Acyclic 4-choosability of planar graphs without adjacent short cycles

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Abstract

The acyclic 4-choosability was proved, in particular, for the following planar graphs: without 3- and 4-cycles (Montassier et al., 2006 [29]), without 4-, 5-, and 6-cycles (Montassier et al., 2006 [29]), either without 4-, 6-, and 7-cycles, or without 4-, 6-, and 8-cycles (Chen, Raspaud, and Wang, 2009), and with neither 4-cycles nor 6-cycles adjacent to a triangle (Borodin et al., 2010 [13]).

There exist planar acyclically non-4-colorable bipartite graphs (Kostochka and Mel'nikov, 1976 [25]). This partly explains the fact that in all previously known sufficient conditions for the acyclic 4-choosability of planar graphs the 4-cycles are completely forbidden. In this paper we allow 4-cycles nonadjacent to relatively short cycles; namely, it is proved that a planar graph is acyclically 4-choosable if it does not contain an i-cycle adjacent to a j-cycle, where $3 \leq j \leq 6$ if $i = 3$ and $4 \leq j \leq 7$ if $i = 4$. In particular, this absorbs all the above-mentioned results.

1. Introduction

By $V(G)$ denote the set of vertices of a graph $G$ and by $E(G)$ its set of edges. A (proper) $k$-coloring of $G$ is a mapping $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ whenever $x$ and $y$ are adjacent in $G$.

A proper vertex coloring of a graph is acyclic if every cycle uses at least three colors [20]. Borodin [2,3] proved Grünbaum’s conjecture that every planar graph is acyclically 5-colorable, improving the earlier bounds 9, 8, 7, and 6 due to Grünbaum [20], Mitchem [26], Albertson and Berman [1], and Kostochka [24], respectively. The bound 5 is best possible; moreover, there are bipartite 2-degenerate planar graphs that are not acyclically 4-colorable [25]. Acyclic colorings turned out to be useful in obtaining results about other types of colorings; for a survey see monographs [23,21].

Now suppose each vertex $v$ of a graph $G$ is given a list $L(v)$ of colors. The list $L$ is choosable if there is a proper vertex coloring of $G$ such that a color of each vertex $v$ belongs to $L(v)$. A graph $G$ is said to be $k$-choosable if every list $L$ is choosable provided that $|L(v)| \geq k$ for each $v \in V(G)$.

It is trivial that each planar graph is 6-choosable, because its every subgraph has a vertex of degree at most 5. Thomassen [31] proved a famous theorem that each planar graph is 5-choosable, and Voigt [32] showed that this bound is best possible.

Borodin et al. [8] proved that every planar graph is acyclically 7-choosable and conjectured a common extension of Borodin’s and Thomassen’s results [3,31]:

Conjecture 1. Every planar graph is acyclically 5-choosable.

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doi:10.1016/j.disc.2012.07.038
However, this challenging conjecture seems to be difficult. As yet, it has been verified only for several restricted classes of planar graphs: those of girth at least 5 [28], without 4- and 5-cycles, or without 4- and 6-cycles [30], with neither 4-cycles nor chordal 6-cycles [34], with neither 4-cycles nor two 3-cycles at a distance less than 3 [19], and without 4-cycles and intersecting 3-cycles [16]. Wang and Chen [33] proved that planar graphs without 4-cycles are acyclically 6-choosable.

Recently, Borodin and Ivanova [10] proved that a planar graph is acyclically 5-choosable if it does not contain an i-cycle adjacent to a j-cycle, where $3 \leq j \leq 5$ if $i = 3$ and $4 \leq j \leq 6$ if $i = 4$, which absorbs the above-mentioned results in [28,30,34]. Also, Borodin and Ivanova [11] proved that every planar graph without 4-cycles is acyclically 5-choosable, which is a common strengthening of the results in [28,30,34,19,16,33].

Some sufficient conditions are also obtained for a planar graph to be acyclically 4- and 3-colorable or choosable. Denote the minimal $k$ with the property that $G$ is acyclically $k$-colorable (acyclically $k$-choosable) by $a(G)$ (by $a'(G)$).

