Europ. J. Combinatorics (1982) **3**, 159–172

Intersection Graphs of k-uniform Linear Hypergraphs

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A finite hypergraph H is said to be linear if every pair of distinct vertices of H is in at most one edge of H. A 2-uniform linear hypergraph is called a graph. The edge-degree of an edge of a graph G is the number of triangles in G containing the given edge. In this paper it is proved that there is a finite family $\mathbb{F}$ of graphs such that any graph G with minimum degree at least 69 is the intersection graph of a 3-uniform linear hypergraph if and only if G has no induced subgraph isomorphic to a member of $\mathbb{F}$. Further, it is shown that there is a polynomial $f(k)$ of degree less than or equal to 3 with the property that given any integer k (≥ 2) there exists a finite family $\mathbb{F}(k)$ of graphs such that any graph G with minimum edge-degree at least $f(k)$ is the intersection graph of a k-uniform linear hypergraph if and only if G has no induced subgraph isomorphic to a member of $\mathbb{F}(k)$.

1. INTRODUCTION

A *k-uniform hypergraph* H is a pair (X, E) such that E is a subset of $\mathbb{P}_k(X)$, the set of all k-subsets of a finite set X, where k is an integer and $k \geq 2$. Elements of X are called the *vertices*, while those of E are called the *edges* of H. A hypergraph H is said to be *linear* if every pair of distinct vertices of H is in at most one edge of H. *Throughout this paper we will consider only linear hypergraphs.* A 2-uniform linear hypergraph is called a graph. For a graph G, the set of its vertices will be denoted by $V(G)$, while that of its edges by $E(G)$. The *intersection graph* of a hypergraph $H = (X, E)$, denoted by $G(H)$, is the graph where $V(G(H)) = E$ and $E(G(H))$ is the set of all unordered pairs $\{e, e'\}$ of distinct elements of E such that $|e \cap e'| = 1$ in H, where $|A|$ for any set A denotes its cardinality. Note that, by linearity of H, $|e \cap e'| \leq 1$ always. The intersection graphs of graphs are called line graphs. Let $\mathcal{I}(k)$ be the set of all graphs which are isomorphic to intersection graphs of k-uniform linear hypergraphs.

The following Theorem 1 is due to Beineke [2] (see Harary [4, p. 74]).

THEOREM 1. *A graph G is a line graph if and only if none of the nine graphs of Figure 1 is an induced subgraph of G.*

We say that a family $\mathcal{M}$ of graphs is characterized by a *family $\mathbb{F}$ of forbidden graphs* if any graph G belongs to $\mathcal{M}$ if and only if G has no induced subgraph isomorphic to a member of $\mathbb{F}$. The above theorem gives a family of nine forbidden graphs for line graphs.

The following infinite family $\mathcal{G}_1 = \{G_1(t), t$ is a positive integer$\}$ of graphs of Figure 2, none of whose members belongs to $\mathcal{I}(3)$ but all the proper vertex-induced subgraphs of every $G_1(t)$ belong to $\mathcal{I}(3)$, shows that the family $\mathcal{I}(3)$ cannot be characterized by a finite family of forbidden graphs (for a proof of this, use Proposition 2.1 or Proposition 2.3). The graph $G_1(t)$ is obtained by arranging $t+2$ copies of $C_4 - e$, the complete graph of order 4 less an edge e, in the form of a chain to get a graph with maximum degree less than or equal to 4 and attaching two pendant edges at each of the two degree two vertices of the graph thus obtained. (See Remark 2.4 for some more infinite families of vertex minimal forbidden subgraphs for $\mathcal{I}(3)$ and see also [1, 5].)

However, in Section 4 we will prove the following Theorem.

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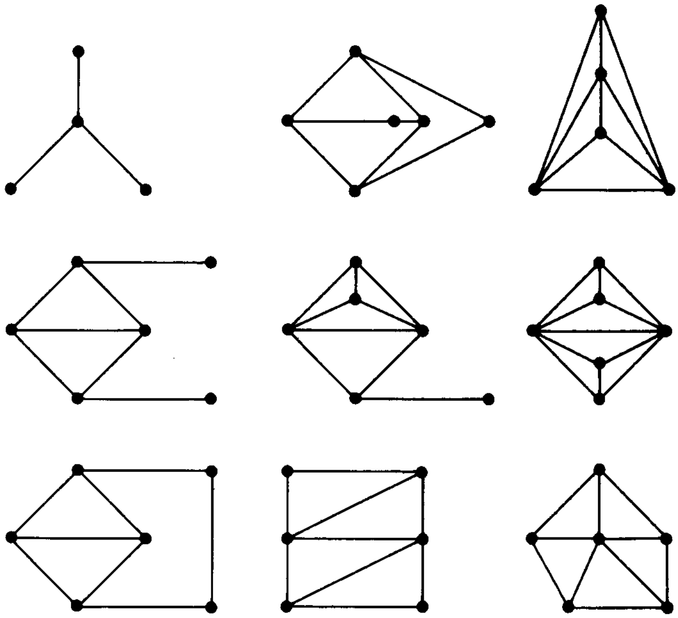

FIGURE 1. The nine forbidden subgraphs for line graphs.

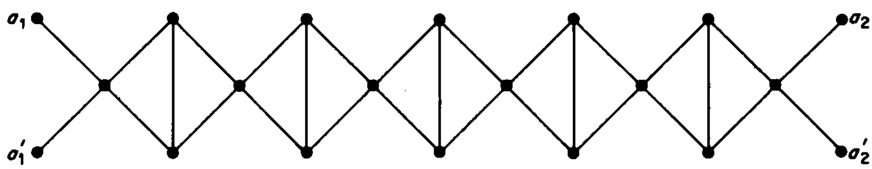

FIGURE 2. The graph $G_1(3)$.

THEOREM 2. *There is a finite family $\mathbb{F}$ of forbidden graphs such that any graph G with minimum degree at least* 69 *belongs to $\mathcal{I}(3)$ if and only if G has no induced subgraph isomorphic to a member of $\mathbb{F}$.*

Of course, one infinite family $\mathcal{G}_k^*$ of vertex minimal forbidden graphs for $\mathcal{I}(k)$, $k \geq 3$, may be obtained inductively as

$$\mathcal{G}_3^* = \mathcal{G}_1 \quad \text{and} \quad \mathcal{G}_k^* = \{G^*/G^*$$

is the graph obtained by attaching one pendant edge at every vertex of degree k in G where $G \in \mathcal{G}_{k-1}^*\}$ (for a proof, use Proposition 2.3). We believe that members of $\mathcal{I}(k)$, $k \geq 4$, with sufficiently large minimum degree cannot be characterized by a finite family of forbidden graphs.

