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Smoothing Properties of Semigroups for Dirichlet Operators of Gibbs Measures*

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In this paper is investigated the special class of elliptic differential second-order operators with an infinite number of variables with the property of a finite radius of dependence for variables. This class is formed by the Dirichlet operators associated with energy forms of Gibbs measures on compact Riemannian manifolds with a finite radius of interaction. Using this property we represent the Dirichlet operator as a finite sum of self-adjoint operators with independent variables and prove that the Dirichlet operator semigroups preserve the specially constructed scales of continuously differentiable functions. We also obtain that these semigroups raise the smoothness of initial functions.

1. INTRODUCTION

The present paper is dedicated to the investigation of semigroups for an important class of elliptic differential operators, which are known as the Dirichlet operators of probability measures on infinite-dimensional spaces. These operators are of particular interest because they are associated with energy Dirichlet forms and, interpretated as quantum Hamiltonians of corresponding physical systems, have a wide range of applications to the problems of mathematical physics, quantum mechanics, and quantum field theory [16, 5, 4].

The theory of energy forms of smooth measures on infinite-dimensional spaces and corresponding differential operators is an intensively developing field of mathematics. Possibly the most complete citations and discussion of different directions in this field can be found in monographs [16, 12, 8] and surveys [5, 9].

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An investigation of Gibbs measures on the infinite product of finite-dimensional spaces not only is of special interest in connection with the important applications to lattice particle systems, but also is motivated by the possibility of deriving information about the properties of corresponding Dirichlet operators. For example, the conditions on the logarithmic Sobolev inequality and its connection with the Dobrushin–Schlosman mixing condition [13, 24, 23], the conditions on its essential self-adjointness, and the ergodicity of its semigroup [2, 3] were obtained.

In this paper we consider the class of Dirichlet operators for which sufficiently complete information about smoothing properties of corresponding semigroups which are analogs of similar properties in the finite-dimensional case can be obtained. We restrict our investigations to differential operators with an infinite number of variables with the additional property of a *finite radius of dependence* for variables appearing in the differential expression.

Such operators are associated with Gibbs measures on the countable product of compact Riemannian manifolds. The main emphasis in this paper is on the development of methods of semigroup theory which are intrinsic for the Dirichlet operators of Gibbs measures.

We provide an approach to the investigation of corresponding semigroups in which it is not necessary to obtain preliminary information about the structure of the Gibbs measures simplex. This is possible because the information about the conditional Gibbs measures in finite volumes of the lattice $\mathbb{Z}^d$ is completely reflected in the coefficients of the Dirichlet operators (see Theorem 3.2).

In this paper we continue the investigations commenced in [7] and initiated by the paper of Roelly and Zessin [22], where the characterization of the Gibbs measure on the space $C[0, 1]^{\mathbb{Z}^d}$ through the special structure integration by parts formula was provided.

In Sections 2 and 3 we outline the results of [7] and propose a characterization of the Gibbs measures simplex in terms of *Hermitian realizations* for a special differential operator (Definition 3.1),

$$H_{\mathbb{Z}^d} = \frac{1}{2} \sum_{k \in \mathbb{Z}^d} \left\{ -\Delta_k + \prec \nabla_k \left(\sum_{A : k \in A} \Phi_A \right), \nabla_k \succ \right\},$$

acting on smooth cylinder functions $C^{\infty}_{\text{cyl}}(M^{\mathbb{Z}^d})$.

In Section 4 we provide a *representation* of the Dirichlet operator $H_{\mathbb{Z}^d}$ as a finite sum of block operators H_i with *independent* variables (see (4.1)). This representation is intimately linked with the special role which systems with a finite radius of interaction play in modern mathematical physics.

The operators H_i have the important property of *smooth solvability* of the parabolic Cauchy problem (Theorem 4.1) in the space $C^{\infty}_{\text{cyl}}(M^{\mathbb{Z}^d})$. This gives the result of essential self-adjointness for these block operators

(Corollary 4.2). Therefore the question of self-adjointness for the operator $H_{\mathbb{Z}^d}$ is reduced to the problem of self-adjointness for the finite sum of self-adjoint operators H_i. The difficulty which arises from the fact that these operators are not comparable is overcome by application of the Da Prato and Grisvard theorem [11] (see Theorem 4.3).

In Sections 5–8 we develop the investigations started in [6] and [1]. We study the action of Dirichlet operator semigroups in the special scales of spaces of smooth functions on the infinite product of manifolds.

In Section 5 we construct these scales and obtain the necessary preliminary information about the *quasi-accretivity* for operator $H_{\mathbb{Z}^d}$ (Theorems 5.6 and 5.8).

The results of Section 6 are rather technical; we apply them to the proof of quasi-accretivity in Section 5 and to the proof of the smoothing properties in Theorems 8.5 and 8.6.

In Section 7 we find the conditions for *essential self-adjointness* of the Dirichlet operator $H_{\mathbb{Z}^d}$ in the space of square integrable functions with respect to the Gibbs measure (Theorem 7.4). Using the *multiplicative* formula for semigroups we reconstruct the properties of dynamics generated by $H_{\mathbb{Z}^d}$ in terms of the properties of local dynamics (Theorem 7.5). As a consequence we obtain the main result of this section that the action of the semigroup $\exp(-tH_{\mathbb{Z}^d})$ *preserves each space from the scales of smooth functions* constructed in Section 5. Note that the similar result was announced in [13, Thm. 2.2] with a brief sketch of the idea of the proof in terms of Ito stochastic differential equations.

In Section 8 we investigate the *smoothing properties* of Dirichlet operator semigroups in special scales of spaces. We prove that the semigroup of a Dirichlet operator acts from the space of finitely differentiable functions into the *Frechet space* of infinitely differentiable functions (Theorems 8.5 and 8.6). Note the interesting intrinsic moment in which the representation of Dirichlet operator $H_{\mathbb{Z}^d}$ again plays a role when we find the conditions on differentiation of the maximum and once again apply the *multiplicative representation* for the semigroup of operator $H_{\mathbb{Z}^d}$.

In the Appendix we give a sketch of the proof for a simple generalization of the Da Prato–Grisvard theorem [11] to the case of a finite number of linear operators.

2. Integration by Parts Characterization of Gibbs Measures

Here we provide a characterization of Gibbs measures on the countable product of manifolds in terms of the integration-by-parts formula.

Consider a d-dimensional lattice $\mathbb{Z}^d$. To each point $k=(k_1, ..., k_d) \in \mathbb{Z}^d$ we let correspond a smooth, compact, complete, connected Riemannian manifold M_k with metric tensor $g^{(k)}$ and Riemannian measure $\sigma_k: d\sigma_k = (\det g^{(k)})^{1/2}\, dx$ in local coordinates on M_k.

The inner scalar product of tensors u and v on manifold M_k is defined by the expression in local coordinates

$$\prec u, v \succ_{T^{p,q}M_k} = \bigtimes_{s=1}^{p} g^{(k)}_{i_s j_s} \bigtimes_{t=1}^{q} g^{r_t l_t}_{(k)} u^{i_1 \cdots i_p}_{r_1 \cdots r_q} \bar{v}^{j_1 \cdots j_p}_{\ell_1 \cdots \ell_q},$$

where $g_{(k)} = \{ g^{i_s j_s}_{(k)} \}$ denotes the inverse to the metric tensor on manifold M_k. The norm of the tensor field $|u| = (\prec u, u \succ)^{1/2}$ is defined in the same way.

Let ∇_k denote the operator of covariant differentiation acting on tensor fields on M_k. Correspondingly, let Δ_k denote the Laplace–Beltrami operator on tensors on M_k.

For a finite or infinite subset $\Lambda \in \mathbb{Z}^d$, $|\Lambda| \leqslant \infty$, we preserve the following notations:

$$M^\Lambda = \bigtimes_{k \in \Lambda} M_k, \qquad x_\Lambda = \{x_k\}_{k \in \Lambda}, \quad x_k \in M_k.$$

We denote by $\mathscr{F}_\Lambda$ the Borel (or Tichonov, for $|\Lambda| = \infty$) σ-algebra on the product of manifolds M^Λ.

Consider the family of interactive potentials $\{\Phi_A, |A| < \infty\}$ in the finite subsets $A \subset \mathbb{Z}^d$ satisfying the following assumptions:

1. Function Φ_A is measurable with respect to the σ-algebra $\mathscr{F}_A$ and is a smooth function $\Phi_A \in C^\infty(M^A)$.

2. The finite radius of interaction

$$\exists r_0 > 0 : \forall A \subset \mathbb{Z}^d : \mathrm{diam}(A) > r_0 \Rightarrow \Phi_A \equiv 0. \tag{2.1}$$

3. The transitional invariantness

$$M_k \equiv M, \qquad \sigma_k \equiv \sigma, \qquad \tau_k \Phi_A = \Phi_{\tau_k A},$$

for every $k, A \subset \mathbb{Z}^d$, where τ_k denotes the shift on vector $k \in \mathbb{Z}^d$.

Now we introduce the Gibbs measure on $M^{\mathbb{Z}^d}$ in the following way. Let V_Λ denote the potential of volume $\Lambda \subset \mathbb{Z}^d$,

$$V_\Lambda(x_\Lambda \mid y) = \sum_{A \cap \Lambda \neq \varnothing} \Phi_A(z), \tag{2.2}$$

where $z = (x_\Lambda, y_{\Lambda^c})$ and $\Lambda^c = \mathbb{Z}^d \backslash \Lambda$. Put

$$Z_\Lambda(y) = \int_{M^{\mathbb{Z}^d}} \exp(-V_\Lambda(x_\Lambda \mid y)) \bigtimes_{k \in \Lambda} d\sigma_k(x_k).$$

We define the Gibbs measure in a finite volume $\Lambda \subset \mathbb{Z}^d$ with fixed boundary condition $y \in M^{\mathbb{Z}^d}$ as

$$d\mu_\Lambda(x \mid y) = \frac{1}{Z_\Lambda(y)} \exp(-V_\Lambda(x \mid y)) \bigtimes_{k \in \Lambda} d\sigma_k(x_k). \tag{2.3}$$

Due to Conditions 1, 2 in (2.1) we have

$$0 < Z_\Lambda(y) < \infty$$

for any $\Lambda \subset \mathbb{Z}^d$, $|\Lambda| < \infty$, and $y \in M_{\mathbb{Z}^d}$, so these measures are correctly defined.

Denote by $\mathbf{E}_\Lambda$ the expectation with respect to the measure μ_Λ. Then due to (2.3) we have the next consistency condition:

$$\mathbf{E}^y_{\Lambda_1} \mathbf{E}^{\cdot}_{\Lambda_2} = \mathbf{E}^y_{\Lambda_1}, \qquad \Lambda_1 \supset \Lambda_2. \tag{2.4}$$

DEFINITION 2.1. The probability measure μ on $M^{\mathbb{Z}^d}$ is a Gibbs one with local specifications $\{\mu_\Lambda, \Lambda \subset \mathbb{Z}^d\}$ iff

$$\mu(\mathbf{E}^{\cdot}_\Lambda) = \mu, \tag{2.5}$$

where $\mu(f) = \int f \, d\mu$.

We adopt the notation $\mu \in \mathscr{G}\{\mu_\Lambda\}$ for Gibbs measures with local specifications $\{\mu_\Lambda\}$.

Remark. The condition (2.5) is equivalent to the assumption that the family $\{\mu_\Lambda\}$ forms the set of conditional measures for measure μ with respect to σ-algebras $\mathscr{F}_{\Lambda^c}$ [14, 18, 19].

Henceforth we suppose that the set of Gibbs measures is *nonempty*:

$$\mathscr{G}\{\mu_\Lambda\} \neq \varnothing.$$

For example, this is satisfied under conditions 1–3 in (2.1) (see [19]).

Let $C^k_{\text{cyl}}(M^{\mathbb{Z}^d})$ denote the set of functions f which are cylindrical on lattice $\mathbb{Z}^d$: there is a finite subset $\Lambda_f \subset \mathbb{Z}^d$ such that f is measurable with respect to the σ-algebra $\mathscr{F}_{\Lambda_f}$ and $f \in C^k(M^{\Lambda_f})$. The space of smooth cylinder functions $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ is defined in the same way.

We call the minimal set Λ_f which satisfies the conditions above the *support of cylindricity* for function f:

$$\operatorname*{supp}_{\text{cyl}} f = \min\{\Lambda : f \text{ is } \mathscr{F}_\Lambda\text{-measurable}\}. \tag{2.6}$$

The next theorem proposes a characterization of the class of Gibbs measures in terms of the integration-by-parts formula [7]. Firstly such characterization of Gibbs measures in the case of linear spin systems was proved in [22].

THEOREM 2.2. *Suppose that the interactive potentials Φ_A satisfy Conditions* 1, 2 *in* (2.1). *The probability measure μ is a Gibbs one with local specifications $\{\mu_A\}$ (i.e., $\mu \in \mathscr{G}\{\mu_A\}$) if and only if the integration-by-parts formula,*

$$\int_{M^{\mathbb{Z}^d}} \sum_{k \in \mathbb{Z}^d} \prec h_k, \nabla_k u \succ d\mu = \int_{M^{\mathbb{Z}^d}} u \sum_{k \in \mathbb{Z}^d} (\prec h_k, \nabla_k V_k \succ - \operatorname{div}_k h_k)\, d\mu, \qquad (2.7)$$

holds for $u \in C^{\infty}_{\mathrm{cyl}}(M^{\mathbb{Z}^d})$ and $h_k \in C^1_{\mathrm{cyl}}(M^{\mathbb{Z}^d}, TM_k)$ with values in the space tangent to M_k, $k \in \mathbb{Z}^d$. Here div_k *denotes the divergence operator on vector fields on M_k and*

$$V_k = \sum_{A : k \in A} \Phi_A \qquad (2.8)$$

denotes the potential of the set $\{k\} \in \mathbb{Z}^d$ as in (2.2).

Proof. (1) First note that all integrals in (2.7) are finite due to the cylindricity of u, $\{h_k\}_{k \in \mathbb{Z}^d}$, and $\mu(M^{\mathbb{Z}^d}) = 1$.

As the measure

$$d\mu_i(x_i \mid y) = \frac{1}{Z_i(y)} \exp(-V_i(x_i \mid y))\, d\sigma_i(x_i)$$

is a perturbation of the Riemannian volume σ_i by smooth density, we can write next the integration-by-parts formula

$$\mathbf{E}_i^y[(\operatorname{div}_i h_i) u] = \mathbf{E}_i^y[\prec h_i(\cdot), \nabla_i V_i(\cdot \mid y) \succ u(\cdot \mid y) - \prec h_i(\cdot), \nabla_i u(\cdot \mid y) \succ].$$

Using the definition of the Gibbs measure we have

$$\mu((\operatorname{div}_i h_i) u) = \mu(\mathbf{E}_i^{\cdot}[(\operatorname{div}_i h_i) u]) = \mu(\prec h_i, \nabla_i V_i \succ u - \prec h_i, \nabla_i u \succ).$$

Summing up on $i \in \mathbb{Z}^d$ we obtain the "if" part proved.