Borodin et al. [14] showed that if $G$ is a planar graph of girth $g$, then $a(G) \leq 4$ if $g \geq 5$ and $a(G) \leq 3$ if $g \geq 7$. Recently, $a'(G) \leq 3$ was proved if $g \geq 7$ [7] or if $G$ has no cycles of length from 4 to 12 (Borodin [4] and, independently, Hocquard and Montassier [22]), which was strengthened to the absence of 4–11-cycles by Borodin and Ivanova [9].

The bound $a'(G) \leq 4$ was proved in the following cases: if $g \geq 5$ [27], or if $G$ has no 4-, 5-, and 6-cycles [29], or no 4-, 6-, and 7-cycles, or else no 4-, 6-, and 8-cycles [18]. Borodin [5] proved $a(G) \leq 4$ for $G$ having neither 4- nor 6-cycles. Recently, Borodin et al. [13] gave a common extension of the results in [27,29,18,5] by proving $a'(G) \leq 4$ under the absence of 4-cycles and triangular 6-cycles (i.e., those adjacent to a 3-cycle).

Furthermore, Montassier et al. [29] proved $a'(G) \leq 4$ for every planar graph without 4-, 5-, and 7-cycles, or without 4-, 5-, and intersecting 3-cycles, while Chen and Raspaud [15] proved this assuming that $G$ has neither 4- and 5-cycles nor 8-cycles with a triangular chord. Borodin [6] proved $a(G) \leq 4$ for $G$ having neither 4- nor 5-cycles. The above-mentioned results in [28,29,15,6] were strengthened by proving that every planar graph $G$ without 4- and 5-cycles is acyclically 4-choosable (Borodin and Ivanova [12] and, independently, Chen and Raspaud [17]).

Recall that there are bipartite planar graphs that are not acyclically 4-colorable [25]. Therefore, while describing acyclically 4-choosable planar graphs, one must impose these or those restrictions on 4-cycles. Note that in all previously known sufficient conditions for the acyclic 4-choosability of planar graphs, the 4-cycles are completely forbidden. In this paper we allow 4-cycles, but disallow them to have a common edge with relatively short cycles.

The purpose of this paper is to prove the following

**Theorem 2.** A planar graph is acyclically 4-choosable if it does not contain an i-cycle adjacent to a j-cycle, where $3 \leq j \leq 6$ if $i = 3$ and $4 \leq j \leq 7$ if $i = 4$.

Clearly, Theorem 2 is a common strengthening of the results in [27,29,18,5,13].

Montassier et al. [29] conjectured that every planar graph without 4-cycles is acyclically 4-choosable. We would like to pose the following stronger conjecture.

**Conjecture 3.** Every planar graph without 4- or 4- or 4-cycles is acyclically 4-choosable.

**2. Proof of Theorem 2**

Suppose a plane graph $G$ with a list $L$ is a counterexample to Theorem 2 on the fewest vertices. Clearly, $G$ is connected and has no pendant vertices. By $F(G)$, $d(v)$, and $r(f)$ denote the set of faces of $G$, the degree of a vertex $v$, and the size of face $f$, respectively.

From Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$, using the well-known relations

$$\sum_{v \in V(G)} d(v) = 2|E(G)| = \sum_{f \in F(G)} r(f),$$

we have

$$\sum_{v \in V(G)} (2d(v) - 6) + \sum_{f \in F(G)} (r(f) - 6) - 12. \quad (1)$$

We set the initial charge of every vertex $v \in V(G)$ and face $f \in F(G)$ to be $ch(v) = 2d(v) - 6$ and $ch(f) = r(f) - 6$, respectively. Note that only 2-vertices and 3-, 4-, and 5-faces have negative initial charge, $-2$, $-3$, $-2$, and $-1$, respectively. Then we use a discharging procedure leading to a final charge $ch^*$ such that

$$\sum_{x \in V(G) \cup F(G)} ch^*(x) = \sum_{x \in V(G) \cup F(G)} ch(x) < 0.$$

Based on the structural properties of $G$, we shall get a contradiction by proving that $ch^*(x) \geq 0$ for every $x \in V(G) \cup F(G)$.
2.1. Structural properties of the minimum counterexample

A vertex or edge is called triangular if it is incident with a 3-face. A vertex of degree at least $k$ or at most $k$ is a $k^+$- or a $k^-$-vertex, respectively, and similar notation is used for the faces. Clearly, $G$ has no triangular 2-vertices. Note that no 3-face can be adjacent to a 6$^+$-face since $G$ has neither pendant vertices, nor triangular 2-vertices.

