Computers and Mathematics with Applications 64 (2012) 3183–3192

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Neumann boundary-value problems for a time-fractional diffusion-wave equation in a half-plane

Yuriy Povstenko

Institute of Mathematics and Computer Science, Jan Długosz University in Częstochowa, al. Armii Krajowej 13/15, 42-200 Częstochowa, Poland

ARTICLE INFO

Keywords:
Non-Fickean diffusion
Fractional calculus
Mittag-Leffler functions
Integral transforms

ABSTRACT

The time-fractional diffusion-wave equation with the Caputo derivative of the order $0 < \alpha < 2$ is considered in a half-plane. Two types of Neumann boundary condition are examined: the mathematical condition with the prescribed boundary value of the normal derivative and the physical one with the prescribed boundary value of the matter flux.

1. Introduction

The classical theory of diffusion is based on the balance equation for mass

$$\rho \frac{\partial c}{\partial t} = -\mathrm{div}\, \mathbf{j} \tag{1}$$

and the Fick law

$$\mathbf{j} = -k \,\mathrm{grad}\, c, \tag{2}$$

where c is a concentration, ρ the mass density, $\mathbf{j}$ the matter flux, and k the diffusion conductivity.

Combination of (1) and (2) leads to the standard parabolic diffusion equation

$$\frac{\partial c}{\partial t} = a\, \Delta c \tag{3}$$

with $a = k/\rho$ being the diffusivity coefficient.

Gurtin and Pipkin [1] and Day [2] proposed the time-nonlocal generalization of the constitutive equation (2)

$$\mathbf{j} = -k \int_0^t K(t-\tau) \mathrm{grad}\, c(\tau)\, \mathrm{d}\tau \tag{4}$$

which gives rise to the diffusion equation with memory [3,4]

$$\frac{\partial c}{\partial t} = a \int_0^t K(t-\tau)\, \Delta c\, \mathrm{d}\tau. \tag{5}$$

Different expressions for the kernel $K(t-\tau)$ describe different types of memory and lead to corresponding generalizations of the classical diffusion equation (3). For example, the time-nonlocal dependence between the matter flux vector $\mathbf{j}$ and the concentration gradient with the "long-tail" power kernel [5,6] (see also [7]):

$$\mathbf{j}(t) = -\frac{k}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t-\tau)^{\alpha-1} \mathrm{grad}\, c(\tau)\, \mathrm{d}\tau, \quad 0 < \alpha \le 1; \tag{6}$$

$$\mathbf{j}(t) = -\frac{k}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-2} \mathrm{grad}\, c(\tau)\, \mathrm{d}\tau, \quad 1 < \alpha \le 2, \tag{7}$$

E-mail address: j.povstenko@ajd.czest.pl.

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doi:10.1016/j.camwa.2012.02.064

where $\Gamma(\alpha)$ is the Gamma function, can be interpreted in terms of fractional calculus:

$$\mathbf{j}(t) = -kD_{\mathrm{RL}}^{1-\alpha}\operatorname{grad} c(t), \quad 0 < \alpha \le 1, \tag{8}$$

$$\mathbf{j}(t) = -kI^{\alpha-1}\operatorname{grad} c(t), \quad 1 < \alpha \le 2. \tag{9}$$

Here $I^{\alpha}f(t)$ and $D_{\mathrm{RL}}^{\alpha}f(t)$ are the Riemann–Liouville fractional integral and the derivative of the order α, respectively [8–10]:

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)\,\mathrm{d}\tau, \quad \alpha > 0. \tag{10}$$

$$D_{\mathrm{RL}}^{\alpha}f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}f(\tau)\,\mathrm{d}\tau\right], \quad n-1 < \alpha < n. \tag{11}$$

It should be noted that in fractional calculus, where integrals and derivatives of arbitrary (not integer) order are considered, there is no sharp boundary between integration and differentiation. For this reason, some authors [11,12] do not use a separate notation for the fractional integral $I^{\alpha}f(t)$. The fractional integral of the order $\alpha > 0$ is denoted as $D^{-\alpha}f(t)$. Using this notation Eqs. (8) and (9) can be rewritten as one dependence:

$$\mathbf{j}(t) = -kD_{\mathrm{RL}}^{1-\alpha}\operatorname{grad} c(t), \quad 0 < \alpha \le 2. \tag{12}$$