(2) Consider the function f, which is measurable with respect to σ-algebra $\mathscr{F}_{\mathbb{Z}^d \setminus \{i\}}$. Put $u = f \cdot u_i$, where $u_i \in C^{\infty}_{\mathrm{cyl}}(M^{\mathbb{Z}^d})$ such that $\operatorname{supp}_{\mathrm{cyl}} u_i \supset \{i\}$ (see (2.6)). Choosing in (2.7) $h_k = 0$, $k \neq i$, we have

$$\int_{M^{\mathbb{Z}^d}} f \prec h_i, \nabla_i u_i \succ d\mu = \int_{M^{\mathbb{Z}^d}} f\{\prec h_i, \nabla_i V_i \succ - \operatorname{div}_i h_i\}\, u_i\, d\mu.$$

As f is an arbitrary smooth cylindrical function on $M^{\mathbb{Z}^d}$ we obtain that the conditional measure $\mu(\cdot \mid \mathscr{F}_{\{i\}^c})$ satisfies the identity

$$\begin{aligned}\mu((\operatorname{div}_i h_i)\, u_i \mid \mathscr{F}_{\{i\}^c}) &= \mu(\prec h_i, \nabla_i V_i \succ u_i - \prec h_i, \nabla_i u_i \succ \mid \mathscr{F}_{\{i\}^c}) \\ &= -\mu(e^{V_i} \prec h_i, \nabla_i \succ (e^{-V_i} u_i) \mid \mathscr{F}_{\{i\}^c})\end{aligned}$$

Let $\hat{\mu}_i^y$ denote the regular image of the conditional measure $\mu(\cdot \mid \mathscr{F}_{\{i\}^c})(y)$ under the projection pr_i on M_i. The procedure for construction of conditional expectations for measure μ implies that for μ-almost all $y \in M^{\mathbb{Z}^d}$ we have

$$\mu(\cdot \mid \mathscr{F}_{\{i\}^c})(y) = \hat{\mu}_i^y \otimes \delta_{y^c}(\cdot),$$

where $y^c = \{y_k\}_{k \neq i}$. Then we obtain that

$$\hat{\mu}_i^y((\operatorname{div}_i h_i)\, u_i) = -\hat{\mu}_i^y(e^{V_i} \prec h_i, \nabla_i \succ (e^{-V_i} u_i)).$$

Setting $u_i = e^{V_i} q_i$, $d\tilde{\mu}_i(\cdot \mid y) = e^{V_i}\, d\hat{\mu}_i^y$, we have

$$\int_{M_i} (\operatorname{div}_i h_i)\, q_i\, d\tilde{\mu}_i(\cdot \mid y) = -\int_{M_i} \prec h_i, \nabla_i q_i \succ\, d\tilde{\mu}_i(\cdot \mid y). \tag{2.9}$$

This identity completely characterizes the Riemannian volume on M_i [10]; therefore

$$d\tilde{\mu}_i(x_i \mid y) = C(y)\, d\sigma_i(x_i)$$

and

$$d\hat{\mu}_i^y(\cdot) = e^{-V_i} C(y)\, d\sigma_i(\cdot).$$

As $\hat{\mu}_i^y$ is a probability measure we have

$$C^{-1}(y) = \int_{M_i} e^{-V_i(\cdot \mid y)}\, d\sigma_i(\cdot);$$

therefore $\hat{\mu}_i^y(\cdot) = \mu_{\{i\}}(\cdot \mid y)$ and the measure μ is a Gibbs one (see Definition 2.1 and (2.3)). ∎

3. Hermitian Realizations for Formal Differential Operators

Definition 3.1. Let $Q \subset \mathbb{Z}^d$, $|Q| \leqslant \infty$, be an arbitrary subset of the lattice $\mathbb{Z}^d$. Consider the family of formal differential expressions acting on $C_{\mathrm{cyl}}^{\infty}(M^{\mathbb{Z}^d})$,

$$H_Q = \sum_{k \in Q} H_k, \tag{3.1}$$

where

$$H_k = -\tfrac{1}{2}\Delta_k + \tfrac{1}{2}\langle \nabla_k V_k, \nabla_k \rangle \tag{3.2}$$

and $V_k = \sum_{A: k \in A} \Phi_A$.
Note that Q can coincide with lattice $\mathbb{Z}^d$.

Under Conditions 1, 2 in (2.1) the operators $\{H_Q, |Q| \leqslant \infty\}$ are correctly defined on the space $C^\infty_{\mathrm{cyl}}(M^{\mathbb{Z}^d})$:

$$\forall f \in C^\infty_{\mathrm{cyl}}(M^{\mathbb{Z}^d}),\ \forall Q \subset \mathbb{Z}^d,\ |Q| \leqslant \infty; \qquad H_Q f \in C^\infty_{\mathrm{cyl}}(M^{\mathbb{Z}^d}).$$

The next theorem is a simple consequence of the integration by parts characterization of Gibbs measures (Theorem 2.2).

THEOREM 3.2. *Suppose that the interactive potentials $\{\Phi_A\}$ satisfy Conditions* 1, 2 *in* (2.1). *Then the family of differential operators $\{H_Q, |Q| \leqslant \infty\}$ in* (3.2) *with domain $C^\infty_{\mathrm{cyl}}(M^{\mathbb{Z}^d})$ is Hermitian in $L_2(M^{\mathbb{Z}^d}, \mu)$ if and only if the measure μ is a Gibbs one with local specifications $\{\mu_A\}$ (i.e., $\mu \in \mathcal{G}\{\mu_A\}$).*
Moreover, the operators $\{H_Q, |Q| \leqslant \infty\}$ are Dirichlet in the sense that

$$(H_Q u, v)_{L_2(\mu)} = \tfrac{1}{2} \int_{M^{\mathbb{Z}^d}} \sum_{k \in Q} \langle \nabla_k u, \nabla_k v \rangle \, d\mu$$

on $u, v \in C^\infty_{\mathrm{cyl}}(M^{\mathbb{Z}^d})$.

Proof. 1. Theorem 2.2 implies that for any $\mu \in \mathcal{G}\{\mu_A\}$ we have

$$\int_{M^{\mathbb{Z}^d}} \langle \nabla_k u, \nabla_k v \rangle \, d\mu = \int_{M^{\mathbb{Z}^d}} v\{-\Delta_k u + \langle \nabla_k V_k, \nabla_k u \rangle\} \, d\mu;$$

therefore each operator $\{H_Q, |Q| \leqslant \infty\}$ is a Dirichlet and Hermitian one in the space $L_2(M^{\mathbb{Z}^d}, \mu)$.

2. The statement of Theorem 2.2 is also valid for the special choice of vector fields $h_k = \nabla_k \psi$ for $\psi \in C^\infty_{\mathrm{cyl}}(M^{\mathbb{Z}^d})$. Equation (2.9) adopts the form

$$\int_{M_k} (\Delta_k \psi)\, v_k \, d\tilde{\mu}_k = -\int_{M_k} \langle \nabla_k \psi, \nabla_k v_k \rangle \, d\tilde{\mu}_k,$$

which is another possible characterization of Riemannian volume. ∎

Remark 3.3. Theorem 3.2 proposes a functional analytic approach to the interpretation of Gibbs measures in terms of differential operators, in contrast to the stochastic approach used in Definition 2.1.

Consider the formal differential operator acting on $C_{\text{cyl}}^{\infty}(M^{\mathbb{Z}^d})$:

$$H_{\mathbb{Z}^d} = -\tfrac{1}{2} \sum_{k \in \mathbb{Z}^d} \Delta_k + \tfrac{1}{2} \sum_{k \in \mathbb{Z}^d} \prec \nabla_k \Big(\sum_{A : k \in A} \Phi_A \Big), \nabla_k \succ. \tag{3.3}$$

The first question which arises in the infinite-dimensional situation is when the operator $H_{\mathbb{Z}^d}$ could be realized as a Hermitian operator in some space $L_2(M^{\mathbb{Z}^d}, \mu)$. In the finite-dimensional situation this question is trivial because there is no problem of uniqueness for the finite-dimensional Lebesgue measure.

The principal infinite-dimensional effect is that the set of all *Hermitian realizations* for operator (3.3) could be completely described by the set of all Gibbs measures $\mathscr{G}\{\mu_A\}$. In the particular case in which $\mathscr{G}\{\mu_A\}$ is an empty set we have that the differential expression $H_{\mathbb{Z}^d}$ could never be realized as a Hermitian operator.

4. Dirichlet Operators of Gibbs Measures as a Finite Sum of Operators with Independent Variables

Below we represent the Dirichlet operator H_Q (3.1) of Gibbs measure as a finite sum of block operators $H_{Q(i)}$ and prove the self-adjointness of these block operators in $L_2(M^{\mathbb{Z}^d}, \mu)$ $\mu \in \mathscr{G}\{\mu_A\}$. This representation of a Dirichlet operator is based on the following special splitting of lattice $\mathbb{Z}^d$.

Let us set $B = [1, a]^d \cap \mathbb{Z}^d$ for a $a \in \mathbb{N}$ such that $a > 3r_0$, where r_0 is the interactive radius (see Condition 2 in (2.1)).

Denote by $\mathscr{U}_{(0)} = \mathscr{U}_{(0, \ldots, 0)}$ the set

$$\mathscr{U}_{(0)} = \bigcup_{k \in (2a\mathbb{Z})^d} \tau_k B,$$

where τ_k is a shift on vector $k \in \mathbb{Z}^d$. (See Fig. 1.)

For $i = (i_1, \ldots, i_d)$, $i_s \in \{0, 1\}$, and $s = 1, \ldots, d$, denote by $\mathscr{U}_{(i)} = \mathscr{U}_{(i_1, \ldots, i_d)}$ the shift of the set $\mathscr{U}_{(0)}$ on the vector $(i_1 a, \ldots, i_d a) \in \mathbb{Z}^d$:

$$\mathscr{U}_{(i)} = \tau_{(i_1 a, 0, \ldots, 0)} \cdots \tau_{(0, \ldots, 0, i_d a)} \mathscr{U}_{(0)}.$$

The sets $\{\mathscr{U}_{(i)}, i \in \{0, 1\}^d\}$ form a finite subdivision of the lattice $\mathbb{Z}^d$ on infinite non-interacting subsets.

The splitting of the lattice $\mathbb{Z}^d$ results in the decomposition of the Dirichlet operators $\{H_Q, |Q| \leqslant \infty\}$ (3.1) as finite sum of block operators $H_{Q(i)}$ on $C_{\text{cyl}}^{\infty}(M^{\mathbb{Z}^d})$,

$$H_Q = \sum_{i \in \{0,1\}^d} H_{Q(i)}, \tag{4.1}$$

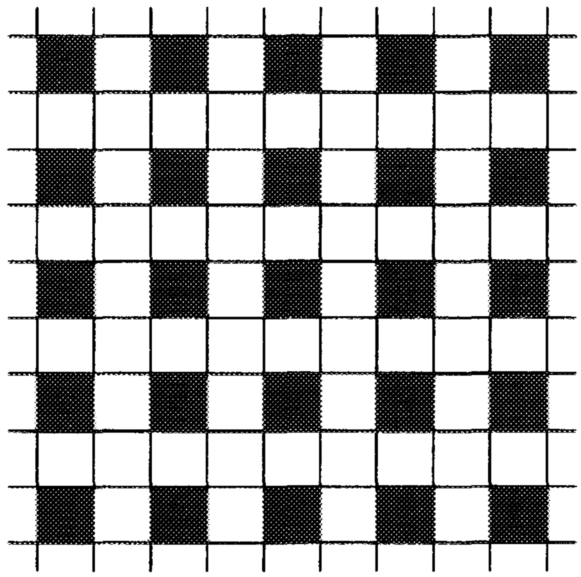

FIG. 1. The set $\mathcal{U}_{(0)}$ for two-dimensional lattice $\mathbb{Z}^d$, $d=2$. This picture describes the sets which split the lattice $\mathbb{Z}^d$. This splitting results in the representation of the Dirichlet operator (4.1) as a finites sum of operators with independent variables.

where

$$H_{Q(i)} = \sum_{k \in \mathcal{U}_{(i)} \cap Q} H_k. \tag{4.2}$$

The formula (4.1) represents the infinite-dimensional operator with *interaction* H_Q as a finite sum of operators $H_{Q(i)}$ with an infinite number of *independent* variables.

THEOREM 4.1. *Let* Φ_A *satisfy Conditions* 1, 2 *in* (2.1). *Then for all* $i \in \{0, 1\}^d$ *and for all* $\{Q \subset \mathbb{Z}^d, |Q| \leqslant \infty\}$ *the Cauchy problem*

$$\begin{cases} \dfrac{\partial f(t, x)}{\partial t} = -H_{Q(i)} f \\ f(0, x) = f_0(x) \end{cases} \tag{4.3}$$

is smoothly solved in the space $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$; i.e., for every $f_0 \in C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ the solution $f(t, \cdot) \in C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$, $t > 0$.

Proof. Put $(i_1, ..., i_d) = (0, ..., 0)$; the other situations are treated similarly.

Due to Condition 2 in (2.1) on the finiteness of the interactive radius, the sub-blocks of operator $H_{Q(0)}$ commute for $k_1 \neq k_2$, $k_1, k_2 \in (2a\mathbb{Z})^d$:

$$[H_{Q \cap \tau_{k_1} B}, H_{Q \cap \tau_{k_2} B}] = 0.$$

This enables us to localize the Cauchy problem (4.3) as follows.

The cylindricity of function $f_0 \in C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ gives the existence of a *finite* set $S \subset (2a\mathbb{Z})^d$ such that for $\tilde{\Lambda} = \bigcup_{k \in S} \tau_k B$ we have

$$\mathscr{U}_{(0)} \supset \tilde{\Lambda} \supset \mathscr{U}_{(0)} \cap Q \cap \operatorname*{supp}_{\text{cyl}} f_0.$$

Therefore the solution of the Cauchy problem (4.3) can be represented on the space $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ as the action of the semigroup for an operator with independent variables:

$$f(t, x) = \prod_{k \in S} \exp\{-tH_{\tau_k B \cap Q}\} f_0(x).$$

Then for all $t > 0$ the support of cylindricity $\text{supp}_{\text{cyl}} f(t, x)$ belongs to the set $\tilde{\Lambda}$ which is in the r_0-vicinity of $\tilde{\Lambda} \cup \text{supp}_{\text{cyl}} f_0$.

Therefore the Cauchy problem is localized on the finite product of compact Riemannian manifolds and could be smoothly solved in the space $C^\infty(M^{\tilde{\Lambda}})$. So assumptions 1 and 2 in (2.1) imply that

$$\forall t > 0, \qquad f(t, \cdot) \in C^\infty(M^{\tilde{\Lambda}})$$

due to the standard finite-dimensional criterions (see, for example, [15, Chap. 9, §6, Thm. 8]). ∎

COROLLARY 4.2. *Under Conditions* 1 *and* 2 *on interactive potentials* $\{\Phi_A\}$ *in* (2.1) *we have that*

> $\forall i \in \{0, 1\}^d$, $\forall Q \subset \mathbb{Z}^d$, $|Q| \leqslant \infty$, *and* $\forall \mu \in \mathscr{G}\{\mu_\Lambda\}$ *the operator* $H_{Q(i)}$ *in* (4.2) *is essentially self-adjoint in space* $L_2(M^{\mathbb{Z}^d}, \mu)$ *with the essential domain of smooth cylinder functions* $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$.

Proof. Consider the Cauchy problem (4.3). Let $P_t^{Q(i)}$ denote the operator which by the initial value $f_0 \in C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ gives the solution $f(t, x) \in C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$.

Due to the inequality

$$\frac{d}{dt}\int_{M^{\mathbb{Z}^d}} |f(t,x)|^2\, d\mu = -2\int_{M^{\mathbb{Z}^d}} \prec H_{Q(i)} P_t^{Q(i)} f_0, P_t^{Q(i)} f_0 \succ d\mu$$

$$= -\int_{M^{\mathbb{Z}^d}} \sum_{k\in Q(i)} |\nabla_k P_t^{Q(i)} f_0|^2\, d\mu \leqslant 0$$

we obtain that

$$\|P_t^{Q(i)} f_0\|_{L_2} \leqslant \|f_0\|_{L_2},$$

so there is a closure $\tilde{P}_t$ from the dense domain $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ in $L_2(M^{\mathbb{Z}^d}, \mu)$, and $\tilde{P}_t$ is a bounded operator.

