\big(y(-\sqrt{\epsilon}w - h)\big)\,dw + \int_{\mathbb{R}\setminus\mathcal{I}} K(w)g\big(y(-\sqrt{\epsilon}w - h)\big)\,dw \\ &\leqslant \kappa\mathcal{D}(s_*) + \big(1 - \mathcal{D}(s_*)\big)\max_{s \in [m,M]} g(s), \quad \text{where } \mathcal{I} = \big[-h/\sqrt{\epsilon}, -(s_* + h)/\sqrt{\epsilon}\,\big]. \end{aligned}$$

In case (II), considering the boundary conditions $y(\hat{s}) = \kappa$, $y'(0) = 0$, setting

$$G(s) = \int_{\mathbb{R}} K(w)g\big(y(s - \sqrt{\epsilon}w)\big)\,dw$$

and then using Lemma 23, we find that

$$\begin{aligned} M = y(0) &= \xi(-\hat{s})\left\{\kappa + \frac{1}{\epsilon(\mu - \lambda)}\int_{\hat{s}}^{0}\big(e^{\lambda(\hat{s}-u)} - e^{\mu(\hat{s}-u)}\big)G(u - h)\,du\right\} \\ &\leqslant \xi(-\hat{s})\left\{\kappa + \frac{1}{\epsilon(\mu - \lambda)}\int_{\hat{s}}^{0}\big(e^{\lambda(\hat{s}-u)} - e^{\mu(\hat{s}-u)}\big)\,du \max_{x \in [m,M]} g(x)\right\} \\ &= \xi(-\hat{s})\kappa + \big(1 - \xi(-\hat{s})\big)\max_{s \in [m,M]} g(s) \leqslant \xi(-s_*)\kappa + \big(1 - \xi(-s_*)\big)\max_{s \in [m,M]} g(s), \end{aligned}$$

since $\xi(-s)$, $s \leqslant 0$, is strictly increasing. Hence, we have proved that

$$M \leqslant \kappa\mathcal{D}(s_*) + \big(1 - \mathcal{D}(s_*)\big)\max_{s \in [m,M]} g(s). \tag{24}$$

Analogously, there exists a solution $z(t)$ such that $z(0) = \min_{s \in \mathbb{R}} z(s) = m$ and $\sup_{s \in \mathbb{R}} z(s) \leqslant M$ so that $z'(0) = 0$, $z''(0) \geqslant 0$. We have again that either (III) $z(t) < \kappa$ for all $t \in [s_*, 0]$ or (IV) there exists some $\hat{s} \in [s_*, 0]$ such that $z(\hat{s}) = \kappa$ and $z(t) < \kappa$ for $t \in (\hat{s}, 0]$. In what follows, we are using the condition

$g(g(\zeta_2)) \geqslant \kappa$ which implies that $g(z(t)) \geqslant \kappa$ once $z(t) \in [g(\zeta_2), \kappa]$. Bearing this last remark in mind, in case (III), we obtain

$$\begin{aligned} m &\geqslant \int_{\mathbb{R}} K(w) g\big(z(-\sqrt{\epsilon} w - h)\big)\, dw \\ &= \int_{-h/\sqrt{\epsilon}}^{-(s_*+h)/\sqrt{\epsilon}} K(w) g\big(z(-\sqrt{\epsilon} w - h)\big)\, dw + \int_{\mathbb{R}\setminus\mathcal{I}} K(w) g\big(z(-\sqrt{\epsilon} w - h)\big)\, dw \\ &\geqslant \kappa \mathcal{D}(s_*) + \big(1 - \mathcal{D}(s_*)\big) \min_{s\in[m,M]} g(s). \end{aligned}$$

