Erratum


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ABSTRACT

In [1], Gu and Tian [Chuanqing Gu, Zhaolu Tian, On the HSS iteration methods for positive definite Toeplitz linear systems, J. Comput. Appl. Math. 224 (2009) 709–718] proposed the special HSS iteration methods for positive definite linear systems $Ax = b$ with $A \in \mathbb{C}^{n \times n}$ a complex Toeplitz matrix. But we find that the special HSS iteration methods are incorrect. Some examples are given in our paper.

1. Introduction

Recently, Gu and Tian [1] proposed the special HSS iteration methods for positive definite linear systems

$$Ax = b$$

with $A \in \mathbb{C}^{n \times n}$ a complex Toeplitz matrix and $x, b \in \mathbb{C}^{n \times n}$. Such systems arise in a variety of applications in mathematics and engineering; see Refs. [2,3] for details. Their aim was to apply the special HSS iteration methods to solve the large sparse non-Hermitian positive definite Toeplitz systems, which is a special version of the HSS iteration method in [4] and the splitting is

$$A = H + S,$$

where the symmetric part $H = \frac{1}{2}(A + A^T)$ is a centrosymmetric matrix, the skew-symmetric part $S = \frac{1}{2}(A - A^T)$ is a skew-centrosymmetric matrix and $T$ denotes the transpose of the matrix. A matrix $A \in \mathbb{C}^{n \times n}$ is said to be centrosymmetric if $JA = A$ and skew-centrosymmetric if $JA = -A$, where $J$ is the permutation matrix with ones on the cross diagonal (bottom left to top right) and zero elsewhere. Then they proposed the modified Hermitian and skew-Hermitian (HSS) splitting iterative method for solving $Ax = b$ with $A \in \mathbb{C}^{n \times n}$: An initial vector $x^{(0)}$ is given. For $k = 0, 1, \ldots$ until $x(k)$ converges, compute

$$
\begin{align*}
(\beta I + H)x^{(k+\frac{1}{2})} &= (\beta I - S)x^{(k)} + b, \\
(\beta I + S)x^{(k+1)} &= (\beta I - H)x^{(k+\frac{1}{2})} + b,
\end{align*}
$$

(3)

where $\beta$ is a given positive constant, $H = \frac{1}{2}(A + A^T)$ and $S = \frac{1}{2}(A - A^T)$.

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2. Main results

We begin this section by recalling some definitions that will be used in this paper.

**Definition 1.** A complex matrix \( A \in \mathbb{C}^{n \times n} \) is called positive definite if \( R(x^H Ax) > 0 \) for all nonzero complex vectors \( x \), where \( x^H \) denotes the conjugate transpose of the vector \( x \) [5, page 399].

A necessary and sufficient condition for a complex matrix \( A \) to be positive definite is that the Hermitian part \( \frac{1}{2}(A^H + A) \) is positive definite.

**Definition 2.** For any complex matrix \( A \), the 2-norm or spectral norm of \( A \) is defined as \( \| A \|_2 = \sqrt{\rho(A^H A)} \), where \( \rho(\cdot) \) represents the spectral radius of a matrix [5, page 295].

**Definition 3.** If a complex matrix \( A \) satisfies \( A^T A = I \), then \( A \) is said to be a complex orthogonal matrix [6, page 63].

It should be noted that a complex orthogonal matrix \( A \) is not orthogonal, i.e., \( A^H A \neq I \). Therefore, \( \| A \|_2 \) may not be equal to one for a complex orthogonal matrix \( A \).

For the special HSS iteration methods, Gu and Tian [1] gave the following theorem:

**Theorem 3** ([1]). Let \( A \in \mathbb{C}^{n \times n} \) be a positive definite matrix, \( H = \frac{1}{2}(A + A^T) \) and \( S = \frac{1}{2}(A - A^T) \) be its centrosymmetric and skew-centrosymmetric parts, respectively. Let

\[
M(\beta) = (\beta I + S)^{-1}(\beta I - H)(\beta I + H)^{-1}(\beta I - S),
\]

be the iteration matrix of the HSS iteration (3) and \( V(\beta) = (\beta I - H)(\beta I + H)^{-1} \). Then the spectral radius \( \rho(M(\beta)) \) is bounded by \( \| V(\beta) \|_2 \) and has the following relation:

\[
\rho(M(\beta)) \leq \| V(\beta) \|_2 < 1 \quad \text{for } \forall \beta > 0;
\]

i.e., the HSS iteration (3) converges to the exact solution \( x^* \in \mathbb{C}^n \) of the system of Toeplitz linear equations (1).