The *edge-degree* $\delta(e)$ in G of an edge e of G is the number of triangles of G containing e. The *minimum edge-degree* of G is the minimum of $\delta(e)$, $e \in E(G)$.

In Section 3 we will prove the following theorem.

THEOREM 3. *There is a polynomial $f(k)$ of degree at most* 3 *with the property that, given any k, there exists a finite family $\mathbb{F}(k)$ of forbidden graphs such that any graph G with minimum edge-degree at least $f(k)$ is a member of $\mathcal{I}(k)$ if and only if G has no induced subgraph isomorphic to a member of $\mathbb{F}(k)$.*

We will prove this result by taking $f(k)=k^3-2k^2+1$. In this paper, we have not tried to find all the minimal families of forbidden graphs for $\mathscr{I}(k)$. The problem of characterizing all the minimal families of forbidden graphs for $\mathscr{I}(3)$ will be discussed in a subsequent communication.

Pertinent definitions are given below and at appropriate junctures of Sections 3 and 4. For graph theoretic terms not defined here and for notation not explained here the reader is referred to Harary [4].

Let $G=(X, E)$ be a graph. The *order* of G is $|X|$. For any $A\subseteq X$, $G[A]$ denotes the induced subgraph of G on A. abc is said to be a *triangle* in G if $G[\{a, b, c\}]$ is the complete graph of order 3. If $e\in E$ and $e=\{a, b\}$, then we will also denote e by ab. For a set $A\subseteq X$, let $N(A)$ be the set of all vertices $y\in X$ such that $yx\in E(G)$ for every $x\in N(A)$. If $A=\{x\}$, we will simply write $N(x)$ instead of $N(\{x\})$. A set

$$A=\{x; a_1, a_2, \ldots, a_r\}, \qquad r\geqslant 2,$$

of G is said to be a *claw* $\langle x; a_1, \ldots, a_r\rangle$ *at* x *of size* r in G if $G[\{a_1, \ldots, a_r\}]$ is the graph without edge and $\{a_1, \ldots, a_r\}\subseteq N(x)$. A claw of size r is called an r-claw. A *clique* A in G is a set of vertices of G such that $G[A]$ is the complete graph. An edge $e=ab$ of G is said to be in a clique A if A is a clique and both a, b belong to A. A triangle is said to be *odd* if there is a vertex n adjacent to exactly an odd number of vertices of the triangle. A triangle is said to be *even* if it is not odd.

2. Preliminaries

In this section, we first present a characterization of the members of the family $\mathscr{I}(k)$, which is a generalization of a criterion of line graphs due to Krausz [6] (see Harary [4, p. 74]) and is global in nature.

PROPOSITION 2.1. *If $G=(X, E)$ is a graph, then $G\in\mathscr{I}(k)$ if and only if in G there exists a set $\mathscr{S}=\{K_1, \ldots, K_r\}$ of cliques with $|K_i|\geqslant 2$, $1\leqslant i\leqslant r$, such that the following two conditions hold:*

(i) *every edge of G is in a unique clique K_i of $\mathscr{S}$.*
(ii) *every vertex of G is in at most k cliques of $\mathscr{S}$.*

PROOF. First suppose that $G\in\mathscr{I}(k)$ and $G=G(H)$. For every $x\in X$, let $K(x)=\{e\in H/x\in e\}$. Then $K(x)$ is a clique in G. Let $\mathscr{S}=\{K_1, \ldots, K_r\}$ be the set of all distinct cliques of size at least 2 of G thus obtained. Then $\mathscr{S}$ satisfies conditions (i) and (ii) of the proposition.

Conversely, given a set $\mathscr{S}$ of cliques of G satisfying conditions (i) and (ii), define, for every $x\in X$, $m(x)$ as the number of cliques of $\mathscr{S}$ containing x. Obtain a new collection $\mathscr{S}'$ of, not necessarily distinct, cliques of G from $\mathscr{S}$ by including $k-m(x)$ copies of the clique $\{x\}$ for every $x\in X$ with $0\leqslant m(x)<k$. Let $\mathscr{S}'=\{K_1, \ldots, K_n\}$. To each element of $\mathscr{S}'$ take a vertex y_i, $1\leqslant i\leqslant n$, and define a hypergraph H on $X=\{y_1, \ldots, y_n\}$ by insisting that a k-subset of X, $\{y_{i_1}, \ldots, y_{i_k}\}$ is an edge of H if and only if $\bigcap_{j=1}^{k} K_{i_j}$ is non-empty. It is then easy to check, using conditions (i) and (ii), that H is a k-uniform linear hypergraph and that $G(H)\simeq G$ under the mapping $\{y_{i_1}, \ldots, y_{i_k}\}\to\bigcap_{j=1}^{k} K_{i}$. Therefore $G\in\mathscr{I}(k)$.

Taking $k=2$ in the above proposition we obtain the characterization of line graphs due to Krause [6].

The following criterion of line graphs, local in nature, is due to Van Rooij and Wilf [7] (see Harary [4, p. 74]) and is a better characterization of line graphs than the above proposition with $k=2$.

PROPOSITION 2.2. *If G is a graph, then G is a line graph if and only if the following two conditions are satisfied in G:*
(i) *G has no 3-claw,*
(ii) *if abc, abd are two distinct odd triangles in G, then $cd \in E(G)$.*

The following Proposition 2.3 gives a characterization of the family $\mathscr{I}(k)$, and is slightly better than the one given Proposition 2.1. For the case $k = 2$, it is similar to, though not as good as, Proposition 2.2.

PROPOSITION 2.3. *If G is a graph, then $G \in \mathscr{I}(k)$ if and only if G has a set T of triangles satisfying the following two conditions:*
(i) *if abc, abd are in T with $c \neq d$, then $cd \in E(G)$ and acd, bcd are also in T;*
(ii) *given any $k+1$ distinct edges of G, all having a vertex in common, at least two of these edges are in a triangle of G which is in T.*

PROOF. First suppose that $G \in \mathscr{I}(k)$. Then by Proposition 2.1, there exists a set $\mathscr{S} = \{K_1, \ldots, K_r\}$ of cliques of G satisfying Proposition 2.1 (i) and (ii). Let T be the set of all triangles *abc* of G with the property the $a, b, c \in K_i$ for some i, $1 \leq i \leq r$. We assert that T satisfies conditions (i) and (ii) of the proposition. Suppose that *abc*, *abd* are in T with $c \neq d$ and let K_i, K_j be cliques in $\mathscr{S}$ containing *abc*, *abd* respectively. Since $ab \in K_i \cap K_j$, it follows by Proposition 2.1 (i) that $i = j$ and this implies that $cd \in E(G)$ and *acd*, *bcd* are also in T. To prove (ii), let $e_1, \ldots, e_{k+1}$ be $k+1$ distinct edges of G incident at the same vertex x. By Proposition 2.1 (i), it follows that $e_i \in K_{j_i}$ (say), where $1 \leq j_i \leq r$; for every $1 \leq i \leq k+1$. By Proposition 2.1 (ii), it then follows that $K_{j_{i1}} = K_{j_{i2}}$ for some $i_1 \neq i_2$, $1 \leq i_1$, $i_2 \leq k+1$. This implies that $e_{i_1}, e_{i_2} \in K_{j_{i1}}$, and hence these two edges are in a triangle of G which is in T.