In case (IV), considering the boundary conditions $z(\hat{s}) = \kappa$, $z'(0) = 0$, and using Lemma 23, we find that

$$\begin{aligned} m = z(0) &= \xi(-\hat{s}) \left\{ \kappa + \frac{1}{\epsilon(\mu - \lambda)} \int_{\hat{s}}^{0} \big(e^{\lambda(\hat{s}-u)} - e^{\mu(\hat{s}-u)}\big) G(u - h)\, du \right\} \\ &\geqslant \xi(-\hat{s}) \left\{ \kappa + \frac{1}{\epsilon(\mu - \lambda)} \int_{\hat{s}}^{0} \big(e^{\lambda(\hat{s}-u)} - e^{\mu(\hat{s}-u)}\big)\, du \min_{s\in[m,M]} g(s) \right\} \\ &= \xi(-\hat{s})\kappa + \big(1 - \xi(-\hat{s})\big) \min_{s\in[m,M]} g(s) \geqslant \xi(-s_*)\kappa + \big(1 - \xi(-s_*)\big) \min_{s\in[m,M]} g(s). \end{aligned}$$

Hence, we have proved that

$$m \geqslant \kappa \mathcal{D}(s_*) + \big(1 - \mathcal{D}(s_*)\big) \min_{s\in[m,M]} g(s). \tag{25}$$

Set $f(s) = \kappa \mathcal{D}(s_*) + (1 - \mathcal{D}(s_*)) g(s)$. From (24) and (25), we deduce that

$$[m, M] \subseteq f\big([m, M]\big) \subseteq f^2\big([m, M]\big) := f\big(f\big([m, M]\big)\big) \subseteq \cdots \subseteq f^j\big([m, M]\big) \subseteq \cdots,$$

where $f : [g(\zeta_2), \zeta_2] \to [g(\zeta_2), \zeta_2]$ is unimodal (decreasing) if g is unimodal (decreasing, respectively) on the interval $[g(\zeta_2), \zeta_2]$. Therefore, as $f(\kappa) = \kappa$ and $Sf = Sg < 0$, the last chain of inclusions and the inequality $|f'(\kappa)| \leqslant 1$ allow to conclude that $m = M = \kappa$, see Proposition 24. □

Remark 16. Theorem 5 follows from Theorem 15 if we take $s_* = -h$ and observe that $0 < e^{-h} \leqslant \xi(h) < 1$. Note that $e^{-h} \leqslant \xi(h)$ amounts to the inequality $\mu(1 - e^{-h(\lambda+1)}) \geqslant \lambda(1 - e^{-h(\mu+1)})$, which holds true since the left-hand side is positive and the right-hand side is negative.

Remark 17. For fixed ϵ, h, and for $s < 0$, consider the following equation

$$\int_{-h/\sqrt{\epsilon}}^{-(s+h)/\sqrt{\epsilon}} K(u)\, du = \xi(-s). \tag{26}$$

It is clear that the left-hand side of (26) is decreasing in $s \in (-\infty, 0)$ from $\vartheta := \int_{-h/\sqrt{\epsilon}}^{+\infty} K(u)\, du \geqslant 0$ to 0 while the right-hand side is strictly increasing from 0 to 1. If $\vartheta > 0$, then (26) has a unique solution $s' \in (-\infty, 0)$ which coincides with the optimal value of s_* in (23).

8. Proof of Theorem 6

For the convenience of the reader, the proof will be divided in several steps. Note that the assumptions of Theorem 6 imply that $\operatorname{supp} K \cap (-h/\sqrt{\epsilon}, \eta) \neq \emptyset$. Let $\varphi(s) = \phi(\bar{c}s)$, $\epsilon := \bar{c}^{-2}$, then $\varphi : \mathbb{R} \to (0, +\infty)$ is a non-constant solution of Eq. (20) satisfying $\varphi(+\infty) = \kappa$. Set $y(t) := \varphi(t) - \kappa$.

Claim I. $|y(t)| > 0$ *is not superexponentially small as* $t \to +\infty$.