Operators $\tilde{P}_t$ are multiplicative, $\tilde{P}_{t+s} = \tilde{P}_t\tilde{P}_s$ (this is checked only on $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$), $\tilde{P}_0 = \text{Id}$, and they are strongly continuous:

> $\forall f_0 \in C^\infty_{\text{cyl}}(M_{\mathbb{Z}^d})$ we have $\tilde{P}_t f_0 \to \tilde{P}_s f_0$ in $C^n(M^{\tilde{A}})$ when $t \to s$, $n \geqslant 0$ (see the Theorem 4.1 proof for the definition of $\tilde{A}$). Due to the criterion of strong convergence and the estimate $\|f\|_{L_2} \leqslant \|f\|_{C^n(M^{\tilde{A}})}$ we obtain that $\tilde{P}_t$ is strongly continuous in $L_2(M^{\mathbb{Z}^d}, \mu)$.

Therefore we obtain the existence of generator A: $\tilde{P}_t = \exp(-tA)$ and, because $\tilde{P}_t$ preserves $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$, we have from the criterion [21, Vol. 2, Theorem X.49] that $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ forms the essential domain for operator A.

But

$$H_{Q(i)} f_0 = C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d}) - \lim_{t\to 0+}\left(-\frac{f(t,x) - f_0(x)}{t}\right)$$

$$= L_2 - \lim_{t\to 0+}\left(-\frac{\tilde{P}_t f_0 - f_0}{t}\right) = Af_0$$

due to $\tilde{P}_t \upharpoonright_{C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})} = P_t^{Q(i)}$; therefore we obtain the essential self-adjointness of $H_{Q(i)}$. ∎

Corollary 4.2 and Theorem 4.1 result in the representation of operator H_Q as a finite sum of essentially self-adjoint operators $H_{Q(i)}$ (4.1). But the operators $H_{Q(i)}$ are not comparable; therefore the standard criterions on the essential self-adjointness for the finite sum of self-adjoint operators do not work.

This difficulty is overcome by a simple generalization of the Da Prato and Grisvard theorem [11].

THEOREM 4.3. *Let $A_1, ..., A_n$ be closed linear operators in Banach space X satisfying the estimates*

$$\|(\lambda + A_i)^{-1}\|_{\mathscr{L}(X)} \leqslant 1/(\lambda - \alpha_i) \tag{4.4}$$

for $\lambda > \alpha_i$, $\alpha_i > 0$, $i = 1, ..., n$.

Suppose that there exists a Banach space Y which is continuously and densely imbedded into X. Let Y also be continuously and densely imbedded into $\mathscr{D}(A_i^2)$ with graph norm, $i = 1, ..., n$.

Suppose that the restrictions of operators A_i on the space Y, $A_i \upharpoonright_Y$, satisfy the estimates

$$\|(\lambda + A_i \upharpoonright_Y)^{-1}\|_{\mathscr{L}(Y)} \leqslant 1/(\lambda - \beta_i) \tag{4.5}$$

for $\lambda > \beta_i$, $\beta_i > 0$, $i = 1, ..., n$.

Then operator $L = A_1 + \cdots + A_n$ with domain $\mathscr{D}(L) = Y$ has closure $\tilde{L}$ in X which is a generator of a strongly continuous semigroup.

Moreover, there is the next multiplicative formula for semigroups,

$$\exp(-t\tilde{L}) = X - s - \lim_{m \to \infty} \left(\prod_{i=1}^{n} \exp\left(-\frac{t}{m} A_i\right)\right)^m \tag{4.6}$$

uniformly on $t \in [0, T]$, $T > 0$.

Proof. See Theorem A.5 in the Appendix (when operator $B \equiv 0$). ∎

5. SPACES OF QUASI-ACCRETIVITY FOR THE DIRICHELT OPERATORS

In this section we construct the scales $\{\mathscr{E}_\Theta\}$, $\{\mathscr{Z}(\mathbf{p}, \alpha)\}$ of spaces of differentiable functions on the infinite product of manifolds $M^{\mathbb{Z}^d}$ in which the operators $\{H_Q, |Q| \leqslant \infty\}$ are uniformly on $Q \subset \mathbb{Z}^d$ quasi-accretive in the sense of the next estimate:

$$\exists C_E : \Re e \prec (H_Q + C_E)u, l_u \succ_E \geqslant 0.$$

Here $u \in C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ and l_u is a tangent functional to u in the space E from these scales.

The spaces $\{\mathscr{E}_\Theta\}$ and $\{\mathrm{Z}(\mathbf{p}, \alpha)\}$ are constructed as a closure of $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ with respect to the certain sequence of seminorms (see Definitions 5.5 and 5.7) and would play the role of spaces X and Y from Theorem 4.3. Further, we obtain different properties for the semigroups of Dirichlet operators $\{\exp(-tH_Q), Q \subset \mathbb{Z}^d, |Q| \leqslant \infty\}$ (see Theorems 5.6, 5.8, 7.2, 7.4, 7.5, 8.4, 8.5, 8.7).

DEFINITION 5.1. The set of nonnegative numbers $\mathbf{p} = \{\mathbf{p}_k\}_{k \in \mathbb{Z}^d}$ is called the *weight* iff for some $r_0 > 0$ there is a constant $M_{\mathbf{p}} \geq 0$ such that $\forall k, j \in \mathbb{Z}^d : |k-j| \leq r_0$ and $\mathbf{p}_k \cdot \mathbf{p}_j \neq 0$:

$$e^{-M_{\mathbf{p}}} \leq |\mathbf{p}_k / \mathbf{p}_j| \leq e^{M_{\mathbf{p}}} \tag{5.1}$$

The *space of multiweights* $\mathbb{P}$ is introduced as a space of *finite unordered* sets of weights

$$\theta = \{\overset{1}{\mathbf{p}}, \ldots, \overset{l}{\mathbf{p}}\} \in \mathbb{P},$$

where $l \geq 1$ is the *length* of multiweight $\theta : l = l(\theta)$. Correspondingly, let $l(\Theta) = \max_{\theta \in \Theta} l(\theta)$ denote the length of the *finite* array of multiweights $\Theta = \{\theta_1, \ldots, \theta_m\} \subset \mathbb{P}$.

We denote by $\operatorname{nul}(\theta)$ the set of points from $\mathbb{Z}^d$ where some weight of multiweight $\theta = \{\overset{1}{\mathbf{p}}, \ldots, \overset{l}{\mathbf{p}}\}$ vanishes:

$$\operatorname{nul}(\theta) = \bigcup_{i=1}^{l(\theta)} \{k \in \mathbb{Z}^d : \overset{i}{\mathbf{p}}_k = 0\}. \tag{5.2}$$

Correspondingly, let $\operatorname{nul}(\Theta)$ denote next set

$$\operatorname{nul}(\Theta) = \bigcup_{\theta \in \Theta} \operatorname{nul}(\theta).$$

Note that the weight $\mathbf{p}_k = e^{c|k|}$ for some $c > 0$ satisfies condition (5.1).

DEFINITION 5.2. We introduce the relation of partial order between two multiweights $\theta = (\overset{1}{\mathbf{p}}, \ldots, \overset{l}{\mathbf{p}}) \ll \pi$ in the following way:

> $\theta \ll \pi$ iff $l(\theta) = l(\pi)$, there is a permutation $\varphi(\pi) = \{\overset{1}{\mathbf{q}}, \ldots, \overset{l}{\mathbf{q}}\}$, and $\exists C_{\theta,\pi}$ such that $\forall k \in \mathbb{Z}^d$, $\forall i = 1, \ldots, l(\theta)$, $\overset{i}{\mathbf{p}}_k \leq C_{\theta,\pi} \overset{i}{\mathbf{q}}_k$.

The finite arrays of multiweights Θ, $\Psi \subset \mathbb{P}$ satisfy $\Theta \ll \Psi$ if

$$\forall \theta \in \Theta \exists \pi = \pi(\theta) \in \Psi \text{ such that } \theta \ll \pi.$$

The corresponding constant $C_{\Theta,\Psi} = \max_{\theta \in \Theta} C_{\theta,\pi(\theta)}$.

DEFINITION 5.3. For $\theta \in \mathbb{P}$, $x \in M^{\mathbb{Z}^d}$, and $u \in C^{\infty}_{\text{cyl}}(M^{\mathbb{Z}^d})$, denote by $|\cdot|_\theta$ the following expression:

$$|u(x)|_\theta = \Bigg(\sum_{k_1, \ldots, k_l \in \mathbb{Z}^d} \overset{l}{\mathbf{p}}_{k_l} \cdots \overset{1}{\mathbf{p}}_{k_1} |\nabla_{k_l} \cdots \nabla_{k_1} u(x)|^2 \Bigg)^{1/2}. \tag{5.4}$$

Here ∇_k denotes the covariant derivative on manifold M_k and $|\cdot|$ denotes the Riemannian norm of the corresponding tensor field.

Let $(\cdot,\cdot)_\theta$ denote the bilinear form induced by $|\cdot|_\theta$:

$$(u(x), v(x))_\theta = \sum_{k_1, ..., k_l \in \mathbb{Z}^d} \overset{l}{\mathbf{p}}_{k_l} \cdots \overset{1}{\mathbf{p}}_{k_1} \cdot \prec \nabla_{k_l} \cdots \nabla_{k_1} u(x), \nabla_{k_l} \cdots \nabla_{k_1} v(x) \succ. \tag{5.5}$$

For $\theta = \varnothing$ we have that $|u(x)|_\varnothing = |u(x)|$ and correspondingly $(u(x), v(x))_\varnothing = u(x)\, v(x)$.

Note that inequality $\theta \ll \pi$ implies

$$|u(x)|_\theta \leqslant C_{\theta,\pi} |u(x)|_\pi \tag{5.6}$$

for all $x \in M^{\mathbb{Z}^d}$, $u \in C^\infty_{\mathrm{cyl}}(M^{\mathbb{Z}^d})$.

DEFINITION 5.4. Consider $\theta \in \mathbb{P}$, $\theta = \{\overset{1}{\mathbf{p}}, ..., \overset{l}{\mathbf{p}}\}$. Denote by $\mathrm{gen}(\theta) \subset \mathbb{P}$ the set of multiweights

$$\mathrm{gen}(\theta) = \bigcup_{\substack{s,t=1 \\ s \neq t}}^{l} \{\overset{s}{\mathbf{p}}\overset{t}{\mathbf{p}}, \overset{1}{\mathbf{p}}, ..., \overset{s-1}{\mathbf{p}}, \overset{s+1}{\mathbf{p}}, ..., \overset{t-1}{\mathbf{p}}, \overset{t+1}{\mathbf{p}}, ..., \overset{l}{\mathbf{p}}\}, \tag{5.7}$$

where weight $\overset{s}{\mathbf{p}}\overset{t}{\mathbf{p}} = \{\overset{s}{\mathbf{p}}_k \overset{t}{\mathbf{p}}_k\}_{k \in \mathbb{Z}^d}$.

In the same way let $\mathrm{gen}(\Theta)$ denote the set

$$\mathrm{gen}(\Theta) = \bigcup_{\theta \in \Theta} \mathrm{gen}(\theta).$$

The finite array of multiweights $\Theta = \{\theta_1, ..., \theta_m\} \subset \mathbb{P}$, $|\Theta| = m$, is called quasiaccretive iff

$$\mathrm{gen}(\Theta) \ll \Theta, \tag{5.8}$$

i.e., $\forall \theta \in \Theta : l(\theta) > 1$ and $\forall \pi \in \mathrm{gen}(\theta)\ \exists \theta'(\pi) \in \Theta$, $l(\pi) = l(\theta'(\pi))$ and there is a permutation $\varphi(\theta'(\pi))$ of θ' such that $\pi \ll \varphi(\theta'(\pi))$.

Later we need the constants

$$M_\theta = \max_{\mathbf{p} \in \theta} M_{\mathbf{p}} \tag{5.9}$$

$$C_\Theta = C_{\mathrm{gen}(\Theta), \Theta}, \tag{5.10}$$

where $M_{\mathbf{p}}$ and $C_{\Theta, \Psi}$ were introduced in (5.1) and (5.6).

DEFINITION 5.5. Let Θ be the array of multiweights $\Theta = \{\theta_1, ..., \theta_m\} \subset \mathbb{P}$. We define the Banach space $\mathscr{E}_\Theta$ of smooth functions on the infinite product of manifolds $M^{\mathbb{Z}^d}$ as a closure of $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ in the norm

$$\|u\|_\Theta = \sup_{x \in M^{\mathbb{Z}^d}} \left(\max_{\theta \in \Theta \cup \{\varnothing\}} |u(x)|_\theta \right), \tag{5.11}$$

where $|u(x)|_\theta$ was introduced in (5.4).

Note that for $u \in C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ $\|u\|_\Theta < \infty$ for $\Theta \subset \mathbb{P}$, $|\Theta| < \infty$ due to the compactness of M_k.

Remarks. (1) The important moment in the definition of $\mathscr{E}_\Theta$ is the absence of factorization for $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ in norm $\|\cdot\|_\Theta$. That is because $\|\cdot\|_\Theta$ restricted on $C^\infty(M^A)$, $|A| < \infty$, is equivalent to the standard Riemannian norm on space $C^n(M^A)$, $n = \max_{\theta \in \Theta} l(\theta)$.

(2) For $\Theta = \varnothing$ the corresponding space $\mathscr{E}_\Theta$ is a closure of $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ in the norm $\|u\|_\varnothing = \sup_{x \in M^{\mathbb{Z}^d}} |u(x)|$.

(3) The set of quasi-accretive arrays is non-empty: fix $\theta = \{\overset{1}{\mathbf{p}}, ..., \overset{l}{\mathbf{p}}\} \in \mathbb{P}$. Then the array

$$\text{Gen}(\theta) = \bigcup_{i=0}^{l(\theta)} \text{gen}^i(\theta) \tag{5.12}$$

is a quasi-accretive one because it contains the set $\text{gen}(\phi)$ for any $\phi \in \text{Gen}(\theta)$. Here $\text{gen}^k(\theta) = \text{gen}^{k-1}(\text{gen}(\theta))$. Correspondingly the set $\text{Gen}(\Theta) = \bigcup_{\theta \in \Theta} \text{Gen}(\theta)$ is defined.

THEOREM 5.6. *Let the interactive potentials $\{\Phi_A\}$ satisfy Conditions 1–3 in (2.1) and let Θ be a quasi-accretive array of multiweights, $|\Theta| < \infty$. Then the Banach space $\mathscr{E}_\Theta$ is a space of uniform on $Q \subset \mathbb{Z}^d \setminus \text{nul}(\Theta)$ quasi-accretivity for the operators $\{H_Q, Q \subset \mathbb{Z}^d \setminus \text{nul}(\Theta), |Q| \leqslant \infty\}$ (3.1):*

$$\exists \mathscr{D}_\Theta \geqslant 0 \; \forall Q \subset \mathbb{Z}^d \setminus \text{nul}(\Theta), |Q| \leqslant \infty \; \forall u \in C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$$

$$\Re e \prec (H_Q + \mathscr{D}_\Theta) u, l_u \succ_{\mathscr{E}_\Theta} \geqslant 0. \tag{5.13}$$

Here l_u denotes the tangent functional to u in $\mathscr{E}_\Theta$.

Remark. For example, for the quasi-accretive array Θ with $\text{nul}(\Theta) = \varnothing$ we have the uniform on lattice $\mathbb{Z}^d$ quasi-accretivity for operators $\{H_Q, |Q| \leqslant \infty\}$. But we give the proof of this result also in the case $Q \subset \mathbb{Z}^d \setminus \text{nul}(\Theta)$ because it is used in the proof of smoothing properties in Theorems 8.5 and 8.6.

Proof. Recall that for $u \in C^{\infty}_{\text{cyl}}(M^{\mathbb{Z}^d})$;

$$\|u\|_{\Theta} = \sup_{x \in M^{\mathbb{Z}^d}} \max_{\theta \in \Theta \cup \{\varnothing\}} |u(x)|_{\theta},$$

where $|\cdot|_{\theta}$ were introduced in (5.4).