They have proved the theorem. Unfortunately, the proof of Theorem 3 in Ref. [1] is incorrect. By similarity transformation, they first note that

\[
\rho(M(\beta)) = \rho((\beta I + S)^{-1}(\beta I - H)(\beta I + H)^{-1}(\beta I - S)) \\
= \rho((\beta I - S)(\beta I + S)^{-1}(\beta I - H)(\beta I + H)^{-1}) \\
\leq \| (\beta I - S)(\beta I + S)^{-1}(\beta I - H)(\beta I + H)^{-1} \|_2 \\
\leq \| (\beta I - S)(\beta I + S)^{-1} \|_2 \| (\beta I - H)(\beta I + H)^{-1} \|_2.
\]

If \( P \in \mathbb{C}^{n \times n} \) is a positive definite matrix, then it holds that \( \| (\beta I - P)(\beta I + P)^{-1} \|_2 < 1 \); see [7]. Since \( A \) is a positive definite matrix, \( H = \frac{1}{2}(A + A^T) \) is also positive definite matrix. It then follows that \( \| V(\beta) \|_2 < 1 \). Letting \( Q(\beta) = (\beta I - S)(\beta I + S)^{-1} \), we see that

\[
Q(\beta)^T Q(\beta) = (\beta I - S)^{-1}(\beta I + S)(\beta I - S)(\beta I + S)^{-1} \\
= (\beta I - S)^{-1}(\beta I - S)(\beta I + S)^{-1} \\
= I.
\]

That is to say, \( Q(\beta) \) is a complex orthogonal matrix for \( \forall \beta > 0 \). So, they think that \( \| Q(\beta) \|_2 = 1 \). It then follows that

\[
\rho(M(\beta)) \leq \| (\beta I - H)(\beta I + H)^{-1} \|_2 < 1.
\]

In fact, \( Q(\beta) = (\beta I - S)(\beta I + S)^{-1} \) is not an orthogonal matrix. So \( \| Q(\beta) \|_2 = 1 \) is incorrect. For example, \( A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \), thus \( Q(\beta) = \begin{pmatrix} \beta & 1 \\ -1 & \beta \end{pmatrix} \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix}^{-1} \). \( \| Q(\beta) \|_2 = 3 \neq 1 \) for \( \beta = 2 \). It is because the definition of 2-norm or spectral norm of \( Q(\beta) \) is \( \| Q(\beta) \|_2 = \sqrt{\rho(Q(\beta)^H Q(\beta))} \). Therefore, even if \( Q(\beta) \) is an complex orthogonal matrix, \( \| Q(\beta) \|_2 \) may not be equal to one.
Table 1
The spectral radius $\rho(M(\beta))$ of the iteration matrices (4).

<table>
<thead>
<tr>
<th>$n$ = 64</th>
<th>$\beta$</th>
<th>0.5</th>
<th>2.5</th>
<th>4.5</th>
<th>6.5</th>
<th>8.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho(M(\beta))$</td>
<td>15.487</td>
<td>12.764</td>
<td>3.367</td>
<td>1.857</td>
<td>0.628</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$ = 128</th>
<th>$\beta$</th>
<th>0.5</th>
<th>2.5</th>
<th>4.5</th>
<th>6.5</th>
<th>8.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho(M(\beta))$</td>
<td>15.360</td>
<td>13.176</td>
<td>3.343</td>
<td>1.888</td>
<td>0.627</td>
<td></td>
</tr>
</tbody>
</table>

3. Examples

Example 1. Now let us give a counterexample of Theorem 3 below. Consider the non-Hermitian positive definite Toeplitz linear systems $Ax = b$ with

\[
A = \begin{pmatrix}
10 & 2i & 3i \\
-1 - 2i & 10 & 2i & 3i \\
-1 - 3i & -1 - 2i & 10 & 2i & 3i \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-1 - 3i & -1 - 2i & 10 & 2i & 3i \\
-1 - 3i & -1 - 2i & 10 & 2i \\
-1 - 3i & -1 & 10 & 2i \\
\end{pmatrix}.
\]

The spectral radius $\rho(M(\beta))$ of the iteration matrices (4) are shown in Table 1. From Table 1, we see that not all positive constants $\beta$ satisfy $\rho(M(\beta)) < 1$ for a definite positive Toeplitz matrix $A$. But a good choice of $\beta$ can make the modified HSS method converge.

References