First we prove that φ cannot be eventually constant. Indeed, if $\varphi(t) = \kappa$ for all $t \geqslant -h$ and $\varphi(t)$ is not constant in some left neighborhood of $t = -h$ then we obtain from (20) that

$$\int_{-\eta}^{\eta} K(s) q(t - \sqrt{\epsilon} s)\, ds \equiv \kappa, \quad t \in [-h, \sqrt{\epsilon}\eta], \tag{27}$$

where $q(t) = g(\varphi(t-h))$. Set $K_1(u) = K(-u/\sqrt{\epsilon} + \eta)/\sqrt{\epsilon}$, $p(u) = q(-u) - \kappa$, $x = \sqrt{\epsilon}\eta - t$, $t \in [-h, \sqrt{\epsilon}\eta]$. Then (27) can be written as a scalar Volterra convolution equation on a finite interval

$$\int_0^x K_1(x-s) p(s)\, ds = \int_{t/\sqrt{\epsilon}}^{\eta} K(s)\big(q(t - \sqrt{\epsilon}s) - \kappa\big)\, ds \equiv 0, \quad x \in [0, \sqrt{\epsilon}\eta + h].$$

In consequence, since $\operatorname{supp} K \cap (-h/\sqrt{\epsilon}, \eta) \neq \emptyset$, a result of Titchmarsh (see [28, Theorem 152]) implies that

$$p(s) = g\big(\varphi(-s-h)\big) - \kappa = 0, \quad s \in [0, \sqrt{\epsilon}\eta + h].$$

Thus $\varphi(t) = \kappa$ for all $t \in [-2h - \sqrt{\epsilon}\eta, -h]$, a contradiction.

Now, when φ is not oscillating around the positive equilibrium, we can see that $y(t) = \varphi(t) - \kappa$ is either decreasing and strictly positive or increasing and strictly negative, for all sufficiently large t. Indeed, if $\varphi(t) \geqslant \kappa$, $t \geqslant -h - \sqrt{\epsilon}\eta$, has a local maximum at $t = b > 0$ then $\varphi(b) > \kappa$, $\varphi'(b) = 0$, $\varphi''(b) \leqslant 0$. In consequence, since $\varphi(+\infty) = \kappa$ and $g'(\kappa) < 0$, we get, for all large b,

$$\kappa < \varphi(b) \leqslant \int_{-\eta}^{\eta} K(s) g\big(\varphi(b - \sqrt{\epsilon}s - h)\big)\, ds \leqslant \int_{-\eta}^{\eta} K(s) g(\kappa)\, ds = \kappa,$$

a contradiction. The same argument works when $\varphi(t) \leqslant \kappa$ for all large t.

Next, observe that $y(t)$ satisfies $\epsilon y''(t) - y'(t) = y(t) + k(t) y(t - \sqrt{\epsilon}\eta - h)$, where, in view of the monotonicity of y, it holds that

$$\begin{aligned} -2g'(\kappa) &\geqslant k(t) \\ &:= -\int_{-\eta}^{\eta} K(s) \frac{g(\varphi(t - \sqrt{\epsilon}s - h)) - g(\kappa)}{\varphi(t - \sqrt{\epsilon}s - h) - \kappa} \cdot \frac{\varphi(t - \sqrt{\epsilon}s - h) - \kappa}{\varphi(t - \sqrt{\epsilon}\eta - h) - \kappa}\, ds \geqslant 0 \end{aligned}$$

for all sufficiently large t. We can use now Lemma 3.1.1 from [13] to conclude that $y(t) > 0$ cannot converge superexponentially to 0.

Claim II. $|y(t)| > 0$ *cannot hold when Eq.* (10) *does not have roots in* $(-\infty, 0)$.

We may suppose that $y(t) = \varphi(t) - \kappa > 0$, $y(+\infty) = 0$. Observe that y verifies

$$\epsilon y''(t) - y'(t) - y(t) + \int_{-\eta}^{\eta} K(s) g_1\big(y(t - \sqrt{\epsilon}s - h)\big)\, ds = 0, \quad t \in \mathbb{R},$$

where $g_1(s) := g(s+\kappa) - \kappa$, $g_1(0) = 0$, $g_1'(0) = g'(\kappa) < 0$. In virtue of Claim I and Lemma 21, we can find a real number $d > 1$ and a sequence $t_n \to +\infty$ such that $y(t_n) = \max_{s \geqslant t_n} y(s)$ and