Then due to the compactness of $M^{\mathbb{Z}^d}$ the norm is attained for some multiweight $\theta = \{\overset{1}{\mathbf{p}}, ..., \overset{j}{\mathbf{p}}\} \in \Theta \cup \varnothing$ and point $x_0 \in M^{\mathbb{Z}^d}$:

$$\|u\|_{\Theta} = |u(x_0)|_{\theta}.$$

It is simple that the tangent in space $\mathscr{E}_{\Theta}$ functional to function $u \in C^{\infty}_{\text{cyl}}(M^{\mathbb{Z}^d})$ equals

$$l_u(f) = (u(x_0), f(x_0))_{\theta},$$

which follows from $l_u(u) = \|u\|^2_{\Theta}$ and the Cauchy inequality (see also (5.5)).

Due to Theorem 6.1 we have that $\forall Q \subset \mathbb{Z}^d \backslash \text{nul}(\Theta)$, $|Q| \leqslant \infty$:

$$\begin{aligned} \mathfrak{Re}\, l_u(H_Q u) &= \mathfrak{Re}(u(x_0), H_Q u(x_0))_{\theta} \\ &\geqslant + \tfrac{1}{2} H_Q |u(x_0)|^2_{\theta} + |u(x_0)|^2_{s_Q(\theta)} - \mathscr{D}_{\theta}\, |u(x_0)|^2_{\text{Gen}(\theta)}, \end{aligned}$$

where $s_Q(\theta) = \{\overset{1}{\mathbf{p}}, ..., \overset{l}{\mathbf{p}}, \mathbf{1}_Q\}$ for weight $\theta = \{\overset{1}{\mathbf{p}}, ..., \overset{l}{\mathbf{p}}\}$ and weight $\mathbf{1}_Q$ is equal to

$$\{\mathbf{1}_Q\}_k = \begin{cases} 1, & k \in Q \\ 0, & k \notin Q. \end{cases}$$

The first and second terms above are non-negative because x_0 is a maximum point. The third term is simply estimated from

$$|u(x_0)|_{\text{Gen}(\theta)} \leqslant C^{l(\Theta)}_{\Theta}\, \|u\|_{\Theta}.$$

So we have that

$$\mathfrak{Re}\, l_u(H_Q u) \geqslant -C^{l(\Theta)}_{\Theta} \max_{\theta \in \Theta} \mathscr{D}_{\theta}\, \|u\|^2_{\Theta}.$$

See the definition of C_{Θ} in (5.10). ∎

Now we give another important example of spaces in which operators $\{H_Q, |Q| \leqslant \infty\}$ are quasi-accretive. We need these spaces later for checking the conditions of Theorem 4.3.

DEFINITION 5.7. Let $\mathbf{p} = \{\mathbf{p}_k\}_{k \in \mathbb{Z}^d}$, $\mathbf{p}_k \geqslant 1$, be some weight (Definition 5.1). Fix vector $\{\alpha_i \in (0, 1),\ i = 1, ..., n,\ \sum_{i=1}^n \alpha_i = 1\}$ for some $n \in \mathbb{N}$.

The space $\mathscr{X}(\mathbf{p}, \alpha)$ is defined as a closure of $C^{\infty}_{\text{cyl}}(M^{\mathbb{Z}^d})$ in the norm

$$\|u\|_{\mathbf{p},\alpha} = \sup_{x \in M^{\mathbb{Z}^d}} |u(x)| \vee \sup_{x \in M^{\mathbb{Z}^d}} \max q_{\psi}(u), \tag{5.14}$$

where max is taken over all possible $k = 1, ..., n$, all possible decompositions of number k,

$$S = \left\{ s_i \in \{1, 2\} \text{ such that } \sum_{i=1}^{j} s_i = k \right\},$$

and subdivisions $\Gamma = \{\gamma_i\}_{i=1}^{j}$ of set $\{1, ..., n\}$ satisfying

$$\gamma_i \cap \gamma_l = \varnothing,\ i \neq l, \qquad \bigcup_{i=1}^{j} \gamma_i = \{1, ..., n\}, \qquad |\gamma_i| \geqslant s_i.$$

Here $q_{\psi}(u)$, $\psi = \psi(k, S, \Gamma)$ admits the representation

$$q_{\psi}(u) = \left(\sum_{k_1, ..., k_j \in \mathbb{Z}^d} \mathbf{p}_{k_1}^{\beta_1} \cdots \mathbf{p}_{k_1}^{\beta_j} |T_{s_1}^{k_1} \cdots T_{s_j}^{k_j} u|^2 (x) \right)^{1/2}, \tag{5.15}$$

where $\beta_i = \sum_{m \in \gamma_i} \alpha_m$.

The operators T_s^k are defined by the formulas

$$T_1^k u = \nabla_k u, \qquad s = 1,$$

$$T_2^k u = \Delta_k u, \qquad s = 2,$$

where ∇_k and Δ_k denote the covariant derivative and the Laplace–Beltrami operator on tensors on M_k. Note that the order of differentiation in $\mathscr{X}(\mathbf{p}, \alpha)$ equals $n = n(\alpha)$.

The next theorem is an analogue of Theorem 5.6 for the spaces $\{\mathscr{X}(\mathbf{p}, \alpha)\}$.

THEOREM 5.8. *Let $\{\Phi_A\}$ satisfy Conditions* 1–3 *in* (2.1). *Then space $\mathscr{X}(\mathbf{p}, \alpha)$ is a space of uniform quasi-accretivity on the lattice for operators $\{H_Q, |Q| \leqslant \infty\}$; i.e.,*

$$\exists \mathscr{D}_{\mathbf{p},\alpha} \geqslant 0 \qquad \forall Q \subset \mathbb{Z}^d,\ |Q| \leqslant \infty \forall u \in C^{\infty}_{\text{cyl}}(M^{\mathbb{Z}^d}),$$

$$\Re e \prec (H_Q + \mathscr{D}_{\mathbf{p},\alpha}) u, l_u \succ_{\mathscr{X}(\mathbf{p},\alpha)} \geqslant 0. \tag{5.16}$$

Here l_u denotes the tangent functional to u in $\mathscr{X}(\mathbf{p}, \alpha)$.

Proof. This is omitted because it completely follows the verification scheme for Theorem 5.6 (see also Theorem 6.1). The proof is contained in [6] and in [1]. ▮

The next theorem controls the condition $Y \subset \mathscr{D}(A_i^2)$ of Theorem 4.3.

THEOREM 5.9. *Let interactive potentials* $\{\Phi_A\}$ *satisfy Conditions* 1–3 *in* (2.1).

Consider the Banach space X *for which we permit one of the following possibilities*:

1. $X = \mathscr{E}_\Theta$ *for quasi-accretive array* $\Theta \subset \mathbb{P}$;
2. $X = \mathscr{L}(\mathbf{p}, \alpha)$ *for some* $\mathbf{p}$, α;
3. $X = L_2(M^{\mathbb{Z}^d}, \mu)$ *for some* $\mu \in \mathscr{G}\{\mu_A\}$.

Then there is space $Y = \mathscr{L}(\mathbf{q}, \beta)$, $Y \subset X$, *such that uniformly on* $Q \subset \mathbb{Z}^d$, $|Q| \leqslant \infty$, *we have the estimate*

$$\max(\|u\|_X, \|H_Q u\|_X) \leqslant C_{X,Y} \|u\|_Y$$

for $u \in C^{\infty}_{\mathrm{cyl}}(M^{\mathbb{Z}^d})$.

Proof. (1) First consider the case $X = \mathscr{E}_\Theta$. Due to the finiteness of array Θ there is an exponentially growing weight $\tilde{\mathbf{p}} = \{\tilde{\mathbf{p}}_k\}_{k \in \mathbb{Z}^d}$ (i.e., $\tilde{\mathbf{p}}_k \sim e^{M|k|}$, $|k| \to \infty$, $M > 0$) that majorizes all weights of the array Θ:

$$\begin{aligned} &\forall \theta = \{\overset{1}{\mathbf{r}}, ..., \overset{j}{\mathbf{r}}\} \in \Theta, \forall i = 1, ..., j(\theta), \\ &\overset{i}{\mathbf{r}}_k \leqslant \tilde{\mathbf{p}}_k, \qquad k \in \mathbb{Z}^d. \end{aligned} \tag{5.17}$$

Put $\alpha = \{\alpha_i = 1/(n+2)\}_{i=1}^{n+2}$ and $\mathbf{p} = (\tilde{\mathbf{p}})^{n+2}$. Then (5.17) and Definitions 5.2, 5.4, and 5.7 imply that

$$\|u\|_\Theta \leqslant \text{const } \|u\|_{\mathbf{p},\alpha}$$

and we only have to prove the statement for $X = \mathscr{L}(\mathbf{p}, \alpha)$.

(2) Consider $X = \mathscr{L}(\mathbf{p}, \alpha)$. Due to (5.14) we have

$$\|H_Q u\|_{\mathbf{p},\alpha} = \sup_{x \in M^{\mathbb{Z}^d}} \{|H_Q u| \vee \max q_\psi(H_Q u)\}.$$

Then $|H_Q u|$ is simply estimated from above by

$$\left(\sum_{k \in \mathbb{Z}^d} 1/\mathbf{p}_k^2\right)^{1/2} \left[\left(\sum_{k \in \mathbb{Z}^d} \mathbf{p}_k^2 |\Delta_k u|^2\right)^{1/2} + \max_{k \in \mathbb{Z}^d} |\nabla_k V_k| \cdot \left(\sum_{k \in \mathbb{Z}^d} \mathbf{p}_k^2 |\nabla_k u|^2\right)^{1/2}\right]. \tag{5.18}$$

In the same way

$$q_\psi(H_Q u) = \left(\sum_{k_1, \ldots, k_j \in \mathbb{Z}^d} \mathbf{p}_{k_1}^{\delta_1} \cdots \mathbf{p}_{k_j}^{\delta_j} \, |T_{s_1}^{k_1} \cdots T_{s_j}^{k_j} (H_Q u)|^2 \right)^{1/2}$$

$$\leqslant \left(\sum_{k \in \mathbb{Z}^d} 1/\mathbf{p}_k \right)^{1/2}$$

$$\cdot \left[\left(\sum_{k_1, \ldots, k_j, k_{j+1} \in \mathbb{Z}^d} \mathbf{p}_{k_1}^{\delta_1} \cdots \mathbf{p}_{k_j}^{\delta_j} \mathbf{p}_{k_{j+1}} \, |T_{s_1}^{k_1} \cdots T_{s_j}^{k_j} \Delta_{k_{j+1}} u|^2 \right)^{1/2} \right.$$

$$+ \left(\sum_{k_1 \cdots k_j, k_{j+1} \in \mathbb{Z}^d} \mathbf{p}_{k_1}^{\delta_1} \cdots \mathbf{p}_{k_j}^{\delta_j} \mathbf{p}_{k_{j+1}} \, |T_{s_1}^{k_1} \cdots T_{s_j}^{k_j} \right.$$

$$\left. \left. \times \prec \nabla_k V_k, \nabla_k u \succ |^2 \right)^{1/2} \right]. \tag{5.19}$$

We introduce the new weight $\mathbf{q} = (\mathbf{p})^{n+2}$ and the power weight β with $\beta_i = 1/(n+2)$, $i = 1, \ldots, n+2$, and construct the space $\mathscr{X}(\mathbf{q}, \beta)$ (see Definition 5.7). It is obvious that $\|u\|_{\mathbf{p}, \alpha}$ and the first terms in (5.18) and (5.19) are estimated by $\|u\|_{\mathbf{q}, \beta}$.

The second terms with $|T_{s_1}^{k_1} \cdots T_{s_j}^{k_j} \prec \nabla_k V_k, \nabla_k u \succ|^2$ can be estimated from above by const $\|u\|_{\mathbf{q}, \beta}$. We only have to repeat the steps from (6.20) to the end of the Theorem 6.1 proof using the relations

$$T_1(fg) = f T_1 g + g T_1 f$$

$$T_2(fg) = f T_2 g + g T_2 f + 2 \prec T_1 f, T_1 g \succ$$

(see Definition 5.7 for $T_\bullet$).

(3) For $X = L_2(M^{\mathbb{Z}^d}, \mu)$ the estimate is trivial, because

$$\left(\int_{M^{\mathbb{Z}^d}} |H_Q u|^2 \, d\mu \right)^{1/2} \leqslant \sup_{x \in M^{\mathbb{Z}^d}} |H_Q u(x)|$$

and the last term has already been estimated. ∎

6. The Estimation of Commutators

The content of this section is rather technical, so it could be omitted in the first reading.

THEOREM 6.1. *Let interactive potentials $\{\Phi_A\}$ satisfy Conditions* 1–3 *in* (2.1). *Then* $\forall\theta\in\mathbb{P}$, $\exists\mathcal{D}_\theta>0$ *such that* $\forall Q\subset\mathbb{Z}^d\backslash \mathrm{nul}(\theta)$, $|Q|\leqslant\infty$, *and for all* $u\in C^\infty_{\mathrm{cyl}}(M^{\mathbb{Z}^d})$ *the estimate*

$$|\Re e(u, H_Q u)_\theta - \tfrac{1}{2}H_Q\,|u|^2_\theta - |u|^2_{s_Q(\theta)}| \leqslant \mathcal{D}_\theta\,|u|^2_{\mathrm{Gen}(\theta)} \tag{6.1}$$

is satisfied pointwise on $x\in M^{\mathbb{Z}^d}$. Here the weight $s_Q(\theta)=\{\overset{1}{\mathbf{p}},...,\overset{l}{\mathbf{p}},\mathbf{1}_Q\}$ for $\theta=\{\overset{l}{\mathbf{p}}\}$ and weight $\mathbf{1}_Q$ is equal to

$$\{\mathbf{1}_Q\}_k=\begin{cases}1, & k\in Q\\ 0, & k\notin Q.\end{cases} \tag{6.2}$$

See the definitions of nul(Θ) and Gen(Θ) in (5.2) and (5.12).

Proof. Using the form of $(\cdot,\cdot)_\theta$ (Definition 5.3) we transform $\Re e(u(x), H_Q u(x))_\theta$ to

$$\Re e(u(x), H_Q u(x))_\theta = \Re e \sum_{k_1,...,k_j\in\mathbb{Z}^d} \overset{j}{\mathbf{p}}_{k_j}\cdots\overset{1}{\mathbf{p}}_{k_1} \cdot \prec \nabla_{k_j}\cdots\nabla_{k_1}u(x), \nabla_{k_j}\cdots\nabla_{k_1}\Big(\sum_{a\in Q}H_a u\Big)(x)\succ. \tag{6.3}$$

Commuting $\sum H_a$ with ∇_k we have the terms

$$\Re e(u, H_Q u)_\theta = \Re e \sum_{\substack{k_1,...,k_j\in\mathbb{Z}^d\\ a\in Q}} \overset{j}{\mathbf{p}}_{k_j}\cdots\overset{1}{\mathbf{p}}_{k_1} \cdot \prec \nabla_{k_j}\cdots\nabla_{k_1}u(x), \Big\{H_a\nabla_{k_j}\cdots\nabla_{k_1}u(x) + \sum_{m=1}^{j}\nabla_{k_j}\cdots\nabla_{k_{m+1}}[\nabla_{k_m}, H_a]\,\nabla_{k_{m-1}}\cdots\nabla_{k_1}u(x)\Big\}\succ. \tag{6.4}$$

The first term in (6.4) can be represented as a sum of the terms

$$\begin{aligned}&\tfrac{1}{2}\Big(\sum_{a\in Q}H_a\Big)\sum_{k_1,...,k_j\in\mathbb{Z}^d}\overset{j}{\mathbf{p}}_{k_j}\cdots\overset{1}{\mathbf{p}}_{k_1}\,|\nabla_{k_j}\cdots\nabla_{k_1}u(x)|^2\\ &\quad+\sum_{\substack{k_1,...,k_j\in\mathbb{Z}^d\\ a\in Q}}\overset{j}{\mathbf{p}}_{k_j}\cdots\overset{1}{\mathbf{p}}_{k_1}\,|\nabla_a\nabla_{k_j}\cdots\nabla_{k_1}u(x)|^2\\ &\quad=\tfrac{1}{2}H_Q\,|u(x)|^2_\theta+|u(x)|^2_{s_Q(\theta)},\end{aligned} \tag{6.5}$$

due to $\Re e\prec f, H_a f\succ = \frac{1}{2}H_a\,|f|^2+|\nabla_a f|^2$ pointwise on any tensor f.