In the case $0 < \alpha \le 1$, as a consequence of Eqs. (1), (8) and (11), we get

$$\frac{\partial c}{\partial t} = a\frac{\partial}{\partial t}\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\Delta c(\tau)\,\mathrm{d}\tau\right] \tag{13}$$

or, after integration with respect to time,

$$c(t) - c(0) = a\,\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\Delta c(\tau)\,\mathrm{d}\tau = aI^{\alpha}\,\Delta c. \tag{14}$$

Now we apply to both sides of (14) the Caputo fractional derivative

$$\frac{\partial^{\alpha}f(t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}\,\mathrm{d}\tau, \quad n-1 < \alpha < n, \tag{15}$$

which is connected with the Riemann–Liouville derivative by the formula obtained in [9]:

$$\frac{\partial^{\alpha}f(t)}{\partial t^{\alpha}} = D_{\mathrm{RL}}^{\alpha}f(t) - \sum_{k=0}^{n-1}\frac{t^{k-\alpha}}{\Gamma(1+k-\alpha)}f^{(k)}(0^+), \quad n-1 < \alpha < n. \tag{16}$$

Thus, we get

$$\frac{\partial^{\alpha}c(t)}{\partial t^{\alpha}} - \frac{\partial^{\alpha}c(0)}{\partial t^{\alpha}} = a\frac{\partial^{\alpha}}{\partial t^{\alpha}}I^{\alpha}\,\Delta c. \tag{17}$$

It should be emphasized that the Caputo derivative of a constant is zero

$$\frac{\partial^{\alpha}c(0)}{\partial t^{\alpha}} = 0 \tag{18}$$

and for $\alpha > 0$ the following equality holds [10]:

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}I^{\alpha}f(t) = f(t). \tag{19}$$

Hence, Eq. (17) is rewritten as

$$\frac{\partial^{\alpha}c}{\partial t^{\alpha}} = a\,\Delta c, \quad 0 < \alpha \le 1. \tag{20}$$

In the case $1 < \alpha \le 2$, Eqs. (1) and (9) give

$$\frac{\partial c}{\partial t} = a\,I^{\alpha-1}\Delta c \tag{21}$$

or, after applying $\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}$ to the both sides of (21),

$$\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}\frac{\partial c}{\partial t} = a\,\Delta c. \tag{22}$$

It should be noted that in the general case

$$\frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\beta f(t)}{\partial t^\beta} \neq \frac{\partial^{\alpha+\beta} f(t)}{\partial t^{\alpha+\beta}}, \tag{23}$$

but for integer $\beta = m$ we have [11]

$$\frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^m f(t)}{\partial t^m} = \frac{\partial^{\alpha+m} f(t)}{\partial t^{\alpha+m}}. \tag{24}$$

From Eqs. (22) and (24) we obtain

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\,\Delta c, \quad 1 < \alpha \le 2. \tag{25}$$

A comparison of Eqs. (20) and (25) shows that the time-nonlocal constitutive equation (12) yields the time-nonlocal diffusion-wave equation with the Caputo fractional derivative of order α:

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\,\Delta c, \quad 0 < \alpha \le 2. \tag{26}$$

Eq. (26) covers the whole spectrum from the so-called localized diffusion (the Helmholtz equation when the order of the time derivative $\alpha \to 0$) through the standard diffusion equation (represented by the particular case $\alpha = 1$) to the ballistic diffusion (the wave equation when $\alpha = 2$). The fractional diffusion-wave equation describes important physical phenomena in amorphous, colloid, glassy and porous materials, in fractals and percolation clusters, comb structures, dielectrics and semiconductors, biological systems, polymers, random and disordered media, and geophysical and geological processes (see [5,12–20] and references therein).

Starting from the pioneering papers [21–25], considerable interest has been shown in solutions to time-fractional diffusion-wave equations. For additional references see [10,26,27].