The number of 3-faces and 4-faces incident with a vertex $v$ is denoted by $\tau_3(v)$ and $\tau_4(v)$, respectively; thus $\tau_3(v) + \tau_4(v) \leq \lfloor \frac{d(v)}{2} \rfloor$.

A triangular 3-vertex joined to a vertex $v$ by a non-triangular edge is a bad neighbor of $v$ (see Fig. 1), and the number of bad neighbors of $v$ is $\beta(v)$. By $\nu_k(v)$ denote the number of $k$-vertices adjacent to $v$. A 2-vertex is quadrangular if it is incident with a 4-face. The number of quadrangular 2-vertices adjacent to $v$ is denoted by $\nu_5^*(v)$.

A weak vertex is either a vertex of degree 3 or a 4-vertex $v$ such that $\nu_2(v) = 1$ and $\nu_3(v) \geq 1$. A 4-vertex $v$ is poor if $\nu_2(v) = \nu_3(v) = \nu_4(v) = 1$.

**Lemma 1** ([5,13]). Each 3-vertex $v$ has $\beta(v) = 0$.

**Lemmas 2 and 3** (in slightly different form) were proved in [14] for acyclic 4-colorings, and in [27] their proofs were transferred to acyclic 4-choosability without substantial changes. These proofs in [14,27] also work without changes in the more general case of Theorem 2, where 3- and 4-cycles are allowed but are disallowed to be adjacent to relatively short cycles.

**Lemma 2** ([14,27]). Each vertex $v$ in $G$ has the following properties:

\begin{itemize}
  \item[(i)] $\nu_2(v) = 0$ if $d(v) \leq 3$;
  \item[(ii)] $\nu_2(v) \leq 1$ if $d(v) = 4$, $\nu_2(v) \leq d(v) - 2$ if $d(v) \leq 9$, and $\nu_2(v) \leq d(v) - 1$ if $d(v) \leq 15$;
  \item[(iii)] if $d(v) = 5$ and $\nu_2(v) = 3$, then the three 2-vertices occur consecutively in cyclic order round $v$, and both of the two faces between consecutive 2-vertices are 6$^+$-faces;
  \item[(iv)] if $d(v) = 5$, $\nu_2(v) = 2$, and $\nu_3(v) = 3$, then $v$ is incident with at least one 6$^+$-face;
  \item[(v)] if $d(v) = 5$ and $\nu_2(v) = 3$, or $d(v) = 6$ and $\nu_2(v) = 4$, then $\nu_3(v) = 0$.
\end{itemize}

**Lemma 3** ([14,27]). Each non-triangular 3-vertex is adjacent to at most one weak vertex.

**Lemma 4** ([29]). No 4-vertex $v$ with $\nu_2(v) = 1$ is adjacent to a triangular 3-vertex.

The idea of the next lemma comes from [14,30].

**Lemma 5** ([18]). No weak 4-vertex $v_4$ is incident with a 5-face $v_1v_2v_3v_4v_5$ such that $d(v_3) = 3$ and $d(v_5) = 2$.