Conversely, suppose that G has a set T of triangles satisfying (i) and (ii). Let $\mathscr{S}' = \{K_1, \ldots, K_l\}$ be the set of all maximal complete subgraphs of G with $|K_i| \geq 3$, $1 \leq i \leq l$ having the property that each triangle of K_i is in T. Let $\mathscr{S}'' = \{K_{l+1}, \ldots, K_r\}$ be the edges of G not contained in any triangle in T and $\mathscr{S} = \mathscr{S}' \cup \mathscr{S}''$. We shall show that $\mathscr{S}$ satisfies the conditions of Proposition 2.1. To prove (i), it is enough to show that each edge *ab* of G is in at most one clique in $\mathscr{S}'$. Suppose that $ab \in K_i \cap K_j$, $i \neq j$, $1 \leq i$, $j \leq l$. By the maximality of K_i and K_j, there exists a vertex $c \in K_i$ such that $c \notin K_j$. Then for any $d \in K_j$ with $d \neq a, b$, since $abc \in K_i$ and $abd \in K_j$, it follows by the definition of K_i and condition (i) that *abc*, *abd* are in T where $c \neq d$. Therefore by (i) of the hypothesis it follows that *acd*, *bcd* are also in T. This being true for every $d \in K_j$ it follows that K_j is not maximal, contradicting its definition. To prove Proposition 2.1 (ii), let, if possible, x be any vertex of G belonging to s distinct cliques $K_{i_1}, \ldots, K_{i_s}$, where $s \geq k+1$. Choose $y_j \in K_{i_j}$, $y_j \neq x$, $1 \leq j \leq s$. Then, by Proposition 2.1 (i), it follows that $y_j \neq y_l$ whenever $j \neq l$. Therefore xy_j, $1 \leq j \leq k+1$, are $k+1$ distinct edges of G having the vertex x in common. Now by condition (ii) of the hypothesis, we infer that two of these edges, xy_1, xy_2 say, lie in a triangle in T and then by the definition of K_{i_1}, K_{i_2} and condition (i) it follows that xy_1y is in T for every $y(\neq x)$ in K_{i_2}, contradicting the maximality of K_{i_1}.

REMARK 2.4. We now describe several infinite families of vertex minimal forbidden graphs for the family $\mathscr{I}(3)$. Let t be a positive integer. Let $H(t)$ be the graph with maximum degree 4 obtained by arranging t copies of $C_4 - e$, the complete graph of order 4 less an edge e, in the form of a chain. Let $G_2(t)$ be the graph obtained from $H(t)$ by attaching two pendant edges at one of the two degree 2 vertices of $H(t)$ and attaching the graph of Figure 3 at the other degree 2 vertex of $H(t)$ (as shown in Figure 4). Let $\mathscr{G}_2 = \{G_2(t)/t \geq 1\}$. Let $G_3(t)$ be the graph obtained from $H(t)$ by attaching a copy of the

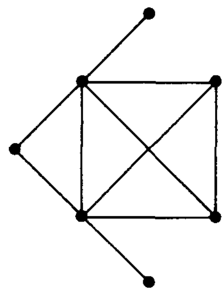

FIGURE 3.

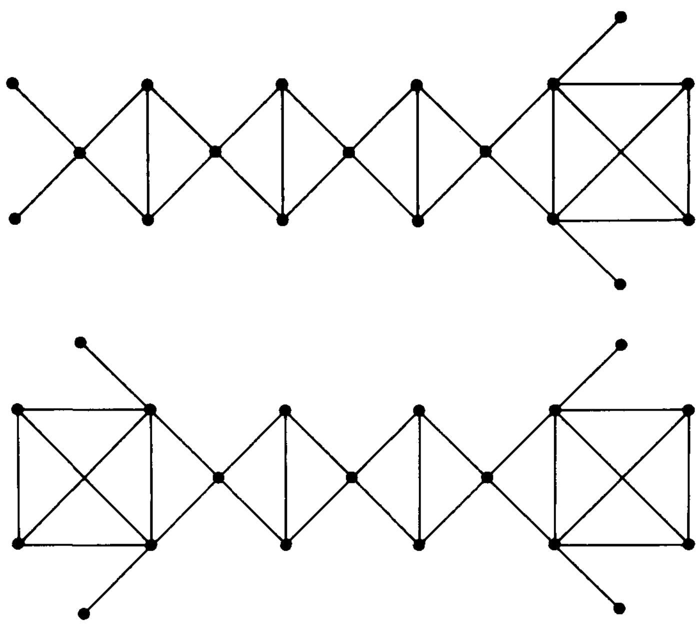

FIGURE 4. The graphs $G_2(3)$ and $G_3(2)$.

graph of Figure 3 at each of the two degree 2 vertices of $H(t)$ (as shown in Figure 4). Let $\mathscr{G}_3=\{G_3(t)/t\geq 1\}$. Typical examples of $G_2(t)$ and $G_3(t)$ are given in Figure 4.

Consider $G_1(t)$ of Figure 2 with t even and $t\geq 2$ and add edges to $G_1(t)$ to get a graph $G_4(t)$ on the same set of vertices as $G_1(t)$, such that the following four conditions are satisfied:

(i) $(a_1, a_2), (a_1', a_2')\in E(G_4)$,

(ii) degrees of a_i and a_i' are less than or equal to 3, for $i=1, 2$,

(iii) the maximum degree in $G_4(t)\leq 4$,

(iv) $G_4(t)$ and $G_1(t)$ have the same set of triangles (for some t there may be several possibilities for $G_4(t)$).

Let $\mathscr{G}_4=\{G_4(t)/t$ is an even integer greater or equal to 2$\}$. See Figure 5 for a typical example.

It is easy to check using Proposition 2.3 that $\mathscr{G}_i$, $i=1, 2, 3, 4$ are infinite families of vertex minimal forbidden graphs for the family $\mathscr{I}(3)$.

The following proposition will be used in Section 4.