$$\max_{s \in [t_n - 3h - 3\eta\sqrt{\epsilon}, t_n]} y(s) \leqslant d y(t_n).$$

Additionally, we can find a sequence $\{s_n\}$, $\lim(s_n - t_n) = +\infty$ such that $|y'(s_n)| \leqslant y(t_n)$. Now, $w_n(t) = y(t + t_n)/y(t_n)$, $t \in \mathbb{R}$, satisfies

$$\epsilon w''(t) - w'(t) - w(t) + \int_{-\eta}^{\eta} K(s) p_n(t - \sqrt{\epsilon}s - h) w(t - \sqrt{\epsilon}s - h)\, ds = 0,$$

where $p_n(t) = g_1(y(t + t_n))/y(t + t_n)$. It is clear that $\lim p_n(t) = g'(\kappa)$ for every $t \in \mathbb{R}$, and that $0 < w_n(t) \leqslant d$ for all $t \geqslant -3(\eta\sqrt{\epsilon} + h)$.

To estimate $|w_n'(t)|$, let $\mathscr{W}_n(t) := \int_{-\eta}^{\eta} K(s) p_n(t - \sqrt{\epsilon}s - h) w_n(t - \sqrt{\epsilon}s - h)\, ds$. Since $z_n(t) = w_n'(t)$ satisfies $z_n(s_n - t_n) = y'(s_n)/y(t_n) \in [-1, 0]$ and

$$\epsilon z_n'(t) - z_n(t) - w_n(t) + \mathscr{W}_n(t) = 0, \quad t \in \mathbb{R},$$

we obtain that

$$w_n'(t) = e^{(t + t_n - s_n)/\epsilon} z_n(s_n - t_n) + \frac{1}{\epsilon} \int_{s_n - t_n}^{t} e^{(t-s)/\epsilon} \big(w_n(s) - \mathscr{W}_n(s)\big)\, ds. \tag{28}$$

Furthermore, for each fixed $t \geqslant -2\eta\sqrt{\epsilon} - 2h$ and sufficiently large n, we have

$$\begin{aligned} \big|w_n'(t)\big| &\leqslant 1 + \frac{1}{\epsilon} \int_{t}^{s_n - t_n} e^{(t-s)/\epsilon} \left(w_n(s) + \sup_{s \neq 0} \frac{|g_1(s)|}{|s|} \int_{-\eta}^{\eta} K(u) w_n(s - \sqrt{\epsilon}u - h)\, du \right) ds \\ &\leqslant 1 + \left(\sup_{s \neq 0} \frac{|g_1(s)|}{|s|} + 1 \right) \frac{d}{\epsilon} \int_{t}^{s_n - t_n} e^{(t-s)/\epsilon}\, ds \leqslant 1 + d + d \sup_{s \neq 0} \frac{|g_1(s)|}{|s|}. \end{aligned}$$

Hence, there is a subsequence $\{w_{n_j}(t)\}$ which converges on $[-2\eta\sqrt{\epsilon} - 2h, +\infty)$, in the compact open topology, to a non-negative decreasing function $w_*(t)$, $w_*(0) = 1$, such that $w_*(t) \leqslant d$ for all $t \geqslant -2\eta\sqrt{\epsilon} - 2h$. By the Lebesgue bounded convergence theorem, we find, for all $t \in [-\eta\sqrt{\epsilon} - h, +\infty)$, that

$$\mathscr{W}_{n_j}(t) \to g'(\kappa) \int_{-\eta}^{\eta} K(s) w_*(t - \sqrt{\epsilon}s - h)\, ds.$$