**Lemma 6** ([29]). There is no 5-vertex $v$ such that $\nu_2(v) = 3$ and $\nu_3(v) = 1$.

**Lemma 7** ([29]). If $xyz$ is a 3-face such that $d(x) = d(y) = 3$, then $d(z) \geq 5$.

2.2. Discharging

We discharge the vertices and faces of $G$ as follows (see Fig. 2):

**R0:** Each 7$^+$-face $f$ gives charge $\xi$ to every incident edge $xy$, where $\xi = \frac{1}{7}$ if $r(f) = 7$ and $\xi = \frac{1}{3}$ otherwise. This $\xi$ further goes to the adjacent 3-face if $xy$ is triangular; otherwise:

\begin{itemize}
  \item[(i)] to $y$ if $d(x) = 2$ or if $x$ is a non-triangular 3-vertex while $d(y) \geq 4$;
  \item[(ii)] to 3-face $ux$ if $d(x) = 3$;
  \item[(iii)] to $x$ and $y$ in portions of $\frac{\xi}{2}$ if $d(x) \geq 4$ and $d(y) \geq 4$.
\end{itemize}

**R1:** Suppose edge $xy$ is incident with faces $f_1$ and $f_2$, where $d(x) = 2$. Then $y$ gives $x$ the following charge:

\begin{itemize}
  \item[(i)] $\frac{6}{5}$ if $r(f_1) = r(f_2) = 5$,
  \item[(ii)] $\frac{11}{10}$ if $r(f_1) = 5$ while $r(f_2) \geq 6$,
\end{itemize}
(iii) 1 if \( r(f_1) \geq 6 \) and \( r(f_2) \geq 6 \) and
(iv) \( \frac{5}{4} \) if \( r(f_1) = 4 \) while \( r(f_2) \geq 8 \).

R2: Suppose edge \( xy \) is incident with faces \( f_1 \) and \( f_2 \), where \( x \) is a non-triangular 3-vertex while \( y \) is non-weak. Then \( y \) gives:

(i) \( \frac{3}{5} \) if \( r(f_1) = r(f_2) = 5 \),
(ii) \( \frac{1}{2} \) if \( r(f_1) = 5 \) while \( r(f_2) \geq 6 \), and
(iii) \( \frac{1}{4} \) if \( x \) is incident with a 4-face.

R3: Every 5-face gets \( \frac{1}{5} \) from every incident vertex.

R4: If \( x \) is a bad neighbor of \( v \) then \( v \) gives \( \frac{1}{2} \) to 3-face \( xyz \).

R5: Every 3-face \( f = uvw \) gets from a 4-v-vertex \( v \):

(i) 1 if \( v \) is incident with a 3-vertex,
(ii) \( \frac{2}{3} \) if \( v \) is poor, and
(iii) \( \frac{1}{6} \) otherwise.

R6: Every 4-face \( f = uxyz \) gets \( \frac{1}{2} \) from every incident vertex, with the following exception: if \( d(w) = 2 \) and \( d(y) \geq 4 \), then \( f \) gets \( \frac{1}{2}, \frac{3}{8}, \frac{1}{4} \), and \( \frac{3}{8} \) from \( w, x, y \), and \( z \), respectively.

2.3. Checking \( ch^{*}(x) \geq 0 \) for every \( x \in V(G) \cup F(G) \)

Case 1. \( f \in F(G) \). If \( r(f) \geq 8 \) then \( ch^{*}(f) = r(f) - 6 - \frac{r(f)}{2} \geq 0 \) by R0. If \( r(f) = 7 \) then \( ch^{*}(f) = r(f) - 6 - \frac{r(f)}{2} = 0 \) by R0.

Suppose \( f = xyz \), where \( d(x) \leq d(y) \leq d(z) \); so, \( ch(f) = r(f) - 6 = -3 \). Recall that \( f \) gets at least \( \frac{3}{5} \) across incident edges from adjacent faces by R0. If \( d(x) = d(y) = 3 \) then \( d(z) \geq 5 \) due to Lemma 7, which implies that \( ch^*(f) \geq -3 + 3 \times \frac{3}{5} + 2 \times \frac{1}{3} + 2 \times \frac{3}{8} + \frac{1}{6} = 0 \) by R6.

If \( r(f) = 4 \) then \( ch^*(f) = 4 - 6 - 4 \times \frac{1}{2} = 0 \) or \( ch^*(f) = 4 - 6 - 2 \times \frac{1}{3} + 2 \times \frac{3}{8} + \frac{1}{6} = 0 \) by R6.

3. Similarly, we have \( ch^*(v) = 0 \) if \( v \) has just one or no incident 5-face by R1(iii) and R1(iii), respectively.