For the empty multiweight $\theta=\varnothing$ the calculations are finished with constant $\mathscr{D}_\theta=0$, but when $\theta\neq\varnothing$ we have to estimate the commutator terms in (6.4).

Now we estimate one of the terms with commutator appearing in (6.4). Using Condition 2 in (2.1),

$$[\nabla_k, H_a]=0 \qquad \text{for} \quad |k-a|>r_0,$$

we have

$$\Bigg|\Re e \sum_{\substack{k_1,\ldots,k_j\in\mathbb{Z}^d\\ a\in Q}} \overset{j}{\mathbf{p}}_{k_j}\cdots\overset{1}{\mathbf{p}}_{k_1} \prec \nabla_{k_j}\cdots\nabla_{k_1}u(x),$$
$$\nabla_{k_j}\cdots\nabla_{k_{m+1}}[\nabla_{k_m}, H_a]\,\nabla_{k_{m-1}}\cdots\nabla_{k_1}u(x)\succ\Bigg|$$
$$\leqslant\Bigg(\sum_{\substack{k_1,\ldots,k_j\in\mathbb{Z}^d\\ a\in Q,\,|a-k_m|\leqslant r_0}} \overset{j}{\mathbf{p}}_{k_j}\cdots\overset{1}{\mathbf{p}}_{k_1}\,|\nabla_{k_j}\cdots\nabla_{k_1}u(x)|^2\Bigg)^{1/2}$$
$$\cdot\Bigg(\sum_{\substack{k_1,\ldots,k_j\in\mathbb{Z}^d\\ a\in Q,\,|a-k_m|\leqslant r_0}} \overset{j}{\mathbf{p}}_{k_j}\cdots\overset{1}{\mathbf{p}}_{k_1}$$
$$\cdot\,|\nabla_{k_j}\cdots\nabla_{k_{m+1}}[\nabla_{k_m}, H_a]\,\nabla_{k_{m-1}}\cdots\nabla_{k_1}u(x)|^2\Bigg)^{1/2}. \tag{6.6}$$

The finite radius of interaction (2.1) leads to the estimate of the first factor in (6.6) by $(2r_0)^d\,|u|_{\mathrm{Gen}(\theta)}$, because the terms under summation do not depend on a.

Therefore we only have to estimate the expression

$$\Bigg(\sum_{\substack{k_1,\ldots,k_j\in\mathbb{Z}^d\\ a\in Q,\,|a-k_m|\leqslant r_0}} \overset{j}{\mathbf{p}}_{k_j}\cdots\overset{1}{\mathbf{p}}_{k_1}\cdot|\nabla_{k_j}\cdots\nabla_{k_{m+1}}[\nabla_{k_m}, H_a]\,\nabla_{k_{m-1}}\cdots\nabla_{k_1}u(x)|^2\Bigg)^{1/2}. \tag{6.7}$$

by $K_\theta\,|u|_{\mathrm{Gen}(\theta)}$:

A. *Form of Commutators in* (6.7). The next commutations are common in Riemannian geometry [10]:

$$[(\nabla_a)_i, (\nabla_a)_j](f_a)_{s_1\cdots s_q}=\sum_{\beta=1}^{q}(R_a)^h_{s_\beta\cdot ji}(f_a)_{s_1\cdots h\cdots s_q} \tag{6.8}$$

$$\begin{aligned}[\Delta_a, (\nabla_a)_i](f_a)_{s_1\cdots s_q}&=(\mathrm{Ric}_a)^k_i\,(\nabla_a)_k\,(f_a)_{s_1\cdots h\cdots s_q}\\ &\quad+2\sum_{\beta=1}^{q}(g_a)^{jt}\,(R_a)^h_{s_\beta\cdot ji}\,(\nabla_a)_t\,(f_a)_{s_1\cdots h\cdots s_q}\\ &\quad+\sum_{\beta=1}^{q}[(g_a)^{jt}\,(\nabla_a)_j\,(R_a)^h_{s_\beta\cdot ti}](f_a)_{s_1\cdots h\cdots s_q}.\end{aligned} \tag{6.9}$$

Here g_a, Ric_a, R_a denote the metric, Ricci, and Riemannian curvature tensors on M_a. Commutations (6.8) and (6.9) mean that on M_a, $a \in \mathbb{Z}^d$, the coordinate system $\{x_i\}$ is taken so $(\nabla_a)_i$ is a covariant derivative on M_a in the ith direction. The objects $(\mathrm{Ric}_a)_i^k$ and $(R_a)^h_{s \cdot ij}$ denote the $\{x_i\}$-coordinates of the R_a and Ric_a tensors. The object (f_a) is supposed to be a q-covariant tensor field on M_a with coordinates $(f_a)_{s_1 \cdots h \cdots s_q}$.

Now let f be of the form $f = \nabla_{k_{m-1}} \cdots \nabla_{k_1} u$; then for some fixed $a \in \mathbb{Z}^d$ there could exist the set $\gamma \subset \{1, ..., m-1\}$ such that for all $i \in \gamma$; $k_i = a$. Then field f is a $|\gamma|$-times covariant tensor field on M_a, so we can rewrite commutations (6.8) and (6.9) in the shorthand form

$$
\begin{aligned}
&[\nabla_a, \nabla_a] f = (R_a) f_{\cdots a \cdots} \\
&[\Delta_a, \nabla_a] f = (\mathrm{Ric}_a) \nabla_a f + 2(R_a) \nabla_a f_{\cdots a \cdots} + (\nabla_a R_a) f_{\cdots a \cdots}
\end{aligned}
\tag{6.10}
$$

It is necessary to stress that expressions (6.10) are only shorter forms of (6.8) and (6.9).

The next lemma gives the form of commutators in (6.7).

LEMMA 6.2. *Consider* $f = \nabla_{k_{m-1}} \cdots \nabla_{k_1} u$ *in* (6.7). *Then for* k, $a \in \mathbb{Z}^d$: $|k - a| > r_0$,

$$
[\nabla_k, H_a] f = 0, \tag{6.11}
$$

and for $|k - a| \leqslant r_0$

$$
[\nabla_k, H_a] f = G^{ka} \nabla_a f + \sum_{q=1}^{m-1} \delta_{k, k_q} \delta_{k, a} \{h_1^a \nabla_a f + h_2^a f\}. \tag{6.12}
$$

Here

$$
G^{ka} = \tfrac{1}{2}(\delta_{ka}(\mathrm{Ric}_a) + \nabla_a \nabla_k V_k) \tag{6.13}
$$

and

$$
\begin{aligned}
&h_1^a \nabla_a f = (R_a) \nabla_a f_{\cdots a \cdots} \\
&\quad h_2^a f = \{(\nabla_a R_a) + \tfrac{1}{2} \langle \nabla_a V_a, R_a \rangle\} f_{\cdots a \cdots},
\end{aligned}
\tag{6.14}
$$

where V_a *denotes the potential of the set* $\{a\} \in \mathbb{Z}^d$ (2.8):

$$
V_a = \sum_{A : a \in A} \Phi_A.
$$

Proof. The proof follows simply from the commutations

$$
\begin{aligned}
[\nabla_k, H_a]f &= -\tfrac{1}{2}[\nabla_k, \Delta_a]f + \tfrac{1}{2}[\nabla_k, \langle \nabla_a V_a, \nabla_a \rangle]f \\
&= -\tfrac{1}{2}[\nabla_k, \Delta_a]f + \tfrac{1}{2}\langle \nabla_k \nabla_a V_a, \nabla_a \rangle f \\
&\quad + \tfrac{1}{2}\langle \nabla_a V_a, [\nabla_k, \nabla_a] \rangle f \\
&= \tfrac{1}{2}\langle \nabla_k \nabla_a V_a, \nabla_a \rangle f \\
&\quad + \delta_{ka}\{\tfrac{1}{2}(\mathrm{Ric}_a)\nabla_a f + (R_a)\nabla_a f_{\cdots a\cdots} + \tfrac{1}{2}(\nabla_a R_a) f_{\cdots a \cdots}\} \\
&\quad + \tfrac{1}{2}\langle \nabla_a V_a, (R_a)\rangle f_{\cdots a \cdots} \\
&= \tfrac{1}{2}\langle [\delta_{ka}(\mathrm{Ric}_a) + \nabla_k \nabla_a V_a], \nabla_a \rangle f \\
&\quad + \delta_{ka}\{h_1^a \nabla_a f_{\cdots a \cdots} + h_2^a f_{\cdots a \cdots}\}. \quad \blacksquare
\end{aligned}
$$

B. *The Estimation of the Commutators in* (6.7). Using Lemma 6.1 and the inequality $\|x+y\| \leqslant \|x\| + \|y\|$ we can estimate term (6.7) the following way:

$$
\Bigg(\sum_{\substack{k_1, \ldots, k_j \in \mathbb{Z}^d \\ a \in Q,\, |a-k_m| \leqslant r_0}} \overset{j}{\mathbf{p}}_{k_j} \cdots \overset{1}{\mathbf{p}}_{k_1} \cdot |\nabla_{k_j} \cdots \nabla_{k_{m+1}} [\nabla_{k_m}, H_a] \nabla_{k_{m-1}} \cdots \nabla_{k_1} u(x)|^2 \Bigg)^{1/2}
$$

$$
\leqslant \Bigg(\sum_{\substack{k_1, \ldots, k_j \in \mathbb{Z}^d \\ a \in Q,\, |a-k_m| \leqslant r_0}} \overset{j}{\mathbf{p}}_{k_j} \cdots \overset{1}{\mathbf{p}}_{k_1} \cdot |\nabla_{k_j} \cdots \nabla_{k_{m+1}} \{G^{k_m a} \nabla_a \nabla_{k_{m-1}} \cdots \nabla_{k_1} u\}|^2 \Bigg)^{1/2} \tag{6.15}
$$

$$
+ \sum_{q=1}^{m-1} \Bigg(\sum_{\substack{k_1, \ldots, k_j \in \mathbb{Z}^d \\ k_q = k_m}} \Big(\prod_{i \neq m, q} \overset{i}{\mathbf{p}}_{k_i} \Big) \overset{m}{\mathbf{p}}_{k_m} \overset{q}{\mathbf{p}}_{k_m} \cdot |\nabla_{k_j} \cdots \nabla_{k_{m+1}} \{h_1^{k_m} \nabla_{k_m} \nabla_{k_{m-1}} \cdots \nabla_{k_1} u\}|^2 \Bigg)^{1/2} \tag{6.16}
$$

$$
+ \sum_{q=1}^{m-1} \Bigg(\sum_{\substack{k_1, \ldots, k_j \in \mathbb{Z}^d \\ k_q = k_m}} \Big(\prod_{i \neq m, q} \overset{i}{\mathbf{p}}_{k_i} \Big) \overset{m}{\mathbf{p}}_{k_m} \overset{q}{\mathbf{p}}_{k_m} \cdot |\nabla_{k_j} \cdots \nabla_{k_{m+1}} \{h_2^{k_m} \nabla_{k_{m-1}} \cdots \nabla_{k_1} u\}|^2 \Bigg)^{1/2}. \tag{6.17}
$$

We can make the change in the indexes in (6.17), omitting the index k_m. Then the multiweight $\pi = \{\overset{m}{\mathbf{p}}\,\overset{q}{\mathbf{p}}, (\overset{i}{\mathbf{p}}, i \neq m, q)\}$ belongs to gen(θ) (see (5.7). Using the definition of Gen(θ) (5.12) we can estimate the (6.17) term from above by

$$\sum_{q=1}^{m-1} \left(\sum_{k_1, \ldots, k_{j-1} \in \mathbb{Z}^d} \overset{j-1}{\mathbf{r}}_{k_{j-1}} \cdots \overset{1}{\mathbf{r}}_{k_1} \cdot |\nabla_{k_{j-1}} \cdots \nabla_{k_m} (h_2^{k_q} \nabla_{k_{m-1}} \cdots \nabla_{k_1} u)|^2 \right)^{1/2}, \tag{6.18}$$

where $\{\overset{j-1}{\mathbf{r}}, \ldots, \overset{1}{\mathbf{r}}\} = \pi \in \mathrm{Gen}(\theta)$ in (5.12).

Interplacing the indexes of summation $\{k_m \to a,\ a \to k_m\}$ in (6.15) and using the relation

$$e^{-M_\theta} \leqslant |\overset{m}{\mathbf{r}}_a / \overset{m}{\mathbf{r}}_{k_m}| \leqslant e^{M_\theta},$$

when $|a - k_m| \leqslant r_0$ (see (5.9)), and the condition that $Q \subset \mathbb{Z}^d \backslash \mathrm{nul}(\theta)$ (5.2), we estimate the (6.15) term from above by

$$e^{M_\theta} \left(\sum_{\substack{k_1, \ldots, k_j \in \mathbb{Z}^d \\ a \in Q,\ |a - k_m| \leqslant r_0}} \overset{j}{\mathbf{p}}_{k_j} \cdots \overset{1}{\mathbf{p}}_{k_1} \cdot |\nabla_{k_j} \cdots \nabla_{k_{m+1}} \{G^{k_m a} \nabla_{k_m} \nabla_{k_{m-1}} \cdots \nabla_{k_1} u\}|^2 \right)^{1/2}. \tag{6.19}$$

The terms in (6.16), (6.18), and (6.19) have the structure

$$\left(\sum_{\substack{k_1, \ldots, k_l \in \mathbb{Z}^d \\ \sigma - \text{cond.}}} \overset{l}{\mathbf{q}}_{k_l} \cdots \overset{1}{\mathbf{q}}_{k_1} \cdot |\nabla_{k_l} \cdots \nabla_{k_t} \{F^b \nabla_{k_{t-1}} \cdots \nabla_{k_1} u\}|^2 \right)^{1/2}. \tag{6.20}$$

The indexes in (6.20) in concrete situations are equal to the following terms.

(6.19) term. $l = j$, $t = m + 1$, $\{\overset{i}{\mathbf{q}} = \overset{i}{\mathbf{p}}\}_{i=1}^{j}$, $F^b = G^{k_m a}$, $b = \{k_m, a\}$, and the *σ-condition* in this situation means that the summation runs on $a \in \mathbb{Z}^d$, $|a - k_m| \leqslant r_0$.

(6.18) term. $l = j - 1$, $t = m$, $\{\overset{i}{\mathbf{q}} = \overset{i}{\mathbf{r}}\}_{i=1}^{j-1}$, $F^b = h_2^{k_q}$, $b = k_q$, and the *σ-condition* is absent.

(6.16) term. $l = j$, $t = m + 1$, $\{\overset{i}{\mathbf{q}} = \overset{i}{\mathbf{p}}\}_{i=1}^{j}$, $F^b = h_1^{k_m}$, $b = k_m$, and the *σ-condition* means that summation runs on points $k_1, \ldots, k_j \in \mathbb{Z}^d$ with $k_q = k_m$ for some fixed $q \in \{1, \ldots, m-1\}$.