If Eq. (26) is considered in a bounded domain, the corresponding boundary conditions should be imposed. The Dirichlet boundary condition specifies the concentration over the surface of the body

$$c|_S = g(\mathbf{x}_S, t). \tag{27}$$

For a fractional diffusion equation, two types of Neumann boundary condition can be considered: the mathematical condition with the prescribed boundary value of the normal derivative

$$\left.\frac{\partial c}{\partial n}\right|_S = G(\mathbf{x}_S, t) \tag{28}$$

and the physical condition with the prescribed boundary value of the matter flux

$$D_{\mathrm{RL}}^{1-\alpha}\left.\frac{\partial c}{\partial n}\right|_S = G(\mathbf{x}_S, t), \quad 0 < \alpha \le 2. \tag{29}$$

In the case of the classical diffusion equation ($\alpha = 1$), these two kinds of boundary conditions are identical, but for the fractional diffusion equation ($\alpha \neq 1$) they are essentially different.

In the present paper, the solutions of a time-fractional diffusion-wave equation are investigated in a half-plane.

The aim of the paper is to show the difference between the solutions with two types of Neumann boundary condition (28) and (29). Several examples of problems are solved using the Laplace integral transform with respect to time and the Fourier transforms with respect to spatial coordinates. The inverse Laplace transform is expressed in terms of the Mittag-Leffler functions. After inversion of Fourier integral transforms the solution is converted into a form amenable to numerical treatment. For the first- and second-time-derivative terms, the solutions obtained reduce to the solutions of the ordinary diffusion and wave equations. Numerical results are illustrated graphically.

2. The fundamental solutions to the Neumann boundary-value problem

Consider the time-fractional diffusion-wave equation in a half-plane

$$\frac{\partial^\alpha c}{\partial t^\alpha} = a\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2}\right), \quad 0 < x < \infty,\ -\infty < y < \infty,\ 0 < t < \infty,\ 0 < \alpha \le 2, \tag{30}$$

under zero initial conditions

$$t = 0: \qquad c = 0, \quad 0 < \alpha \le 2, \tag{31}$$

$$t = 0: \qquad \frac{\partial c}{\partial t} = 0, \quad 1 < \alpha \le 2. \tag{32}$$

We adopt as a boundary condition the physical Neumann boundary condition in the form

$$x = 0: \qquad j = P_0\, \delta(y)\, \delta_+(t), \tag{33}$$

where $\delta(y)$ is the Dirac delta function, and the matter flux j is defined according to Eq. (12):

$$j = -kD_{\mathrm{RL}}^{1-\alpha} \frac{\partial c}{\partial x}, \quad 0 < \alpha \le 2. \tag{34}$$

In Eq. (33), the constant multiplier P_0 has been introduced to obtain the nondimensional quantity $\bar{c}$ (see Eq. (49)) displayed in the figures.

The zero conditions at infinity are also assumed:

$$\lim_{x\to\infty} c(x, y, t) = 0, \qquad \lim_{y\to\pm\infty} c(x, y, t) = 0. \tag{35}$$

To solve the problem considered the Laplace integral transform will be used. Recall that the Laplace transform rules for fractional integrals and derivatives have the following form [9,10]:

$$\mathcal{L}\{I^\alpha f(t)\} = \frac{1}{s^\alpha} f^*(s), \tag{36}$$

$$\mathcal{L}\left\{D_{\mathrm{RL}}^\alpha f(t)\right\} = s^\alpha f^*(s) - \sum_{k=0}^{n-1} D^k I^{n-\alpha} f(0^+) s^{n-1-k}, \quad n-1 < \alpha < n, \tag{37}$$

$$\mathcal{L}\left\{D_{C}^\alpha f(t)\right\} = s^\alpha f^*(s) - \sum_{k=0}^{n-1} f^{k}(0^+) s^{\alpha-1-k}, \quad n-1 < \alpha < n, \tag{38}$$

where the asterisk denotes the transform, and s is the transform variable.