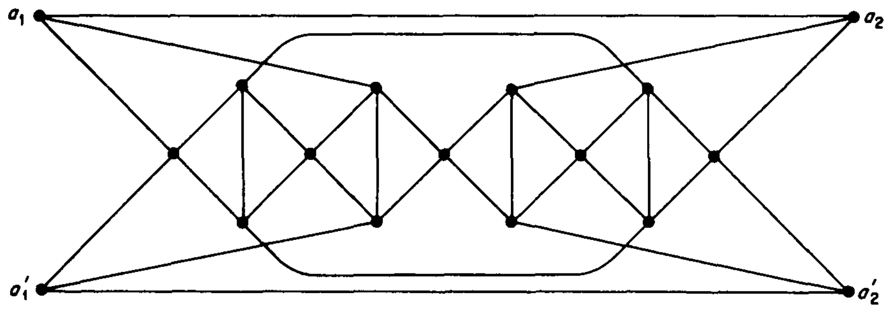

FIGURE 5. A $G_4(2)$ graph.

PROPOSITION 2.5. *If G is a graph such that G has a set S, possibly empty, of triangles satisfying*
(i) *G has no 3-claw,*
(ii) *all odd triangles of G are in S,*
(iii) *abc, abd* $\in S$ *with* $c \neq d$, *implies that* $cd \in E(G)$ *and acd, bcd are also in S; then there exists a set T, possibly empty, of triangles in G with the property that T contains S, and T satisfies Proposition* 2.3 (i) *and* (ii) *with* $k = 2$. *Further, no triangle in S has an edge in common with a triangle in* $T - S$.

PROOF. By Proposition 2.2, it follows that G is a line graph. Let $G_1, \ldots, G_s$ be the connected components of G. Consider a given G_i, $1 \leq i \leq s$, and let a, b, c be vertices of G_i such that $abc \in S$. If abc is an even triangle of G, then either $G_1 \simeq H_1$ of Figure 6,

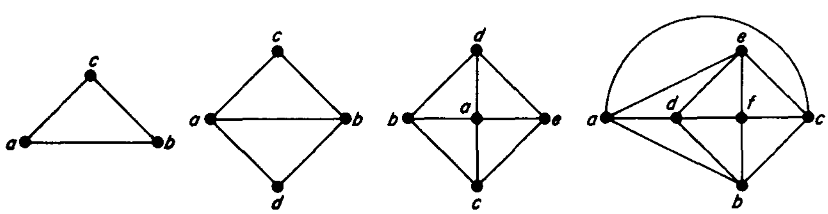

FIGURE 6. The graphs H_1, H_2, H_3 and H_4 respectively.

or there is a vertex d of G_i such that d is joined to exactly two vertices of abc, say $da, db \in E(G)$. Then by (iii), $abd \notin S$ and therefore in particular abd is an even triangle of G. Now it is easy to check that $G_i \simeq H_j$, $2 \leq j \leq 4$, of Figure 6 (see Harary [3, p. 77]). Now if S_i, $1 \leq i \leq s$, denotes the subset of triangles in S of the graph G_i, and if $G_i \not\simeq H_j$, $1 \leq j \leq 4$, $1 \leq i \leq s$, then S_i is precisely the set of all odd triangles of G_i; further, if G_i is isomorphic to H_1, H_2, H_3 or H_4, then S_i is a subset of $T_1' = \{abc\}$, $T_2' = \{abc\}$, $T_3' = \{abc, ade\}$ and $T_4' = \{abc, ade, cef, bdf\}$, respectively. Define

$$T_i = \begin{cases} S_i \text{ if } G_i \not\simeq H_j, & 1 \leq j \leq 4 \\ T_j' \text{ if } G_i \simeq H_j, & 1 \leq j \leq 4; \end{cases}$$

and $T = \bigcup_{i=1}^{s} T_i$. Then it is easy to check that T contains S and T satisfies Proposition 2.3 (i) and (ii) with $k = 2$. Further, no triangle in S has an edge in common with a triangle in $T - S$.

3. Proof of Theorem 3

Before embarking on the proof of Theorem 3 we assume for Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5 that $G \in \mathcal{I}(k)$ and $G \simeq G(H)$ under the identity map.

Let $\mathcal{A}(k)$ be the finite family of all graphs Γ of order k^2+3 in which there exist two distinct vertices a, b in $V(\Gamma)$ such that $ab \notin E(\Gamma)$ and

$$N(\{a, b\}) = V(\Gamma) - \{a, b\}.$$

Lemma 3.1. *G has no induced subgraph isomorphic to a member of $\mathcal{A}(k)$.*

Proof. Suppose that $G = G(H)$ has an induced subgraph $\Gamma \in \mathcal{A}(k)$ with a, b as specified in the definition of Γ; and $a = \{x_1, \ldots, x_k\}$, $b = \{y_1, \ldots, y_k\}$ in H. Now $a \cap b = \varnothing$, and if $c \in N(\{a, b\})$ in Γ, then in H, c must contain an x_l and a y_m. Since H is linear, a pair $\{x_l, y_m\}$ can occur in at most one edge of H. Hence $|N(\{a, b\})| \leq k^2$ in Γ, contradicting the definition of Γ.

Lemma 3.2. *G has no $(k+1)$-claw.*

Proof. Suppose that $\langle e, e_1, \ldots, e_r\rangle$ is an r-claw in $G = G(H)$. Then in H, $|e \cap e_i| = 1$, $e_i \cap e_j = \varnothing$ for $i \neq j$ and $|e| = k$. Therefore $r \leq k$.

Before describing another forbidden family for $\mathcal{I}(k)$ we prove a lemma, which is a particular case of a theorem of Deza [3], using the pigeon-hole-principle for the sake of completeness.

Lemma 3.3. *If G has a clique K with $|K| \geq k^2 - k + 2$, then there is a vertex x of H such that $x \in e$ in H for all $e \in K$.*

Proof. Let $e' \in K$ and $e' = \{x_1, \ldots, x_k\}$ in H. Since K is a clique in G, for every vertex $e \in K$ we have that $|e \cap e'| = 1$ in H. Since, $|K| \geq k^2 - k + 2$, it follows by the pigeon-hole principle that some vertex x_i of e' is in at least $k+1$ vertices of K. Also, since any e in K intersects these $k+1$ vertices, and $|e| = k$, this implies that $x_i \in e$ for every $e \in K$.

Let $\mathbb{B}(k)$ be the finite family of all graphs Γ of order less than or equal to $2(k^2 - k + 1)$ such that

(i) there exists a set $A = \{a, b, c, d\}$ with $|A| = 4$ in Γ such that $\Gamma[A] = C_4 - x$, the complete graph of order 4 less the edge x, with $x = ad$ and

(ii) the triangle abc is in a clique L of G and the triangle bcd is in a clique M of G with $|L| \geq k^2 - k + 2$ and $|M| \geq k^2 - k + 2$ (note that L and M may have some vertices $\neq b, c$ in common).

