A Characterization of Finite Orthogonal Simple Groups*

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INTRODUCTION

Suppose \( F_q \) is the finite field of \( q \) elements, where \( q \) is odd, and \( V \) is a quadratic space over \( F_q \), i.e., a finite-dimensional vector space over \( F_q \) with a nondegenerate symmetric bilinear form. If \( \Omega(V) \) denotes the commutator subgroup of the orthogonal group \( O(V) \) of \( V \), then the corresponding projective group \( P\Omega(V) \) is simple if \( \dim V \geq 5 \). We are concerned with characterizing this group among all simple groups, by the structure of the centralizer of an involution. This has already been done when \( \dim V = 5 \) or \( 6 \), the group then being isomorphic with \( PSp(4, q) \), \( PSL(4, q) \), or \( PSU(4, q) \) \([12, 8, 9]\). In this paper we deal with the high-dimensional cases (see Hypothesis 6 below).

By examining the centralizer of an element of \( P\Omega(V) \) corresponding to an involution of \( \Omega(V) \) whose fixed-point subspace \( U \) has codimension 2 in \( V \), we are led to the following

**HYPOTHESIS 1.** \( G \) is a finite group, containing an involution \( t \) whose centralizer \( C(t) \) has a normal subgroup \( M \) isomorphic with \( \Omega(U) \), where \( U \) is a quadratic space over \( F_q \), \( q \) odd.

**HYPOTHESIS 2.** \( C(t) \) has two involutions \( u, u' \), which induce the automorphisms of \( M \) which correspond to automorphisms of \( \Omega(U) \) induced by involutions \( z, z' \) of \( \Omega(U) \), one from each of the cosets of \( \Omega(U) \) which do not lie in the subgroup \( O^+(U) \) of elements of determinant 1 in \( O(U) \).

**HYPOTHESIS 3.** \(| C(t) : \langle M, u, u' \rangle | \) is odd.

**HYPOTHESIS 4.** \([u, u']h^{-1} \) has even order, where \( h \) is the element of \( M \) corresponding to the element \( [z, z'] \) of \( \Omega(U) \).

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HYPOTHESIS 5. Either \(| C(t) \cap C(M)| < 3(q - e)/2, \) or \(| C(t) \cap C(M)|' < 3(q - e)', \) where \(q \equiv e \pmod 4, e = \pm 1, \) and \(m'\) denotes the greatest odd divisor of the integer \(m.\)

HYPOTHESIS 6. If \(e = 1,\) either \(\dim U \geq 9, \) or \(\dim U = 8\) and \(U\) has square discriminant. If \(e = -1,\) either \(\dim U \geq 14, \) or \(\dim U = 12\) and \(U\) has square discriminant, or \(\dim U = 10\) and \(U\) has nonsquare discriminant.

THEOREM. Suppose \(G\) satisfies Hypotheses 1-6. Then one of the following holds.

(a) \(G \cong C(t) \cdot O(G).\)

(b) \(\dim U\) is odd, \(M\) is normal in \(G,\) and there is a series of normal subgroups of \(G,

\[ M \subseteq G_1 \subseteq G_2 \subseteq G, \]

such that \(G_1/M\) and \(G_2/G_1\) have odd order, and \(G_2/G_1\) is isomorphic with \(\text{PGL}(2, r),\) for some odd prime power \(r.\)

(c) \(G\) has a normal subgroup \(G_0\) isomorphic with \(\text{PQ}(V)\) or \(\Omega(V),\) where \(V\) is a quadratic space over \(F_q\) having the same discriminant as \(U,\) and \(\dim V = \dim U + 2,\) and \(G\) is a semi-direct product

\[ G = W G_0, \; W \cap G_0 = 1, \]

where \(W\) is a cyclic group of odd order acting faithfully on \(G_0 \) "by field automorphisms."

The dimensional restrictions of Hypothesis 6 are made in order to apply earlier work of the author [13] on generating \(\text{PQ}(V)\) and \(\Omega(V),\) to construct the group \(G_0\) in case (c). The bulk of the present paper is devoted to an analysis of the fusion of involutions which is also valid for lower-dimensional cases. Thus any relaxation of the dimensional restrictions of [13] would lead to a corresponding relaxation here. However, the theorem cannot be precisely correct in the case when \(\dim U = 6\) and \(U\) has square discriminant, because of the existence of the triality automorphism of \(\text{PQ}(V)\) in the 8-dimensional case.

If the structure of \(C(t)\) were assumed to be exactly what it is in \(\text{PQ}(V),\) then, in case (c), \(G\) is isomorphic with \(\text{PQ}(V).\) We have chosen our hypotheses to allow for odd order cyclic extensions by field automorphisms, as this appears to be the degree of generality required for use in characterizations of classical simple groups by their Sylow 2-subgroups. In this connection, one may remark that the second form of Hypothesis 5 amounts to a condition on the order of \(O(C(t))\), since this can be shown to be the normal 2-complement of \(C(t) \cap C(M).\) In a general classification problem, this sort of in-
formation on $O(C(t))$ is what one might expect to obtain by the use of the signalizer functor methods of Gorenstein and his collaborators.

The existence of the nonsimple case $(b)$ presumably arises from the fact that, when $\dim U$ is odd, $C(t)$ is not in standard form in the sense of Gorenstein and Walter [6], i.e., $C(t) \cap C(M)$ contains a four-group. A similar phenomenon appears in a more complicated way in Phan's characterization of $PSL(n, q)$ [10]. In the odd-dimensional case, Olsson has characterized $PQ(V)$ (at least in the case $q = 1 \pmod{4}$) by the centralizer of an involution which is in standard form [7].

1. Properties of Orthogonal Groups

We begin with a few remarks concerning the involutions and semi-involutions of the orthogonal group $O(V)$ of a quadratic space $V$ over a finite field $F_q$ of odd characteristic, and the automorphisms of the commutator group $\Omega(V)$. Details may be found in Dieudonné's book [2]. We assume $\dim V > 5$.

If $x$ is an involution of $O(V)$, then $V$ is the orthogonal direct sum of the positive and negative subspaces $V^+, V^-$ of $x$, which are the kernels of $x - 1$ and $x + 1$ respectively. The centralizer of $x$ in $O(V)$ is naturally isomorphic with the direct product of $O(V^+)$ and $O(V^-)$. Its centralizer in $\Omega(V)$ is a subgroup whose commutator subgroup is isomorphic with $\Omega(V^+) \times \Omega(V^-)$ (except for some cases when $q = 3$, when $\Omega(V^+) \times \Omega(V^-)$ is the only proper normal subgroup of the centralizer having 2-power index).

We call $\dim V^-$ the type of $x$, and the discriminant of $V^-$ the norm of $x$. The latter is defined to within a nonzero square factor in $F_q$. The involution $x$ lies in the rotation group $O^+(V)$ if and only if its type is even, and then its norm is its spinor norm. Thus $x$ lies in $\Omega(V)$ if and only if its type is even and its norm is a square. Two involutions lie in the same coset of $\Omega(V)$ in $O(V)$ if and only if their types have the same parity and their norms are the same, modulo squares. By the fact that the equivalence class of a quadratic space over $F_q$ is determined by its dimension and discriminant, we see that two involutions are conjugate in $O(V)$ if and only if they have the same type and the same norm, modulo squares. They are then conjugate by an element of $\Omega(V)$, since the centralizer of an involution contains elements from each of the cosets of $\Omega(V)$ in $O(V)$.

If $y$ is a semi-involution of $O(V)$, i.e., $y^2 = -1$, then $V$ has even dimension $2m$ and square discriminant, since $y^2 \in \Omega(V)$. If $q \equiv 1 \pmod{4}$, so that $F_q$ contains a square root $\eta$ of $-1$, then the positive and negative subspaces $V^+, V^-$ of the involution $\eta y$ are totally isotropic of dimension $m$. The centralizer of $y$ in $O(V)$ is naturally isomorphic with $GL(V^+) = GL(m, q)$,
acting on $V-$ via the contragredient representation. We remark that an involution in this centralizer has type $2r$, where $r$ is its type when it is regarded as an element of $GL(m, q)$. The centralizer of $y$ in $\Omega(V)$ has commutator subgroup isomorphic with $SL(m, q)$. By taking a basis of $V-$ and the dual basis of $V^-$, we see that $V$ can be expressed as an orthogonal direct sum of hyperbolic planes $V_1, ..., V_m$ invariant under $y$. All semi-involutions are conjugate in $O(V)$.

If $q = -1 \pmod{4}$, $V$ can be made into a vector space of dimension $m$ over $F_2$, with a nondegenerate Hermitian form relative to the automorphism of $F_2$ of order 2, in such a way that the centralizer of $y$ in $O(V)$ is naturally isomorphic with the unitary group $GU(V) = GU(m, q)$. An involution in this centralizer has type $2r$, where $r$ is its type when it is regarded as an element of $GU(m, q)$. The centralizer of $y$ in $\Omega(V)$ has commutator subgroup isomorphic with $SU(m, q)$. By taking an orthonormal basis of $V$ as a unitary $F_2$-space, we see that $V$ can be expressed as an orthogonal direct sum of planes $V_1, ..., V_m$ of square discriminant, invariant under $y$. Again, all semi-involutions are conjugate in $O(V)$.

Each automorphism of $\Omega(V)$ is obtained by restriction from an automorphism of $O(V)$, which in turn is induced by a semi-similitude of $V$. This is a semi-linear transformation $z$ of $V$ defined relative to an automorphism $\sigma$ of $F_2$, such that

$$(\nu z, w) = (\nu, w)^{\sigma} a,$$

for all $\nu, w$ in $V$, where $a$ is a nonzero scalar, called the multiplier of $z$. Since such a transformation maps a subspace of $V$ on a subspace of the same dimension, it follows that automorphisms of $\Omega(V)$ preserve the type of any involution. Thus the image of an involution $x$ of $\Omega(V)$ under an automorphism is conjugate to $x$ in $\Omega(V)$. (A similar argument shows that automorphisms of $SL(m, q)$ or $SU(m, q)$ map involutions into conjugate involutions, since types are preserved.) More generally, if $x_1, ..., x_r$ is a set of commuting involutions in $\Omega(V)$, and $y_1, ..., y_r$ are their images under an automorphism, then there is an element of $\Omega(V)$ conjugating each $x_i$ into the corresponding $y_i$.