In consequence, integrating (28) between 0 and t and then taking the limit as $n_j \to \infty$ in the obtained expression, we establish that $w_*(t)$ satisfies

$$\epsilon w''(t) - w'(t) - w(t) + g'(\kappa)\int_{-\eta}^{\eta} K(s)w(t-\sqrt{\epsilon}s-h)\,ds = 0 \tag{29}$$

for all $t \geqslant -\eta\sqrt{\epsilon}-h$. We claim that $w_*(t)$ is positive for $t \geqslant -\eta\sqrt{\epsilon}-h$. Indeed, if $w_*(t')=0$ for some t' then $t'>0$ since $w_*(0)=1$ and $w_*(t)$ is decreasing. Next, if t' is the leftmost positive point where $w_*(t')=0$, then (29) implies

$$\int_{\max\{-\eta,-h/\sqrt{\epsilon}\}}^{\eta} K(s)w(t'-\sqrt{\epsilon}s-h)\,ds = 0.$$

However, this contradicts to the following two facts: (i) due to the definition of t', it holds $w(t'-\sqrt{\epsilon}s-h)>0$ for all $s \in (\max\{-\eta,-h/\sqrt{\epsilon}\},\eta)$; (ii) $K(s)\geqslant 0$ and $\operatorname{supp} K \cap (-h/\sqrt{\epsilon},\eta) \neq \emptyset$. Hence, $w_*(t)>0$ and we can use Lemma 3.1.1 from [13] to conclude that $w_*(t)>0$ is not a small solution. Then Lemma 22 implies that there exists $b_0<0$ such that $w_*(t)=v(t)+O(\exp(b_0t))$, $t\to+\infty$, where v is a *non-empty* finite sum of eigensolutions of (29) associated to the eigenvalues $\lambda_j \in F = \{b_0 < \Re\lambda_j \leqslant 0\}$. Now, since the set F does not contain any real eigenvalue by our assumption, we conclude that $w_*(t)$ should be oscillating on $\mathbb{R}_+$ (see e.g. [14, Lemma 2.3]), a contradiction.

Remark 18. To establish the non-monotonicity of wavefronts in [8], the hyperbolicity of Eq. (29) and C^2-smoothness of g at κ were assumed. However, as we have shown, the first condition can be removed and it suffices to assume that g is a continuous function which is differentiable at κ.

Remark 19. For Eq. (1), Liang and Wu found numerically that the wavefronts may exhibit unsteady multihumps. As it is observed in [12,17] for these cases, the first hump (its shape, size and location) remains stable on the front of the waves, but the second hump expands in width to the positive direction as the number of iteration is increasing. However the multihump waves of [12,17] may appear due to the numerical instability of the algorithms. In fact, (H) can be used to prove that, for a fixed $\alpha>\kappa$, neither wavefront $\phi(t)$ can satisfy $\phi(t)\geqslant\alpha$ during 'sufficiently large' time interval J (the maximal admissible length of J depends on α: $|J|=2q_*(\alpha)>0$). For example, analyzing the multihumps profiles from [12,17], we may suppose that $J=(-q_*,q_*)$ and that $y(0)=\max_{s\in J} y(s)$. Then, if q_* is sufficiently large, we easily get a contradiction:

$$\begin{aligned}
\alpha \leqslant y(0) &\leqslant \int_{\mathbb{R}} K(w)g\big(y(-\sqrt{\epsilon}w-h)\big)\,dw \\
&= \int_{-(q_*+h)/\sqrt{\epsilon}}^{(q_*-h)/\sqrt{\epsilon}} K(w)g\big(y(-\sqrt{\epsilon}w-h)\big)\,dw + \int_{\mathbb{R}\setminus J} K(w)g\big(y(-\sqrt{\epsilon}w-h)\big)\,dw \\
&\leqslant g(\alpha)\mathcal{D}_1(q_*) + \big(1-\mathcal{D}_1(q_*)\big)\max_{x\geqslant 0} g(x) < \kappa < \alpha,
\end{aligned}$$

since

$$\lim_{q_*\to+\infty}\mathcal{D}_1(q_*) := \lim_{q_*\to+\infty}\int_{-(q_*+h)/\sqrt{\epsilon}}^{(q_*-h)/\sqrt{\epsilon}} K(w)\,dw = 1.$$

Appendix A

Consider $\psi(z,\epsilon) = \epsilon z^2 - z - q + p\exp(-zh)\int_{\mathbb{R}} K(s)\exp(-\sqrt{\epsilon}zs)\,ds$, where $p > q$ and $K(s)$ satisfies condition (2).

