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The Genus Problem for Cubic Graphs

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We prove that the following problem is NP-complete: Given a cubic graph G and a natural number g, is it possible to draw G on the sphere with g handles added?

The graph genus problem was listed as unsolved problem 3 by Garey and Johnson [1]. The NP-completeness of this problem was established in [5]. Ringel [4] raised the problem of characterizing those graphs which triangulate some surface. (If the surface is orientable, a triangulation is of minimum genus, by Euler's formula). The NP-completeness of Ringel's question was established in [6]. In this paper we consider a dual version of Ringel's question, namely the genus problem in the cubic case. This question was raised by Richter [3] who observed that vertices of large degree play an important role in [5]. (They are even more important in [6].) The complexity of a combinatorial problem may change when restricted to the cubic case. Thus unsolved problem 1 in [1], the graph isomorphism problem, is solvable in polynomial time for cubic graphs [2], whereas its complexity for general graphs is unknown. If $\varphi\colon G \to H$ is an isomorphism, then it is easy to replace every vertex of G or H by an appropriate subgraph such that φ can be modified to an isomorphism of the two resulting cubic graphs. Also, if G is drawn on a surface of minimum genus it is easy to modify G into a cubic graph drawn on the same surface of minimum genus, see Figure 2, below. However, these modifications work only if we already know the isomorphism or the minimum genus embedding, respectively. Also, it does no seem feasible to derive the cubic case from [6] using duality. Firstly, we do not know the dual graph before we know the embedding. Secondly, if an embedding is of minimum genus we cannot conclude that the dual graph embedding is of minimum genus. (To see this, take a cubic graph G and embed it on a surface which is not of minimum genus and such that all faces are bounded by cycles no two of which have more than one edge in common. Then the dual graph G^* is a triangulation

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and therefore a minimum genus embedding. But, its dual G is not a minimum genus embedding.)

We shall reduce the independence number problem to the genus problem for cubic graphs using ideas similar to those in [5] combined with an estimate of the genus of a graph G expressed in terms of (generalized) genera of G_1, G_2 where G_1, G_2 are the components of G minus an edge-cut.

2. AN INEQUALITY FOR THE GENUS

The terminology and notation is the same as in [5]. In particular, embeddings of a connected graph G may be treated purely combinatorially by a rotation system $\Pi = (\pi_1, \pi_2. ..., \pi_n)$ where the vertices are labelled v_1, $v_2, ..., v_n$ and π_i is a cyclic permutation of the edges incident with v_i for $i = 1, 2, ..., n$. We call π_i the *clockwise orientation* of the edges incident with v_i. As in [5], the *Π-genus* $g(G, \Pi)$ of G is defined by Euler's formula, and the *genus* $g(G)$ of G is the minimum Π-genus taken over all embeddings Π. If $A \subseteq V(G)$ and Π is an embedding of G, then we let m denote the smallest number of facial walks $W_1, W_2, ..., W_m$ such that each vertex of A is contained in some W_i, $1 \leqslant i \leqslant m$. We write

$$g(G, \Pi, A) = g(G, \Pi) + m/2$$

and we define $g(G, A)$ as the minimum $g(G, \Pi, A)$ taken over all embeddings Π of G. With this terminology we now have:

PROPOSITION 2.1. *Let G_1, G_2 be disjoint connected graphs and let $A_1 \subseteq V(G_1)$, $A_1 \neq \emptyset$. Let $\varphi: A_1 \to V(G_2)$ be $1-1$. Let G_0 be the graph obtained from $G_1 \cup G_2$ by adding all edges from x to $\varphi(x)$, where $x \in A$. Put $A_2 = \varphi(A_1)$. Then*

$$g(G_0) \geqslant g(G_1, A_1) + g(G_2, A_2) - 1. \tag{1}$$

If φ can be extended to a isomorphism of G_1 onto G_2, then (1) *is an equality, that is,*

$$g(G_0) = 2g(G_1, A_1) - 1. \tag{2}$$

Proof. Let Π_0 be an embedding of G_0 such that $g(G_0, \Pi) = g(G_0)$. Let Π_i be the induced embedding of G_i for $i = 1, 2$. For each $i \in \{0, 1, 2\}$ we let n_i, q_i, and f_i denote the number of vertices, edges, and Π_i-facial walks of G_i, respectively. Let m_0 denote the number of Π_0-facial walks containing at least one edge between G_1 and G_2, and let m_i denote the number of those

Π_i-facial walks in G_i which are not Π_0-facial walks in G_0. By Euler's formula,

$$n_i - q_i + f_i = 2 - 2g(G_i, \Pi_i) \tag{3}$$

for $i = 0, 1, 2$.

