Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 222 (2008) 274–283
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Simulation of the continuous time random walk of the space-fractional diffusion equations

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Received 6 January 2007; received in revised form 10 September 2007

Abstract

In this article, we discuss the solution of the space-fractional diffusion equation with and without central linear drift in the Fourier domain and show the strong connection between it and the α-stable Lévy distribution, $0 < \alpha < 2$. We use some relevant transformations of the independent variables x and t, to find the solution of the space-fractional diffusion equation with central linear drift which is a special form of the space-fractional Fokker–Planck equation which is useful in studying the dynamic behaviour of stochastic differential equations driven by the non-Gaussian (Lévy) noises. We simulate the continuous time random walk of these models by using the Monte Carlo method.

MSC: 26A33; 45K05; 60J60; 60G50; 60G15; 65N06; 80-99

Keywords: Fractional diffusion; Space-Fractional derivative; Fokker–Planck equation; Stochastic processes; α-stable distribution; Continuous time random walk; Monte Carlo method

1. Introduction

In the continuous time random walk (*CTRW*), see Montroll and Weiss [21], a walker (particle) is starting at time $t = 0$ and at a given point x. At every time step t_n, $n \geq 0$, the particle is staying fixed with a waiting time of random length then makes an instantaneous space step in a random direction which may be also of random length, depending on the considered model. There are two different types of CTRW. The first type considers independent time and space steps (*decoupled or separable CTRW)* in which the time and space steps are independent identical distributed random (*iid*) variables, also the waiting time and the jumps are independent on each others. The second type is called the non-separable (*coupled CTRW*) in which the space and time steps are dependent on each others. In the long time and large distance limit the decoupled model of iid variables goes over into the fractional diffusion equation by a properly scaled passage to the limit of vanishing space and time steps, see Gorenflo et al. [12–14,9].

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.10.052

We focus in the decoupled CTRW and its relation to the fractional diffusion equation (*space-FDE*) and henceforth to the space-fractional diffusion equation with central linear drift (*space-FDECLD*), being an especial form of the space-fractional Fokker–Planck equation (*space-FFPE*). The Fokker–Planck equation (*FPE*) is one of the most widely used equations of statistical physics for studying the dynamic behaviour of stochastic differential equations driven by the Gaussian noises. Usually the FPE can be derived following the Langevin stochastic differential equation because it describes how a collection of initial data evolves in time, for information about the FPE and the Brownian motion see [7,22]. However, it turns out that many physical phenomena are outside this frame work. Especially solutes that move through fractal media commonly exhibit large deviations from the stochastic processes of Brownian motion and do not require finite velocity, e.g. diffusion by geophysical turbulence and chaotic systems, see [23,25]. The extension to Lévy stable motion is a straightforward generalization due to the common properties of Lévy stable motion and Brownian motion, see [24], but the Lévy flights differ from the regular Brownian motion by the occurrence of extremely long jumps whose length is distributed according to the Lévy long tail $\sim |x|^{-1-\alpha}$, $0 < \alpha < 2$. We are interested in such processes in which the probability distributions of the waiting time is characterized by e^{-t}, i.e. a Markov Process, and the probability distribution of the jumps has fat tails $\in (0, 2)$. We will simulate the continuous time random walk of the *space-FDE* and *space-FDECLD* for different values of the fractional order $\alpha \in (0, 2]$.

Let $u(x, t)$ be the solution of the space-fractional diffusion equation (*space-FDE*)

$$\frac{\partial u(x,t)}{\partial t} = a \underset{x\ \theta}{D^{\alpha}} u(x,t), \quad 0 < \alpha \le 2, \qquad u(x,0) = \delta(x), \tag{1.1}$$

where $u(x, t)$ represents a transition probability density function satisfying the conservation property

$$u(x,t) \ge 0, \qquad \int_{-\infty}^{\infty} u(x,t)\mathrm{d}x = 1, \quad \forall t \ge 0.$$

In Eq. (1.1), $a > 0$ is the diffusion constant and $\underset{x\ \theta}{D^{\alpha}}$ is the asymmetric Riesz–Feller fractional derivative of order α and skewness θ. This pseudo-differential operator has the symbol $-|\kappa|^{\alpha}\mathrm{i}^{\theta \operatorname{sig}(\kappa)}$, where $\operatorname{sig}(\kappa) = \kappa/|\kappa| = -1, 0,$ or 1, depending on $\kappa < 0, = 0$, or > 0 respectively and the skewness parameter θ is restricted to the region $|\theta| \le \min\{\alpha, (2-\alpha)\}$, depending on α, see [5] and [26], as

$$\begin{cases} |\theta| \le \alpha, & \text{if } 0 < \alpha \le 1, \\ |\theta| \le 2-\alpha, & \text{if } 1 < \alpha \le 2. \end{cases} \tag{1.2}$$