Lemma 20. *Assume that $p > q > 0$. Then there exist extended positive real numbers $\epsilon_0 \leqslant \epsilon_1$, $\epsilon_i = \epsilon_i(h, p, q)$, such that, for every $\epsilon \in (0, \epsilon_0) \cup (\epsilon_1, \infty)$, equation $\psi(\lambda, \epsilon) = 0$ has exactly two real roots $\lambda_1(\epsilon) < \lambda_2(\epsilon)$. Furthermore, $\lambda_1(\epsilon)$, $\lambda_2(\epsilon)$ are positive if $\epsilon < \epsilon_0$ and are negative if $\epsilon > \epsilon_1$. If $\epsilon \in (\epsilon_0, \epsilon_1)$, then $\psi(z,\epsilon) > 0$ for all $z \in \mathbb{R}$. Next, $\epsilon_0 = \epsilon_1$ if and only if $\epsilon_0 = \epsilon_1 = \infty$. Furthermore, $\epsilon_1 = \infty$ if $\int_{\mathbb{R}} sK(s)\,ds \geqslant 0$ and ϵ_1 is finite if the equation*

$$z^2 - q + p\int_{\mathbb{R}} \exp(-zs)K(s)\,ds = 0 \tag{30}$$

has two negative roots. Finally, if $\int_{\mathbb{R}} xK(x)\,dx \leqslant 0$ then ϵ_0 is finite and

$$c_* := 1/\sqrt{\epsilon_0} > \left|\int_{\mathbb{R}} sK(s)\,ds\right| / (h + 1/p). \tag{31}$$

Proof. Observe that $\psi''_z(z,\epsilon) > 0$, $z \in \mathbb{R}$, so that $\psi(z,\epsilon)$ is strictly concave with respect to z. This guaranties the existence of at most two real roots. Next, since $\psi(z,0)$ has a unique real (positive) root z_0, where $\psi'_z(z_0,0) < 0$, we find that $\psi(z,\varepsilon)$ possesses exactly two positive roots for all small $\epsilon > 0$.

After introducing a new variable $w = \sqrt{\epsilon}z$, we find that equation $\psi(z,\epsilon) = 0$ takes the following form

$$\left(q + \frac{w}{\sqrt{\epsilon}} - w^2\right)\exp\left(\frac{wh}{\sqrt{\epsilon}}\right) = p\int_{\mathbb{R}} \exp(-ws)K(s)\,ds\ \big(:= G(w)\big). \tag{32}$$

As we have seen, Eq. (32) may have at most two real roots and, for small $\epsilon > 0$, it possesses two positive roots $w_1(\epsilon) < w_2(\epsilon)$. Furthermore, we have that $G(0) = p$, $G''(w) > 0$. An easy analysis of (32) shows that positive $w_1(\epsilon) < w_2(\epsilon)$ exist and depend continuously on ϵ from the maximal interval $(0, \epsilon_0)$, where ϵ_0, when finite, is determined by the relation $w_1(\epsilon_0) = w_2(\epsilon_0)$. To prove that Eq. (32) does not have any real positive root for $\epsilon > \epsilon_0$, it suffices to note that $G(w)$ does not depend on ϵ while the left-hand side of (32) decreases with respect to ϵ at every positive point w where $q + w/\sqrt{\epsilon} - w^2 > 0$.

Similarly, for $\epsilon > \epsilon_0$, the left-hand side of (32) increases to $q - w^2$ with respect to ϵ at every $w < 0$ where $q + w/\sqrt{\epsilon} - w^2 > 0$. In consequence, ϵ_1 is finite if and only if Eq. (30) has two simple negative roots. It is evident that this may happen only if $G'(0) = -\int_{\mathbb{R}} sK(s)\,ds > 0$ and that in this case $\epsilon_1 > \epsilon_0$.