Lemmas 3.4 and 3.5 follow easily from Lemma 3.3.

Lemma 3.4. *G has no induced subgraph isomorphic to a member of $\mathbb{B}(k)$.*

Let $\mathcal{C}(k)$ be the finite family of all graphs Γ of order $k^2 - k + 3$ such that

(i) there exists $a \in V(\Gamma)$ with the property that $V(\Gamma) - \{a\}$ is a clique in Γ and

(ii) $k + 1 \leq \deg a \leq k^2 - k + 1$.

Lemma 3.5. *G has no induced subgraph isomorphic to a member of $\mathcal{C}(k)$.*

Define $\mathbb{F}(k)=\mathscr{A}(k)\cup\mathbb{B}(k)\cup\mathscr{C}(k)\cup\{k+1\text{-claw}\}$. Thus we have shown so far that if $G\in\mathscr{I}(k)$, then G has no induced subgraph isomorphic to a member of the finite family $\mathbb{F}(k)$.

We will now complete the proof of Theorem 3. *For the remainder of this section we will assume that G is a graph with at least one edge which has no induced subgraph isomorphic to a member of $\mathbb{F}(k)$ and the minimum edge-degree of G is at least $f(k)=k^3-2k^2+1$ a polynomial of degree* 3. We then show that $G\in\mathscr{I}(k)$. To this end we need the following four Lemmas 3.6, 3.7, 3.8 and 3.9 which together culminate in the fact that the set $M(G)$ of all maximal cliques in G of size at least k^2-k+2 satisfies the conditions of Proposition 2.1.

Lemma 3.6. *Every edge of G is in a member of $M(G)$.*

Proof. Let $e=xy\in E(G)$ and $S=\langle x; y, w_1, \ldots, w_r\rangle$ be a maximal claw at x, containing y, in G. Since G has no $(k+1)$-claw we have that $0\leq r\leq k-1$. We consider two cases.

Case 1. $r>0$.

Let $T=\langle x; w_1, \ldots, w_r, s_1, \ldots, s_t\rangle$ be a claw at x of the maximum possible size which has $\langle x; w_1, \ldots, w_r\rangle$ as a subclaw. Note that if $y=s_i$, for some i, then $t=1$. By the maximality of S, we have that $ys_i\in E(G)$ for every i, $1\leq i\leq t$, with $y\neq s_i$; and also $r+t\leq k$. Define $z=s_t$. Now $|N(\{z, g\})|\leq k^2$ for $g=w_j$, s_i whenever $g\neq y$ where $1\leq j\leq r$, $1\leq i\leq t-1$, for otherwise G will have an induced subgraph isomorphic to a member of $\mathscr{A}(k)$. Since the edge-degree of xy is at least $f(k)=k^3-2k^2+1$ it follows that there exists a set A, not containing y and contained in $N(\{x, z\})$ of at least $f(k)-(k^2-1)(k-1)-1=k^2-k-1$ vertices of G or $f(k)-(k^2-1)(k-1)=k^2-k$ vertices of G according as $z\neq y$ or $z=y$, such that none of w_j, s_i, $1\leq j\leq r$, $1\leq i\leq t-1$, is joined to a vertex in A. By the maximality of S it follows that $y\in N(A)$ and also any two of the vertices in A are joined to each other, for otherwise we get a claw of size at least $r+t+1$ which has $\langle x; w_1, \ldots, w_r\rangle$ as a subclaw. Thus all the vertices in A together with x, y, z form a clique of size at least k^2-k+2. Let M be a maximal clique containing the above clique. Then $M\in M(G)$ and $xy\in M$.

Case 2. $r=0$.

This implies that $yw\in E(G)$ whenever $xw\in E(G)$. If now y_1, y_2 are vertices joined to x such that $y_1y_2\notin E(G)$, then arguing as in Case 1, with xy_1 instead of xy, we get a member M of $M(G)$ containing xy_1. Clearly, $xy\in M$. Thus we may assume $M=N(x)\cup\{x\}$ is a complete graph in G. Since the edge-degree of xy is at least k^3-k, it follows that $|M|\geq k^3-k+2>k^2-k+1$ since $k>1$.

Lemma 3.7. *Every edge of G is in at most one member of $M(G)$.*

Proof. Suppose that there are two distinct elements $K_1, K_2\in M(G)$ containing the edge xy. Then there are vertices $c\in K_1$ and $d\in K_2$ such that for $A=\{x, y, c, d\}$, $G[A]=C_4-e'$ the complete graph of order 4 less the edge e' with $e'=cd$. Then for $B=K_1\cup K_2$, the graph $G[B]$ will have an induced subgraph isomorphic to a member of $\mathbb{B}(k)$, contradicting the hypothesis.

Lemma 3.8. *If $K\in M(G)$ and $x\notin K$, then x is joined to at most k vertices of K.*

Proof. In the other case, G will have a member of $\mathbb{B}(k)$ as an induced subgraph.

Lemma 3.9. *Every vertex of G is in at most k distinct members of $M(G)$.*

PROOF. Suppose that the result is not true and let x be a vertex of G which is in $k+1$ distinct elements $K_1, \ldots, K_{k+1}$ of $M(G)$. By Lemma 3.7, $|K_i \cap K_j| = \{x\}$, $i \neq j$. Now let $a_1 \in K_1$ and $a_1 \neq x$. By Lemma 3.8, it follows that a_1 is joined to at most k vertices of K_2. Hence there exists an $a_2 \in K_2$ such that $\langle x; a_1, a_2\rangle$ is a 2-claw. Suppose that we have constructed an r-claw $\langle x; a_1, \ldots, a_r\rangle$, $r < k+1$, in G such that $a_i \in K_i$. Then each a_i is joined to at most $k-1$ vertices distinct from x of K_{r+1} and, since $|K_{r+1}| \geq k^2-k+2 > r(k-1)+1$, there exists an $a_{r+1} \in K_{r+1}$ such that $\langle x; a_1, \ldots, a_{r+1}\rangle$ is an $(r+1)$-claw in G. Taking $r = k$, we get a $(k+1)$-claw in G, contradicting the hypothesis.

REMARK 3.10. We note that the hypothesis that every edge of G has edge-degree at least k^3-k is used only in the proof of Lemma 3.6 (and Lemma 3.6 is not used in the proofs of Lemmas 3.7, 3.8 and 3.9).

PROOF OF THEOREM 3. The necessity follows from Lemmas 3.1 through 3.5. To prove the sufficiency, let $\mathcal{S} = M(G)$. Then by Lemmas 3.6 through 3.9, it follows that $\mathcal{S}$ satisfies Proposition 2.1 (i) and (ii) and therefore $G \in \mathcal{I}(k)$.