In the Fourier domain $\underset{x\ \theta}{D^{\alpha}}$ reads

$$\mathcal{F}\left\{ \underset{x\ 0}{D^{\alpha}} f(x); \kappa \right\} = -|\kappa|^{\alpha}\mathrm{i}^{\theta \operatorname{sig}(\kappa)} \widehat{f}(\kappa), \quad \kappa \in \mathbb{R}, \tag{1.3}$$

where the Fourier transform of a (generalized) function $f(x)$, $x \in \mathbb{R}$, is

$$\mathcal{F}\{f(x); \kappa\} = \widehat{f}(\kappa) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\kappa x} f(x)\mathrm{d}x, \quad x \in \mathbb{R}. \tag{1.4}$$

The space-fractional operator $\underset{x\ \theta}{D^{\alpha}}$ is a generalization of the space-fractional operator $\underset{x\ 0}{D^{\alpha}}$ by admitting the parameter θ of asymmetry (skewness), see Feller [5] and Gorenflo and Mainardi [20,12,13,11]. $\underset{x\ 0}{D^{\alpha}}$ is called the *Riesz* symmetric space-fractional derivative operator, namely the linear pseudo-differential operator with symbol $-|\kappa|^{\alpha}$, see e.g. [5], i.e.

$$\mathcal{F}\left\{ \underset{x\ 0}{D^{\alpha}} f(x); \kappa \right\} = -|\kappa|^{\alpha} \widehat{f}(\kappa), \quad \kappa \in \mathbb{R}, \tag{1.5}$$

We have used the definition of the pseudo-differential operator A which acts with respect to the variable $x \in \mathbb{R}$ on a sufficiently well-behaved function $\phi(x)$ and is defined through its Fourier representation, see [15], namely

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\kappa x} A((\phi)(x))\mathrm{d}x = \widehat{A}(\kappa)\widehat{\phi}(\kappa).$$

Here $\widehat{A}(\kappa)$ represents the symbol of A, given as $\widehat{A}(\kappa) = (A\mathrm{e}^{-\mathrm{i}\kappa x})\mathrm{e}^{+\mathrm{i}\kappa x}$. We define $F(x)$ as $F(x) = -\frac{\mathrm{d}}{\mathrm{d}x}U(x)$ to be the drift directed towards the origin where $U(x)$ is defined as a symmetric differentiable potential (i.e. $U(x) = U(-x)$) and increasing for $x > 0$. The most essential restriction is that $F(x)$ must be an odd and non-negative function for $x > 0$. Many forms of $U(x)$ can affect the diffusive particle to be attracted by the origin so they can also be studied. For different types of Lévy oscillators characterized by the potentials, see [3]. We consider only the harmonic oscillator, namely $U(x) = -b\frac{x^2}{2}$, $b > 0$, where b is called the drift term. To avoid confusing with Eq. (1.1) we use (ξ, τ), to be the pair of the independent variables, v to be the dependent variable and the asymmetric Riesz pseudo-differential operator ${}_{\xi}D_{\theta}^{\alpha}$. Therefore Eq. (1.1) takes the form

$$\frac{\partial v(\xi,\tau)}{\partial \tau} = b\frac{\partial}{\partial \xi}(\xi v(\xi,\tau)) + a\,{}_{\xi}D_{\theta}^{\alpha} v(\xi,\tau), \quad 0 < \alpha \le 2. \tag{1.6}$$

This equation represents the action of a linear force acting towards the origin on the motion of a free particle. Eq. (1.6) is a special form of the space-fractional Fokker–Planck equation (*space-FFPE*) and is also known as the space-fractional diffusion equation with central linear drift (*space-FDECLD*), see [1,10] where the stochastic process modelled by it is still Markovian. Eq. (1.1) represents the case $U(x) = c$, where c is a positive constant, i.e. $F(x) = 0$. *This paper is organized as follows*: In Section 2, we discuss the relation between the solutions of the space-FDE and the space-FDECLD in the Fourier domain. In Section 3, the simulation of the CTRW is interpreted for different fractional orders of the FDE and of the FDECLD. In Section 4 the numerical results are displayed and interpreted.