Applying the Laplace transform to Eq. (30) while taking into account the zero initial conditions (31) and (32) and the rules (36)–(38) leads to

$$s^\alpha c^* = a\left(\frac{\partial^2 c^*}{\partial x^2} + \frac{\partial^2 c^*}{\partial y^2}\right), \tag{39}$$

$$x = 0: \qquad -ks^{1-\alpha} \frac{\partial c^*}{\partial x} = P_0 \delta(y), \quad 0 < \alpha \le 2. \tag{40}$$

Next, the cosine Fourier transform with respect to the spatial coordinate x and the exponential Fourier transform with respect to the spatial coordinate y result in

$$c^{***} = \frac{aP_0}{\sqrt{2\pi}k} \frac{s^{\alpha-1}}{s^\alpha + a(\xi^2 + \eta^2)}, \tag{41}$$

where the standard formula for the cosine Fourier transform of the second derivative has been used:

$$\mathcal{F}\left\{\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2}\right\} = -\xi^2 f^*(\xi) - \left.\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right|_{x=0}. \tag{42}$$

In Eq. (41) each of the integral transforms is denoted by an asterisk, and s, ξ and η are the transform variables.

To invert the Laplace transform the following formula [9,10] is used:

$$\mathcal{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^\alpha + b}\right\} = t^{\beta-1} E_{\alpha,\beta}(-bt^\alpha), \tag{43}$$

where $E_{\alpha,\beta}(z)$ is the generalized Mittag-Leffler function in two parameters α and β:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0,\ \beta > 0,\ z \in C. \tag{44}$$

Inversion of all the integral transforms gives

$$c = \frac{aP_0}{\pi^2 k} \int_{-\infty}^{\infty} \cos(y\eta)\, \mathrm{d}\eta \int_0^{\infty} E_\alpha[-a(\xi^2 + \eta^2)t^\alpha]\, \cos(x\xi)\, \mathrm{d}\xi. \tag{45}$$

Solution (45) is inconvenient for numerical treatment. To obtain a solution amenable for numerical calculations, we pass to the polar coordinates in the (ξ, η)-plane and in the (x, y)-plane: $\xi = \rho\cos\vartheta$, $\eta = \rho\sin\vartheta$, $x = r\cos\varphi$, $y = r\sin\varphi$.

Thus we get

$$c = \frac{aP_0}{\pi^2 k}\int_0^\infty E_\alpha(-a\rho^2 t^\alpha)\,\rho\,\mathrm{d}\rho\int_{-\pi/2}^{\pi/2}\cos(x\rho\cos\vartheta)\cos(y\rho\sin\vartheta)\,\mathrm{d}\vartheta. \tag{46}$$

Substitution of $v = \sin\vartheta$ taking into account the following integral [28]:

$$\int_0^1 \frac{1}{\sqrt{1-v^2}}\cos(p\sqrt{1-v^2})\cos qv\,\mathrm{d}v = \frac{\pi}{2}J_0\left(\sqrt{p^2+q^2}\right) \tag{47}$$

gives

$$\bar{c} = \frac{1}{\pi}\int_0^\infty E_\alpha(-\sigma^2)J_0(\bar{r}\sigma)\,\sigma\,\mathrm{d}\sigma, \tag{48}$$

where $J_0(r)$ is the Bessel function of the first kind, and the nondimensional quantities

$$\bar{c} = \frac{kt^\alpha}{P_0}c, \qquad \bar{r} = \frac{r}{\sqrt{a}t^{\alpha/2}}, \qquad \sigma = \rho\sqrt{a}t^{\alpha/2} \tag{49}$$

have been introduced.

Let us analyze several particular cases. For the Helmholtz equation ($\alpha \to 0$) we have $E_0(-\sigma^2) = \frac{1}{1+\sigma^2}$ and

$$\bar{c} = \frac{1}{\pi}K_0(\bar{r}). \tag{50}$$

Here $K_0(r)$ is the modified Bessel function of the third kind.

The well-known solution to the classical diffusion equation corresponding to $\alpha = 1$ and $E_1(-\sigma^2) = \exp(-\sigma^2)$ reads

$$\bar{c} = \frac{1}{2\pi}\exp\left(-\frac{\bar{r}^2}{4}\right). \tag{51}$$

The subdiffusion with $\alpha = 1/2$ leads to

$$\bar{c} = \frac{1}{2\pi^{3/2}}\int_0^\infty \frac{1}{u}\exp\left(-u^2 - \frac{\bar{r}^2}{8u}\right)\mathrm{d}u, \tag{52}$$

where the integral representation

$$E_{1/2}(-\sigma^2) = \frac{2}{\sqrt{\pi}}\int_0^\infty \exp\left(-u^2 - 2\sigma^2 u\right)\mathrm{d}u$$

has been used.