4. PROOF OF THEOREM 2

In this section we give a proof of Theorem 2. First we prove the following simple but interesting

LEMMA 4.1. *If $\Gamma \in \mathcal{I}(k)$, $k \geq 3$, and every vertex of Γ is in a clique of size k^2-k+2, then the graph Γ_1, obtained from Γ by removing those edges of Γ which are in cliques of size k^2-k+2, is a member of $\mathcal{I}(k-1)$.*

PROOF. Let $\mathcal{S} = \{K_1, \ldots, K_r\}$ be a set of cliques of Γ satisfying the conditions of Proposition 2.1. If L is any maximal clique of Γ containing a vertex x with $|L| \geq k^2-k+2$, then we assert that $L \in \mathcal{S}$. For otherwise, if $K_{i_1}, \ldots, K_{i_s}$ are the cliques of $\mathcal{S}$ containing x, then $s \leq k$ and since $|L| \geq k^2-k+2$, we may assume that $|L \cap K_{i_1}| \geq k+1$ and $|L \cap K_{i_2}| \geq 2$. Let $y \in L \cap K_{i_2}$, $y \neq x$, and $x, y_1, \ldots, y_k$ be $k+1$ distinct vertices in $L \cap K_{i_1}$. Since x, y_i, $1 \leq i \leq k$, belong to K_{i_1} it follows, by Proposition 2.1 (i), that yx, yy_i, $1 \leq i \leq k$, belong to distinct cliques in $\mathcal{S}$ contradicting Proposition 2.1 (ii) for $\mathcal{S}$.

Let $\mathcal{S}_1 \subseteq \mathcal{S}$ be the cliques of size at most k^2-k+1. Then from the hypothesis it follows that $\mathcal{S}_1$ satisfies the conditions of Proposition 2.1 for $k-1$ implying that $\Gamma_1 \in \mathcal{I}(k-1)$.

To provide three new finite families of forbidden graphs for the family $\mathcal{I}(3)$, we prove the following crucial Lemma 4.2. To this end we need the following definition.

Let Γ be a graph. Let Γ_1 be the graph obtained from Γ by removing those edges of Γ which are in cliques of size 8. We say that a triangle abc in Γ is a *good triangle* in Γ if abc satisfies one of the following conditions:

(i) abc is contained in a clique of size 8 in T,
(ii) abc is an odd triangle in T_1,
(iii) there are vertices d, e of Γ such that $\Gamma[\{a, b, c, d, e\}]$ is the graph of Figure 7 and abc is a triangle in Γ_1.

From now onwards whenever we write $\Gamma = G(H_1)$ we implicitly assume that H_1 is a 3-uniform linear hypergraph.

Now we prove the following important lemma.

LEMMA 4.2. *If $\Gamma \in \mathcal{I}(3)$ with $\Gamma = G(H_1)$ and every vertex of Γ is in a clique of size 8, then for any good triangle abc of Γ there is a vertex A of H_1 such that $A \in a \cap b \cap c$ in H_1.*

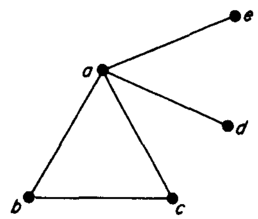

FIGURE 7.

PROOF. For triangles of type (i) the result follows by Lemma 3.3 with $k=3$. Suppose that the result is false for a triangle abc of type (ii). Relabel a by a_1, b by a_2 and c by a_3. Let $a_1=\{x_1, x_2, x_3\}$, $a_2=\{x_1, y_2, y_3\}$ and $a_3=\{x_2, y_2, z_3\}$. Since $a_1a_2a_3$ is an odd triangle in Γ_1, there exists a vertex a_4 (say) which is joined in Γ_1 to exactly one vertex a_1 (say) or the three vertices of the triangle $a_1a_2a_3$. Let K_i be a maximal clique of size at least 8 containing a_i, $1 \leq i \leq 4$. Since a_1a_2, a_2a_3, a_3a_1, $a_4a_1 \in E(\Gamma_1)$ it follows that, if $i \neq j$, $1 \leq i, j \leq 3$, then $a_i \notin K_j$; further $a_1 \notin K_4$, $a_4 \notin K_1$ and K_4 cannot contain both a_2 and a_3. Also by Lemma 3.3 we have that $x_3 \in e$ for every $e \in K_1$, $y_3 \in e'$ for every $e' \in K_2$ and $z_3 \in e''$ for every $e'' \in K_3$. Now we consider two cases.

Case I. $a_4a_1 \in E(\Gamma_1)$ *and* a_4a_2, $a_4a_3 \notin E(\Gamma_1)$.

We consider three subcases depending upon the possibilities for K_4.

Subcase (i) $K_4 = K_2$ *or* $K_4 = K_3$.

Assume without loss of generality that $K_4 = K_2$. This implies that $y_3 \in a_4$. Since $a_2 \neq a_4$ and $a_1a_4 \in E(\Gamma_1)$, it then follows by linearity that either $x_2 \in a_4$ or $x_3 \in a_4$. If $x_3 \in a_4$, then by the maximality of K_1, $a_4 \in K_1$ which implies that $a_1a_4 \notin E(\Gamma_1)$. Thus we may assume that $x_2 \in a_4$. Therefore, $a_4a_3 \in E(\Gamma)$ and our assumption then implies that a_4a_3 belongs to a maximal clique K_5 of Γ of size at least 8. Then by Lemma 3.3, $x_2 \in e^*$ for every $e^* \in K_5$ in H_1 and this implies that $a_1 \in K_5$ in Γ and hence $a_1a_3 \notin E(\Gamma_1)$, a contradiction.

Subcase (ii) $K_4 \neq K_2, K_3$ *but* K_4 *contains* a_2 *or* a_3.

Assume without loss of generality that $a_2 \in K_4$. Then $a_3 \notin K_4$. By Lemma 3.3 there exists a vertex x (say) such that $x \in e$ for every $e \in K_4$. Since a_1a_4, $a_2a_4 \in E(\Gamma)$, a_4 has a non-empty intersection with a_1 as well as with a_2. If $x_3 \in a_4$, then $a_4 \in K_1$; and if $y_3 \in a_4$, then since $a_2, a_4 \in K_4$ it follows that $x = y_3$ and hence $K_4 = K_2$. If $x_1 \notin a_4$, then both $x_2, y_2 \in a_4$ which implies by linearity of H_1 that $a_4 = a_3$. Thus we may assume that $x_1 \in a_4$. Since $a_2 \in K_4$ and $x_1 \in a_2 \cap a_4$, it follows by Lemma 3.3 that $x_1 \in e$ for every $e \in K_4$ which implies that $a_1 \in K_4$, a contradiction.