The relation $\nabla_k(fg) = (\nabla_k f)g + f(\nabla_k g)$ and the inequality $\|x+y\| \leqslant \|x\| + \|y\|$ permit us to estimate (6.20) as a finite sum of the expressions

$$\left(\sum_{\substack{k_1, \ldots, k_l \in \mathbb{Z}^d \\ \sigma - \text{cond}}} \overset{l}{\mathbf{q}}_{k_l} \cdots \overset{1}{\mathbf{q}}_{k_1} \cdot \left| \left(\prod_{i \in I} \nabla_{k_i} F^b \right) \cdot \left[\prod_{i \in J} \nabla_{k_i} \right] \nabla_{k_{t-1}} \cdots \nabla_{k_1} u \right|^2 \right)^{1/2}, \tag{6.21}$$

where the sets I and J form the subdivision of $\{t, \ldots, l\}$ on two non-intersecting subsets,

$$I \cup J = \{t, \ldots, l\}, \qquad I \cap J = \varnothing.$$

The field F^b has a finite diameter of support of cylindricity $2r_0$ (see (6.13), (6.14), and Condition 2 in (2.1)).

Therefore term (6.21) can be estimated by

$$\sup_{i \in I} \sup_{b \subset \mathbb{Z}^d} \left| \left(\prod_{i \in I} \nabla_{k_i} \right) F^b \right|$$
$$\cdot \left(\sum_{\substack{k_1, \ldots, k_l \in \mathbb{Z}^d \\ \sigma - \text{cond} \\ |k_i - k_s| \leqslant 2r_0,\ i \in I}} \overset{l}{\mathbf{q}}_{k_l} \cdots \overset{1}{\mathbf{q}}_{k_1} \left| \left(\prod_{i \in J} \nabla_{k_i} \right) \nabla_{k_{t-1}} \cdots \nabla_{k_1} u \right|^2 \right)^{1/2}. \tag{6.22}$$

Here $s = m$ for (6.18), (6.19) and $s = q$, $q \in \{1, \ldots, t-1\}$, for (6.16).

The factor at (6.22) is finite due to Assumptions 1–3 on $\{\Phi_A\}$ in (2.1) and due to the finite length of array $\text{Gen}(\theta) \subset \mathbb{P}$, so $|I| \leqslant l(\theta) < \infty$.

We only have to estimate (6.22) by $|u(x)|_\psi$ for some $\psi \in \text{Gen}(\theta)$. But $|u(x)|_\psi$ is completely determined by its multiweight ψ (5.4), so we only have to control the reduction of multiweights.

Using relations (5.1) and (5.9) we can add all weights $\{\overset{i}{\mathbf{q}}_{k_i}\}_{i \in I}$ to the weight $\{\overset{s}{\mathbf{q}}_{k_s}\}$ in (6.22) and estimate it by

$$e^{|I| \cdot M_0} \left(\sum_{\substack{k_1, \ldots, k_l \in \mathbb{Z}^d \\ \sigma - \text{cond} \\ |k_i - k_s| \leqslant 2r_0,\ i \in I}} \left(\prod_{i \notin I \cup \{s\}} \overset{i}{\mathbf{q}}_{k_i} \right) \left(\prod_{i \in I \cup \{s\}} \overset{i}{\mathbf{q}}_{k_s} \right) \right.$$
$$\left. \cdot \left| \left(\prod_{i \in J} \nabla_{k_i} \right) \nabla_{k_{t-1}} \cdots \nabla_{k_1} u(x) \right|^2 \right)^{1/2}. \tag{6.23}$$

Now we can omit the σ-condition under the sum in (6.23) because for the (6.18) term it is absent, and for the (6.16) term the sum of the diagonal

$\{k_m = k_q\}$ is less than the sum of the independent variables $\{k_m, k_q\}$. For the (6.19) term the sum

$$\sum_{\substack{k_1, ..., k_j \in \mathbb{Z}^d \\ a \in \mathbb{Z}^d, |a - k_m| \leqslant r_0}} \alpha_{k_1, ..., k_j}$$

is less than

$$(2r_0)^d \sum_{k_1, ..., k_j \in \mathbb{Z}^d} \alpha_{k_1, ..., k_j}$$

as $\alpha_{k_1, ..., k_j}$ do not depend on a.

Moreover, the terms under the summation in (6.23) do not depend on $\{k_i\}_{i \in I}$ (recall that $I \cap J = \varnothing$). So the term (6.23) is less than

$$(2r_0)^{d(1+l(\theta))} e^{M_\theta |I|} \left(\sum_{k_i \in \mathbb{Z}^d, i \in \{1, ..., l\} \setminus I} \left(\prod_{i \notin I \cup \{s\}} \overset{i}{\mathbf{q}}_{k_i} \right) \right.$$

$$\left. \cdot \left(\prod_{i \in I \cup \{s\}} \overset{i}{\mathbf{q}}_{k_s} \right) \left| \left(\prod_{i \in J} \nabla_{k_i} \right) \nabla_{k_{l-1}} \cdots \nabla_{k_1} u(x) \right|^2 \right)^{1/2}. \tag{6.24}$$

The final estimate of the (6.24) term is trivial because the multiweight

$$\pi = \left\{ \prod_{i \in I \cup \{s\}} \overset{i}{\mathbf{q}}; (\overset{t}{\mathbf{q}}, t \in \{1, ..., l\} \setminus (I \cup \{s\})) \right\}$$

belongs to $\text{Gen}(\theta)$.

The final estimate on the quasi-accretive constant $\mathscr{D}_\theta$ (6.1),

$$\mathscr{D}_\theta \leqslant l_\theta (1 + l_\theta)(2r_\theta e^{M_\theta})^{l_\theta + 1} \, 2^{l_\theta} \max(F_1, F_2), \tag{6.25}$$

follows from (6.4), (6.15)–(6.19), (6.24), (6.23), and the calculation of the maximal number of terms appearing after the successive application of $\nabla(fg) = (\nabla f)g + f(\nabla g)$ to (6.20). Here $l(\theta)$ is a length of weight $\theta \in \mathbb{P}$, constant M_θ appears in (5.9), and r_0 denotes the interactive radius. The constants F_1, F_2 are equal to

$$F_1 = \sup_{k, j \in \mathbb{Z}^d} |||\delta_{kj} \operatorname{Ric}_k + \nabla_k \nabla_j V_{\{k,j\}}|||_{l(\theta)-1}$$
$$F_2 = \sup_{k \in \mathbb{Z}^d} |||R_k|||_{l(\theta)-1}, \tag{6.26}$$

where

$$|||F|||_n = \sup_{x \in M^{\mathbb{Z}^d}} \{|\nabla_{k_1} \cdots \nabla_{k_j} f(x)| : j \leqslant n, k_i \in \mathbb{Z}^d\}. \tag{6.27}$$

At last, recall that for $\theta = \varnothing$ the proof is finished on (6.5) with constant $\mathscr{D}_\varnothing = 0$. ∎

7. Dirichlet Operators Semigroups in the Scale $\mathscr{E}_\theta$: The Property of Invariantness

Definition 7.1. We denote by $\hat{H}^\mu_Q$, $\tilde{H}^\Theta_Q$, and $\tilde{H}^{\mathbf{p},\alpha}_Q$ the closures of operator $\{H_Q, |Q| \leqslant \infty\}$ (4.1) from domain $\mathscr{D}(H_Q) = C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ in the spaces $\{L_2(M^{\mathbb{Z}^d}, \mu), \mu \in \mathscr{G}\{\mu_A\}$, $\mathscr{E}_\Theta$, and $\mathscr{Z}(\mathbf{p}, \alpha)$, respectively.

Below we prove that operators $\hat{H}^\mu_Q$, $\tilde{H}^\Theta_Q$, and $\tilde{H}^{\mathbf{p},\alpha}_Q$, $|Q| \leqslant \infty$, are generators of strongly continuous semigroups in the corresponding spaces.

In other words, the operators $\{H_Q, |Q| \leqslant \infty\}$ are *essentially selfadjoint* in $\{L_2(M^{\mathbb{Z}^d}, \mu), \mu \in \mathscr{G}\{\mu_A\}\}$ and are *essentially maximally quasi-accretive* in scales $\{\mathscr{E}_\Theta\}$ and $\{\mathscr{Z}(\mathbf{p}, \alpha)\}$ (see Appendix, Definition A.3 for the definition of maximal quasi-accretivity).

Due to the smooth solvability of the Cauchy problem (4.3) in $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ and the quasi-accretivity of $H_{Q(i)}$ (4.2) in spaces $\{\mathscr{E}_\Theta\}$ and $\{\mathscr{Z}(\mathbf{p}, \alpha)\}$ (Theorems 5.6 and 5.8), we have the next statement as in Corollary 4.2.

Theorem 7.2. *Under Conditions* 1–3 *on interactive potentials* $\{\Phi_A\}$ *in* (2.1) *we have that*

1. $\forall i \in \{0, 1\}^d$ *and* $\forall Q \subset \mathbb{Z}^d$, $|Q| \leqslant \infty$, *the operator* $H_{Q(i)}$ *is essentially maximally quasi-accretive in space* $\mathscr{Z}(\mathbf{p}, \alpha)$ *with essential domain* $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ *for every* $\mathbf{p}, \alpha$;

2. $\forall i \in \{0, 1\}^d$ *and* $\forall Q \subset \mathbb{Z}^d \backslash \text{nul}(\Theta)$, $|Q| \leqslant \infty$, *for the quasi-accretive array of multiweights* Θ *the operator* $H_{Q(i)}$ *is essentially maximally quasi-accretive in space* $\mathscr{E}_\Theta$ *with essential domain* $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$.

Proof. The proof completely repeats the proof of Corollary 4.2; therefore it is omitted. The only difference lies in the estimate

$$\|P^{Q(i)}_t f\|_X \leqslant e^{\mathscr{D}_X t} \|f\|_X$$

for $X = \mathscr{E}_\Theta$ or $X = \mathbb{Z}(\mathbf{p}, \alpha)$. This estimate simply follows from the quasi-accretivity of operator $H_{Q(i)}$ in the space X (see Theorems 5.6 and 5.8). ∎

COROLLARY 7.3. *Under Conditions* 1–3 *on interactive potentials* $\{\Phi_A\}$ *in* (2.1) *we have that corresponding closures of* $H_{Q(i)}$, $|Q| \leqslant \infty$, *satisfy the next estimates* (*for* Θ *to be quasi-accretive we ask* $Q \cap \operatorname{nul}(\Theta) = \varnothing$):

$$\|(\lambda + \hat{H}^{\mu}_{Q(i)})^{-1}\|_{\mathscr{L}(L_2(\mu))} \leqslant 1/\lambda, \qquad \lambda > 0 \tag{7.1}$$

$$\|(\lambda + \tilde{H}^{\Theta}_{Q(i)})^{-1}\|_{\mathscr{L}(\mathscr{E}_\Theta)} \leqslant 1/(\lambda - \mathscr{D}_\Theta), \qquad \lambda > \mathscr{D}_\Theta \tag{7.2}$$

$$\|(\lambda + \tilde{H}^{\mathbf{p},\alpha}_{Q(i)})^{-1}\|_{\mathscr{L}(\mathscr{X}(\mathbf{p},\alpha))} \leqslant 1/(\lambda - \mathscr{D}_{\mathbf{p},\alpha}), \qquad \lambda > \mathscr{D}_{\mathbf{p},\alpha}. \tag{7.3}$$

Proof. Due to Theorem 7.2 and Corollary 4.2 the required estimate follows from the Hille–Yosida theorem in the Lumer–Phillips form (see Remark A.4 in the Appendix). ∎

THEOREM 7.4. *Under Conditions* 1–3 *in* (2.1) *we have that* $\forall\, |Q| \leqslant \infty$ *the Dirichlet operator* H_Q *with domain* $\mathscr{D}(H_Q) = C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ *is*

1. *essentially self-adjoint in the space* $L_2(M^{\mathbb{Z}^d}, \mu)$, *for* $\mu \in \mathscr{G}\{\mu_A\}$ (*see* (2.5));

2. *essentially maximaly quasi-accretive in* $\mathscr{X}(\mathbf{p}, \alpha)$ *and in* $\mathscr{E}_\Theta$ *for* Θ *quasi-accretive* (*Definitions* 5.4 *and* 5.5), $Q \cap \operatorname{nul}(\Theta) = \varnothing$.

Proof. Here we apply Theorem 4.3 to the operators $\{A_i = H_{Q(i)}\}_{i \in \{0,1\}^d}$. Theorem 5.9 and Corollary 7.2 give the conditions of Theorem 4.3 in the cases $Y = \mathscr{X}(\mathbf{r}, \gamma)$ and $X \in \{L_2(M^{\mathbb{Z}^d}, \mu), \mathscr{E}_\Theta, \mathscr{X}(\mathbf{p}, \alpha)\}$, so for $\sum A_i = H_Q$ (see (4.1)) we obtain the essential domain $Y = \mathscr{X}(\mathbf{r}, \gamma)$. We only have to ensure that $\tilde{A}^X_i \upharpoonright_Y = \tilde{A}^Y_i$, which follows from the solvability of the Cauchy problem (4.3) in $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$.

Due to the density of $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ in Y and the estimates

$$\|u\|_X \leqslant \|u\|_Y \qquad \text{and} \qquad \|H_Q u\|_X \leqslant \|u\|_Y,$$

we obtain that $C^\infty_{\text{cyl}}(M^{\mathbb{Z}^d})$ is the essential domain for H_Q. ∎

Now we begin to investigate the action of $\{\exp(-t\hat{H}^\mu_Q), \mu \in \mathscr{G}\{\mu_A\}\}$ semigroups in the scales of spaces $\{\mathscr{E}_\Theta\}$ and $\{\mathscr{X}(\mathbf{p}, \alpha)\}$.

THEOREM 7.5. *Under Conditions* 1–3 *in* (2.1) *on potentials* $\{\Phi_A\}$ *we have that* $\forall \mu \in \mathscr{G}\{\mu_A\}$ *the semigroups of Hermitian realizations of a formal Dirichlet operator of the Gibbs measure* $\{\exp(-t\hat{H}^\mu_Q), \mu \in \mathscr{G}\{\mu_A\}, Q \subset \mathbb{Z}^d, |Q| \leqslant \infty\}$ *have the following properties*:

1. *preserve each space* $\mathscr{X}(\mathbf{p}, \alpha)$ *or* $\mathscr{E}_\Theta$ *for the quasi-accretive array* $\Theta \in \mathbb{P}$, $Q \cap \operatorname{nul}(\Theta) = \varnothing$;

2. *coincide for different* $\mu \in \mathscr{G}\{\mu_A\}$ *on these spaces at fixed* $Q \subset \mathbb{Z}^d$, $|Q| \leqslant \infty$;

3. *for* $f_0 \in C^{\infty}_{\text{cyl}}(M^{\mathbb{Z}^d})$ $\forall Q \subset \mathbb{Z}^d$, $|Q| \leqslant \infty$,

$$\exp(-t\hat{H}^{\mu}_{Q}) f_0 \in \bigcap_{\mathbf{p},\alpha} \mathscr{X}(\mathbf{p}, \alpha)$$

and

$$\exp(-t\hat{H}^{\mu}_{Q}) f_0 \in \bigcap_{\substack{Q \cap \text{nul}(\Theta) = \varnothing \\ \Theta \text{ is } q\text{-accretive}}} \mathscr{E}_{\Theta}.$$

Proof. We use Theorem 4.3 in two different cases:

1. $X = E$, in which the space E belongs to the scales $\{\mathscr{E}_{\Theta}\}$ or $\{\mathscr{X}(\mathbf{p}, \alpha)\}$, and $Y = \mathscr{X}(\mathbf{q}, \beta)$. Due to Theorem 5.9 we can choose $(\mathbf{q}, \beta)$ so that $Y \subset \mathscr{D}_E(H^2_{\mathbb{Z}^d})$;

2. $X = L_2(M^{\mathbb{Z}^d}, \mu)$, $\mu \in \mathscr{G}\{\mu_A\}$, and $Y = \mathscr{X}(\mathbf{q}, \beta)$ as above.