While the multiplier of the product of two semisimilitudes $z_1$, $z_2$ may not be the product of the multipliers of $z_1$ and $z_2$, it is but for a square factor. Thus the group $\Gamma O(V)$ of semisimilitudes of $V$ has a normal subgroup $\Gamma O^q(V)$ of index 1 or 2, consisting of the semisimilitudes whose multipliers are squares in $F_2$. (The index is 1 if and only if dim $V$ is odd.) The group $\Gamma O^q(V)$ is a product of the group $H$ of homotheties of $V$ with the group $\Gamma O^+(V)$ of semisimilitudes of $V$ of multiplier 1. Of course $H$ induces only the identity automorphism of $\Omega(V)$. If $x \in O(V)$, $y \in \Gamma O^+(V)$, then the com-
mutator \([x, y]\) has determinant 1. Further, if \(x\) is the product of symmetries relative to lines \(L_1, \ldots, L_m\) in \(V\), then \(x^y\) is the product of symmetries relative to the lines \(L_x y, \ldots, L_m y\), which have the same discriminants as \(L_1, \ldots, L_m\). Hence \([x, y] = x^{-1}x^y\) has spinor norm 1, so that \([x, y] \in \Omega(V)\). Thus \(O(V)/\Omega(V)\) is central in \(\Gamma O^+(V)/\Omega(V)\). Since \(\Gamma O^+(V)/\Omega(V)\) is a cyclic group isomorphic with the automorphism group of \(F_q\), it follows that \(\Gamma O^+(V)/\Omega(V)\) is Abelian. Hence the outer automorphism group of \(\Omega(V)\) has an Abelian normal subgroup of index 1 or 2, and so has a normal 2-complement, which is in fact isomorphic with the 2-complement of the automorphism group of \(F_q\).

If \(\sigma\) is an automorphism of \(F_q\) of odd order \(d\), then nonsquares of the fixed field \(E\) of \(\sigma\) are still nonsquares of \(F_q\). It follows that a basis of \(V\) can be found for which the matrix of the bilinear form on \(V\) has coefficients in \(E\). The semilinear transformation \(z\) on \(V\) defined relative to \(\sigma\) which leaves the elements of this basis fixed is a semisimilitude of multiplier 1, having order \(d\). This transformation \(z\) is unique in the sense that any other semisimilitude of multiplier 1 defined relative to \(\sigma\) which has order \(d\) is conjugate to \(z\) by an element of \(O(V)\), since the cohomology set \(H^1(\langle \sigma \rangle, O(V))\) is trivial [11, p. 162]. We say the automorphism of \(\Omega(V)\) induced by \(z\) is a field automorphism of \(\Omega(V)\) corresponding to \(\sigma\). We have shown that this is unique to within conjugacy in the automorphism group of \(\Omega(V)\). We also call the induced automorphism of \(PO(V)\) a field automorphism.

2. Structure of \(C(t)\)

We now assume Hypotheses 1-6 and begin the proof of the Theorem. In this section we set up some notation and obtain some information on the structure of \(C(t)\).

**Lemma (2A).** Hypotheses 2, 3, 4 of the Theorem are independent of the choice of \(z, z'\).

**Proof.** Suppose the hypotheses are satisfied with involutions \(z, z'\), and another choice of involutions \(z_1, z_1'\) is made from the cosets \(\Omega(U)z, \Omega(U)z'\). We write

\[z_1 = xz, \quad z_1' = z'x',\]

where \(x, x' \in \Omega(U)\), and let \(y, y'\) be the elements of \(M\) corresponding to \(x, x'\) respectively. Let \(u_1 = yu, u_1' = u'y'\). Since \(z, z_1, z', z_1'\) are involutions, \(z\) inverts \(x\) and \(z'\) inverts \(x'\). Thus \(u\) inverts \(y\) and \(u'\) inverts \(y'\), so that \(u_1, u_1'\) are involutions. They induce the automorphisms of \(M\) which correspond to
the automorphisms of $O(U)$ induced by $z_1, z_1'$, and $\langle M, u_1, u_1' \rangle = \langle M, u, u' \rangle$, so that Hypotheses 2 and 3 are satisfied with $z_1, z_1'$ in place of $z, z'$, and $u_1, u_1'$ in place of $u, u'$. A computation shows that

$$[z_1, z_1'] = x[x, z'](x'x)^{z'z'},$$

so that the element $h_1$ of $M$ corresponding to $[z_1, z_1']$ is given by

$$h_1 = yh(y'y)^{u'u'} y'.$$

We then compute that $[u_1, u_1'] h_1^{-1} = y[u, u'] h^{-1} y^{-1}$, which has the same order as $[u, u'] h_1^{-1}$, so that Hypothesis 4 is satisfied by $z_1, z_1', u_1, u_1'$. This proves the lemma.

We identify the group $M$ with $O(U)$ in Hypothesis 1. If $W$ is a non-degenerate subspace of $U$, we write $O(W)$ for the subgroup of $O(U)$ which acts as the identity on the orthogonal complement of $W$ in $U$, and $O^+(W) = O(W) \cap O^+(U)$, $Q(W) = O(W) \cap M$. We denote the central involution of $O(W)$ by $t(W)$. This lies in $O(W)$ if and only if $W$ has even dimension and square discriminant.

The space $U$ can be decomposed as an orthogonal direct sum

$$U = V_0 \oplus V_1 \oplus \cdots \oplus V_{n-1},$$

where $V_1, \ldots, V_{n-1}$ are 2-dimensional subspaces of square discriminant, $\dim V_0 \leq 2$, and $V_0$ has nonsquare discriminant when $\dim V_0 = 2$[1,p. 158]. By scaling the bilinear form on $U$ if necessary, we may assume also that $V_0$ has nonsquare discriminant when $\dim V_0 = 1$. The center $Z(M)$ is nontrivial if and only if $\dim V_0 = 0$, in which case $Z(M) = \langle t(U) \rangle$.

We note that Hypothesis 6 is equivalent with saying that $n \geq 5$ and $q \equiv 1 \pmod{4}$, or $n \geq 8$ and $q \equiv -1 \pmod{4}$, or $n = 7$ and $\dim V_0 \neq 1$, or $n = 5$ and $\dim V_0 = 2$. These are precisely the dimensional conditions under which the results of [13] have been proved. Except for the application of these results in Sections 4, 5, all the work in this paper will be valid under the less stringent

**Hypothesis 6'.** $n \geq 5$.

For $i = 1, \ldots, n - 1$, and also for $i = 0$ when $\dim V_0 = 2$, we choose nondegenerate 1-dimensional subspaces $L_i, L_i'$ of $V_i$ such that $L_i$ has square discriminant, $L_i'$ has nonsquare discriminant, and the product $t(L_i) t(L_i')$ is a generator of the Sylow 2-subgroup of $O^+(V_i)$. We now fix on a choice of $z, z'$ as follows:

(i) If $\dim V_0 = 0$, let $z = t(L_1), z' = t(L_2')$.

(ii) If $\dim V_0 = 1$, let $z = t(L_2), z' = t(U)$.

(iii) If $\dim V_0 = 2$, let $z = t(L_0), z' = t(L_0) t(U)$.
In particular, \([z, z'] = 1\), so that \(h = 1\) in Hypothesis 4. We choose involutions \(u, u'\) as in Hypotheses 2, 3, 4, and set

\[ K = C(M) \cap C(t), \quad s = [u, u']. \]

Then we see that \(s \in K\). Also we note that in cases (ii) and (iii), \(K\) contains \(u', uu'\) respectively.

**Lemma (2B).** We may assume \(s\) is a 2-element \(\not= 1\).

**Proof.** The dihedral group \(\langle u, u' \rangle\) has Sylow 2-subgroup \(\langle u_1, u' \rangle\), where \(u_1 = us^k\), for a suitable integer \(k\). By Hypothesis 4, \(|\langle u, u' \rangle|\) is divisible by 8, so that \([u_1, u'] \not= 1\). Since \(s \in K\), \(u_1\) induces the same automorphism of \(M\) that \(u\) does. The index \(|\langle M, u, u' \rangle : \langle M, u_1, u' \rangle|\) is odd. Thus we may replace \(u\) by \(u_1\) without disturbing Hypotheses 2, 3, and 4, and then \(s\) becomes a 2-element \(\not= 1\).

**Lemma (2C).** A Sylow 2-subgroup of \(K\) is given by \(\langle s \rangle Z(M), \langle s, u' \rangle, \langle uu' \rangle\), according as \(\dim V_0 = 0, 1, 2\). In any case, \(t \in \langle s \rangle Z(M)\).

**Proof.** Since \(\langle M, u, u' \rangle\) contains a Sylow 2-subgroup of \(C(t)\) and \(K \triangleleft C(t)\), \(K \cap \langle M, u, u' \rangle\) contains a Sylow 2-subgroup of \(K\). As \(u\) induces an outer automorphism of \(M\), and so do \(u', uu'\) when \(\dim V_0 = 0\), we obtain the first statement. Since \(t \in Z(K)\) and \(t\) commutes with \(u\), we obtain the second statement. This proves the lemma.

Now we set

\[ H = \langle M, u, u' \rangle, \quad d = | C(t) : HK |. \]

By Hypothesis 3, \(d\) is an odd number.

**Proposition (2D).** (a) There exists an element \(w\) of odd order in \(C(t)\) which induces a field automorphism of order \(d\) on \(M\), and \(C(t) = HK\langle w \rangle\).

(b) \(C(t)/KM\) is Abelian.

(c) \(C(t)/M\) has a normal 2-complement.