Clearly,

$$f_0 = (f_1 - m_1) + (f_2 - m_2) + m_0, \tag{4}$$

$$n_0 = n_1 + n_2, \tag{5}$$

$$q_0 = q_1 + q_2 + |A|, \tag{6}$$

and

$$m_0 \leqslant |A|. \tag{7}$$

(7) follows from the fact that every edge is traversed precisely twice by facial walks (which may be identical), and every facial walk which traverses an edge from G_1 to G_2 also traverses an edge from G_2 to G_1. For $i = 1, 2$, and for each vertex z in A_i, at least one of the m_i facial walks defined above contains z. Hence

$$g(G_i, \Pi_i, A_i) \leqslant g(G_i, \Pi_i) + m_i/2 \tag{8}$$

for $i = 1, 2$.

We now combine (3)–(8):

$$\begin{aligned} 2 - 2g(G_0) = 2 - 2g(G_0, \Pi_0) &= n_0 - q_0 + f_0 \\ &= n_1 + n_2 - q_1 - q_2 - |A| + f_1 - m_1 + f_2 - m_2 + m_0 \\ &= (2 - 2g(G_1, \Pi_1)) + (2 - 2g(G_2, \Pi_2)) - |A| - m_1 - m_2 + m_0 \\ &\leqslant 2 - 2g(G_1, \Pi_1, A_1) + 2 - 2g(G_2, \Pi_2, A_2) \\ &\leqslant 4 - 2g(G_1, A_1) - 2g(G_2, A_2) \end{aligned}$$

which implies (1).

We now prove (2). Let Π_1 be an embedding of G_1 and let $W_1, ..., W_{m_1}$ be facial walks such that

$$g(G_1, A_1) = g(G_1, \Pi_1, A_1) = g(G_1, \Pi_1) + m_1/2.$$

Let Π_2 be the embedding of G_2 obtained from Π_1 by changing "clockwise" to "anticlockwise", and let $W'_1, ..., W'_{m_2}$ be the facial Π_2-walks corresponding to $W_1, W_2, ..., W_{m_1}$ (where $m_2 = m_1$). We form an embedding Π_0 of G_0 which we describe topologically as follows (and we leave it to the reader to translate it to the combinatorial terminology): For each $i=1, 2, ..., m_1$ we cut out a disc in the faces bounded by W_i and W'_i, and we connect these two faces by a tube (or cylinder or handle). We add all edges from x to $\varphi(x)$, where $x \in A \cap W_i$ across that tube. Then (7) becomes an equality. Combining (3)–(8) now gives

$$g(G_0, \Pi_0) = g(G_1, A_1) + g(G_2, A_2) - 1.$$

Combining this with (1) gives (2). ∎

3. REDUCTION OF MAXIMUM INDEPENDENCE TO GENUS OF CUBIC GRAPHS

Let H be a connected graph of minimum degree at least 3 in which we want to find $\alpha(H)$, that is, the maximum cardinality of a set of independent vertices. We let $\beta(H)$ denote the smallest number of vertices covering (meeting) all edges. Let n, q denote the number of vertices and edges of H, respectively. Clearly,

$$\alpha(H) + \beta(H) = n.$$

Now let Π be any embedding of H. Let M be obtained from H by replacing each edge xy by the graph of Fig. 1.

(Note that, if we identify x and y in Fig. 1, then the resulting graph is nonplanar and has therefore no embedding with more than 5 facial walks. In Fig. 1 we also put $y_i = x_i$ for $i = 5, 6, 7$.)

Let Π_0 denote the embedding of M such that Π_0 agrees with Π in H and is as shown in Fig. 1 for all vertices not in H. (In particular, the edges x_6x_7, x_6y_2, x_6x_2 occur in clockwise order around x_6.) Then M has $n + 11q$ vertices, $17q$ edges, and $n + 4q$ Π_0-facial walks. Hence

$$(n + 11q) - 17q + (n + 4q) = 2 - 2g(M, \Pi_0)$$

which implies that

$$g(M, \Pi_0) = q - n + 1.$$

Let A denote the set of vertices of the form x_7. If S is a set of vertices of H, then the $|S|$ Π_0-facial walks of M containing the vertices of S also contain all vertices of A, if and only if S meets all edges of H. Hence

$$g(M, \Pi_0, A) = q - n + 1 + \beta(H)/2.$$

Let G_1 be obtained from M by modifying each vertex and its clockwise orientation as indicated in Fig. 2. Let Π_1 denote the resulting embedding of G_1.