The solution to the standard wave equation ($\alpha = 2$ and $E_2(-\sigma^2) = \cos\sigma$) has the following form [29]:

$$c = \frac{P_0}{\pi\sqrt{a}}\frac{\partial}{\partial t}\frac{H(\sqrt{a}t - r)}{\sqrt{at^2 - r^2}}, \tag{53}$$

where $H(x)$ is the Heaviside step function.

For better understanding of the behavior of solution (48) consider the representation of the Mittag-Leffler function $E_\alpha(x)$ for large negative values of the argument [9,12]:

$$E_\alpha(-x) \sim \frac{1}{\Gamma(1-\alpha)}\frac{1}{x}, \quad x \to \infty,\ 0 < \alpha < 2,\ \alpha \neq 1. \tag{54}$$

Hence, only the fundamental solution to the classical diffusion equation ($\alpha = 1$) has no singularity at the origin. To investigate the type of singularity we rewrite Eq. (48) in the following form:

$$\bar{c} = \frac{1}{\pi}\int_0^\infty \left[E_\alpha(-\sigma^2) - \frac{1}{\Gamma(1-\alpha)}\frac{1}{(1+\sigma^2)}\right]J_0(\bar{r}\sigma)\,\sigma\,d\sigma + \frac{1}{\pi\Gamma(1-\alpha)}\int_0^\infty \frac{1}{1+\sigma^2}J_0(\bar{r}\sigma)\,\sigma\,d\sigma. \tag{55}$$

The first integral in (55) has no singularity at the origin, while the second one can be calculated analytically and yields the logarithmic singularity at the origin:

$$\bar{c} \sim \frac{1}{\pi\Gamma(1-\alpha)}K_0(\bar{r}), \quad 0 < \alpha < 2,\ \alpha \neq 1, \tag{56}$$

or

$$\bar{c} \sim -\frac{1}{\pi \Gamma(1-\alpha)} \ln \bar{r}, \quad 0 < \alpha < 2, \ \alpha \neq 1. \tag{57}$$

Comparison of (50) and (56) allows us to substitute the condition $0 < \alpha < 2$ by $0 \leq \alpha < 2$.

It should be noted that the fundamental solution to the Cauchy problem for the diffusion-wave equation in the whole plane with the initial condition

$$t = 0: \qquad c = \frac{p_0}{2\pi r} \delta_+(r)$$

is half of the solution (48) (see [30]). The corresponding estimation

$$c \sim -\frac{p_0}{2\pi a t^{\alpha} \Gamma(1-\alpha)} \ln \bar{r} \tag{58}$$

is consistent with the behavior of the solution for small $\bar{r}$ obtained by Schneider [31].

Next we consider the fundamental solution to the initial–boundary-value problem (30)–(32) with the following boundary condition:

$$x = 0: \qquad \frac{\partial c}{\partial x} = -W_0 \, \delta(y) \, \delta_+(t), \tag{59}$$

where $W_0 = \text{const}$.

This problem was discussed in detail in [26]. For completeness and comparison of the solutions, we briefly recall the main result. The integral transforms allow us to obtain

$$c^* = \frac{a W_0}{\sqrt{2\pi}} \frac{1}{s^{\alpha} + a(\xi^2 + \eta^2)}, \tag{60}$$

and after inversion we get

$$\bar{c} = \frac{1}{\pi} \int_0^{\infty} E_{\alpha,\alpha}(-\sigma^2) J_0(\bar{r}\sigma) \, \sigma \, d\sigma \tag{61}$$

with $\bar{c} = tc/W_0$.

In the case of the standard diffusion equation, the solution (51) is obtained. For the wave equation we have

$$\bar{c} = \begin{cases} \dfrac{1}{\pi} \dfrac{1}{\sqrt{1-\bar{r}^2}}, & 0 \leq \bar{r} < 1, \\ 0, & 1 < \bar{r} < \infty. \end{cases} \tag{62}$$

The dependence of the nondimensional function $\bar{c}$ on the nondimensional radial coordinate $\bar{r}$ calculated from Eq. (48) for various values of the order of the fractional derivative α is shown in Fig. 1. Similarly, Fig. 2 presents the curves obtained from Eq. (61).