**Proof.** The group \(\bar{C} = C(t)/KM\) is isomorphic with a subgroup of the outer automorphism group of \(M\), having \(HK/KM\) as Sylow 2-subgroup. By the remarks of Section 1, \(\bar{C}\) has a normal 2-complement, generated by the coset of an element \(w\) which induces a field automorphism of order \(d\) on \(M\). Thus \(C(t) = HK\langle w \rangle\). By replacing \(w\) by \(w^m\), where \(m\) is a suitable power of 2, we may assume that \(w\) has odd order. Since \(C(t) = KM\langle u, u', w \rangle\), and each of the elements \(u, u', w\) induces an automorphism on \(M\) equal to that given by an orthogonal semisimilitude of multiplier 1, the group \(\bar{C}\) is Abelian. By
(2C) and the fact that $t$ is central, we see easily that $K \langle w \rangle$ has a normal 2-complement $R$ (and so $K$ has normal 2-complement $K \cap R$, as remarked in the Introduction). Then $RM/M$ is a normal 2-complement of $C(t)/M$. This proves the proposition.

In order to analyse the fusion of involutions of $C(t)$, we shall make computations with centralizers of four-subgroups containing $t$. Since the structures of these centralizers are not known precisely, we shall use two functors to obtain groups whose structure can be exactly determined.

For each finite group $X$ we define a subgroup $X_{\infty}$ as follows. If $q > 3$, we take $X_{\infty}$ as the intersection of the terms of the lower 2-series of $X$. (By the Feit-Thompson theorem, this is the terminal member of the derived series of $X$, but we do not need this fact.) If $q = 3$, we set $X_{\infty} = O^2(X)$, the group generated by the elements of odd order in $X$. Thus $X = X_{\infty}$ when $X = \Omega(W)$ for any quadratic space $W$ over $F_q$ of dimension at least 3, even when $\dim W = 3$ or 4 and $q = 3$. A homomorphism of $X$ into a group $Y$ must map $X_{\infty}$ into $Y_{\infty}$.

We also define a subgroup $X_{\infty,2}$ of $X$ to be the subgroup generated by $X_{\infty}$ and a Sylow 2-subgroup of $X$, so that $X_{\infty,2}/X_{\infty}$ is a Sylow 2-subgroup of $X/X_{\infty}$. This is unique to within conjugacy in $X$. A homomorphism of $X$ into a group $Y$ maps $X_{\infty,2}$ into a conjugate of $Y_{\infty,2}$.

**Lemma (2E).** If $X$ is a subgroup of $C(t)$, then $X_{\infty} = (X \cap M)_{\infty}$, and there is a conjugate $Y$ of $X$ in $C(t)$ such that $Y_{\infty,2} = (Y \cap H)_{\infty,2}$.

**Proof.** If $q > 3$, $C(t) \subseteq M$ by (2D) (c). If $q = 3$, then $C(t) = HK$ in (2D) (a), since $d = 1$. By Hypothesis 5, $K$ has order 2 or 4, while $H/M$ is a 2-group. Hence $C(t) \subseteq M$ in this case also. (Since $M_{\infty} = M$, actually $C(t)_{\infty} = M$.) Thus $X_{\infty} \subseteq X \cap M$, so that $X_{\infty} = (X_{\infty})_{\infty} \subseteq (X \cap M)_{\infty}$. The reverse inclusion is clear. Since $H$ contains a Sylow 2-subgroup of $C(t)$, some conjugate of a Sylow 2-subgroup of $X$ is contained in $H$, and so we have the second statement of the lemma.

We conclude this section with some more notation. For $i = 1, \ldots, n - 1$, set $t_i = t(V_i)$, and let $t_n$ be the involution in $\langle s \rangle$. By (2C), $t = t_n$, except possibly in the case that $\dim V_0 = 0$ and $Z(M) \subseteq \langle s \rangle$, when it may happen that $t = t_1t_2 \ldots t_{n-1}$, or $t = t_1t_2 \ldots t_{n-1}t_n$. More generally, if $\emptyset \neq A \subseteq \{1, \ldots, n - 1\}$, we set

$$V_A = \sum_{i \in A} V_i,$$

and set $t_A = t(V_A)$, the central involution of $\Omega(V_A)$. If $A = \emptyset$, we set $t_A = 1$. Then $t_A$ is the product of all the $t_i$ with $i \in A$. If $A = \{i, j, k, \ldots\}$, we also write $t_A$ as $t_{ijk \ldots}$. Finally, we set

$$u_{ij} = t(L_i) t(L_j), \quad u_{ij}' = t(L_i') t(L_j'),$$
for distinct $i, j$ in $\{1, \ldots, n-1\}$, or in $\{0, 1, \ldots, n-1\}$ if $\dim V_0 = 2$. We note that

$$[u_{ij}, u_{mk}] = \begin{cases} 1, & \text{if } i, j, m, k \text{ are distinct}, \\ s_i, & \text{if } i = m, j \neq k, \\ s_{ij}, & \text{if } i = m, j = k, \end{cases}$$

where $s_i$ is a generator of the Sylow 2-subgroup of $\Omega(V_i)$. If $2^x$ is the greatest power of 2 dividing $q - \epsilon$, the elements $s_1, \ldots, s_{n-1}$ have order $2^{x-1}$, while $s_0 = 1$.

3. The Case $Z(M) \subseteq \langle s \rangle$; Fusion

For the next two sections we assume that $Z(M) \subseteq \langle s \rangle$. (The contrary case will be considered in Section 5.) In this section we obtain the following result.

**Proposition (3A).** Assume $Z(M) \subseteq \langle s \rangle$ and suppose the conclusions (a), (b) of the Theorem do not hold.

(i) The involutions of the elementary group $D = \langle t_1, t_2, \ldots, t_n \rangle$ which are conjugate in $G$ to $t$ are precisely $t_1, t_2, \ldots, t_n$.

(ii) At most three conjugate classes of $C(t)$ not containing $t$ are fused to $t$ in $G$.

(iii) If $x$ is an involution of $C(t)$ distinct from $t$, such that $t$ is conjugate in $G$ to $x$ but not to $xt$, then $x$ is conjugate to $t_1$ in $C(t)$.

(iv) The order of $s$ is $2^{x-1}$.

**Proof.** First suppose $\dim V_0 = 0$, so that $Z(M) = \langle t \rangle$, $t = t_n = t_{12} \ldots n_1$. By (2B), there exists an element $v$ of order 4 in $\langle uu' \rangle$. Since $uu'$ induces an outer automorphism on $M$ and $(uu')^2 = s$ centralizes $M$, $v$ does not lie in $M$. Every involution in $C(t)$ is conjugate to an element of $H$. Since $H/M = \langle uM, u'M \rangle$ is dihedral, every involution in $H$ is conjugate in $H$ to an element of $M, uM, u'M$, or $vM$. Since

$$M = \Omega(U), \langle u, M \rangle \cong \langle z, \Omega(U) \rangle, \langle u', M \rangle \cong \langle z', \Omega(U) \rangle,$$

our remarks on involutions in Section 1 show that the involutions of $M$, $uM, u'M$ are conjugate in $H$ to involutions of form

$$t_A, \quad \emptyset \neq A \subseteq N,$$

$$ut_A, \quad A \subseteq P,$$

$$u't_A, \quad A \subseteq Q,$$
where \( N = \{1, 2, \ldots, n - 1\} \), \( P = N \setminus \{1\} \), \( Q = N \setminus \{2\} \), and in each case we need take only one subset \( A \) of each cardinality. Indeed, \((u'w)k\) commutes with \( t_A \) and transforms \( u \) into \( u's_k \), so, by taking a suitable \( k \), we see that \( u't_A \) is conjugate in \( H \) to \( u't_A \). If \( A \subseteq P \), \( u't_A = u't_A \), which is conjugate in \( \langle u, M \rangle \) to \( u't_A \). Thus it is necessary only to take those \( u't_A \) with \( 2 \mid A \mid \leq n - 2 \). Similarly, if \( A \subseteq Q \), \( u't_A \) is conjugate in \( H \) to \( u't_A \), and so we may again restrict to the \( A \) with \( 2 \mid A \mid \leq n - 2 \). Using (2E), we find

\[
C(t, t_A)_m = \begin{cases} 
\Omega(V'_{N-A}), & \text{if } |A| = 1, \\
\Omega(V_A) \Omega(V'_{N-A}), & \text{if } 2 \leq |A| \leq n - 3, \\
\Omega(V_A), & \text{if } |A| = n - 2,
\end{cases}
\]

\[
C(t, u't_A)_m = \begin{cases} 
\Omega(L \oplus V_P), & \text{if } |A| = 0, \\
\Omega(L' \oplus V_A) \Omega(L \oplus V_{P-A}), & \text{if } 1 \leq |A| \leq \frac{1}{2}(n - 2),
\end{cases}
\]

\[
C(t, u't_A)_m = \begin{cases} 
\Omega(L' \oplus V_Q), & \text{if } |A| = 0, \\
\Omega(L \oplus V_A) \Omega(L' \oplus V_{Q-A}), & \text{if } 1 \leq |A| \leq \frac{1}{2}(n - 2),
\end{cases}
\]

where \( A \) is a subset of \( N, P, Q \) in the three cases, and \( L \) and \( L' \) are the orthogonal complements of \( L \) and \( L' \) in \( V_1 \) and \( V_2 \), respectively.

If \( y = vx \) is an involution in \( vM \), then \( x^v = v^{-1}(vx)^2 = t \). Since \( v \in \langle uw' \rangle \) and \((zz')^2 = 1\), the automorphism of \( M \) induced by \( v \) is the same as the automorphism of \( \Omega(U) \) induced by an element \( g \) of \( O(U) \) such that \( g^2 = 1 \). Then

\[
(gx)^2 = x^g x = x^g x = t.
\]

Thus \( gx \) is a semiinvolution of \( O(U) \), and the automorphism of \( M \) induced by \( y = vx \) is the same as the automorphism of \( \Omega(U) \) induced by \( gx \). Our remarks on semiinvolutions in Section 1 show that, after replacing \( y \) by its conjugate by a suitable element of \( M \), we can assume that \( gx \) leaves each subspace \( V_i \) invariant. Since a semiinvolution of \( O(V_i) \) is inverted by elements of determinant \(-1\) in \( O(V_i) \), we see that, for distinct \( i, j \) in \( N \),

\[
[y, u_{ij}] = t_{ij}.
\]

Since \( y \) acts as a semiinvolution on \( M = \Omega(U) \), it follows from (2E) that

\[
C(t, y)_\infty \cong SL(n - 1, q) \quad \text{or} \quad SU(n - 1, q),
\]

according as \( \epsilon = 1 \) or \(-1\).