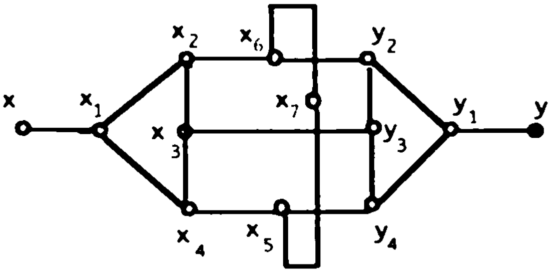

FIGURE 1

Clearly

$$g(G_1, \Pi_1) = g(M, \Pi_0) = q - n + 1$$

and

$$g(G_1, A) \leqslant g(G_1, \Pi_1, A) = g(M, \Pi_0, A) = q - n + 1 + \beta(H)/2.$$

With this notation we shall now prove

PROPOSITION 3.1.

$$g(G_1, A) = q - n + 1 + \beta(H)/2.$$

Proof. Let Π_2 be an embedding of G_1, and let $X = \{W_1, W_2, ..., W_m\}$ be a collection of Π_2-facial walks such that

$$g(G_1, A) = g(G_1, \Pi_2, A) = g(G_1, \Pi_2) + m/2.$$

Subject to this we choose Π_2 such that it agrees with Π_1 around as many vertices as possible.

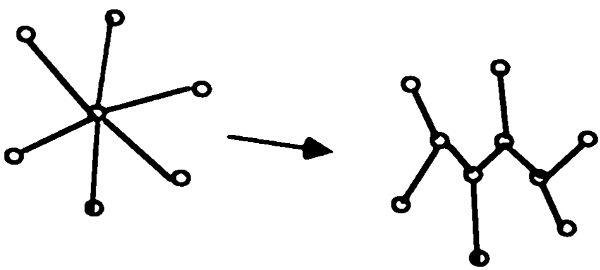

FIGURE 2

Consider an edge xy in H. Let W be the Π_2-facial walk containing xx_1. If W leaves the graph of Fig. 1 along x_1x (after having entered along xx_1), then there is a Π_2-facial W' containing yy_1 and y_1y. Possibly $W' = W$. We now change Π_2 so that it agrees with Π_1 on all vertices $x_1, ..., x_7, y_1, ..., y_7$. The new embedding has at least as many facial walks as Π_2 (because Π_2 does not have more than 4 facial walks that are contained in the graph of Fig. 1). Also, the new embedding has a collection of m facial walks containing all vertices of A. By the maximality of Π_2 we may assume that Π_2 agrees with Π_1 on the subgraph of Fig. 1.

Suppose next that Π_2 has a facial walk W containing xx_1 and y_1y. Then there is also a Π_2-facial walk W' containing yy_1 and x_1x. Possibly $W' = W$. Then there are at most $3\Pi_2$-facial walks contained entirely in the subgraph of Fig. 1. Again, we change Π_2 so that it agrees with Π_1 in the subgraph of Fig. 1. If $W \neq W'$, then W and W' correspond to only one facial walk in the new embedding. But we get instead 4 facial walks in the subgraph of Fig. 1. Also, m is not increased. Hence the new embedding also realizes $g(G_1, A)$ contradicting the maximality of Π_2. If $W = W'$ then the new embedding has at least two facial walks more than Π_2 and m increases by at most one. Again, we get a contradiction.

We have proved that Π_2 agrees with Π_1 on each subgraph shown in Figure 1. Hence Π_2 and Π_1 have the same number of facial walks. Also, $m \geqslant \beta(H)$. This implies that

$$g(G_1, A) \leqslant g(G_1, \Pi_1, A) \leqslant g(G_1, \Pi_2, A) = g(G_1, A). \quad \blacksquare$$

THEOREM 3.2. *The following problem is NP-complete. Given a cubic graph G_0 and a natural number k. Is $g(G_0) \leqslant k$?*

Proof. Let H be any connected graph of minimum degree at least 3, and let m be any natural number. Now form G_1 and A as in Proposition 3.1. Then form G_0 as in Proposition 2.1 where G_2 is isomorphic to G_1, and $A_1 = A$ is the set of vertices of degree 2 in G_1. Note that G_0 is cubic. By Propositions 2.1, 3.1

$$g(G_0) = 2g(G_1, A) - 1 = 2(q - n + 1) + \beta(H) - 1 = 2q - n - \alpha(H) + 1.$$

Hence the inequality $\alpha(H) \geqslant m$ is equivalent to the inequality

$$g(G_0) \leqslant 2q - n - m + 1.$$

As it is an *NP*-complete problem to decide if the former inequality holds, see [1], Theorem 3.2. follows. $\blacksquare$

The proof of Theorem 3.2 easily extends to the nonorientable case and to r-regular graphs where $r = 4, 5$. It probably extends to r-regular graphs for each fixed r. If c is a fixed constant, $0 < c < 1$, one may also consider the genus problem for $\lfloor cn \rfloor$-regular graphs with n vertices. The complexity may depend on c. Perhaps the problem is NP-complete for $c = 1/100$, but in P for $c = 99/100$.

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