Subcase 2.2. \( d(v) = 3 \). If \( v \) is triangular, then \( v \) does not participate in discharging, so \( ch^*(v) = ch(v) = 0 \).

Suppose \( v \) is surrounded by faces \( f_1, f_2, \) and \( f_3 \), where \( 5 \leq r(f_1) \leq r(f_2) \leq r(f_3) \). Note that each of at least two non-weak neighbors of \( v \) gives \( v \) either \( \frac{1}{5} \), or \( \frac{1}{6} \) by R2(i, ii). If \( r(f_1) = 5 \) then \( ch^*(v) = ch(v) = 0 \); if \( r(f_1) = 5 < r(f_2) \) then \( ch^*(v) = \frac{3}{5} \times \frac{2}{5} = 0 \) by R2(ii); if \( r(f_2) = 5 < r(f_3) \) then \( ch^*(v) = \frac{2}{5} \times \frac{2}{5} = 0 \) by R2(ii). Finally, if \( r(f_3) = 5 \) then \( ch^*(v) = \frac{3}{5} \times \frac{2}{5} = 0 \).

Subcase 2.3. \( d(u) = 4 \). Now \( ch^*(v) = 2 \), while \( v_2(v) \leq 1 \) by Lemma 2(ii).

Subsubcase 2.3.1 There is a 4-face \( f = uvwx \).

First suppose that \( d(w) = 2 \). Recall that \( v \) neither gives charge to adjacent 3-vertices by R2, nor participates in R4 due to Lemma 4. Note that \( f \) causes the total expenditure \( \mu = \frac{3}{5} \) for \( v \) by R0(iii), R1(iv), and R6. Indeed, if \( d(w) = 3 \) then \( \mu = \frac{5}{6} + \frac{1}{3} - \frac{1}{2} = \frac{5}{6} \); otherwise, \( \mu = \frac{5}{6} + \frac{1}{3} - \frac{1}{2} \). Furthermore, \( v \) can only give either \( \frac{1}{2} \) to a triangle by R5(ii) if \( v \) is poor, or at most \( \frac{1}{2} \) to another incident 4-face by R6, or else \( \frac{1}{2} \) to a 5-face by R3. Therefore, \( ch^*(v) \geq 2 \times 4 - 6 \leq \frac{5}{6} - \frac{1}{2} = 0 \).

Now we can assume by symmetry that \( v_2(v) = 0 \), i.e. \( v \) is not adjacent to a quadrangular 2-vertex. In this case, \( f \) causes the total expenditure at most \( \frac{1}{2} \) for \( v \) by R0(iii), R2(ii), and R6. Indeed, it suffices to note that if \( v \) gives \( \frac{1}{2} \) to \( f \), then \( v \) receives \( \frac{1}{2} \) by R0(iii). (If, say, \( d(w) = 3 \), then \( v \) gives \( \frac{1}{2} \) to \( u \) by R2(iii)) but gets \( \frac{1}{2} \) from edge \( uv \) by R0(i). The same is true for \( u \). Let \( f_1 = \ldots u_1 w_1 v \) be the face at \( v \) opposite to \( f \). In addition to the donation of at most \( \frac{1}{2} \) caused by \( f \), our \( v \) can give at most 1 to \( f_1 \) if \( r(f_1) = 3 \) by R5, or at most \( \frac{1}{2} \) to \( f \) if \( r(f_1) = 4 \) (since \( f_1 \) is non-incident with 2-vertices adjacent to \( v \) by the assumption that \( v_2(v) = 0 \)), or else \( \frac{1}{2} \) by R3 if \( r(f_1) = 5 \). Furthermore, \( v \) gives either at most \( \frac{11}{10} \) to a 2-vertex \( u_1 \) or \( w_1 \) by R1(ii), R3(ii), or at most \( \frac{1}{2} \) to the 3-vertices in \( \{u_1, w_1\} \) by R2(ii), (ii). However, if \( r(f_1) = 5 \) then \( \beta(v) = 0 \), which means that \( d \) does not participate in R4. Therefore, in all these cases \( v \) gives at most \( \frac{11}{10} + \frac{1}{2} + \frac{1}{2} = \frac{3}{5} - 2 \times \frac{1}{4} = 1 \) to \( f_1, u_1, \) and
Subsubcase 2.3.2. There are no 4-faces at $v$.