If \( x \) is an involution of \( H \), then \( x \) is conjugate in \( C(t) \) to another involution \( x' \) of \( H \) such that \( C(t, x')_{\omega_2} = C_H(x')_{\omega_2} \), by (2E). Then \( C_M(x) \) must be isomorphic with \( C_M(x') \). An examination of cases shows that \( |C_H(x)| = |C_H(x')| \), from which it follows that

\[
C(t, x)_{\omega_2} = C_H(x)_{\omega_2}.
\]
This remark holds in all the cases which we shall consider, and will be used without further reference.

Since conclusion (a) of the Theorem is assumed to fail, it follows by Glauberman’s $\mathbb{Z}^*$-theorem [3] that $t$ is fused in $G$ to some other involution $x$ of $C(t)$. If $x^g = t$, $g \in G$, then $t^g = y$ is an involution of $C(t)$ distinct from $t$. We may assume that $x$ and $y$ are among the representative involutions of $H$ which we have mentioned. Since $g$ transforms $C(t, x)$ into $C(t, y)$, the groups $C(t, x)_\infty$ and $C(t, y)_\infty$ are isomorphic. For each possible $x$ this determines a small number of possibilities for $y$.

First we show that $x \neq t$, for $A \subseteq N$, $2 \leq |A| \leq n - 2$. Otherwise we can take $y = t_A$ or $t_{N-A}$, and $g$ maps $C(t, t_A)_\infty$ on $C(t, t_A)_\infty$ or $C(t, t_{N-A})_\infty$. Since $\langle t_A \rangle = Z(\Omega(V_A))$, $g$ cannot map $\Omega(V_A)$ on itself. Using the Krull–Schmidt theorem, we can conclude that $|A| < n - 2$ and $g$ maps $\Omega(V_A)$ to $\Omega(V_{N-A})$, so that $t_A$ maps on $t_{N-A}$, a contradiction.

Next we show that $x \neq ut_A$, for $A \subseteq P$, $1 \leq |A| \leq \frac{1}{2}(n - 2)$. Otherwise we can take $y = ut_A$ or $u't_B$, where $B \subseteq Q$, $|B| = |A|$. If $y = ut_A$, the Krull–Schmidt theorem implies $g$ maps $\Omega(L_A \oplus V_A)$ and $\Omega(L \oplus V_{P-A})$ on themselves, or possibly interchanges them (when $|A| = \frac{3}{2}(n - 2)$). Since automorphisms of a group $\Omega(W)$ map involutions into conjugate involutions, we see that, choosing $i \in A$, $j \in P - A$, we may assume $g$ maps $t_i$ and $t_j$ on themselves or interchanges them. Then $ut_At_{ij}$ is mapped on $t_{ij}$. Since $ut_At_{ij} - ut_C$, where $C = (A - \{i\}) \cup \{j\}$ (so that $|C'| = |A|$), and $tt_{ij} = t_D$, where $D = N - \{i, j\}$, we see that $t_D$ is conjugate to $t$. Since $|D| = n - 3 \geq 2$ by Hypothesis 6', we have a contradiction. If $y = u't_B$, the Krull–Schmidt theorem shows that $\Omega(L_A \oplus V_A)$ and $\Omega(L \oplus V_{P-A})$ map on $\Omega(L_A' \oplus V_B)$ and $\Omega(L_A' \oplus V_{O-B})$, and a contradiction is obtained in a similar way.

Similarly, $x \neq u't_A$, for $A \subseteq Q$, $1 \leq |A| \leq \frac{1}{2}(n - 2)$.

Also, $x$ cannot be an involution in $vM$. Otherwise we can take $y$ in $vM$ as well. Then $g$ maps $C(t, x)_\infty$ on $C(t, y)_\infty$, and these groups are isomorphic with $SL(n - 1, q)$ or $SU(n - 1, q)$. We can assume that $x$ and $y$ both act on $M$ like a semiinvolutions leaving all the $V_i$ invariant. Then $t_{12}$ is contained in both $C(t, x)_\infty$ and $C(t, y)_\infty$, and has type 2 when these groups are considered as $SL(n - 1, q)$ or $SU(n - 1, q)$. Since automorphisms of $SL(n - 1, q)$ and $SU(n - 1, q)$ map involutions into conjugate involutions, we can assume that $g$ maps $t_{12}$ on itself, so that $xt_{12}$ is mapped on $tt_{12}$. But we have seen that $xt_{12} = x[x, u_{12}] = x^u_{12}$. Thus $tt_{12} = t_{84\ldots n-1}$ is conjugate to $t$, a contradiction, since $n \geq 5$.

Thus the only possibilities for $x$ are the elements of the conjugacy classes of $t_1$, $u$, $u'$ in $C(t)$, and we have proved statement (ii) of the proposition in this case.

We remark that $u$ is conjugate to $ut$ in the dihedral group $\langle u, u' \rangle$, and
similarly $u'$ is conjugate to $u't$. Thus, if $t_1$ is not conjugate to $t$, then every four-subgroup containing two involutions conjugate to $t$ actually has all its involutions conjugate to $t$. However, $u$ is conjugate to $ut_1$, $u'$ is conjugate to $u't_2$, and $t_2$ is conjugate to $t_1$, so that one of the four-subgroups $\langle u, t_1 \rangle$, $\langle u', t_2 \rangle$ would have exactly two involutions conjugate to $t$. Thus $t_4$ must be conjugate to $t$, and we have proved (i) and (iii).

By (iii), we can find an element $a$ of $G$ which normalizes $\langle t, t_1 \rangle$ and interchanges $t$ with $t_4$. We may compute $E := C(t, t_4)_{\chi, 2}$, obtaining

$$E = \langle \Omega(V_p), s_1, u_{12}, u_{2}^{'}, u, u' \rangle.$$  

We may assume that $a$ normalizes this and so also normalizes $[E, C_G(E_a)]$. Computation shows that this last group is $\langle s, s_1 \rangle$. Since $t \in \langle s \rangle, t_1 \in \langle s_1 \rangle$ and $t$ and $t_1$ are interchanged by $a$, it follows that the order of $s$ equals the order of $s_1$, which is $2^{n-1}$. This proves the proposition in the case $\dim V_0 = 0$.

Next suppose that $\dim V_0 = 2$. Since $Z(M) \lhd 1$, $N$ is the semi-direct product $\langle u, u' \rangle M$. The involutions of $H$ are conjugate to elements of $M$, $tM, uM, u'M$, and thus to elements of form

$$t_A, tt_A, ut_A, u't_A \ (A \subseteq N),$$

where $A \neq \emptyset$ in the first case, and in each case we need take only one subset $A$ of each cardinality. We have

$$C(t, t_A)_{\chi} = \begin{cases} 
\Omega(V_A \oplus V_{N-A}), & \text{if } |A| = 1, \\
\Omega(V_A) \Omega(V_0 \oplus V_{N-A}), & \text{if } 2 \leq |A| \leq n-2, \\
\Omega(V_N), & \text{if } A = N,
\end{cases}$$

$$C(t, ut_A)_{\chi} = \begin{cases} 
\Omega(L_0 \oplus V_N), & \text{if } A = \emptyset, \\
\Omega(L_0 \oplus V_A) \Omega(L_0 \oplus V_{N-A}), & \text{if } 1 \leq |A| \leq n-2, \\
\Omega(L_0 \oplus V_N), & \text{if } A = N,
\end{cases}$$

where $L_0$ is the orthogonal complement of $L_0$ in $V_0$; and

$$C(t, tt_A)_{\chi} = C(t, t_A)_{\chi}, \quad C(t, u't_A)_{\chi} = C(t, ut_A)_{\chi}.$$  

By Glauberman's $Z^*$-theorem, we have involutions $x, y$ of $C(t)$, distinct from $t$, such that $x^g = t, y^g = y$, for some $g$ in $G$. If $A \subseteq N, |A| > 1$, then $x \neq t_A, x \neq tt_A$. Otherwise we may take $y = t_A$ or $tt_A$, and $g$ normalizes $C(t, t_A) = C(t, tt_A)$. We can assume that $g$ normalizes $E = C(t, t_A)_{\chi, 2}$, where we compute that

$$E = \langle C(t, t_A)_{\chi}, u_{0i}, u_{0i}', u', u' \rangle.$$
for any \( i \) in \( A \). The Krull–Schmidt theorem determines \( Q(V_A) \) as that indecomposable direct factor of \( C(t, t_A)_\infty \) which has nontrivial center. Then \( \langle s \rangle \) is the center of the derived group of \( C_L(Q(V_A)) \), and must be normalized by \( g \). This implies that \( g \) centralizes \( t \), a contradiction.

Also \( x \neq t t_A \) if \( |A| = -1 \). Otherwise we may take \( y = t_A \) or \( t t_A \), so that \( g \) normalizes \( C(t, t_A)_\infty = Q(V_0 \oplus V_{N-A}) \). If \( i \in N - A \), then \( t_i^g \) is conjugate in \( Q(V_0 \oplus V_{N-A}) \) to \( t_i \), and we may assume \( t_i^g = t_i \). Then \( t t_A t_i \) is conjugate to \( t t_i \), which is conjugate in \( H \) to \( t t_A \), and so conjugate to \( t \). This contradicts the previous paragraph.