2. Solutions of the space-FDE and space-FDECLD in the Fourier domain

The aim of this section is to prove that $\widehat{u}(\kappa, t;\theta)$ is a characteristic function of an α-stable probability density. To do so, we take the Fourier transform of Eq. (1.1) and solve the resulting ordinary differential equation, we obtain

$$\widehat{u}(\kappa,t;\theta) = \exp[-t|\kappa|^{\alpha}\mathrm{e}^{\frac{\mathrm{i}\theta\pi}{2}\operatorname{sig}(\kappa)}], \quad \widehat{u}(\kappa,0) = 1, \tag{2.1}$$

for the strictly stable distributions, we use the parameterization of Feller for the characteristic functions, see Appendix A, namely

$$\widehat{p}_{\alpha}(\kappa;\theta) = \exp[-|\kappa|^{\alpha}\mathrm{e}^{\frac{\mathrm{i}\theta\pi}{2}\operatorname{sig}(\kappa)}]. \tag{2.2}$$

We recognize that $p_{\alpha}(x;\theta) = p_{\alpha}(-x;-\theta)$. Substituting $\theta = 0$ in (1.2), we obtain the Cauchy distribution if $\alpha = 1$ and the Gaussian distribution if $\alpha = 2$. In general we assume the restriction (1.2) to be satisfied.

To prove that the characteristic function of the solution of the space-FDECLD also belongs to the class of α-stable probability densities and is related to the solution of Eq. (1.1), we take the Fourier transform of both sides of Eq. (1.6), and obtain the ordinary differential equation

$$\frac{\partial \widehat{v}(\kappa,\tau)}{\partial \tau} = -b\kappa\frac{\partial \widehat{v}(\kappa,\tau)}{\partial \kappa} - a|\kappa|^{\alpha}\mathrm{e}^{\frac{\mathrm{i}\theta\pi}{2}\operatorname{sig}(\kappa)}\widehat{v}(\kappa,\tau). \tag{2.3}$$

To solve this equation, we use the method of characteristics, we obtain the chain of equations

$$\frac{\mathrm{d}\tau}{1} = \frac{\mathrm{d}\kappa}{b\kappa} = \frac{\mathrm{d}\widehat{v}(\kappa,\tau)}{(-a|\kappa|^{\alpha}\widehat{v}(\kappa,\tau)\mathrm{e}^{\mathrm{i}\theta\frac{\pi}{2}\operatorname{sig}(\kappa)})}. \tag{2.4}$$

This equation can easily be solved using the initial condition $v(\xi,0) = \delta(\xi)$, so we obtain the characteristic function

$$\widehat{v}(\kappa,\tau) = \exp\left[-|\kappa|^{\alpha}\frac{a}{b\alpha}(1-\mathrm{e}^{-b\alpha\tau})\mathrm{e}^{\mathrm{i}\theta\frac{\pi}{2}\operatorname{sig}(\kappa)}\right]. \tag{2.5}$$

Setting $\tau' = \frac{1}{b\alpha}(1-\mathrm{e}^{-b\alpha\tau}) = \tau - \frac{b\alpha}{2!}\tau^2 + \frac{(b\alpha)^2}{3!}\tau^3 \mp \cdots$, we notice that $\tau' \to \tau$ as $b \to 0$ which is the same as in Eq. (2.1) (i.e. the fractional diffusion without drift). By this abbreviation we rewrite Eq. (2.5) to take the standard form

$$\widehat{v}(\kappa,\tau) = \exp[-|\kappa|^{\alpha}a\tau'\mathrm{e}^{\mathrm{i}\theta\frac{\pi}{2}\operatorname{sig}(\kappa)}]. \tag{2.6}$$

For the strictly stable distributions of this equation, see Appendix A, we have

$$\widehat{h}_\alpha(\kappa, \theta) = \exp[-|\kappa|^\alpha e^{\frac{i\theta\pi}{2}\operatorname{sig}(\kappa)}], \quad a = 1, \tag{2.7}$$

clearly we notice that $\widehat{h}_\alpha(\kappa, \theta) = \widehat{p}_\alpha(\kappa, \theta)$ (see [28,27]). Comparing the structures of Eqs. (2.5) and (2.6) gives rise to the conjecture that the solution of the space-FDE and the solution of the space-FDECLD can be transformed to each other with some nonlinear re-scaling for the space coordinate and for the time coordinate. Actually Biler et al. [2] have given without proof a *transformation theorem* for these solutions in the symmetric case after choosing, without losing generality, the constant drift $b = 1$ and $\theta = 0$. In the thesis of the second author [1], we have given a generalization to this theorem in the asymmetric case (i.e. $\theta \neq 0$). We state here the theorem and its generalization without proof. The transformations (2.9) and (2.8) between the two pairs of independent variables (x, t) and (ξ, τ) are