(A) $v$ is non-triangular. Suppose $v_3(v) = 1$; due to Lemma 4 we have $\beta(v) = 0$. Recall that if $v_3(v) \geq 1$, then our $v$ is weak, so it does not give charge to 3-vertices by R2, which means that whatever $v_3(v)$, we have $ch^*(v) \geq 2 - \frac{6}{5} - 4 \times \frac{1}{5} = 0$ by R1 and R3.

Now suppose $v_3(v) = 0$, and let $v_5(v)$ be the number of 5-faces at $v$. Note that $v_5(v) + \beta(v) \leq 4$, since no bad neighbor can be incident with a 5-face; this implies that $ch^*(v) \geq 2 - \frac{\beta(v)}{2} - \frac{3(4-\beta(v))}{10} - \frac{v_5(v)}{5} \geq 2 - \frac{\beta(v)}{2} - 4 - \frac{4-\beta(v)}{2} = 0$ by R2–R4.
(B) $v$ is triangular. If $\tau_3(v) = 2$ then $ch^*(v) \geq 2 - 2 \times 1 = 0$ by R5. Suppose $\tau_3(v) = 1$. Recall that if $v_2(v) \geq 1$ then $v_2(v) = 1$ and $\beta(v) = 0$ due to Lemma 2(ii) and Lemma 4.

Suppose $v_2(v) = 1$. Now $v$ gives at most 1 to its 3-face and nothing to the other 3-vertex adjacent to $v$ along a nontriangular edge by R2. If $v$ is not incident with a 5-face, then $ch^*(v) \geq 2 - 2 \times 1 = 0$ by R5 and R1(iii) since $v$ gives at most 1 to its 2-vertex. If $v$ is incident with 5-face $f$, then $f$ lies opposite the triangle incident with $v$. Then Lemma 5 ensures that $v$ gets $\frac{5}{14}$ by R0 and gives $\frac{3}{2}$ to the incident triangle by R5(iii) and Lemma 4. Also, $v$ gives $\frac{3}{7}$ to its 5-face and $\frac{11}{10}$ to its 2-neighbor. Thus, $ch^*(v) \geq 2 + \frac{5}{7} + \frac{3}{14} - \frac{6}{14} - \frac{3}{14} - \frac{11}{10} = 1 + \frac{5}{14} - 1 - \frac{3}{10} > 0$.

Now assume $v_2(v) = 0$. If $\beta(v) \geq 1$ then $v$ is not incident with 5-faces, which implies $ch^*(v) \geq 2 - 1 - 2 \times \frac{5}{4} = 0$ by R3, R4, and R5; otherwise, $ch^*(v) \geq 2 - 2 \times \frac{5}{4} = 0$. Hence $ch^*(v) \geq 2d(v) - 6 - v_2(v) \times \frac{5}{4} - v_2^*(v) \times \frac{3}{7} - (d(v) - v_2(v)) \times \frac{3}{7} = \rho(v)$.

For example, suppose that there is a 4-face $f = uwwv$. If $d(w) \geq 4$, then $v$ gets $\frac{1}{5}$ by R0(iii) from a 8-face incident with edge $uw$. Similarly, if $d(w) \leq 3$, then our $v$ gets $\frac{1}{4}$ by R0(i). These two examples already show us that in many cases $ch^*(v) > \rho(v)$. Moreover, we can easily check that the actual expenditure of $v$ on $f$, $u$, and $w$ is at least $\frac{1}{2}$ less than that included in the formula for counting $\rho(v)$. In other words, we can say informally:

(•) Each 4-face saves at least $\frac{1}{2}$ with respect to $\rho(v)$.

Subcase 2.4. $d(w) = 5$. Now $ch^*(v) = 5$, and $v_2(v) \leq 3$ due to Lemma 2(ii).