Subcase (iii) K_4 *contains none of the vertices* a_2, a_3.

Since $a_1a_4 \in E(\Gamma_1)$, it follows that a_1 and a_4 have non-empty intersections and $x_3 \notin a_4$. If $x_2 \in a_4$, then $a_4a_3 \in E(\Gamma)$ and since $a_4a_3 \notin E(\Gamma_1)$, there exists a maximal clique K_6 (say) of size at least 8 containing a_4a_3, and since $x_2 \in a_4 \cap a_3$, it follows by Lemma 3.3 that $x_2 \in e$ every $e \in K_6$ which implies that $a_1 \in K_6$ and therefore $a_1a_3 \notin E(\Gamma_1)$. Thus we may assume that $x_1 \in a_4$, then $a_4a_2 \in E(\Gamma)$ and as above it can be shown that $a_1a_2 \notin E(\Gamma_1)$, a contradiction.

Case II. a_4a_1, a_4a_2, $a_4a_3 \in E(\Gamma_1)$.

This implies that a_4 has a non-empty intersection with each of a_1, a_2 and a_3. If x_3 or y_3 or $z_3 \in a_4$, then $a_4 \in K_1$ or K_2 or K_3, respectively. Thus we may assume that none of x_3, y_3 and z_3 is in a_4. This then implies that a_4 contains at least two points of x_1, x_2 and y_2, contradicting the linearity of H_1.

This completes the proof of the assertion for triangles of type (ii).

Suppose now that the result is false for a triangle $a_1a_2a_3$ of type (iii) and relabel d by a_4 and e by a_5. If a_1a_5 or $a_1a_4 \in E(\Gamma_1)$, then $a_1a_2a_3$ is an odd triangle of Γ_1 and the result follows by (ii). Thus we may assume that $a_1a_5, a_1a_4 \notin E(\Gamma_1)$. Let L, M be maximal cliques of size at least 8 containing a_1a_5, a_1a_4 respectively. Since $a_4a_5 \notin E(\Gamma)$, $L \neq M$. This implies that one of L, M is different from K_1, say $L \neq K_1$. By Lemma 3.3, there is an x such that $x \in e$, for every $e \in L$, in particular $x \in a_1$ and since $K_1 \neq L$, $x \neq x_3$. If $x = x_1$ or x_2, then a_2 or $a_3 \in L$, and this is a contradiction since $a_1 \in L$ and $a_1a_2, a_1a_3 \in E(\Gamma_1)$.

Now let $\mathbb{D}(3)$ be the finite family of all graphs Γ such that the following three conditions hold:

(i) $|V(\Gamma)| \leq 64$ and every vertex of Γ is in a clique of size 8.
(ii) There exists a set $D = \{a, b, c, d\}$ of vertices such that $\Gamma[D]$ is the complete graph of order 4 less the edge cd.
(iii) Triangles abc, abd are good triangles in Γ.

LEMMA 4.4. *If $G \in \mathscr{I}(3)$, then G has no induced subgraph isomorphic to a member of* $\mathbb{D}(3)$.

PROOF. Suppose that G has an induced subgraph, $\Gamma \in \mathbb{D}(3)$. Clearly $\Gamma \in \mathscr{I}(3)$ and let $\Gamma = G(H_1)$. Then by Lemma 4.3 there are vertices A and B of H_1 such that $A = a \cap b \cap c$ and $B = a \cap b \cap d$ in H_1. Then $\{A, B\} \subseteq a \cap b$, and by linearity of H_1, $A = B$, contradicting the hypothesis that $cd \notin E(G)$.

Let $\mathbb{E}(3)$ be the finite family of all graphs Γ such that the following conditions hold:

(i) $|V(\Gamma)| \leq 64$ and every vertex of Γ is in a clique of size 8.
(ii) There exists a set $D = \{a, b, c, d\}$ such that $\Gamma[D]$ is the complete graph of order 4.
(iii) Triangles abc, abd are both good triangles in Γ.
(iv) The edge cd is contained in a clique K of size 8 such that there is a vertex $e \in K$ such that $ea, eb \notin E(\Gamma)$.

Proofs of Lemmas 4.5 and 4.6 are similar to that of Lemma 4.4.

LEMMA 4.5. *If $G \in \mathscr{I}(3)$, then G has no induced subgraph isomorphic to a member of* $\mathbb{E}(3)$.

Let $\mathbb{J}(3)$ be the finite family of all graphs Γ on at most 11 vertices such that the following three conditions hold:

(i) there is a set $D = \{a, b, c, d, e\}$ of vertices in Γ such that $\Gamma[D]$ is the graph of Figure 7;
(ii) there is a clique K of size 8 in Γ containing the edge bc;
(iii) there is a vertex $d' \in K$ $(d' \neq a)$ such that $d'a \notin E(\Gamma)$.

LEMMA 4.6. *If $G \in \mathscr{I}(3)$, then G has no induced subgraph isomorphic to a member of* $\mathbb{J}(3)$.

Let $\mathbb{F} = \mathbb{F}(3) \cup \mathbb{D}(3) \cup \mathbb{E}(3) \cup \mathbb{J}(3)$. Note that $\mathbb{F}$ is a finite family of graphs. We have shown so far that if $G \in \mathscr{I}(3)$, then G has no induced subgraph isomorphic to a member of $\mathbb{F}$. *For the remainder of this section we assume that G is a graph such that G has no induced subgraph isomorphic to a member of $\mathbb{F}$ and that the minimum degree of G is at least* 69; and also G_1 is the graph obtained from G by removing those edges which occur in cliques of size 8 in G (see Lemma 4.7 below). To complete the proof of Theorem 2 we need the following six lemmas.

LEMMA 4.7. *Every vertex of G is in a clique of size* 8.

PROOF. Let x be a vertex of G and $n(x)=|N(x)|$. Let r be the maximum size of a claw at x in G. Then, since the 4-claw is in $\mathbb{F}(3)$, $r\leq 3$. If $r=1$, then $N(x)$ is a clique in G of size at least 69. Thus we may assume that $r=2$ or 3. Let $S=\langle x; y_1, \ldots, y_r\rangle$ be an r-claw at x. By the maximality of r it follows that, if $y\in N(x)$, with $y\neq y_i$, $1\leq i\leq r$, then y is joined to at least one of these y_is. Let $n(x, y_i)=|N(\{x, y_i\})|$. Hence there is an i, $1\leq i\leq r$, such that

$$n(x, y_i)\geq\frac{n(x)-r}{r}.$$

Without loss of generality assume that $i=1$. Now using the forbidden graphs of $\mathscr{A}(3)$, we have that $n(y_1, y_i)\leq 9$, $i\neq 1$. Hence there are at least

$$\frac{n(x)}{r}-1-8(r-1)=\frac{n(x)}{r}+7-8r$$

vertices in G which are joined to both x and y_1 but are joined to none of y_i, $i\neq 1$. By the maximality of the claw S, it follows that all these vertices together with x and y_1 form a clique of size at least

$$\frac{n(x)}{r}+9-8r,$$

which is at least 8 since $r=2$ or 3 and $n(x)\geq 69$.