$$\xi = x(\alpha t + 1)^{-1/\alpha}, \qquad \tau = \alpha^{-1}\log(\alpha t + 1), \tag{2.8}$$

and

$$x = \xi e^{\tau}, \qquad t = \frac{1}{\alpha}(e^{\alpha\tau} - 1). \tag{2.9}$$

By the transformation (2.9), a solution $u(x, t)$ of the space-FDE (1.1) goes over into a solution $v(\xi, \tau)$ of the space-FFPE (1.6). By the transformation (2.8) the solution $v(\xi, \tau)$ of (1.6) goes over into the solution $u(x, t)$ of (1.1). These transformations are inverse to each other, and we have the relation

$$v(\xi, \tau) = (\alpha t + 1)^{1/\alpha} u(x, t),$$

with its inverse

$$u(x, t) = e^{-\tau} v(\xi, \tau).$$

We use also without proof the following lemma stating that: if $x = a\xi$, $f(x) = g(\xi)$, for $a > 0$ then

$$\underset{\xi\ \theta}{D^\alpha} g(\xi) = a^\alpha \underset{x\ \theta}{D^\alpha} f(x).$$

By applying this lemma we obtain

$$\underset{\xi\ \theta}{D^\alpha} v(\xi, \tau) = \underset{\xi\ \theta}{D^\alpha}\left(e^\tau v\left(x(\alpha t + 1)^{-1/\alpha}, \alpha^{-1}\log(\alpha t + 1)\right)\right) = e^{(\alpha+1)\tau} \underset{x\ \theta}{D^\alpha} u(x, t). \tag{2.10}$$

Notice here as $\tau = 0 \Longrightarrow \xi = x$, the solutions of the two considered equations with the same initial condition are equivalent.

$$v(\xi, \tau) = \frac{\alpha^{1/\alpha}}{(1 - e^{-\alpha\tau})^{1/\alpha}} p_\alpha\left(\frac{\alpha^{1/\alpha}(\xi - \xi_0 e^{-\tau})}{(1 - e^{-\alpha\tau})^{1/\alpha}}\right). \tag{2.11}$$

For $\tau \to \infty$ this solution becomes stationary $v(\xi, \tau) \to \alpha^{1/\alpha} p_\alpha(\alpha^{1/\alpha}\xi)$ in contrast to the stationary solution $u(x, t)$ of Eq. (1.1) which leads to zero everywhere for $t \to \infty$. If we put $\alpha = 2$ and $\beta = 1$ in this formula, we obtain the well-known solution of the classical diffusion with central linear drift. For more details see the thesis of the second author [1].

3. Simulations of the space-FDE and space-FDECLD

For the purpose of simulations of the approximate particle paths for space-FDE and henceforth to space-FFPE, we recall the notations of the theory of CTRW or compound (*cumulative*) renewal process, see [21] and Cox [4]. In the CTRW, a particle starts at the origin ($x_0 = 0$), and waits a period of time T_n, at a particular location x_{n-1}, before moving instantaneously to the next location with jump width X_n, $\forall n = 0, 1, 2, \ldots$. We call T_n, the waiting times, where $T_n = t_n - t_{n-1}$, $t_n > t_{n-1}$. The waiting times T_n are iid and likewise the jumps X_n are iid. Furthermore the waiting times and the jumps are independent of each others. The new position at $t_n = t_{n-1} + T_n$ which is equivalent to

$$t_0 = 0, \qquad t_n = \sum_{n=1}^{n} T_n, \tag{3.1}$$

is $x_n = x_{n-1} + X_n$, which equals to

$$x_0 = 0, \qquad x_n = \sum_{n=1}^{n} X_n, \tag{3.2}$$

and the particle remains resting at $x = x_{n-1}$ in the time interval $t_n < t < t_{n+1}$.

For generating the random variable X, i.e. the jump, having the Cauchy or the Gaussian distributions, we use the method of inversion which seems to be simple and most effective [19]. We also use the same method to simulate the random variable T, i. e. the waiting time, having an exponential distribution with mean 1. For simulating the jump X corresponding stable distribution, $\alpha \in (0, 2)$, we use the method given in the book of Janicki [16]. In this paper the simulation of a random variable X having symmetric and nonsymmetric α-stable distribution is given.