Clearly, $\psi'_z(0,\epsilon_0) < 0$. For $\int_{\mathbb{R}} xK(x)\,dx \leqslant 0$, the latter inequality amounts to (31). It is easy to see that the equality $\epsilon_0 = +\infty$ actually can happen when $\int_{\mathbb{R}} xK(x)\,dx > 0$. □

Next propositions are crucial in the proof of Theorem 15.

Lemma 21. *Let $y : \mathbb{R}_+ \to (0, +\infty)$ satisfy $y(+\infty) = 0$. Given an integer $d > 1$ and a real $\rho > 0$, we define $\alpha = (\ln d)/\rho > 0$. Then either* (a) *$y(t) = O(e^{-\alpha t})$ at $t = +\infty$, or* (b) *there exists a sequence $t_j \to +\infty$ such that $y(t_j) = \max_{s \geqslant t_j} y(s)$ and $\max_{s \in [t_j - \rho, t_j]} y(s) \leqslant dy(t_j)$.*

Proof. See [31, Lemma 23]. □

The proof of the next lemma follows that of Proposition 7.1 from [22]. When $K(s) = \delta(s)$ is a Dirac delta function, the obtained asymptotic estimates for y are uniform in ϵ, see [1, Lemma 4.1].

Lemma 22. *Let $y \in C^2(\mathbb{R}, \mathbb{R})$ verify the equation*

$$y''(t) + \alpha y'(t) + \beta y(t) + p \int_{\mathbb{R}} K(s) y(t + qs + h)\, ds = f(t), \quad t \geqslant 0, \tag{33}$$

where K satisfies (2)*, $\alpha, \beta, p, q, h \in \mathbb{R}$ and $f(t) = O(\exp(-bt))$, $t \to +\infty$ for some $b > 0$. Suppose further that $\sup_{t \geqslant 0} |y(t)|$ is finite and, in the case of K with non-compact support, it holds $|y(t)| \leqslant c \exp(\gamma t)$, $t \leqslant 0$, for some $\gamma \leqslant 0$. Then, given $\sigma \in (0, b)$, we have that*

$$y(t) = w(t) + \exp\bigl(-(b - \sigma)t\bigr) o(1), \quad t \to +\infty,$$

where $w(t)$ is a finite sum of eigensolutions of (33) *associated to the eigenvalues $\lambda_j \in \{-(b - \sigma) < \Re \lambda_i \leqslant 0\}$.*

Proof. Remark that the conditions of Lemma 22 imply that $\sup_{t \geqslant 0} |y''(t)|$ is finite and that $|y'(t)| = O(1)$ at $t = +\infty$ (if $\alpha \neq 0$) or $|y'(t)| = O(t)$ (if $\alpha = 0$). The proof of this observation is based on deriving estimations similar to (17) and is omitted here. Applying the Laplace transform $\mathcal{L}$ to (33), we obtain that $\chi(z)\tilde{y}(z) = \tilde{f}(z) + r(z)$, where $\tilde{y} = \mathcal{L}y$, $\tilde{f} = \mathcal{L}f$ and

$$r(z) = y'(0) + zy(0) + \alpha y(0) + pe^{zh} \int_{\mathbb{R}} K(s) e^{zqs}\, ds \int_0^{h+ps} e^{-zu} y(u)\, du,$$

$$\chi(z) = z^2 + \alpha z + \beta + pe^{zh} \int_{\mathbb{R}} K(s) e^{qzs}\, ds.$$