First suppose $v_2(v) = 3$. Due to Lemma 6, $v$ is non-triangular, and by Lemma 2(iii), $\rho(v)$ we know that $\tau_3(v) = \beta(v) = 0$ and the central 2-neighbor of $v$ is surrounded by two 6-faces. If $\tau_4(v) = 0$, then this implies that $ch^*(v) \geq 4 - 1 - 2 \times \frac{3}{10} - 3 \times \frac{5}{4} > 0$ by R1 and R3. Suppose $\tau_4(v) = 1$; it follows that $v$ gets at least $2 \times \frac{1}{4}$ by R0(i). Furthermore, $v$ gives 1 to the central 2-vertex and at most $2 \times \frac{5}{4} + 2 \times \frac{1}{4}$ to the other 2-vertices, and at most $2 \times \frac{1}{4}$ to the incident 4- and 5-faces. This implies that $ch^*(v) \geq 4 + 2 \times \frac{5}{4} - 1 - 2 \times \frac{3}{4} - \frac{2}{2} = 0$.

If $v_2(v) \leq 1$ then $ch^*(v) \geq \rho(v) > 0$, so suppose $v_2(v) = 2$. If $v$ is incident with a 4-face, then it gets $\frac{1}{2}$ by (•) and $ch^*(v) \geq \rho(v) + \frac{1}{2} \geq 4 - 2 \times \frac{3}{4} - \frac{3}{4} + \frac{1}{2} = 0$. Suppose that $\tau_4(v) = 0$. Now $\rho(v) = -\frac{3}{10}$, but we can improve the lower bound $ch^*(v) \geq -\frac{1}{10}$ by arguing more carefully.

If $v$ is adjacent to a 4-vertex $z$ along a non-triangular edge, then the actual modified donation of $v$ to $z$ is at most $2 \times \frac{1}{10}$ rather than $\frac{1}{2}$ included into the formula for $\rho(v)$, which implies that $ch^*(v) \geq \rho(v) + \frac{1}{2} - 2 \times \frac{1}{10} = 0$.

Thus, from now on we can assume that every nontriangular edge from $v$ leads to a 3-vertex. Let us subdivide the neighbors of $v$ into two subsets. We say that a neighbor of $v$ is of type 1 either if edge $uv$ is triangular or if $u$ is a bad neighbor of $v$ (note that edge $uv$ cannot be incident with a 5- or 6-face). Otherwise, edge $uv$ is nontriangular and $d(u) \leq 3$, in which case $u$ is said to be a vertex of type 2. A 7-face ... $uww$ is special if $u$ and $w$ belong to different types. A 5-face ... $uww$ is non-special if $u$ and $w$ are of the same type.

It is not hard to see that if there is a 3-face $T = xuv$ or $v$ has a bad neighbor $b$, then there exist at least two special faces at $v$. Indeed, consider the longest clockwise sequence $S_1$ of non-special 7-faces around $v$, starting from face ... $xuv$ (where $T$ is oriented clockwise) or ... $bv$, respectively. Since $v_2(v) = 2$ by assumption, it follows that our $S_1$ will end in a special face. The same is true for the counter-clockwise sequence $S_2$ that starts from a 7-face ... $yuv$ or ... $bv$. Clearly, the two terminal special faces obtained this way are distinct. Note that every special face saves $\frac{1}{10}$ on edge $uw$ and also brings $\frac{1}{2}$ to $v$ by R0(i), so $ch^*(v) \geq \rho(v) + 2 \times \frac{1}{2} + 2 \times \frac{1}{10} = \frac{3}{5} + \frac{3}{10} + \frac{1}{2} > 0$.

Finally, assume that $v_2(v) = 2$, $v_3(v) = 3$ and $\tau_3(v) = \beta(v) = 0$. By Lemma 2(iv), there is a 6-face $f = \cdots uww$ at $v$, which means that we have a rough estimate $ch^*(v) \geq 2 \times 5 - 6 - 2 \times \frac{5}{4} - 3 \times \frac{1}{4} - 4 \times \frac{1}{2} = -\frac{1}{10}$. However, this bound should be strengthened by $2 \times \frac{1}{2}$ due to the fact that each of the vertices $u$ and $w$ takes from $v$ at most $\frac{1}{10}$ if it is a 2-vertex or at most $\frac{1}{3}$ if it is a 3-vertex. Thus in fact $ch^*(v) \geq -\frac{1}{10} + 2 \times \frac{1}{2} > 0$.