First for the symmetric α-stable random variable X, i.e. $\beta = 0$, we follow the steps

(1) generate a random variable V uniformly distributed on $-\pi/2, \pi/2$ and an exponential random variable W with mean 1,
(2) compute

$$X = \frac{\sin(\alpha V)}{(\cos(V))^{1/\alpha}} \left(\frac{\cos(V - \alpha V)}{W} \right)^{(1-\alpha)/\alpha}. \tag{3.3}$$

Second for the nonsymmetric α-stable random variables X with $\beta \in [-1, 1]$, see Eqs. (A.1) and (A.4) for the relation between θ and β, we follow the steps

(1) generate a random variable V uniformly distributed on $-\pi/2, \pi/2$ and an exponential random variable W with mean 1,
(2) compute

$$X = D_{\alpha,\beta} \frac{\sin(\alpha(V + C_{\alpha,\beta}))}{(\cos V)^{1/\alpha}} \left(\frac{\cos(V - \alpha(V + C_{\alpha,\beta}))}{W} \right)^{(1-\alpha)/\alpha}, \tag{3.4}$$

where

$$C_{\alpha,\beta} = \frac{\arctan(\beta \tan(\pi\alpha/2))}{\alpha},$$
$$D_{\alpha,\beta} = [\cos(\arctan(\beta \tan(\tan \pi\alpha/2)))]^{-1/\alpha}.$$

For simulating the random variables ξ and τ which represent the jump and the waiting time of the space-FDECLD process, we calculate the position x_n, Eq. (3.2), and the time instant t_n, Eq. (3.1), of the space-FDE (1.1). After that we use the transformation theorem (2.8) to transfer the pair (x_n, t_n) to the pair (ξ_n, τ_n). This procedure is done at every $n \geq 1$.

4. Numerical results

Fig. 1 represents the well-known analytical solution as $\alpha = 2$ for the standard diffusion equation, while the diffusion equation with central linear drift is plotted at Fig. 2, for different values of t. The figures show how the coordinates representing the space and the time in the diffusion with drift are compressed and the width of the curve is also narrower than the corresponding one on the left.

We will see the same observation in the simulation of the CTRW of our models. We simulate the CTRW of our diffusion models using the Monte Carlo methods for 10 000 steps and with seed 100. We display the space-FDE in the first column and the space-FDECLD in the second column. In all our figures the waiting time is exponentially distributed with mean 1 that means all the processes are Markov.

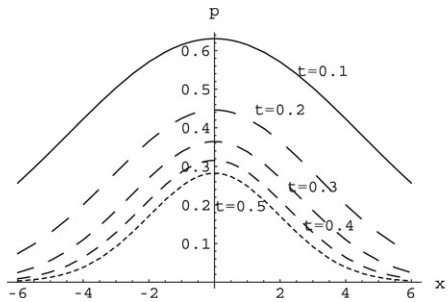

Fig. 1. Standard diffusion.

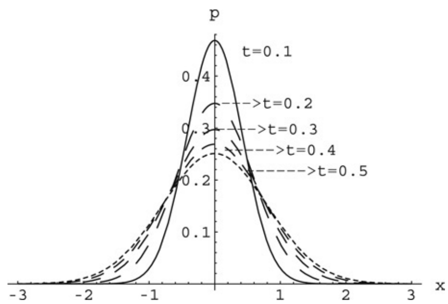

Fig. 2. Diffusion with central linear drift.

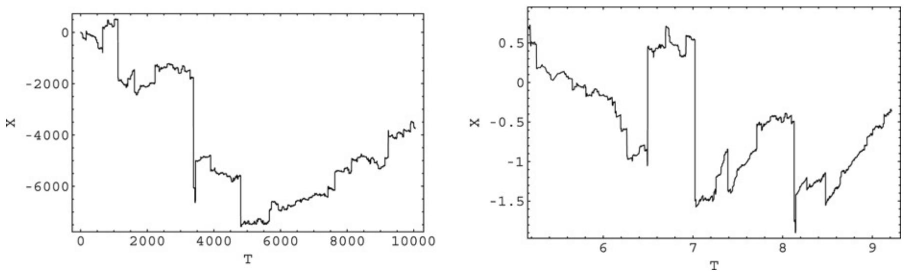

Fig. 3. Symmetric Cauchy distribution.