Since y is bounded on $\mathbb{R}_+$, we conclude that $\tilde{y}$ is analytic in $\Re z > 0$. Moreover, from the growth restrictions on y, f, K we obtain that r is an entire function and $\tilde{f}$ is holomorphic in $\Re z > -b$. Therefore $H(z) = (\tilde{f}(z) + r(z))/\chi(z)$ is meromorphic in $\Re z > -b$. Observe also that $H(z) = O(z^{-1})$, $z \to \infty$, for each fixed strip $\Pi(s_1, s_2) = \{s_1 \leqslant \Re z \leqslant s_2\}, s_1 > -b$. Now, let $\sigma > 0$ be such that the vertical strip $-b < \Re z < -b + 2\sigma$ does not contain any zero of $\chi(z)$. By the inversion formula, for some sufficiently small $\delta > 0$, we obtain that

$$y(t) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} e^{zt} \tilde{y}(z)\, dz = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} e^{zt} H(z)\, dz = w(t) + u(t), \quad t > 0,$$

where

$$w(t) = \sum_{-b+\sigma < \Re \lambda_j \leqslant 0} \operatorname{Res}_{z=\lambda_j} \frac{e^{zt}(\tilde{f}(z) + r(z))}{\chi(z)} = \sum_{-b+\sigma < \Re \lambda_j \leqslant 0} e^{\lambda_j t} P_j(t),$$

$$u(t) = \frac{1}{2\pi i} \int_{-b+\sigma - i\infty}^{-b+\sigma + i\infty} e^{zt} H(z)\, dz.$$

Now, observe that on any vertical line in $\Re z > -b$ which does not pass through the poles of $\chi(z, \epsilon)$ and $0 \in \mathbb{C}$, we have

$$H(z) = a(z) + \frac{y(0)}{z}, \quad \text{where } a(z) = O\left(z^{-2}\right),\ z \to \infty.$$

Therefore, for $a_1(s) = a(-b + \sigma + is)$, we obtain

$$u(t) = \frac{e^{(-b+\sigma)t}}{2\pi i}\left\{\int_{-\infty}^{+\infty} e^{ist} a_1(s)\,ds\right\} + \frac{y(0)}{2\pi i}\int_{-b+\sigma-i\infty}^{-b+\sigma+i\infty} \frac{e^{zt}}{z}\,dz, \quad t > 0.$$

Next, since $a_1 \in L_1(\mathbb{R})$, we have, by the Riemann–Lebesgue lemma, that

$$\lim_{t\to+\infty}\int_{\mathbb{R}} e^{ist} a_1(s)\,ds = 0.$$

For $t > 0$, a direct computation shows that $\int_{-b+\sigma-i\infty}^{-b+\sigma+i\infty} z^{-1}e^{zt}\,dz = 0$. Thus we get $u(t) = e^{-(b-\sigma)t}o(1)$, and the proof is completed. □

Lemma 23. *If y verifies* (20) *and the conditions $y(a) = y_0$, $y'(b) = 0$, then*

$$y(b) = \xi(b-a)\left\{y_0 + \frac{1}{\epsilon(\mu-\lambda)}\int_a^b \left(e^{\lambda(a-u)} - e^{\mu(a-u)}\right)(Gy)(u-h)\,du\right\},$$

where ξ is defined in (22)*.*

Proof. It suffices to consider the variation of constants formula for (20):

$$y(t) = Ae^{\lambda t} + Be^{\mu t} + \frac{1}{\epsilon(\mu-\lambda)}\left\{\int_a^t e^{\lambda(t-s)}g(s)\,ds + \int_t^b e^{\mu(t-s)}g(s)\,ds\right\},$$

where $g(s) := (Gy)(s-h)$. □

The following proposition can be deduced from Singer's results (see, e.g., [18, Proposition 3.3] or [6, Proposition 1]).

Proposition 24. *Assume that $f : [\zeta_*, \zeta^*] \to [\zeta_*, \zeta^*]$, $f \in C^3[a, b]$, is either strictly decreasing function or it has only one critical point s_M (maximum) in $[\zeta_*, \zeta^*]$. Let also $|f'(\kappa)| \leqslant 1$, where κ is the unique fixed point of f. If the Schwarzian derivative satisfies $(Sf)(s) < 0$ for all $s \neq s_M$ then κ is globally asymptotically stable.*

The condition of the negativity of Sg (which requires C^3-smoothness of g) can be weakened with the use of a generalized Yorke condition introduced in [18] and analyzed in [19] from the biological point of view.

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