Theorem 4.3 results in the following two multiplicative formulas for the semigroups (see (4.6)):

1. In the space $L_2(M^{\mathbb{Z}^d}, \mu)$, $\mu \in \mathscr{G}\{\mu_A\}$,

$$\exp(-t\hat{H}^{\mu}_{Q}) = L_2 - s - \lim_{m \to \infty} \left\{ \prod_{i \in \{0,1\}^d} \exp\left(-\frac{t}{m} \hat{H}_{Q(i)}\right)\right\}^m. \tag{7.4}$$

2. In the space $E \in \{\mathscr{E}_{\Theta}, \mathscr{X}(\mathbf{p}, \alpha)\}$,

$$\exp(-t\tilde{H}^{E}_{Q}) = E - s - \lim_{m \to \infty} \left\{ \prod_{i \in \{0,1\}^d} \exp\left(-\frac{t}{m} \tilde{H}^{E}_{Q(i)}\right)\right\}^m. \tag{7.5}$$

The representations (7.4) and (7.5) permit us to obtain the properties of an operator with interaction H_Q through the properties of operators with independent variables $H_{Q(i)}$.

Consider $f_0 \in C^{\infty}_{\text{cyl}}(M^{\mathbb{Z}^d})$. Then (7.4), (7.5), and the smooth solvability of the Cauchy problem (Theorem 4.1) give the next line of identities:

$$\begin{aligned} \exp(-t\hat{H}^{\mu}_{Q}) f_0 &= L_2 - \lim_{m \to \infty} \left\{ \prod_{i \in \{0,1\}^d} \exp\left(-\frac{t}{m} \hat{H}^{\mu}_{Q(i)}\right)\right\} f_0 \\ &= \{\text{sequence of the smooth cylindric Cauchy problems}\} \\ &= L_2 - \lim_{m \to \infty} \left\{ \prod_{i \in \{0,1\}^d} \exp\left(-\frac{t}{m} \tilde{H}^{E}_{Q(i)}\right)\right\} f_0 \\ &= E - \lim_{m \to \infty} \left\{ \prod_{i \in \{0,1\}^d} \exp\left(-\frac{t}{m} \tilde{H}^{E}_{Q(i)}\right)\right\} f_0 \\ &= \exp(-t\tilde{H}^{E}_{Q}) f_0. \end{aligned} \tag{7.6}$$

So the action of the Dirichlet operator semigroup on smooth cylinder function can be represented as a sequence of functions which are obtained as the successive solutions for the Cauchy problems for operators $H_{Q(i)}$ (4.2). This sequence of functions converges not only in the topology of space L_2 but also in the topology of space E, because of (7.5). Therefore we have that $f_0 \in C_{\text{cyl}}^{\infty}(M^{\mathbb{Z}^d})$ implies $\exp(-t\hat{H}_Q^{\mu}) f_0 \in E$ for any E from the scale $\{\mathscr{E}_\Theta\}$ or $\{\mathscr{L}(\mathbf{p}, \alpha)\}$. So we have proved that $\exp(-t\hat{H}_Q^{\mu})$ preserves the spaces of scales $\{\mathscr{E}_\Theta\}$ and $\{\mathscr{L}(\mathbf{p}, \alpha)\}$. The formula (7.6) gives that the semigroups $\exp(-t\hat{H}_Q^{\mu})$ coincide for different $\mu \in \mathscr{G}\{\mu_A\}$ on spaces $\mathscr{E}_\Theta$ or $\mathscr{L}(\mathbf{p}, \alpha)$. ▮

Remark 7.6. Under the conditions of Theorem 7.5 we also obtain that $\exp(-t\tilde{H}_Q^A)$ restricted onto the space $B \subset A$ coincides with $\exp(-t\tilde{H}_Q^B)$. Here the spaces A, B are from the scales $\{L_2(M^{\mathbb{Z}^d}, \mu), \mu \in \mathscr{G}\{\mu_A\}\}$, $\{\mathscr{E}_\Theta\}$, or $\{\mathscr{L}(\mathbf{p}, \alpha)\}$. The inclusion $B \subset A$ is understood in the sense of a corresponding estimate for the norms $\|\cdot\|_A \leqslant \|\cdot\|_B$ on $C_{\text{cyl}}^{\infty}(M^{\mathbb{Z}^d})$.

8. The Smoothing Properties of Dirichlet Operators Semigroups

Below we investigate the action of semigroups for Dirichlet operators of Gibbs measures $\{\exp(-tH_Q^{\mu}), \mu \in \mathscr{G}\{\mu_A\}, Q \subset \mathbb{Z}^d, |Q| \leqslant \infty\}$ in the scale $\{\mathscr{E}_\Theta\}$. We provide the conditions on array $\Theta \in \mathbb{P}$ when for the space $\mathscr{E}_\Theta$ there is a directed sequence of Banach spaces $\{\mathscr{E}_{\Theta_i}\}_{i \geqslant 0}$;

$$\mathscr{E}_\Theta = \mathscr{E}_{\Theta_0} \supset \cdots \supset \mathscr{E}_{\Theta_i} \supset \mathscr{E}_{\Theta_{i+1}} \supset \cdots,$$

such that for every $t > 0$, $i \geqslant 1$ we have the estimate

$$\|\exp(-t\hat{H}_{\mathbb{Z}^d}^{\mu}) f_0\|_{\Theta_i} \leqslant \text{const}(i, t)\, \|f_0\|_{\Theta}.$$

Here $i = l(\Theta_i) - l(\Theta)$ denotes the increment of the order of differentiation.

In other words, we obtain that the solution $f(t, x)$ of the Cauchy problem

$$\begin{cases} \dfrac{\partial f(t, x)}{\partial t} = -\hat{H}_{\mathbb{Z}^d}^{\mu} f(t, x) \\ f(0, x) \in \mathscr{E}_\Theta \end{cases}$$

is infinitely differentiable: $\forall t > 0$, $f(t, x) \in \bigcap_{i \geqslant 0} \mathscr{E}_{\Theta_i}$.

DEFINITION 8.1. For a finite array of multiweights $\Theta \in \mathbb{P}$ we define the array

$$s_Q(\Theta) = \bigcup_{\theta = (\overset{1}{\mathbf{p}}, \ldots, \overset{l}{\mathbf{p}}) \in \Theta} \{\overset{1}{\mathbf{p}}, \ldots, \overset{l}{\mathbf{p}}, \mathbf{1}_Q\}, \tag{8.1}$$

where $\mathbf{1}_Q$ denotes the following weight:

$$\{\mathbf{1}_Q\}_k = \begin{cases} 1, & k \in Q \\ 0, & k \in \mathbb{Z}^d \setminus Q. \end{cases}$$

Let also $\Theta_i(Q)$ denote the next array,

$$\Theta_i(Q) = \bigcup_{l=0}^{i} s_Q^l(\Theta). \tag{8.2}$$

with $s_Q^l = s_Q(s_Q^{l-1})$.

DEFINITION 8.2. The array of multiweights is said to be *consistent* if both Θ and $\Theta_1(\mathbb{Z}^d)$ are quasi-accretive arrays (see Definition 5.3). We say that array Θ is k-consistent if for $i = 1, \ldots, k$ $(i \geqslant 1)$ the arrays $\Theta_i(\mathbb{Z}^d)$ are quasi-accretive.

Remark 8.3. Let the quasi-accretive array $\Theta \subset \mathbb{P}$ satisfy the condition

$$s_{\mathbb{Z}^d}(\text{gen}(\Theta)) \ll \Theta. \tag{8.3}$$

Then array Θ is ∞-consistent. It is simply proved by induction that each $\Theta_i(\mathbb{Z}^d)$ is quasi-accretive and satisfies Property (8.3).

THEOREM 8.4. *Suppose that the interactive potentials $\{\Phi_A\}$ satisfy Conditions 1–3 in (2.1). Let $\Theta \in \mathbb{P}$ be a consistent array. Then $\forall Q \subset \mathbb{Z}^d \setminus \text{nul}(\Theta)$, $|Q| \leqslant \infty$, we have the estimate*

$$\forall f \in \mathscr{E}_\Theta : \|\exp(-t\hat{H}_Q^\mu) f\|_{\Theta_1(Q)} \leqslant C(t) \, \|f\|_\Theta$$

with the uniform on Q constant $C(t)$, $t > 0$.

Proof. Introduce on $\mathscr{E}_{\Theta_1(Q)}$ the next sequence of seminorms

$$\rho_t(f) = \sup_{x \in M^{\mathbb{Z}^d}} \left\{ \sum_{\theta \in \Theta} (f, f)_\theta + 2t \sum_{\theta' \in s_Q(\Theta)} (f, f)_{\theta'} \right\}.$$

Then

1. for $t = 0$, $\rho_0(f)$ is equivalent to $\|f\|_\Theta^2$;
2. for $t > 0$, $\rho_t(f)$ is equivalent to $\|f\|_{\Theta_1(Q)}^2$.

Therefore the statement of Theorem 8.4 can be obtained from the estimate

$$\rho_t(\exp(-t\hat{H}^{\mu}_Q f)) \leqslant e^{\alpha t}\rho_0(f)$$

with some constant $\alpha > 0$.

To obtain the above estimate it is sufficient to prove that

$$\frac{\partial \rho_t(\exp(-t\hat{H}^{\mu}_Q)f)}{\partial t} \leqslant \alpha \rho_t(\exp(-t\hat{H}^{\mu}_Q)f). \tag{8.4}$$

Theorem 7.5 gives for $f_0 \in C^{\infty}_{\mathrm{cyl}}(M^{\mathbb{Z}^d})$ that

$$\exp(-t\hat{H}^{\mu}_Q)f_0 \in \bigcap_{\mathbf{p},\alpha} Z(\mathbf{p},\alpha)$$

and

$$\exp(-t\hat{H}^{\mu}_Q)f_0 \in \bigcap_{\substack{\mathrm{nul}(\Theta)\cap Q=\varnothing \\ \Theta \text{ is } q\text{-accretive}}} \mathscr{E}_{\Theta}.$$

The countable product of compact manifolds is a Tichonov metrizable compact with metric

$$\rho(x, y) = \sum_{k \in \mathbb{Z}^d} \frac{1}{e^{c|k|}} \rho_i(x_i, y_i),$$

where $c > 0$ and $\rho_i(x_i, y_i)$ denotes the geodetic distance between points x_i, y_i on manifold M_i. The next statement proposes the conditions on differentiability of the maximum function.

LEMMA [20, Chap. 2, Thm. 3.13]. *Let K be a compact metric space. Suppose that the function $F(t, x)$ satisfies the following assumptions*:

1. *$F(t, x)$, $(\partial F(t, x)/\partial t)$, are continuous on $[0, T] \times K$;*
2. $\max_{x \in K, t \in [0, T]} |\partial^2 F(t, x)/\partial t^2| < \infty$.

Then $\forall t \in [0, T]$ we have that

$$\frac{\partial}{\partial t} \max_{x \in K} F(t, x) = \max_{x \in X(t)} \frac{\partial F(t, x)}{\partial t}, \tag{8.5}$$

where $X(t) = \{x : F(t, x) = \max_{y \in K} F(t, y)\}$.

Consider $f_0 \in C^{\infty}_{\mathrm{cyl}}(M^{\mathbb{Z}^d})$. Put

$$F(t, x) = \sum_{\theta \in \Theta} (P_t f_0, P_t f_0)_{\theta} + 2t \sum_{\theta' \in s_Q(\Theta)} (P_t f_0, P_t f_0)_{\theta'},$$

where $P_t = \exp(-t\hat{H}^{\mu}_Q)$. It is obvious from Definition 5.3 of $(\cdot, \cdot)_\theta$ (see (5.5)) that the derivatives of $F(t, x)$ are represented as a finite sum of terms $(P_t H^i_Q f_0, P_t H^m_Q f_0)_\theta$ for some powers $i, m \geqslant 0$. The cylindricity of f_0 and $\{H^i_Q f_0\}_{i \geqslant 0}$ implies that $\partial^k F(t, x)/\partial t^k$ are bounded on $[0, T] \times M^{\mathbb{Z}^d}$, $k \geqslant 0$.

Due to Remark 7.6 and uniform convergence on $[0, T]$ in (7.5) we have that $\{\partial^k F(t, x)/\partial t^k\}_{k \geqslant 0}$ can be represented as the *uniform limits* on $[0, T] \times M^{\mathbb{Z}^d}$ of terms like

$$(T_{m,t} H^i_Q f_0, T_{m,t} H^k_Q f_0)_\theta, \tag{8.6}$$

where operator $T_{m,t}$ equals

$$T_{m,t} = \left(\prod_{i \in \{0,1\}^d} \exp\left(-\frac{t}{m} \hat{H}^{\mu}_{Q(i)} \right) \right)^m \tag{8.7}$$

as in (7.6).

But $T_{m,t} H^i_Q f_0$ is a *cylinder* function on $M^{\mathbb{Z}^d}$; therefore the term (8.6) is continuous in the topology of $[0, T] \times M^{\mathbb{Z}^d}$.

So we have obtained that $\{\partial^k F(t, x)/\partial t^k\}_{k \geqslant 0}$ are *bounded continuous* on $[0, T] \times M^{\mathbb{Z}^d}$ as the uniform limits of continuous functions.

For $x \in X(t) = \{x : \max_{y \in M^{\mathbb{Z}^d}} F(t, y) = F(t, x)\}$ we have

$$\frac{\partial F(t, x)}{\partial t} = 2 \sum_{\theta' \in s_Q(\Theta)} |P_t f_0|^2_{\theta'} + 2 \sum_{\theta \in \Theta} \Re e(-H_Q P_t f_0, P_t f_0)_\theta$$
$$+ 4t \sum_{\theta' \in s_Q(\Theta)} \Re e(-H_Q P_t f_0, P_t f_0)_{\theta'}.$$

Theorem 6.1 enables us to estimate this expression from above by

$$\frac{\partial F(t, x)}{\partial t} \leqslant 2 \sum_{\theta' \in s_Q(\Theta)} |P_t f_0|^2_{\theta'}$$
$$- H_Q \left[\sum_{\theta \in \Theta} |P_t f_0|^2_\theta + 2t \sum_{\theta' \in s_Q(\Theta)} |P_t f_0|^2_{\theta'} \right]$$
$$- 2 \sum_{\theta \in \Theta} |P_t f_0|^2_{s_Q(\theta)} - 4t \sum_{\theta' \in s_Q(\Theta)} |P_t f_0|^2_{s_Q(\theta')}$$
$$+ \sum_{\theta \in \Theta} \mathscr{D}_\theta |P_t f_0|^2_{\mathrm{Gen}(\theta)} + 4t \sum_{\theta' \in s_Q(\Theta)} \mathscr{D}_{\theta'} |P_t f_0|^2_{\mathrm{Gen}(\theta')}$$
$$\leqslant -H_Q F(t, x) + \sum_{\theta \in \Theta} \mathscr{D}_\theta |P_t f_0|^2_{\mathrm{Gen}(\theta)}$$
$$+ 4t \sum_{\theta' \in s_Q(\Theta)} \mathscr{D}_{\theta'} |P_t f_0|^2_{\mathrm{Gen}(\theta')}.$$

The expression $H_Q F(t, x)$ is correctly defined due to the estimate

$$
\begin{aligned}
&|H_Q(P_t f_0, P_t f_0)_\theta| \\
&\quad \leqslant |P_t f_0|^2_{s_Q(\theta)} + \Big(\sum_{k \in \mathbb{Z}^d} 1/\mathbf{q}_k\Big) |P_t f_0|_\theta \\
&\qquad \cdot \Big(\sum_{k_1, \dots, k_{j+1} \in \mathbb{Z}^d} \overset{j}{\mathbf{p}}_{k_j} \cdots \overset{1}{\mathbf{p}}_{k_1} \mathbf{q}_{k_{j+1}} |H_{k_{j+1}} \nabla_{k_j} \cdots \nabla_{k_1} P_t f_0|^2\Big)^{1/2}.
\end{aligned}
\tag{8.8}
$$

Here we use that $H_Q \prec f, f \succ = 2\Re e \prec H_Q f, f \succ + |\nabla f|^2$.