Subcase 2.5. $d(v) = 6$. Now $ch(v) = 6$, and $v_2(v) \leq 4$ due to Lemma 2(ii). First suppose that $v_2(v) = 4$. Then this lemma also says that $v_3(v) = 0$. Here, $\rho(v) = 6 - 4 \times \frac{1}{2} - 2 \times \frac{1}{2} = -1$, so we are done by (•) if $\tau_4(v) \geq 2$. If $\tau_4(v) = 1$, then $ch^*(v) \geq 6 - 2 \times \frac{5}{4} - 2 \times \frac{1}{2} - 3 \times \frac{1}{2} + 2 \times \frac{1}{4} = 0$ by R1(iv), R1(i), R6, R3, and R0, unless $\tau_5(v) = 1$, in which case $ch^*(v) \geq 6 - 2 \times \frac{5}{4} - 2 \times \frac{1}{2} - \frac{3}{10} - \frac{9}{7} - \frac{1}{2} + \frac{3}{4} + \frac{1}{8} > 0$ by the same rules augmented by R5(iii).
Suppose that $τ_d(v) = 0$. If $τ_3(v) = 1$ then $φ_3(v) ≤ 3$ and $v$ gets $1/7$ at least twice from incident $7^+$-faces by $R_0(j)$, which implies that $ch^*(v) ≥ 6 − 1 − 2 × 11/10 − 2 × 6/5 − 3 × 1/2 + 2 × 1/5 > 0$. If $τ_3(v) = 0$ then $ch^*(v) ≥ 6 − 4 × 6/5 − 6 × 1/5 = 0$ by Lemma 2(v). Finally, if $v_2(v) ≤ 3$ then $ch^*(v) ≥ ρ(v) ≥ 6 − 3 × 2/3 − 3 × 1/2 = 0$.

Subcase 2.6. 7 ≤ $d(v) ≤ 9$. Due to Lemma 2(iii), we have $v_2(v) ≤ d(v) − 2$. Suppose that $τ_4(v) ≥ 1$, then by (⋆) we can improve the bound $ρ(v)$ by at least $1/2$, which implies that $ch^*(v) ≥ ρ(v) + 1/2 = 2d(v) − 6 − (d(v) − 2) × 7/5 − 2 × 1/2 + 1/2 = 2(d(v) − 7) − 7/2 ≥ 0$. If $τ_d(v) = 0$, then $v_2^*(v) = 0$ and so $ch^*(v) ≥ ρ(v) = 2d(v) − 6 − (d(v) − 2) × 7/5 − 2 × 1/2 + 1/2 = 2(d(v) − 7) − 7/2 ≥ 0$ by Remark 1.

Subcase 2.7. $d(v) ≥ 10$. If $τ_d(v) = 0$, then $ch^*(v) ≥ 2d(v) − 6 − 7d(v)/5 = 3/5(d(v) − 10) ≥ 0$. Suppose that $τ_d(v) ≥ 1$; now (⋆) works again. If $d(v) ≥ 11$, then $ch^*(v) ≥ ρ(v) + 1/2 = 2d(v) − 6 − d(v) × 7/5 + 1/2 = ρ(v) ≥ d(v) − 11/2 ≥ 0$. If $d(v) = 10$, then $v_2(v) ≤ 9$ by Lemma 2(ii), so $ch^*(v) ≥ 14 − 9 × 2/3 − 1/2 + 1/2 = 0$.

Hence, after discharging according to rules $R_0$–$R_6$ the charge of each vertex and face of $G$ is non-negative, which contradicts (1).

Acknowledgments

This work was supported by the Ministry of Education and Science of the Russian Federation (contract number 14.740.11.0868). The second author was supported by grants 12-01-00631, 12-01-00448, and 12-01-98510 of the Russian Foundation for Basic Research.

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