An argument like that used in the case \( \dim V_0 = 0 \) shows that \( x \neq u t_A \), \( x \neq u t_A \), for \( 1 \leq |A| \leq n - 2 \). Thus the only possibilities for \( x \) are the elements of the conjugacy classes of \( t_1, u, u', u t_N, u' t_N \) in \( C(t) \). The same arguments as before show that \( t_1 \) is conjugate to \( t \), and that statements (i), (iii), (iv) of the proposition hold. Further, if \( u \) and \( u t_N \) were both conjugate to \( t \), then there would exist an element \( g \) of \( G \) transforming \( \langle u, u t_N \rangle \) into \( \langle t, t_1 \rangle \), by (iii). Then \( g \) must map \( t_N \) on \( t t_1 \). However, \( g \) maps \( C(t, u, u t_N)_\infty = \Omega(V_N) \), which contains \( t_N \), on a subgroup of \( C(t, t_1)_\infty = \Omega(V_0 \oplus V_p) \), which does not contain \( t t_1 \), a contradiction. Thus \( u \) and \( u t_N \) are not both conjugate to \( t \). Similarly \( u' \) and \( u' t_N \) are not both conjugate to \( t \), so that statement (ii) of the proposition holds. This proves the proposition in the case \( \dim V_0 = 2 \).

Finally suppose that \( \dim V_0 = 1 \). Again \( Z(M) = 1 \), and the involutions of \( H \) are conjugate to elements of form

\[
C(t_A, t t_A, u t_A, u' t_A (A \subseteq N),
\]

where \( A \neq \emptyset \) in the first case and \( 1 \notin A \) in the third case, and in each case we take only one subset \( A \) of each cardinality. We have

\[
C(t, t_A)_\infty =\begin{cases} 
Q(V_0 \oplus V_{N-A}), & \text{if } |A| = -1, \\
Q(V_A) Q(V_0 \oplus V_{N-A}), & \text{if } 2 \leq |A| \leq n - 2, \\
Q(V_N), & \text{if } A = N,
\end{cases}
\]

\[
C(t, u t_A)_\infty =\begin{cases} 
Q(V_0 \oplus L \oplus V_p), & \text{if } A = \emptyset, \\
Q(L_1 \oplus V_A) Q(V_0 \oplus L \oplus V_{p-N}), & \text{if } 1 \leq |A| \leq n - 3, \\
Q(L_2 \oplus V_A), & \text{if } |A| = n - 2,
\end{cases}
\]

where \( L \) is the orthogonal complement of \( L_1 \) in \( V_1 \); and \( C(t, t t_A)_\infty = C(t, u t_A)_\infty = C(t, t_n)_\infty \) for \( A \neq \emptyset \), \( C(t, u')_\infty = M \).

By methods similar to those used in the case \( \dim V_0 = \emptyset 2 \), we may prove that the conjugacy classes of \( C(t) \) which are fused in \( G \) to \( t \) are at most the classes of \( t_1, u, u', u t_p, u' t_N \), and that \( u \) and \( u t_p \) cannot both be conjugate to \( t \), and \( u' \) and \( u' t_N \) cannot both be conjugate to \( t \). If \( t_1 \) is conjugate to \( t \), we see as before that the conclusions of the proposition hold. We omit the details.
Now suppose that $t_1$ is not conjugate to $t$. Then no four-subgroup of $G$ can have just two involutions conjugate to $t$. Since $ut_1$ is conjugate to $u$, $u't_1$ is conjugate to $ut_1$, and $u't_N$ is conjugate to $u't_1u't_1$, and $u't_N$ is conjugate to $t_1$, $t$ cannot be conjugate to $u$, $ut_1$ or $u't_N$. By Glauberman's $Z$*-theorem, $t$ is conjugate in $G$ to $u'$. There is an element of $G$ interchanging $t$ and $u'$. Since $C(t, u') = M$, $t$ and $u'$ are conjugate in the normalizer $N(M)$. Since $t$ is not conjugate in $G$ to any other element of $tM$,

$$C_{N(M)/M}(tM) = C(t)/M,$$

whose Sylow 2-subgroup $S := \langle uM, u'M \rangle$ is dihedral of order at least 8. It follows that $S$ is a Sylow subgroup of $N(M)/M$. Also $t$ is not conjugate to an element of $uM$. We may now apply the result of Gorenstein and Walter [5] to $N(M)/M$ to conclude that there is a series of normal subgroups of $N(M)$,

$$M \subseteq G_1 \subseteq G_2 \subseteq N(M),$$

such that $G_1/M$ and $N(M)/G_2$ have odd order, and $G_2/G_1$ is isomorphic with $PGL(2, r)$ for some odd prime power $r$.

It remains to show in this case that $N(M) = G$, so that case (b) of the Theorem holds. We show first that $C(tt_A) \subseteq N(M)$, for $A \subseteq N$. This is clear when $A = \emptyset$. Suppose that $t$ is conjugate in $C(tt_A)$ to some other element of $C(t, tt_A)$. We may calculate

$$C(t, tt_A)^{\infty} = C_{M}(tt_A)^{\infty} \langle u, u' \rangle$$

for a suitable choice of $C_{M}(tt_A)^{\infty}$, and find that elements of $C(t, tt_A)^{\infty}$ which are conjugate to $u'$ in $H$ are already conjugate to $u'$ in $C(t, tt_A)^{\infty}$. Since the conjugacy class of $u'$ in $C(t)$ is the only one fused to $t$, $t$ must be conjugate to $u'$ in $C(tt_A)$. Since $C(t, tt_A)^{\infty} \subseteq M$ and $u'$ centralizes $M$, $C(t, tt_A)^{\infty} = C(t, tt_A, u')^{\infty}$, which is contained in $C(u', tt_A)^{\infty}$. Since $t$ and $u'$ are conjugate in $C(tt_A)$, $C(t, tt_A)^{\infty} = C(u', tt_A)^{\infty}$. It follows easily that

$$C(t) \cap C(u', tt_A)^{\infty} = C(u') \cap C(t, tt_A)^{\infty},$$

for a suitable choice of $C(u', tt_A)^{\infty}$. If we denote this group by $R$, then

$$R := C_{M}(tt_A)^{\infty} \langle t, u' \rangle,$$

a proper subgroup of $C(t, tt_A)^{\infty}$. Thus $R/C(u', tt_A)^{\infty}$ is a proper subgroup of the 2-group $C(u', tt_A)^{\infty}C(u', tt_A)^{\infty}$. Hence there exists an element $g$ of $C(u', tt_A)^{\infty}$ which normalizes $R$ but does not lie in $C(t)$. Since the only elements of $R$ which are conjugate in $G$ to $t$ are $t$, $u'$ and $u't$. It follows that $t^g = u't$, so that $[t_A, g] = u'$, since $g$ centralizes $tt_A$. However, we can
compute that $t_A \in R'$, the derived group of $R$, which is normalized by $g$, so that $u' \in R' \subseteq M$, a contradiction.

Thus $t$ is not conjugate in $C(tt_A)$ to any other element of $C(t, tt_A)$. By Glauberman's $Z^*$-theorem,

$$C(tt_A) = C(t, tt_A) \cap O(C(tt_A)),$$

where the first factor is in $N(M)$. Now, $O(C(tt_A))$ is normalized by the subgroup $\langle t, u' \rangle$, whose involutions are conjugate in $N(M)$. Since $C(t) \subseteq N(M)$, it follows that $C(u') \subseteq N(M)$, $C(u't) \subseteq N(M)$, and so $O(C(tt_A)) \subseteq N(M)$ [4, p. 555]. Hence $C(tt_A) \subseteq N(M)$.

Since $u'$ is conjugate to $t$ in $N(M)$ and involutions of $M$ which are conjugate in $N(M)$ are already conjugate in $M$, $u't_A$ is conjugate to $tt_A$ in $N(M)$, and so $C(u't_A) \subseteq N(M)$, for all $A \subseteq N$.

If $\varnothing \neq B \subseteq N$, then $\langle t, u' \rangle$ is a four-subgroup of $C(t_B)$ whose involutions are conjugate in $N(M)$ and thus in $C(t_B)$. A four-subgroup of $C(tt_A)$ whose involutions are conjugate to $t$ in $G$ must be conjugate in $C(tt_A)$ to $\langle t, u' \rangle$, whose involutions are not all conjugate in $C(tt_A)$. Hence $t_B$ and $tt_A$ are not conjugate in $G$.

Since we can choose

$$C(t, ut_A)_{\alpha, 2} = C_M(ut_A)_{\alpha, 2} \langle t, u \rangle,$$

which contains no involution conjugate in $H$ to $u'$, this group contains a Sylow 2-subgroup of $C(ut_A)$, and there is no four-subgroup in $C(ut_A)$ whose involutions are all conjugate in $G$ to $t$. Thus $t_B$ and $ut_A$ are not conjugate in $G$.

Let $x \in G$, and choose any nonempty subset $B$ of $N$. Since $t$ and $t_B^{x}$ are not conjugate in $G$, there is an involution $y$ which commutes with both $t$ and $t_B^{x}$, such that $ty$ is conjugate in $G$ to $t$ or $t_B$. Then we see that $ty$ is conjugate in $C(t)$ to $t_A$ or $u't_A$, for some $A \subseteq N$, so that $y$ is conjugate to $tt_A$ or $u'tt_A$. Since $u'tt_A$ is conjugate to $ut_A$, we see in any case that $t_B^{x} \in C(y) \subseteq N(M)$. Thus $N(M)$ contains a normal subgroup $M_1$ of $G$ such that $t_B \in M_1$. Since $M$ is simple, $M \subseteq M_1$. Also, $N(M)/M$ does not have a subgroup isomorphic with $M$, so that $M$ must be characteristic in $M_1$. Thus $M$ is normal in $G$, $G = N(M)$, and conclusion (b) of the Theorem holds. This finishes the proof of the proposition.  

4. THE CASE $Z(M) \subseteq \langle s \rangle$; CONCLUSION

We now show that the Theorem holds in the case $Z(M) \subseteq \langle s \rangle$. We recall that $D$ is the elementary group $\langle t_1, t_2, \ldots, t_n \rangle$, and $t = t_B$.  