3. The constant boundary value in a local domain

Of special interest is the initial–boundary-value problem (30)–(32) with the constant boundary value of a flux in the domain $|y| < l$:

$$x = 0: \qquad j = \begin{cases} j_0, & |y| < l, \\ 0, & |y| > l. \end{cases} \tag{63}$$

The integral transforms technique results in the following expression:

$$c^* = \frac{2 a j_0 \sin(l\eta)}{\sqrt{2\pi} k \eta} \frac{s^{\alpha-2}}{s^{\alpha} + a(\xi^2 + \eta^2)}. \tag{64}$$

Inverting the transforms, we obtain

$$c = \frac{2 a j_0 t}{\pi^2 k} \int_{-\infty}^{\infty} \frac{\sin(l\eta)}{\eta} \cos(y\eta) \, d\eta \int_0^{\infty} E_{\alpha,2}[-a(\xi^2 + \eta^2) t^{\alpha}] \cos(x\xi) \, d\xi. \tag{65}$$

After passing to the polar coordinates and taking into account the integral

$$\int_0^1 \frac{1}{v\sqrt{1-v^2}} \cos(p\sqrt{1-v^2}) \sin qv \, dv = \frac{\pi}{2} \int_0^q J_0\left(\sqrt{p^2 + u^2}\right) du, \quad q > 0, \tag{66}$$

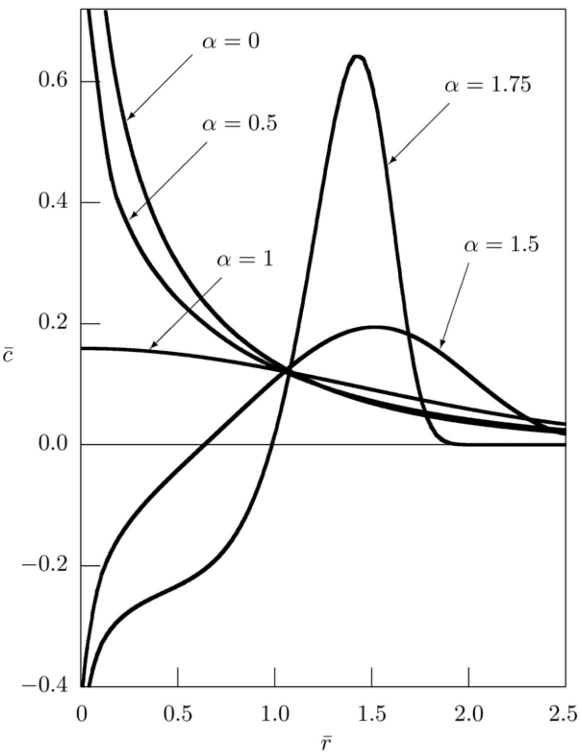

Fig. 1. Dependence of the solution on the distance for various values of the order of the fractional derivative (the fundamental solution with the prescribed boundary flux).

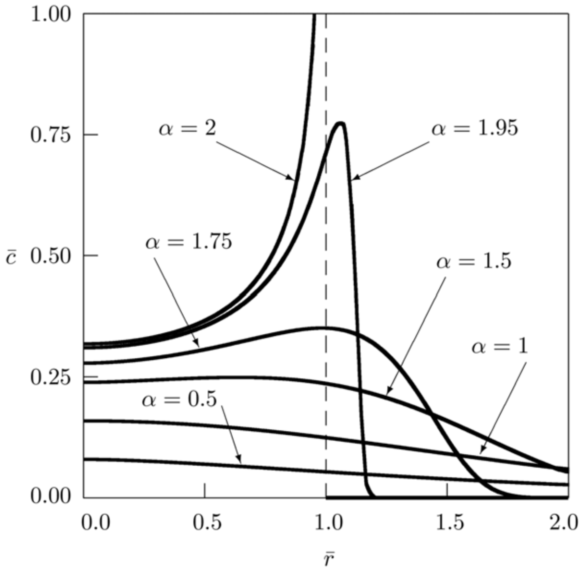

Fig. 2. Dependence of the solution on the distance for the various values of the order of the fractional derivative (the fundamental solution with the prescribed normal derivative).