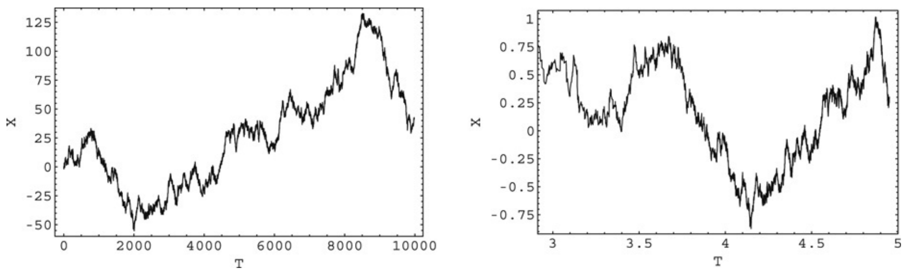

Fig. 4. Symmetric Gauss distribution.

Beginning in Fig. 3 by the simulation of the CTRW corresponding to the Cauchy distribution, i.e. $\alpha = 1$, in the symmetric case, i.e. $\beta = 0$, shows the long jump of the diffusive particle but the width of the jump is longer on the right due to the effect of the central linear drift. The CTRW corresponding to the Gaussian distribution, i.e. $\alpha = 2$, also in the symmetric case is simulated in Fig. 4 and the jump is typically Brownian motion.

We simulate the CTRW of our diffusion models corresponding to the standard α-stable distribution that means we put $c = 1$, $\mu = 0$ in Eq. (A.1). Fig. 5 represents the symmetric distribution with $\alpha = 1.5$. The nonsymmetric

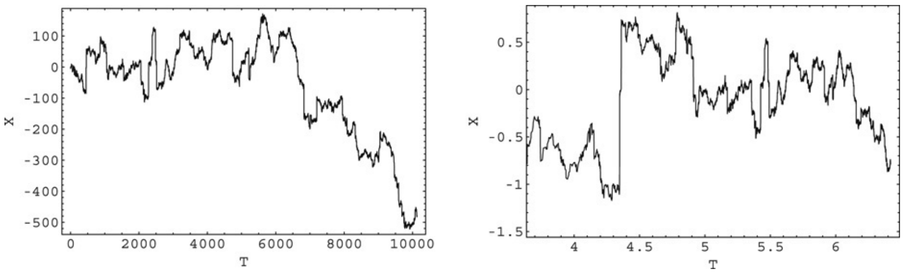

Fig. 5. Stable distribution for $\alpha = 1.5$, $\beta = 0$.

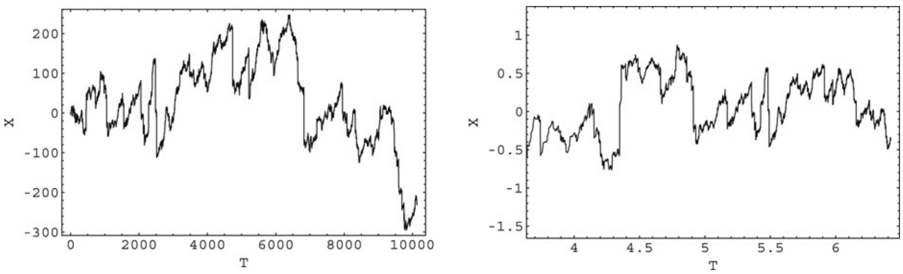

Fig. 6. Stable distribution for $\beta = 0.3$, $\alpha = 1.5$.

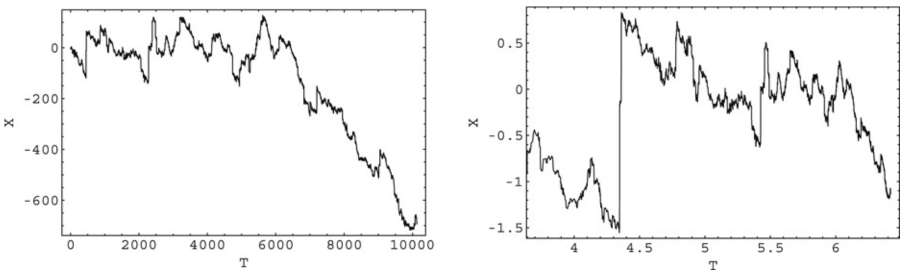

Fig. 7. Stable distribution for $\beta = -0.3$, $\alpha = 1.5$.