PROPOSITION (4A). Assume $Z(M) \subseteq \langle s \rangle$ and suppose the conclusions (a), (b) of the Theorem do not hold. Then there is an element $\tau$ of $N(D)$ such that the subgroup $G_0 = \langle M, \tau \rangle$ is isomorphic with $P\Omega(V)$, where $V$ is a quadratic space over $F_q$ having the same discriminant as $U$, and $\dim V = \dim U + 2$.

Proof. By (3A), elements of $N(D)$ permute the involutions $t_1, t_2, \ldots, t_n$ among themselves. The elements of $N(D) \cap M$ induce the symmetric group on $t_1, t_2, \ldots, t_{n-1}$, since $M$ contains elements permuting $V_1, V_2, \ldots, V_{n-1}$ in any desired manner. There is an element $y$ of $G$ which transforms $t_1$ to $t_n$, and we may assume that $y$ interchanges $t_1$ and $t_n$, by (3A) (iii). Then $y$ normalizes

$$C(t_1, t_n)_\infty = \Omega(V_0 \oplus V_\mu).$$

By a remark in Section 1, this group contains an element $c$ such that $t_i^x = t_i$ for $i = 2, \ldots, n - 1$. Then $yc$ lies in $N(D)$ and interchanges $t_1$ and $t_n$. Thus $N(D)$ induces the full symmetric group on $t_1, \ldots, t_n$. We have a surjective homomorphism $\lambda$ of $N(D)$ on the symmetric group $\Sigma_n$ on $\{1, \ldots, n\}$, such that

$$t_i^x = t_{\lambda(x)}(i),$$

for $i = 1, \ldots, n$, $x \in N(D)$. If $A \subseteq N$, and $|A| \geq 2$, then

$$C(\{t_i \mid i \notin A\})_\infty = \Omega(V_0 \oplus V_\lambda(A)).$$

Thus, if $A\lambda(x) \subseteq N$ also, then

$$\Omega(V_0 \oplus V_A)^x = \Omega(V_0 \oplus V_{A\lambda(x)}).$$

As in [13, Lemma 3], this will hold also if $|A| \leq 1$, and also $\Omega(V_A)^x = \Omega(V_{A\lambda(x)})$.

If $A$ is any proper subset of $\{1, \ldots, n\}$, we may choose a subset $B$ of $N$ and an element $y$ of $N(D)$ such that $A = B\lambda(y)$, and define

$$\Omega(A) = \Omega(V_B)^y, \quad \Omega'(A) = \Omega'(V_B)^y.$$

As in [13, Lemma 4], these are well-defined, and

$$\Omega(A)^x = \Omega(A\lambda(x)), \quad \Omega'(A)^x = \Omega'(A\lambda(x)),$$

for every proper subset $A$ of $\{1, \ldots, n\}$, $x \in N(D)$. Also, if $A$ and $B$ are disjoint proper subsets of $\{1, \ldots, n\}$, then $[\Omega'(A), \Omega'(B)] = 1$.

We now choose an element $g$ of $N(D)$ such that

$$\lambda(g) = (n - 3, n - 2, n - 1, n)$$

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For $i = 1, \ldots, n - 2$, we fix an isometry $v \mapsto v_i$ of $V_1$ on $V_i$. If $\alpha$ is an element of the symmetric group $\Sigma_{n-2}$ on $\{1, \ldots, n - 2\}$, we obtain an element $\sigma(\alpha)$ of $\Omega(\{1, \ldots, n - 2\})$ such that $v_i \sigma(\alpha) = v_{i\alpha}$, for every $v \in V_i$, $i = 1, \ldots, n - 2$. Then $\sigma$ is a homomorphism of $\Sigma_{n-2}$ into $N(D)$, with $\lambda(\sigma(\alpha)) = \alpha$, for $\alpha \in \Sigma_{n-2}$.

We may extend $\sigma$ to a homomorphism of $\Sigma_{n-1}$ into $N(D)$ as follows. Since $\sigma((n - 3, n - 2))$ is an element of $Q({n - 3, n - 2})$ interchanging $t_{n-2}$ and $t_{n-3}$, $\sigma((n - 3, n - 2))$ is an element of $Q({n - 2, n - 1})$ interchanging $t_{n-2}$ and $t_{n-1}$. Then $\sigma((n - 3, n - 2))\tau$ defines an isometry $v_{n-2} \mapsto v_{n-1}$ between $V_{n-2}$ and $V_{n-1}$. Now the equation $v_i \sigma(\alpha) = v_{i\alpha}$ defines $\sigma(\alpha)$ for all $\alpha \in \Sigma_{n-1}$, in such a way that

$$\sigma((n - 2, n - 1)) = \sigma((n - 3, n - 2))\tau.$$

Setting $\tau = \sigma((n - 2, n - 1))\tau$, we see that $\tau^3 = 1$, $(\tau\sigma((n - 2, n - 1)))^3 = (\sigma((n - 2, n - 1)))^3 = 1$, and $\tau$ commutes with $\sigma((i, i + 1))$ when $i = 1, \ldots, n - 3$, since $\tau \in Q({n - 2, n - 1})\tau = Q({n - 1, n})$ and $\sigma((i, i + 1)) \in Q((i, i + 1))$. Since $\lambda(\tau) = (n - 1, n)$, $\tau$ normalizes $Q(V_0 \oplus V_i \oplus V_j)$, whenever $1 \leq i < j \leq n - 2$.

By the results of [13], $G_0 = \langle M, \tau \rangle$ is isomorphic with $P\Omega(V)$ or $\Omega(V)$, for a quadratic space $V$ with the asserted properties. (Here we are using Hypothesis 6.) In fact $G_0$ is isomorphic with $P\Omega(V)$ for otherwise we must have the case $\dim V_0 = 0$, $t \in M$, and we find that $C_{G_0}(M)$ contains a four-subgroup, contradicting the fact that $K = C(M)$ has cyclic Sylow 2-subgroup, by (2C). This proves the proposition.

As remarked in [13], $G_0$ contains $M$ in the “natural” way, i.e., $U$ can be identified with a nondegenerate subspace of codimension 2 in $V$ so that $M$ is the subgroup of $G_0 = P\Omega(V)$ corresponding to elements of $\Omega(V)$ which are the identity on the orthogonal complement of $U$ in $V$. Also, $t$ is the involution corresponding to the involution of $\Omega(V)$ with positive subspace $U$.

**PROPOSITION (4B).** In (4A), let $G_1 = C(t)G_0$. Then $G_1$ is a subgroup of $G$, $G_0$ is normal in $G_1$, and $G_1$ is a semidirect product

$$G_1 = WG_0, \quad W \cap G_0 = 1,$$

where $W$ is cyclic of odd order $d$, and $W$ acts faithfully on $G_0$ by field automorphisms.

**Proof.** By the structure of $P\Omega(V)$, the intersection of $K = C(M) \cap C(t)$ with $G_0$ has order $\frac{1}{2}(q - \epsilon)$ if $\dim V_0 = 0$, $q - \epsilon$ otherwise, By (2C), (3A) (iv) and Hypothesis 5, $K \subseteq G_0$. In particular, $C(G_0) = Z(G_0) = 1$. Also the structure of $C_{G_0}(t)$ shows that $G_0$ contains a Sylow 2-subgroup of $C(t)$.

We can assume that the field automorphism of $M$ induced by the element $w$ of (2D) fixes the involutions $t_1, \ldots, t_{n-1}$, so that $w \in C(D)$. Thus $w$ nor-
malizes every Ω(A), where A ⊂ \{1, ..., n\}, and so \(w\) normalizes \(G_0\). It follows from (2D) that \(C(t)\) normalizes \(G_0\), so that \(G_1\) is a subgroup of \(G\) with \(G_0\) as a normal subgroup, \(G_1 = \langle w \rangle G_0\), and \(G_1/G_0\) has odd order. Since \(C(G_0) = 1\), \(G_1\) is isomorphic with a subgroup of the automorphism group of \(G_0\). By the remarks of Section 1, \(G_1\) has the structure asserted.

**Proposition (4C).** \(G_1 = G\).

**Proof.** By the structure of \(G_1\), exactly three conjugacy classes of \(C(t)\) not containing \(t\) are fused to \(t\) in \(G_1\). By (3A) (ii), it follows that an element \(x\) of \(G_1\) is conjugate to \(t\) in \(G\) only if it is already conjugate to \(t\) in \(G_1\). Also, since \(C(t) \subseteq G_1\), if \(x\) is an element of \(G_1\) conjugate to \(t\) and \(y\) is an element of \(G\) such that \(x^y \in G_1\), then \(y \in G_1\).

We shall now show that \(C(t_1) \subseteq G_1\) when \(A \subseteq \{1, ..., n\}, |A| = 2\). We may assume \(A = \{1, 2\}\). If \(G_0\) is identified with \(PΩ(V)\) in (4A), \(t_1\) and \(t_2\) are the elements corresponding to involutions in \(Ω(V)\) with orthogonal negative subspace \(V_1\) and \(V_2\) of dimension 2. Let \(E\) be the set of involutions of \(C(t_1, t_2) = C_{G_1}(t_1, t_2)\) which are conjugate to \(t\) in \(G\). If \(x \in E\), then \(x \in G_0\) since \(G_1/G_0\) is odd, and \(x\) is the element corresponding to an involution \(\bar{x}\) of \(Ω(V)\) of type 2 which leaves \(V_1\) and \(V_2\) invariant. For \(i = 1, 2\), we write \(d_i(x)\) for the dimension of the intersection of \(V_i\) with the negative subspace of \(\bar{x}\). Since \(x\) is conjugate to \(t\), \(C(x, t_{12})\) may be calculated within \(G_1\). In particular, we may calculate \(C(x, t_{12})_{\bar{x}}\) within \(G_0\). If \(d_1(x) + d_2(x) < 2\), then \(C(x, t_{12})_{\bar{x}} \cong Ω(W') \times Ω(Y)\), where \(\dim W = \dim V + d_1(x) + d_2(x) - 6\), \(\dim Y = 4 - d_1(x) - d_2(x)\). If \(d_1(x) + d_2(x) = 2\), then \(C(x, t_{12})_{\bar{x}} \cong Ω(W')\), where \(\dim W = \dim V - 4\). It follows that if \(x' \in E\) and \(x\) is conjugate to \(x'\) in \(C(t_{12})\), then \(d_1(x') + d_2(x') = d_1(x') + d_2(x')\). Also, for \(i = 1, 2\), \(t_i x\) is conjugate in \(G\) to \(t\) if \(d_i(x) = 1\), while \(t_{12} x\) is conjugate to \(t_{12}\) if \(d_1(x) + d_2(x) = 1\).