we arrive at

$$\bar{c} = \frac{1}{\pi} \int_0^\infty E_{\alpha,2}(-\kappa^2\sigma^2)\,\sigma\,\mathrm{d}\sigma \int_{\bar{y}-1}^{\bar{y}+1} J_0\left(\sigma\sqrt{u^2+\bar{x}^2}\right)\mathrm{d}u, \tag{67}$$

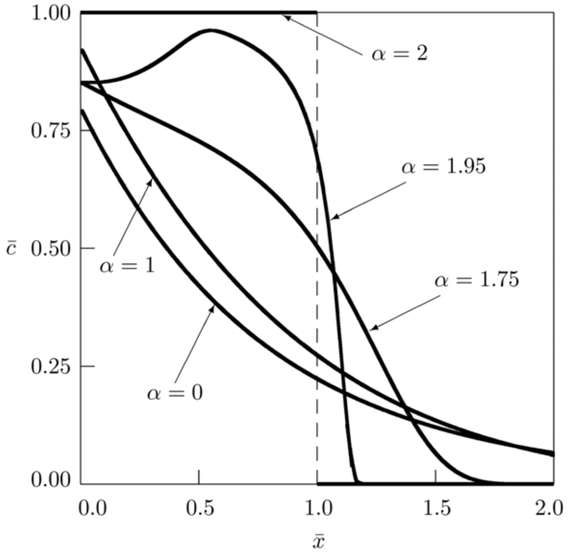

Fig. 3. Dependence of the solution on the distance for various values of the order of the fractional derivative (the constant boundary value of flux in the local domain).

where

$$\bar{c} = \frac{kl}{aj_0 t}\, c, \qquad \bar{x} = \frac{x}{l}, \qquad \bar{y} = \frac{y}{l}, \qquad \kappa = \frac{\sqrt{a} t^{\alpha/2}}{l}. \tag{68}$$

The Neumann problem with the constant boundary value of the normal derivative in the local domain $|y| < l$

$$x = 0: \qquad \frac{\partial c}{\partial x} = \begin{cases} -w_0, & |y| < l, \\ 0, & |y| > l, \end{cases} \tag{69}$$

was considered in [26]. Here we present only the final result. The integral transforms lead to

$$c^* = \frac{2aw_0 \sin(l\eta)}{\sqrt{2\pi}\,\eta} \, \frac{1}{s[s^\alpha + a(\xi^2 + \eta^2)]}. \tag{70}$$

As

$$\frac{1}{s[s^\alpha + a(\xi^2 + \eta^2)]} = \frac{1}{a(\xi^2 + \eta^2)} \left[\frac{1}{s} - \frac{s^{\alpha - 1}}{s^\alpha + a(\xi^2 + \eta^2)} \right], \tag{71}$$

the solution reads

$$c = \frac{2w_0}{\pi^2} \int_{-\infty}^{\infty} \frac{\sin(l\eta)}{\eta} \cos(y\eta)\, d\eta \int_0^\infty \frac{\cos(x\xi)}{\xi^2 + \eta^2} \left\{ 1 - E_\alpha[-a(\xi^2 + \eta^2) t^\alpha] \right\} d\xi \tag{72}$$

or, after passing to the polar coordinates,

$$\bar{c} = \frac{1}{\pi} \int_0^\infty \left[1 - E_\alpha(-\kappa^2 \sigma^2) \right] \frac{1}{\sigma}\, d\sigma \int_{\bar{y}-1}^{\bar{y}+1} J_0\left(\sigma \sqrt{u^2 + \bar{x}^2} \right) du, \tag{73}$$

where $\bar{c} = c/(w_0 l)$, and $\bar{x}$, $\bar{y}$ and κ are described by (68).

The dependence of the nondimensional solution $\bar{c}$ on the nondimensional distance $\bar{x}$ calculated from Eq. (67) is shown in Fig. 3. The plots of $\bar{c}$ versus $\bar{x}$ are depicted in Fig. 4 according to Eq. (73). In both cases $\bar{y} = 0$ and $\kappa = 1$.