LEMMA 4.8. *G_1 has no 3-claw.*

PROOF. Suppose that G_1 has a 3-claw $\langle x; y_1, y_2, y_3\rangle$. By Lemma 4.7 there exists a maximal clique K containing x, with $|K|\geq 8$. Since $xy_i\in E(G_1)$, the vertex $y_i\notin K$, $i=1, 2, 3$. By Lemma 3.8, y_i is joined to at most two vertices of K, in G, distinct from x. Thus there exists a $y_4\in K$ such that $\langle x; y_1, y_2, y_3, y_4\rangle$ is a 4-claw in G, a contradiction.

Let T be the set of all good triangles in G and T_1 be the set of all triangles of G_1 which are in T.

LEMMA 4.9. *The set T_1 satisfies the conditions of Proposition 2.5 for the graph G_1 with S replaced by T_1 and k by 2.*

PROOF. Proposition 2.5 (i) follows from Lemma 4.8. From the definition of good triangles, T_1 contains all odd triangles of G_1 implying Proposition 2.5 (ii). Suppose now that T_1 does not satisfy Proposition 2.5 (iii). Then there are good triangles abc and abd in T_1 with $c\neq d$ such that Proposition 2.5 (iii) is not satisfied. Now if $cd\in E(G_1)$, then clearly acd, bcd are odd triangles in G_1 and condition (iii) will be satisfied. Hence we can assume that $cd\notin E(G_1)$. Now if $cd\notin E(G)$, then by Lemma 4.7 G has a member of $\mathbb{D}(3)$ as an induced subgraph. Thus we may assume that $cd\in E(G)$. Then there exists a clique K of size at least 8 containing cd and not containing a, b. Then, as in the proof of Lemma 4.8, we can find a vertex $e\in K$ such that ea, $eb\notin E(G)$. Clearly G has a member of $\mathbb{E}(3)$ as an induced subgraph, a contradiction.

Now by Proposition 2.5 with $k=2$ for the graph G_1, we can find a set T' of triangles of G_1 such that $T\subseteq T'$ and T' satisfies the conditions of Proposition 2.3 with $k=2$. Define $\bar{T}=T\cup T'$. We will show that $\bar{T}$ satisfies the conditions of Proposition 2.3 with $k=3$ for G.

LEMMA 4.10. *If abc, abd* $\in \bar{T}$, *with* $c \neq d$, *then* $cd \in E(G)$ *and abc, bcd are also in* $\bar{T}$.

PROOF. Let abc, $abd \in \bar{T}$ with $c \neq d$. Then by the definition of good triangles, both the triangles must be in T' or both not in T'. If both are in T' the result follows from the definition of T'. Thus we may assume that abc, $abd \notin T'$. This implies that both abc, abd are good triangles of type (i). Therefore there exist maximal cliques K_1 and K_2 of size at least 8 such that $abc \in K_1$ and $abd \in K_2$. If now $c \notin K_2$, then there exists a vertex $e \in K_2$ such that $ce \notin E(G)$. Then clearly G has an induced subgraph isomorphic to a member of $\mathbb{B}(3)$. Thus $c \in K_2$ and therefore $cd \in E(G)$ and acd, bcd are good triangles.

LEMMA 4.11. *If there are three distinct maximal cliques* K_1, K_2, K_3 *in G containing a given vertex x of G each of size at least* 8, *then* $N(x) \subseteq K_1 \cup K_2 \cup K_3$.

PROOF. Suppose that $z \in N(x)$ and $z \notin K_1 \cup K_2 \cup K_3$. Then as in the proof of Lemma 3.9, it may be proved that G has a 4-claw $\langle x; y_1, y_2, y_3, y_4 \rangle$ with $y_i \in K_i$, $i = 1, 2, 3$, a contradiction.

LEMMA 4.12. *If* e_1, e_2, e_3 *and* e_4 *are four distinct edges of G having a vertex x in common, then at least two of these edges are in a triangle in* $\bar{T}$.

PROOF. Let $e_i = xy_i$, $1 \leq i \leq 4$. If at least three e_is are edges in G_1, then the result follows from the definition of $T' \subseteq \bar{T}$. If none of the four edges is in G_1, then by Lemma 3.9 with $k = 3$ and the definition of G_1, it follows that at least two of these e_is are contained in a clique of size at least 8 and hence are in a triangle in $\bar{T}$.

Now by Lemma 4.11 we may assume that exactly two e_is, e_1 and e_2 say, are not edges in G_1, and e_3, e_4 are edges in G_1. Let K_i be a maximal clique of size at least 8 in G containing e_i, $i = 1, 2$. If $K_1 = K_2$, then e_1, e_2 are in a triangle of $\bar{T}$. Thus we may assume that $K_1 \neq K_2$. By Lemma 3.7 it follows that $K_1 \cap K_2 = \{x\}$. Then, using an argument similar to that used in the proof of Lemma 3.9, we can find vertices $x_1 \in K_1$, $x_2 \in K_2$ such that

$$x_1x_2, x_1y_3, x_1y_4, x_2y_3, x_2y_4 \notin E(G).$$

Now if $y_3y_4 \in E(G_1)$, then clearly $xy_3y_4 \in T \subseteq \bar{T}$. Thus we may assume that $y_3y_4 \notin E(G_1)$. If $y_3y_4 \notin E(G)$, then $\langle x; x_1, x_2, y_3, y_4 \rangle$ is a 4-claw in G. Therefore, we may suppose that $y_3y_4 \in E(G)$ and $y_3y_4 \notin E(G_1)$. This implies that there is a clique K of size 8 containing the edge y_3y_4. Since $xy_3 \in E(G_1)$, we have that $x \notin K$. Thus G has an induced subgraph isomorphic to a member of $\mathbb{J}(3)$ with $a = x$, $b = y_3$, $c = y_4$, $d = x_1$, $e = x_2$, a contradiction.

PROOF OF THEOREM 2. The necessity that G has no induced subgraph isomorphic to $\mathbb{F}$ follows from Lemmas 3.1, 3.2, 3.4, 3.5, 4.4, 4.5, 4.6. The sufficiency that $G \in \mathcal{I}(3)$ follows from Lemmas 4.10, 4.12 and Proposition 2.3.

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Received 27 September 1979 and in revised form 23 July 1981

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