For $f_0 \in C^\infty_{\mathrm{cyl}}(M^{\mathbb{Z}^d})$ both terms in the estimate above are finite due to the third statement of Theorem 7.5.

The convergence (7.5) and estimate (8.8) imply that $H_Q(P_t f_0, P_t f_0)_\theta$ can be represented as a uniform limit on $[0, T] \times M^{\mathbb{Z}^d}$ of the continuous cylinder on $M^{\mathbb{Z}^d}$ functions

$$
H_Q(P_t f_0, P_t f_0)_\theta = \lim_{m \to \infty} H_Q(T_{m,t} f_0, T_{m,t} f_0)_\theta,
$$

where $T_{m,t}$ is defined in (8.7). Therefore $H_Q F(t, x)$ is a continuous function on $[0, T] \times M^{\mathbb{Z}^d}$. As $x \in X(t)$ is a maximum point for $F(t, x)$ we see that

$$
-H_Q F(t, x) \leqslant 0.
$$

The condition of consistency for array Θ implies that $\mathrm{gen}(\Theta) \ll \Theta$ and $\mathrm{gen}(\Theta_1(Q)) \ll \Theta_1(Q)$. So we can write the next estimate,

$$
\begin{aligned}
\frac{\partial F(t, x)}{\partial t} &\leqslant \mathscr{D}_{\Theta_1(Q)} C^{l(\Theta_1(Q))}_{\Theta_1(Q)} 2\, |\Theta_1(Q)| \cdot (|P_t f_0|^2_\Theta + 4t\, |P_t f_0|^2_{\Theta_1(Q)}) \\
&\leqslant \mathrm{const}(1 + 2t)\, F(t, x).
\end{aligned}
$$

Due to (8.5) we have the required estimate (8.4) for $t \in [0, T]$.

The density of $C^\infty_{\mathrm{cyl}}(M^{\mathbb{Z}^d})$ in $\mathscr{E}_\Theta$ and the estimate $\|\exp(-t\hat{H}^\mu_Q) f_0\|_{\Theta_1(Q)} \leqslant C(t)\, \|f\|_\Theta$ for $f \in C^\infty_{\mathrm{cyl}}(M^{\mathbb{Z}^d})$ give the statement of Theorem 8.4. ∎

COROLLARY 8.5. *Let the interactive potentials* $\{\Phi_A\}$ *satisfy Conditions* 1–3 *in* (2.1).

Suppose that array Θ *is k-consistent for* $1 \leqslant k \leqslant \infty$. *Then for* $i = 1, \dots, k$ *and* $\forall Q \subset \mathbb{Z}^d \backslash \mathrm{nul}(\Theta)$, $|Q| \leqslant \infty$, *we have the estimate*

$$
\|\exp(-t\hat{H}^\Theta_Q) f\|_{\Theta_i(Q)} \leqslant C_i(t)\, \|f\|_\Theta
$$

for $f \in \mathscr{E}_\Theta$. *Therefore the semigroup of Dirichlet operators raises the smoothness of the initial function.*

Remark 8.6. As a consequence of Theorem 8.5 we have that the semigroup of Dirichlet operators $H_{\mathbb{Z}^d}$ acts from the space $\mathscr{E}_{\varnothing}$ with sup $|\cdot|$ norm into the Frechet space $\bigcap_{i\geq 0}\mathscr{E}_i$, where $\mathscr{E}_i$ is equipped with the norm

$$\|u\|_i = \sup_{x\in M^{\mathbb{Z}^d}} \max_{0\leq j\leq i} \left(\sum_{k_1,\ldots,k_j\in\mathbb{Z}^d} |\nabla_{k_j}\cdots\nabla_{k_1}u(x)|^2 \right)^{1/2}.$$

Below we investigate the role which Conditions 1–3 play in (2.1). We show that only Condition 2 has principal importance.

THEOREM 8.7. *Let the interactive potentials $\{\Phi_A\}$ satisfy Condition* 2 *in* (2.1). *Suppose that for some $n\geq 0$, $\Phi_A\in C^{n+5}(M^A)$ and we have the estimates*

$$\sup_{k,j\in\mathbb{Z}^d} \left|\!\left|\!\left| \delta_{k_j}\,\mathrm{Ric}_k + \nabla_k\nabla_j\left(\sum_{A:k,j\in A}\Phi_A\right)\right|\!\right|\!\right|_{n+3} < \infty \tag{8.9}$$

and

$$\sup_{k\in\mathbb{Z}^d} |\!|\!| R_k |\!|\!|_{n+3} < \infty. \tag{8.10}$$

Here $|\!|\!|\cdot|\!|\!|$ *denotes the expressions*

$$|\!|\!|F|\!|\!|_n = \sup_{x\in M^{\mathbb{Z}^d}} \max_{k_1,\ldots,k_j\in\mathbb{Z}^d} \left\{ |T_{k_1}^{s_1}\cdots T_{k_j}^{s_j}F(x)| : \sum_{i=1}^{j} s_i\leq n,\ s_i = 1, 2\right\}$$

and

$$T_k^1 f = \nabla_k f$$

$$T_k^2 f = \Delta_k f$$

as in (5.15).

Then

1. *Operators $\{H_Q, |Q|\leq\infty\}$ are essentially self-adjoint in $L_2(M^{\mathbb{Z}^d},\mu)$, $\mu\in\mathscr{G}\{\mu_A\}$, with essential domain of smooth cylinder functions $C_{\mathrm{cyl}}^{\infty}(M^{\mathbb{Z}^d})$.*

2. *The semigroup* $\exp(-t\hat{H}_Q^{\mu})$ *preserves spaces $\mathscr{E}_\Theta$ for* $Q\cap\mathrm{nul}(\Theta)=\varnothing$, *$l(\Theta)\leq n$ and Θ, to be quasi-accretive.*

3. *The semigroup* $\exp(-t\hat{H}_Q^{\mu})$ *acts from $\mathscr{E}_\Theta$ into $\mathscr{E}_{\Theta_i(Q)}$ as a continuous map. Here $i+l(\Theta)\leq n$, $Q\cap\mathrm{nul}(\Theta)=\varnothing$, and array Θ should be i-consistent.*

Proof. We only sketch the general scheme, because it completely follows the scheme of this paper. First recall that conditions (8.9) and (8.10) appear in (6.26).

The Cauchy problem (4.3) is smoothly solved in $C^{n+4}_{\text{cyl}}(M^{\mathbb{Z}^d})$ as in Theorem 4.1. The uniform quasi-accretivity in Theorem 5.6 for $l(\Theta) \leqslant n+4$ gives the multiplicative formula for semigroups in the space $\mathscr{E}_\Theta$ with $l(\Theta) = n$. Therefore we have Theorem 8.4 only for $i \leqslant n - l(\Theta)$. ∎

9. Appendix

Here we provide some facts and definitions of semigroup theory in Banach space X. It can be found in more details in [17], where the wide citation on this and connected questions is also given.

We also give a simple generalization of the Da Prato–Grisvard theorem [11] for the case of a finite number of operators sum (see Theorem A.5).

Definition A.1. The *strongly continuous semigroup* in Banach space X is a family of bounded operators $T = \{T(t), t \in \mathbb{R}_+\} \subset \mathscr{L}(X)$ which satisfies the conditions

1. $\forall t, s \in \mathbb{R}_+ : T(t)\, T(s) = T(t+s)$;
2. $T(0) = \mathrm{Id}$;
3. for every $f \in X$ the function $T(\cdot)f : \mathbb{R}_+ \to X$ is continuous.

Definition A.2. The strongly continuous semigroup T is said to be *contractive* if $\|T(t)\|_{\mathscr{L}(X)} \leqslant 1$, $t \in \mathbb{R}_+$. Correspondingly, T is said to be *quasi-contractive* if $\|T(t)\|_{\mathscr{L}(X)} \leqslant e^{Mt}$, $M > 0$.

Definition A.3. *Let* $\langle \cdot, \cdot \rangle$ denote the duality between X and dual space X^*. For every element $f \in X$ put

$$J(f) = \{\phi \in X^* : \|\phi\|^2 = \|f\|^2 = \langle f, \phi \rangle\}.$$

The set $J(f) \neq \varnothing$ for every $f \in X$ due to the Hann–Banach theorem.

Let j be some section of the set J so $j : X \to X^*$ and $j(f) \in J(f)$ for every $f \in X$. The map j is said to be the *duality mapping.*

Operator A is said to be *quasi-accretive* with respect to the duality mapping j if there is a constant $M \in \mathbb{R}^1$ such that

$$\forall A \in \mathscr{D}(A) : \Re e \langle (A+M) f, j(f) \rangle \geqslant 0.$$

The quasi-accretive operator is said to be *maximally quasi-accretive* if A does not have quasi-accretive extensions.

Remark A.4. The Hille-Yosida theorem in the Lumer–Phillips form implies that the maximally quasi-accretive operator A is a generator of the quasi-contractive semigroup $\exp(-tA)$ with no dependence on concrete duality mapping.

Moreover, the resolvent of operator A satisfies the estimate

$$\|(\lambda + A)^{-1}\|_{\mathscr{L}(X)} \leqslant 1/(\lambda - M)$$

for some constant $M > 0$.

THEOREM A.5. *Let X be a Banach space. Suppose that*

1. *Operators $A_1, ..., A_p$, B are closed operators in X and there are constants $\alpha_1, ..., \alpha_p$, $\beta > 0$ such that*

$$\begin{aligned} \|(\lambda + A_i)^{-1}\|_{\mathscr{L}(X)} &\leqslant 1/(\lambda - \alpha_i), \qquad \lambda > \alpha_i \\ \|(\lambda + B)^{-1}\|_{\mathscr{L}(X)} &\leqslant 1/(\lambda - \beta), \qquad \lambda > \beta. \end{aligned} \tag{9.1}$$

2. *There is a Banach space Y which is densely and continuously imbedded into X, $Y \subset \mathscr{D}(B)$ such that Y is densely and continuously imbedded into $\mathscr{D}(A_i^2)$ for $i = 1, ..., p$ with graph norm in X:*

$$\forall y \in Y: \|A_i^2 y\|_X \leqslant \text{const } \|y\|_Y.$$

3. *There are constants $\gamma_1, ..., \gamma_p$, $\delta > 0$ such that the restrictions of operators A_i, B onto Y satisfy the estimates*

$$\begin{aligned} \|(\lambda + A_i \upharpoonright_Y)^{-1}\|_{\mathscr{L}(Y)} &\leqslant 1/(\lambda - \gamma_i), \qquad \lambda > \gamma_i \\ \|(\lambda + B \upharpoonright_Y)^{-1}\|_{\mathscr{L}(Y)} &\leqslant 1/(\lambda - \delta), \qquad \lambda > \delta. \end{aligned}$$

Then the operator $L = A_1 + \cdots + A_p + B$ with domain $\mathscr{D}(L) = Y$ an essentially maximally quasi-accretive operator in space X.

Proof. We briefly outline the main steps of the proof.

1. Let A be a quasi-accretive generator of the strongly continuous semigroup $\exp(-tA)$ in space X. For sufficiently large $n \in \mathbb{N}$ the Yosida approximation of operator A is defined:

$$A^n = nA(n + A)^{-1} = n - n^2(n + A)^{-1}.$$

Then the estimate

$$\sup_{\|x\| + \|Ax\| + \|A^2x\| \leqslant R} \|A^n x - Ax\|_X = \sup_{\cdots} \|(n+A)^{-1} A^2 x\|_X$$
$$\leqslant R/(n-M) \to 0, \qquad n \to \infty$$

implies the convergence of the Yosida approximation uniformly on balls from $\mathscr{D}(A^2)$. Here $M > 0$ denotes the constant of quasi-accretivity for operator A.

2. Let $L_{n_1, \ldots, n_p}$ denote the operator

$$\begin{cases} \mathscr{D}(L_{n_1, \ldots, n_p}) = \mathscr{D}(B) \\ L_{n_1, \ldots, n_p} = A_1^{n_1} + \cdots + A_p^{n_p} + B, \end{cases} \tag{9.2}$$

where $A_i^{n_i}$ denotes the Yosida approximations of operators A_i.

Put $\omega_k = \beta + \alpha_1 + \cdots + \alpha_k$, then for every $\varepsilon > 0$ there is a constant $C_{\varepsilon,p} > 0$ such that for every $k \leqslant p$, $\lambda > \omega_k + \varepsilon$, and $n_1, \ldots, n_p \geqslant C_{\varepsilon,p}$ we have the estimate

$$\|(L_{n_1, \ldots, n_k} + \lambda)^{-1}\|_{\mathscr{L}(X)} \leqslant 1/(\lambda - (\omega_k + \varepsilon)).$$

This estimate is obtained easily from the iterative estimation of the equation

$$(L_{n_1, \ldots, n_{k+1}} + \lambda)^{-1} = (L_{n_1, \ldots, n_k} + \lambda + n_{k+1})^{-1}$$
$$\cdot \{1 - n_k^2 (A_k + n_k)^{-1} (L_{n_1, \ldots, n_k} + \lambda + n_k)^{-1}\}^{-1}.$$

3. As in Step 2 we have that $\forall \varepsilon > 0$ $\exists K_\varepsilon$ such that for $\lambda > \theta + \varepsilon$, $\theta = \gamma_1 + \cdots + \gamma_p + \beta$, we have the estimate

$$\|(L_{n_1, \ldots, n_p} + \lambda)^{-1}\|_{\mathscr{L}(Y)} \leqslant 1/(\lambda - (\theta + \varepsilon)). \tag{9.3}$$

4. Now we prove that $(L + \lambda) Y$ is dense in X for $\lambda > \theta + \varepsilon$. From (9.3) for $n_1, \ldots, n_p > K_\varepsilon$ we obtain that

$$(L_{n_1, \ldots, n_p} + \lambda)^{-1} Y \subset Y \cap \mathscr{D}(B) \subset \mathscr{D}(L). \tag{9.4}$$

So for $y \in Y$,

$$(L + \lambda)(L_{n_1, \ldots, n_p} + \lambda)^{-1} y - y$$
$$= \{(A_1 - A_1^{n_1}) + \cdots + (A_p - A_p^{n_p})\}(L_{n_1, \ldots, n_p} + \lambda)^{-1} y. \tag{9.5}$$

The estimate (9.3) implies that for fixed $y \in Y$ the sequence $\{(L_{n_1, \ldots, n_p} + \lambda)^{-1} y\}_{n_1 \cdots n_p \geqslant K_\varepsilon}$ is bounded in $\mathscr{D}(A_i^2)$, $i = 1, \ldots, p$, with graph norm.

Step 1 of the proof implies that

$$(A_i - A_i^{n_i})(L_{n_1, \ldots, n_p} + \lambda)^{-1} y \to 0$$

uniformly in X when $n_i \to \infty$ for every $i = 1, \ldots, p$.

Uniform convergence and (9.5) imply that for every $y \in Y$,

$$(L + \lambda)(L_{n_1, \ldots, n_p} + \lambda)^{-1} y \to y, \qquad \text{for} \quad n_1, \ldots, n_p \to \infty.$$

Using (9.4) we have that $\widetilde{(L+\lambda)\,Y} \supset Y$; therefore $(L+\lambda)\,Y$ is dense in X for some $\lambda > \theta + \varepsilon$. Here $\widetilde{(L+\lambda)\,Y}$ denotes the closure of set $(L+\lambda)\,Y$ in the space X.

The density of Y and so of $(L+\lambda)\,Y$ in X and the quasi-accretivity of operator L give the essential maximal quasi-accretivity of operator L, $\mathscr{D}(L) = Y$ (see the Hille–Yosida theorem in the Lumer–Phillips form [17]). ∎

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