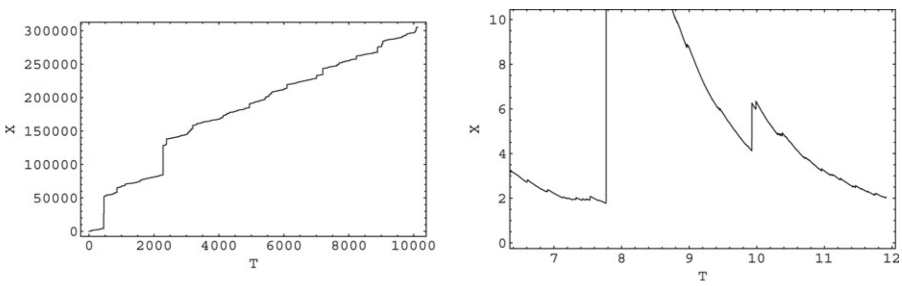

Fig. 8. Stable distribution for $\alpha = 1$, $\beta = \alpha$.

distribution with $\alpha = 1.5$ is plotted in Fig. 6 with $\beta = 0.3$ while Fig. 7 is devoted to the extremal nonsymmetric distribution with $\beta = -0.3$. Fig. 8 shows the nonsymmetric distribution for $\alpha = 1$ and $\beta = \alpha$. This figure is a good example for the one-tail long jump.

5. Conclusions

We gave here the solution of the space-fractional diffusion equation with central linear drift by using the solution of the space-fractional diffusion equation using a transformation to their independent variables to each others. We simulated the CTRW of these fractional diffusion equations for the α-stable distribution. The simulation shows how jumps and waiting times are somehow compressed with respect to the case of no drift. The simulation shows also how the width of the jump is dependent on the values of *alpha* and the skewness of the distribution β.

Appendix A. The characteristic function

The properties of many distributions are more easily investigated in terms of their characteristic functions. The characteristic function is a variant of the Fourier transform of the applied probability density function. Let us denote the characteristic function of a random variable X with density $p(x) = \frac{d}{dx}P(X \le x)$ by $\widehat{p}(\kappa)$, defined as

$$\widehat{p}(\kappa) = E[e^{i\kappa X}] = \int_{-\infty}^{\infty} e^{i\kappa x} p(x)dx, \quad \kappa \in \mathbb{R}.$$

The probability density function is said to be symmetric ($P(x) = 1 - P(-x - 0)$) iff $\widehat{p}(-\kappa) = \widehat{p}(\kappa)$, where $P(x)$ is the probability distribution function (see [18]).

In our survey of the theorem of stable probability distributions, we follow Gnedenko and Kolmogorov [8], which is based on the results of Khintchine and Lévy. By using their notation, the characteristic function $\widehat{p}(\kappa)$ belongs to an α-stable distribution, $\alpha \in (0, 2]$, if and only if it has the form

$$\log \widehat{p}(\kappa) = i\mu\kappa - c|\kappa|^{\alpha}\left\{1 + i\beta\frac{\kappa}{|\kappa|}w(|\kappa|, \alpha)\right\}, \tag{A.1}$$

where $\kappa \in \mathbb{R}$, $c \ge 0$, $\mu > 0$, $|\beta| \le 1$, and the function $w(\kappa, \alpha)$ is defined as

$$w(|\kappa|, \alpha) = \begin{cases} \tan\dfrac{\pi\alpha}{2} & \text{if } \alpha \ne 1, \\ (2/\pi)\log|\kappa| & \text{if } \alpha = 1, \in \mathbb{R}, \end{cases} \tag{A.2}$$

(see [18] and the references therein). For $\alpha = 2$, we have $w(|\kappa|, \alpha) = 0$ which is the special case of the normal distribution.

Note here that the authors who follow Lévy [17] assign the opposite sign to β in the canonical form (A.1) where β is the symmetry parameter and determines the skewness of the distribution. If $\beta = 0$, we have a symmetric distribution. c is the scale parameter and determines the spread of the samples from a distribution around the mean. μ is the location parameter and $\exp(i\mu\kappa)$ basically corresponds to a shift in the *x-axis* of the probability density function. For $1 < \alpha \le 2$, μ represents the mean and for $0 < \alpha \le 1$, it represents the median. A stable distribution is said to be standard if $\mu = 0$ and $c = 1$.