Now suppose \(g \in C(t_{12})\). We choose \(x \in E\) such that \(d_1(x) = 0, d_2(x) = 1\). (For example, \(x = u_{2n}\) will do.) Since \(d_1(t_1) = 2, d_2(t_1) = 0, t_1\) and \(x^g\) are not conjugate in \(C(t_{12})\). Thus \(t_1 x^g\) contains an involution \(v\), \(v\) commutes with \(t_1\) and \(x^g\), and \(t_1 v\) is conjugate in \(C(t_{12})\) to \(t_1\) or \(x^g\). In particular, \(t_1 v \in E\). Also, \(d_1(t_1 v) + d_2(t_1 v) \neq 0\). Since \(t_1 v \neq t_1, d_1(t_1 v) \neq 2\). If \(d_1(t_1 v) = 1\), then \(v = t_1(t_1 v)\) is conjugate in \(G\) to \(t\), so that \(x^g \in C(v) \subseteq G_1\), and so \(g \in G_1\). If \(d_1(t_1 v) = 0\) and \(d_2(t_1 v) = 1\), then \(t_1 v\) is conjugate in \(G\) to \(t_1\), and \(x^g \in C(t_{12}) \cap C(v) \subseteq C(t_{12} t_1 v) \subseteq G_1\), so again \(g \in G_1\). Finally, if \(d_2(t_1 v) = 2\), then \(t_1 v \in t_2\), which is not conjugate to \(x\) in \(C(t_{12})\) since \(d_1(t_1 v) + d_2(t_1 v) = 2 \neq d_1(x) + d_2(x)\). Thus \(x v x^g\) must be conjugate to \(x\) in \(C(t_{12})\). Since \(v = t_1 v, t_{12} x\) is conjugate to \(x\) in \(C(t_{12})\). But \(t_{12} x\) is conjugate in \(G\) to \(t_{12}\) since \(d_1(x) + d_2(x) = 1\). Then \(t_{12}\) is conjugate in \(G\) to \(x\) and thus to \(t\), contradicting (3A) (i). Thus \(C(t_{12}) \subseteq G_1\).

If \(y \in G\) and \(t_{12} y \in C(t)\), then \(t y^{-1} \in C(t_{12}) \subseteq G_1\), and so \(y \in G_1\). By the
structure of $G_1$; every involution of $G_1$ is conjugate in $G_1$ to some element of $C(t)$. It follows that an element of $G_1$ is conjugate in $G$ to $t_{13}$ only if it is already conjugate in $G_1$ to $t_{12}$. In particular, if $x$ is an involution of $C(t_1)$ which is conjugate to $t_{12}$ in $G$, then $t_1x$ is conjugate to $t$, $t_{12}$, or $t_{123}$ in $G_1$.

Next we show that $C(t_{123}) \subseteq G_1$. We may assume $t_{123}$ is not conjugate to $t_{12}$ in $G$. Choose an element $x$ of $G_1$ such that $x$ is conjugate in $G$ to $t$ and $xt_{123}$ is conjugate in $G$ to $t_{123}$. (For example, $x = u_{34}$ will do.) Then $x$ cannot be conjugate to $t_3$ in $C(t_{123})$. If $g \in C(t_{123})$, there exists an involution $v$ commuting with both $t_3$ and $x^a$, such that $t_3v$ is conjugate in $C(t_{123})$ to $t_3$ or $x^a$. Then $t_3v$ is conjugate in $G$ to $t$, and $v \in C(t_3) \subseteq G_1$. By (3A) (iii), $v = t_3(t_3v)$ is conjugate in $G_1$ to $t$ or $t_{12}$. Then $x^a \in C(v) \subseteq G_1$, so that $g \in G_1$, $C(t_{123}) \subseteq G_1$.

Finally, let $g$ be any element of $G$. Since $t_4$ and $t_{12}$ are not conjugate in $G$, there exists an involution $v$ commuting with $t_4$ and $t_{12}$, such that $t_4v$ is conjugate in $G$ to $t$ or $t_{12}$. If $t_4v$ is conjugate to $t$, then $v = t_4(t_4v)$ is conjugate in $G_1$ to $t$ or $t_{12}$, by (3A) (iii). If $t_4v$ is conjugate to $t_{12}$, then $v$ is conjugate in $G_1$ to $t$, $t_{12}$ or $t_{123}$. In any case, $t_{12}^g \in C(v) \subseteq G_1$, and so $g \in G_1$. This proves the proposition, and completes the proof of the Theorem in the case that $Z(M) \subseteq \langle s \rangle$.

5. THE CASE $Z(M) \subseteq \langle s \rangle$

We now suppose that $Z(M) \subseteq \langle s \rangle$, so that $\dim V_0 = 0$, and $t$ is one of the involutions $t_N$, $t_n$, $t_Nt_n$, where $N = \{1, \ldots, n-1\}$. In this case $H$ is the semidirect product of $M$ with $\langle u, u' \rangle$, and we see that the involutions of $H$ are all conjugate in $H$ to involutions of the form

$$t_A(\emptyset \subset \emptyset \subseteq N), \quad t_A(\emptyset \subset A \subseteq N), \quad ut_A(\emptyset \subset P), \quad u't_A(\emptyset \subset Q),$$

where $P = N - \{1\}$, $Q = N - \{2\}$, and in each case we need take only one subset $A$ of each cardinality. Using (2E), we find $C(t, t_A(\emptyset \subset N))$, $C(t, ut_A(\emptyset \subset P))$, $C(t, u't_A(\emptyset \subset Q))$ as in the case $\dim V_0 = 0$ of (3A), while $C(t, t_A(\emptyset \subset N)) = C(t, t_A(\emptyset \subset A)) = C(t, t_A(\emptyset \subset N))$ for $A \neq \emptyset$, and $C(t, t_n(\emptyset \subset N)) = M$.

Setting $G_1 = C(t_Nt_n)$, we note that $G_1 \supseteq H$, so that $G_1$ satisfies the hypotheses of the Theorem. Each involution of $C_G(t)$ is conjugate to one of the elements of $H$ mentioned above.

LEMMA (5A). *The element $w$ of (2D) may be assumed to lie in $G_1$.*

*Proof.* $w$ normalizes the group $K$, which has Sylow 2-subgroup $T = \langle Z(M), s \rangle$. By the Frattini argument,

$$K \langle w \rangle = KN_{K \langle w \rangle}(T) = KO(N_{K \langle w \rangle}(T)) = KO(C_{K \langle w \rangle}(T)),$$
since $K\langle w \rangle$ has a normal 2-complement (see the proof of (2D)), and

$$[T, O(N_{K\langle w \rangle}(T))] \subseteq T \cap O(N_{K\langle w \rangle}(T)) = 1.$$  

Hence $w = xw'$, where $x \in K$ and $w'$ is an element of odd order centralizing $T$. Then we may replace $w$ by $w'$ in (2D). This proves the lemma.

**Proposition** (5B). Assume $Z(M) \subseteq \langle s \rangle$, and suppose $G \neq C(t) O(G)$. Let $G_1 = C(t_n t_n)$. Then, $t = t_n$ or $t_n$, and the involutions of $D = \langle t_1, \ldots, t_n \rangle$ which are conjugate in $G_1$ to $t_n$ are precisely $t_1, \ldots, t_n$.

**Proof.** By Glauberman's $Z^*$-theorem, there exist involutions $x, y$ among the elements $t_A, t_A t_n, u t_A, u' t_A$ which we have mentioned, such that $x \neq t$, $y \neq t$, $x^g = t$, $t^g = y$, for some $g$ in $G$. We investigate the possible $x, y$, omitting details of calculations where they resemble those used earlier, in Section 3.

First, $x \neq t_A$, for $A \subseteq N$, $|A| > 1$. Otherwise we may assume $y = t_A, t_A t_n, t_{N-A} t_n$. In each case, since $g$ transforms $C(t, x)_x$ to $C(t, y)_x$, we obtain a contradiction as in the proof of (3A).

If $x = t_1$, then we may take $y = t_1, t_1 t_n, t_p$ or $t_p t_n$. We may take $C(t, x)_{x,2} = C(t, y)_{x,2} = E$, where

$$E = \langle \Omega(V_p), s_1, u_{12}, u'_{12}, u, u' \rangle,$$

as in (3A). We calculate that

$$[E, C_E(E_{s_1})] = \langle s, s_1 \rangle,$$

$$C_{E_E}(E_{s_1}) = \langle s_1 s, s_1^2 \rangle,$$

and may assume that $g$ normalizes $E$. Since $t_1 \in \langle s, s_1 \rangle$, $t$ must lie in $\langle s, s_1 \rangle$, so that $t = t_n$, and $s$ and $s_1$ must have equal orders. Then $s_1 s$ has larger order than $s_1^2$, so that $\langle t_1 t_n \rangle$ is characteristic in $\langle s_1 s, s_1^2 \rangle$, and $g$ must centralize $t_1 t_n$, so that $y = t_1$. Since $g$ normalizes $Z(E_{s_1}) = \langle t_p \rangle$, $g$ centralizes $t_1 t_p t_n = t_n t_n$, so that $t_1, \ldots, t_n$ are conjugate in $G_1$.