4. Numerical results and discussion

The solutions of time-fractional diffusion-wave equation in a half-plane have been studied under the mathematical and physical Neumann boundary conditions. The solutions obtained satisfy the appropriate conditions at the boundary $x = 0$ and reduce to the solutions of the classical diffusion equation in the limit $\alpha = 1$. In the case $0 < \alpha < 1$ the time-fractional diffusion equation interpolates between the Helmholtz equation and the diffusion equation. In this case, when it is possible to consider the limit $\alpha \to 0$, the solutions obtained reduce to the solutions of the Helmholtz equation. In the case $1 < \alpha < 2$

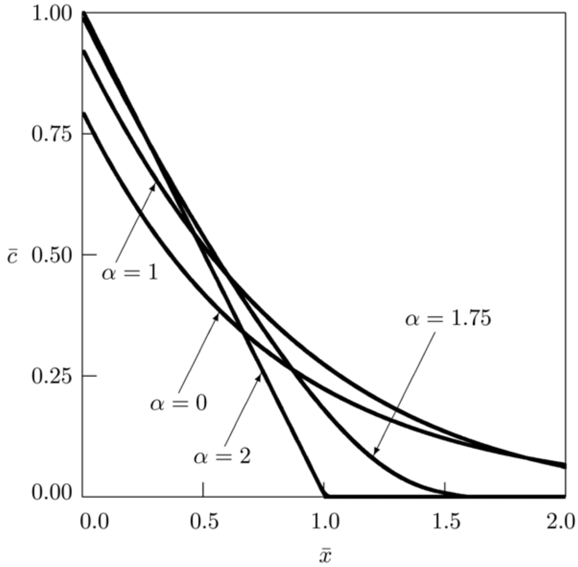

Fig. 4. Dependence of the solution on the distance for various values of the order of the fractional derivative (the constant boundary value of the normal derivative in the local domain).

the time-fractional diffusion equation interpolates between the diffusion equation and the wave equation. In the limit $\alpha = 2$ the solutions obtained reduce to the solutions of the wave equation.

Numerical calculations were carried out using a computer program written in FORTRAN. For calculations of the Mittag-Leffler functions of negative argument, algorithms proposed in [9,32,33] have been used:

$$E_{\alpha,\alpha}(-x) = \begin{cases} x^{1/\alpha-1}\,\Phi_\alpha^{(0)}(x), & 0 < \alpha < 1, \\ x^{1/\alpha-1}\left[\Phi_\alpha^{(0)}(x) - \Psi_\alpha^{(0)}(x)\right], & 1 < \alpha < 2; \end{cases}$$

$$E_{\alpha}(-x) = \begin{cases} \Phi_\alpha^{(1)}(x), & 0 < \alpha < 1, \\ \Phi_\alpha^{(1)}(x) + \Psi_\alpha^{(1)}(x), & 1 < \alpha < 2; \end{cases}$$

$$E_{\alpha,2}(-x) = x^{-1/\alpha}\left[-\Phi_\alpha^{(2)}(x) + \Psi_\alpha^{(2)}(x)\right], \quad 1 < \alpha < 2,$$

where

$$\Phi_\alpha^{(m)}(x) = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty e^{-ux^{1/\alpha}}\,\frac{u^{\alpha-m}}{u^{2\alpha} + 2u^\alpha\cos(\alpha\pi) + 1}\,dx, \quad m = 0, 1, 2,$$

$$\Psi_\alpha^{(m)}(x) = \frac{2}{\alpha}\,e^{x^{1/\alpha}\cos(\pi/\alpha)}\cos\left[x^{1/\alpha}\sin\left(\frac{\pi}{\alpha}\right) - (m-1)\frac{\pi}{\alpha}\right], \quad m = 0, 1, 2.$$

It can be seen from the figures that the solutions of the fractional diffusion equation in the case $1 < \alpha < 2$ approximate propagating steps and humps typical for the standard wave equation in contrast to the shape of curves describing the slow diffusion regime. In particular, it is evident from the figures how wave fronts arising in the case of the wave equation are approximated by the solutions of the time-fractional diffusion-wave equation with α approaching the value 2. Comparison of Figs. 3 and 4 shows that for $0 < \alpha \le 1$ the difference between the solutions under mathematical and physical Neumann boundary conditions is practically negligible, but in the case $1 < \alpha \le 2$ the solutions become quite different. From Figs. 1 and 2 it is clear that in the case of fundamental solutions, the curves are also quite different, both for $0 < \alpha < 1$ and for $1 < \alpha < 2$.

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