In our work we use another parameterization of the characteristic function. According to Feller the characteristic function of strictly stable densities $p(x; \theta)$ are denoted by $\widehat{p_\alpha}(\kappa; \theta)$ and are defined as

$$\widehat{p_\alpha}(\kappa; \theta) = \exp[-|\kappa|^{\alpha} e^{\frac{i\theta\pi}{2}\operatorname{sig}(\kappa)}]. \tag{A.3}$$

The range of the parameters α and θ is restricted to: $0 < \alpha \le 2$, $|\theta| \le \min\{\alpha, 2 - \alpha\}$, and is visualized by the Feller–Takayasu diamond [20]. The relation between Gnedenko–Kolmogorov and Feller on the other side is related to the skewness θ of $\widehat{p_\alpha}(\kappa; \theta)$ in Eq. (A.3) and the skewness β of $\widehat{p}(\kappa)$ in Eq. (A.2) as follows

$$\beta = \frac{\tan(\theta\frac{\pi}{2}\operatorname{sig}(\kappa))}{\tan(\frac{\alpha\pi}{2})}, \quad \alpha \ne 1 \tag{A.4}$$

with $\mu = 0$, $c = 1$. It is important to say here that in most cases, the inverse Fourier transform of the general canonical form (A.1) and (A.2) cannot be carried out with elementary functions. The most known ones (see [18]) are corresponding to $\alpha = 1$, $\beta = 0$, and $c = 1$ giving the Cauchy distribution,

$$p_1(x; 0) = \frac{1}{\pi}\frac{1}{1 + x^2},$$

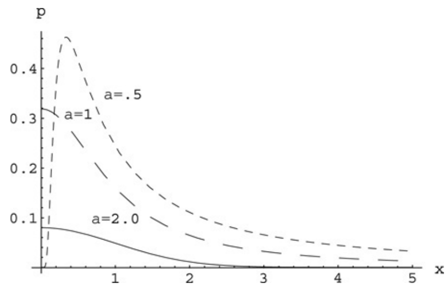

Fig. A.1. The comparison of the probability density of Gaussian, Cauchy and the one corresponds to $\alpha = 1/2, \theta = -1$ Eq. (A.5).

$\alpha = 2$, $\beta = 0$, and $c = 1$ giving the Gaussian distribution

$$p_2(x; 0) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4},$$

and $\alpha = 1/2$ which has been shown by Lévy with $\mu = 0$, $c = 1$, and $\beta = -1$ as

$$P_{1/2}(x; -1) = \begin{cases} \frac{1}{\sqrt{2\pi}} x^{-3/2} e^{-1/(2x)} & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{A.5}$$

For $\alpha = 1$, $\theta \neq 0$, the stable distributions according to Feller are different from those of Gnedenko–Kolmogorov. In fact in this case Feller's distributions are strictly stable whereas those of Gnedenko–Kolmogorov are not. Feller has shown for $\alpha = 1$, $0 < |\theta| < 1$ that

$$p_1(x; \theta) = \frac{1}{\pi} \frac{\cos(\theta\pi/2)}{[x + \sin(\theta\pi/2)]^2 + [\cos(\theta\pi/2)]^2} \quad -\infty < x < \infty,$$

and for $\alpha = 1$, $\theta = \pm 1$,

$$p_1(x; \pm 1) = \delta(x \pm 1), \quad -\infty < x < \infty.$$

Apart from these three cases no stable distribution functions are known whose density functions are elementary functions. The other stable distributions according to Feller can be obtained from Eq. (A.3) in terms of convergent power series valid for $x > 0$ (see for example: [5,6,20]),

(a) $0 < \alpha < 1$ (negative powers), $|\theta| \leq \alpha$

$$p_\alpha(x; \theta) = \frac{1}{\pi x} \sum_{n=1}^{\infty} (-x^{-\alpha})^n \frac{\Gamma(n\alpha + 1)}{n!} \sin\left[\frac{n\pi}{2}(\theta - \alpha)\right], \tag{A.6}$$

(b) $1 < \alpha \leq 2$, $|\theta| \leq 2 - \alpha$

$$p_\alpha(x; \theta) = \frac{1}{\pi x} \sum_{n=1}^{\infty} (-x)^n \frac{(1 + \frac{n}{\alpha})}{n!} \sin\left[\frac{n\pi}{2\alpha}(\theta - \alpha)\right]. \tag{A.7}$$

The values for $x < 0$ can be obtained from (A.6) and (A.7) by using the *symmetry relation* $p_\alpha(-x; \theta) = p_\alpha(x; -\theta)$. Fig. A.1 exhibits $p_\alpha(x; \theta)$ for different values of $a = \alpha$ and θ, see [20].

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