Next, $x \neq t_A t_n$, for $A \subseteq N$, $|A| \neq n - 2$. Otherwise we may take $y = t_A t_n$ or $t_{N-A} t_n$, by what we have already proved. We may take $C(t, x)_{x,2} = C(t, y)_{x,2} = F$, a certain subgroup of $H$, and we may assume $g$ normalizes $F$. If $A = \emptyset$ or $N$, then $F = H, F = M$, and $g$ normalizes $Z(M) = \langle t_n \rangle$, so that $t$ differs from $x, y$ and $t_n$. Since $t, x, y$, $t_n \in \langle t_n \rangle$, we must have $x - y$. Now $g$ normalizes $C_H(M) = \langle t_n \rangle$, but does not centralize $t_n$. Thus $s$ must have order 2, $s = t_n$, and $\langle u, u' \rangle$ is dihedral of order 8. Since

$$(u' u^g)^2 = t_n^g = t_n^2.$$
we have $(uu')^a = uu'a$ or $uu't_n a$, where $a \in M$ and $(uu'a)^2 = t_N t_n$. This implies that $a^u u'a = t_N$. Since the automorphism of $M = \Omega(U)$ induced by $uu'$ is the same as that given by an involution $b$ of type 2 in $O(U)$,

$$C_M((uu')^a) \cong C_M(uu') \cong \Omega(W),$$

where $W$ is a nondegenerate subspace of codimension 2 in $U$. However, the automorphism of $M$ induced by $uu'a$ or $uu't_n a$ is the same as that given by $ba$, where $a^b a = t_N$, i.e., $ba$ is a semi-involution. This implies that $C_M((uu')^a)$ is isomorphic with $SL(n - 1, q)$ or $SU(n - 1, q)$, a contradiction.

If $2 \leq |A| \leq n - 3$, we may assume $1 \in A$, $2 \not\in A$. Then we may take

$$F = \langle \Omega(V_A) \Omega(V_{N-A}), u_{12}, u'_{12}, u, u' \rangle.$$

Then $g$ must map $\Omega(V_A)$ and $\Omega(V_{N-A})$ on themselves, or interchange them. Computation shows that

$$Z(\langle \Omega(V_A) \rangle) \cap Z(\langle \Omega(V_{N-A}) \rangle) = \langle s \rangle.$$

Thus $g$ maps $t_n$ on itself. Also $\langle t_A \rangle = Z(\Omega(V_A))$ is mapped on itself or $\langle t_{N-A} \rangle = Z(\Omega(V_{N-A}))$. Thus $g$ maps $x$ on $x$ or $t_{N-A} t_n$, a contradiction.

If $|A| = 1$, say $A = \{1\}$, then $g$ normalizes $[F, C_p(F_\omega)] = \langle s, s_1 \rangle$, so that $t - (t_1 t_n)^{q} - t_n$. Also $y = t_1 t_n \cdots x$, and the order of $s$ does not exceed that of $s_1$. Since $g$ normalizes $C_p(F_\omega)^\prime = \langle s_1, s_1^2 \rangle$, but $g$ does not centralize $t_1 t_n$, the order of $s$ must be less than that of $s_1$. Now $F$ is a semidirect product

$$F := \Omega(V_\rho) R,$$

$$R = \langle uu'u_{13}, uu_{13} \rangle \times \langle u_{13} u'_{12}, u_{13} \rangle,$$

where the factors of $R$ are dihedral groups of different orders (at least 8), since $(uu'u_{13})^2 = s$, $(u_{13} u'_{12})^2 = s_1$. By applying the Krull–Schmidt theorem, we see easily that an automorphism of $R$ must fix $t_n$. Thus $g$ must normalize $\Omega(V_\rho) \langle t_n \rangle$, which does not contain $x$, a contradiction since $x^g = t_n$. This completes the proof that $x \neq t_1 t_n$ when $|A| \neq n - 2$.

Now suppose $x = t_A t_n$, where $A \subseteq N$, $|A| = n - 2$. We may assume $A = P$. By what we have proved, we may take $y = x$, so that $g$ normalizes $F = C(t, x)_{\omega, 2}$. This is the same group as in the case $x = t_1 t_n$ just considered, and a similar argument leads to the conclusion that $t = t_N$ and $g$ interchanges $t_i$ with $t_n$, so that $g$ maps $x t_i \Rightarrow t_N t_n$ on itself.

At this point we have shown that if an element $x$ of $D = \langle t_1, \ldots, t_n \rangle$ other than $t$ is fused in $G$ to $t$, then either $t = t_n$ and $x = t_i$ for some $i$, or $t = t_N$ and $x = t_A t_n$, where $|A| = n - 2$. In both cases, $t_1, \ldots, t_n$ are conjugate in $C(t_N t_n) \cdots G_1$. If some other element $a = t_A (|A| > 1)$ or $a$
$t_A t_n(|A| > 0)$ is conjugate to $t_n$ in $G_1$, then $at_N t_n = t_{N-A} t_n(|N - A| < n - 2)$ or $at_N t_n = t_{N-A}(N - A \neq N)$ is conjugate to $t_N$, leading to a contradiction whether $t$ is $t_n$ or $t_N$. Thus the proposition holds in this situation.

The method used in (3A) shows that $x \neq u t_A$ when $A < P$, $1 \leq |A| \leq n - 3$, and $x \neq u't_B$ when $B \subseteq Q$, $1 \leq |B| \leq n - 3$.

If $x = u$, then we may take $y = u$, $u t_A$, $u'$ or $u' t_O$. In each case there are the three possibilities $t = t_n$, $t_N$ or $t_N t_n$ to consider, making 12 cases in all. Since the arguments are much the same in every case, we illustrate with the case $y = u'$, $t = t_N$. We may take

$$C(t, u)_{x_2} = \Omega(tL \oplus V_P) \times \langle t_N, t_n, u \rangle,$$
$$C(t, u')_{x_2} = \Omega(tL' \oplus V_Q) \times \langle t_N, t_n, u \rangle,$$

and we may assume $g$ maps $C(t, u)_{x_2}$ on $C(t, u')_{x_2}$. Since $g$ maps $u$ on $t_N$ and $t_n$ on $u'$, $g$ maps $t_n$ on $t_n$, $t_N t_n$, $t_n u'$, or $t_N t_n u'$. In the first case $g$ maps $ut_n$ on $t_N t_n$, but $ut_n$ is conjugate to $u$, and so $t_N t_n$ is conjugate to $t$, a contradiction. In the second case the same argument shows that $t_n$ is conjugate to $t$, a contradiction. In the third case, since $t_n u'$ is conjugate to $u'$, we see that $t_n$ is conjugate to $t$, a contradiction. In the fourth case, we note that $g$ maps $\Omega(tL \oplus V_P)$ on $\Omega(tL' \oplus V_Q)$, and we may assume $g$ maps $t_p$ on $t_q$, by the remarks of Section 1. Then $g$ maps $t_p t_n$ on $t_q t_n u'$, since $t_p t_n u'$ is conjugate to $u'$, $t_p t_n$ is conjugate to $t$. In this way we see that some element of $D$ other than $t$ is fused to $t$ in $G$, and thus the result holds in this case.

Similar arguments prove the proposition when $x = u'$. Since the hypotheses of the theorem are still satisfied with $u t_p$ in place of $u$ or $u' t_O$ in place of $u'$, we have the result when $x = u t_p$ or $x = u' t_O$. Since this exhausts the possibilities for $x$, the proposition is proved.

**PROPOSITION (5C).** Assume $Z(M) \subseteq \langle s \rangle$ and $G \neq C(t) O(G)$. Then $G_1 = C(t, t_n)$ has a normal subgroup $G_0$ isomorphic with $\Omega(V)$, where $V$ is a quadratic space over $F_2$, $\dim V = \dim U + 2$, and $V$ has the same discriminant as $U$, and $G_1$ is a semi-direct product

$$G_1 = \langle W G_0 \rangle,$$

where $W$ is a cyclic group of odd order $d$, which acts faithfully on $G_0$ by field automorphisms.

**Proof.** The group $G_0 \cong \Omega(V)$ is constructed as in (4A), using the results of [13]. We see also that $G_1 = G_0 / \langle t_n t_n \rangle$ satisfies the hypotheses of the Theorem, and the results of Section 4 show that $G_1$ is a split extension of $\Omega(V)$ by a cyclic group of odd order acting faithfully by field automorphisms. The structure of $G_1$ follows easily.
PROPOSITION (5D). $G = G_1$.

Proof. By the structure of $G_1$, the intersection of $K = C(M) \cap C(t)$ with $G_1$ has order $q - 1$, and this intersection contains the Sylow 2-subgroup $\langle Z(M), s \rangle$ of $K$. By Hypothesis 5, $K \subseteq G_0$. Now (5A) and (2D) imply that $C(t) \subseteq G_1$.

If $x$ is an involution of $G_1$ distinct from $t_{nt_n}$, then $t_{nt_n}$ does not lie in an indecomposable direct factor of $C(t_{nt_n}, x)$. If $x$ is not conjugate in $G_1$ to $t_n$, then $x$ does lie in an indecomposable direct factor of $C(t_{nt_n}, x)$, and it follows easily that $x$ is not conjugate to $t_{nt_n}$ in $G$. If $t_n$ is conjugate to $t_{nt_n}$ in $G$, then $t = t_n$. There must exist an element $g$ of $G$ interchanging $t_n$ and $t_{nt_n}$. Then $g$ centralizes $t_n$, contradicting the fact that $C(t) \subseteq C(t_{nt_n})$. Thus $t_{nt_n}$ is not fused in $G$ to any other element of $G_1$.

By Glauberman’s $Z^*$-theorem, $G = G_1O(G)$. Since $O(G) = 1$, $t_{nt_n}$ and $t$ invert $O(G)$. Then all the conjugates of $t_{nt_n}$ in $G_1$ centralize $O(G)$. Since these conjugates generate $G_1$, $G_1$ centralizes $O(G)$. Hence $O(G) = 1$, $G = G_1$. This proves the proposition, and completes the proof of the Theorem in all cases.

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