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Traveling wave solutions in delayed nonlocal diffusion systems with mixed monotonicity ☆

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ABSTRACT

This paper deals with the existence of traveling wave solutions in delayed nonlocal diffusion systems with mixed monotonicity. Based on two different mixed-quasimonotonicity reaction terms, we propose new definitions of upper and lower solutions. By using Schauder's fixed point theorem and a new cross-iteration scheme, we reduce the existence of traveling wave solutions to the existence of a pair of upper and lower solutions. The general results obtained have been applied to type-K monotone and type-K competitive nonlocal diffusive Lotka–Volterra systems.

1. Introduction

Recently, the following nonlocal diffusion system is widely discussed in population dynamics [5,8]:

$$\frac{\partial U(x,t)}{\partial t} = \int_{\mathbb{R}} J(x-y)\big[U(y,t) - U(x,t)\big]\,dy + f\big(U(x,t)\big), \tag{1.1}$$

where $t \in \mathbb{R}$, $x, y \in \mathbb{R}$, $U \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and J is a kernel function. As mentioned in Murray [10, pp. 408–413], nonlocal diffusion systems of the form (1.1) are more accurate than reaction–diffusion systems in modeling the spatial diffusion of the individuals in some biology areas, such as embryological development process. Moreover, the nonlocal diffusion systems are also proposed in other practical fields, for example, phase transition model [2], material science [1], network model [6] and lattice dynamical systems [3,4].

Similar to reaction–diffusion systems, time delay is inevitable in modeling nonlocal diffusion phenomena. Due to its importance in determining the long time behavior of the corresponding initial value problem, many researchers have studied the existence of traveling wave solutions for the delayed nonlocal diffusion systems. Particularly, Pan et al. [12] and Pan [11] considered the traveling wavefronts of the following delayed nonlocal diffusion system

$$\frac{\partial u_i(x,t)}{\partial t} = \int_{\mathbb{R}} J_i(x-y)\big[u_i(y,t) - u_i(x,t)\big]\,dy + f_i\big(u_t(x)\big), \quad i = 1,\dots,n, \tag{1.2}$$

where the nonlinear reaction terms f_i $(i = 1,\dots,n)$ satisfy the quasimonotonicity (QM) condition or the exponential quasi-monotonicity (EQM) condition. As stated in our paper [13], it is quite common that the reaction terms in a virtual model

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may not satisfy either the QM condition or the EQM condition, such as type-K Lotka–Volterra systems. For this reason, we will consider the existence of traveling wave solutions of (1.2) with mixed quasimonotonicity reaction terms and this constitutes the purpose of the current paper. That is, we will consider three-dimensional nonlocal diffusion systems of the form

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = \displaystyle\int_{\mathbb{R}} J_1(x-y)\big[u_1(y,t)-u_1(x,t)\big]\,dy + f_1\big(u_1(x,t-\tau_{11}),u_2(x,t-\tau_{12}),u_3(x,t-\tau_{13})\big),\\ \dfrac{\partial u_2(x,t)}{\partial t} = \displaystyle\int_{\mathbb{R}} J_2(x-y)\big[u_2(y,t)-u_2(x,t)\big]\,dy + f_2\big(u_1(x,t-\tau_{21}),u_2(x,t-\tau_{22}),u_3(x,t-\tau_{23})\big),\\ \dfrac{\partial u_3(x,t)}{\partial t} = \displaystyle\int_{\mathbb{R}} J_3(x-y)\big[u_3(y,t)-u_3(x,t)\big]\,dy + f_3\big(u_1(x,t-\tau_{31}),u_2(x,t-\tau_{32}),u_3(x,t-\tau_{33})\big), \end{cases} \tag{1.3}$$

where $t,x,y\in\mathbb{R}$, τ_{ij} $(1\leqslant i,j\leqslant 3)$ denote time delays, $J_i(x)$ is even and $\int_{\mathbb{R}} J_i(x)\,dx$ is finite, $f_i\in C(\mathbb{R}^3,\mathbb{R})$ and satisfies mixed quasimonotonicity condition which will be specified later, $i=1,2,3$. By using the operators H,F and a new cross-iteration scheme, we will construct a subset in the Banach space $C(\mathbb{R},\mathbb{R}^3)$ equipped with the exponential decay norm and reduce the existence of traveling wave solutions to the existence of a new admissible pair of upper and lower solutions, which are different from Pan [11] and Pan et al. [12].

The rest of this paper is organized as following. In Section 2, we reduce the existence of traveling wave solutions to the existence of fixed point of the operator F. In Section 3, we obtain the existence of traveling wave solutions if the nonlinear reaction term of the delayed nonlocal diffusion system satisfies the mixed-quasimonotonicity case 1 (MQM-1) condition. In Section 4, we get similar results for the MQM-2 case. In the last section, we apply our main results to the three-dimensional delayed K-type Lotka–Volterra nonlocal diffusion systems and prove the existence of traveling wave solutions.

2. Preliminaries

Throughout this paper, we employ the usual notations for the standard ordering in $\mathbb{R}^3$. That is, for $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$, we denote $u\leqslant v$ if $u_i\leqslant v_i$, $i=1,2,3$; $u<v$ if $u\leqslant v$ but $u\neq v$; and $u\ll v$ if $u\leqslant v$ but $u_i\neq v_i$, $i=1,2,3$. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^3$ and $\|\cdot\|$ denote the supremum norm in $C([-\tau,0],\mathbb{R}^3)$, where $\tau\triangleq\max\{\tau_{ij}\mid 1\leqslant i,j\leqslant 3\}$.

A traveling wave solution of (1.3) is a special translation invariant solution of the form $u_1(x,t)=\varphi_1(x+ct)$, $u_2(x,t)=\varphi_2(x+ct)$, $u_3(x,t)=\varphi_3(x+ct)$, where $\varphi_1,\varphi_2,\varphi_3\in C^1(\mathbb{R},\mathbb{R})$ are the profiles of the wave that propagates through the one-dimensional spatial domain at a constant speed $c>0$. Substituting $u_1(x,t)=\varphi_1(x+ct)$, $u_2(x,t)=\varphi_2(x+ct)$, $u_3(x,t)=\varphi_3(x+ct)$ into (1.3) and denoting $x+ct$ by t, we find that (1.3) has a traveling wave solution if and only if the following wave equations

$$\begin{cases} c\varphi_1'(t) = \displaystyle\int_{\mathbb{R}} J_1(y-t)\big[\varphi_1(y)-\varphi_1(t)\big]\,dy + f_1^c(\varphi_{1t},\varphi_{2t},\varphi_{3t}),\\ c\varphi_2'(t) = \displaystyle\int_{\mathbb{R}} J_2(y-t)\big[\varphi_2(y)-\varphi_2(t)\big]\,dy + f_2^c(\varphi_{1t},\varphi_{2t},\varphi_{3t}),\\ c\varphi_3'(t) = \displaystyle\int_{\mathbb{R}} J_3(y-t)\big[\varphi_3(y)-\varphi_3(t)\big]\,dy + f_3^c(\varphi_{1t},\varphi_{2t},\varphi_{3t}) \end{cases} \tag{2.1}$$

with asymptotic boundary conditions

$$\lim_{t\to\pm\infty}\varphi_i(t)=\varphi_{i\pm},\quad i=1,2,3 \tag{2.2}$$

have a solution $(\varphi_1(t),\varphi_2(t),\varphi_3(t))$ on $\mathbb{R}$, where $f_i^c(\varphi_{1t},\varphi_{2t},\varphi_{3t}): C([-c\tau,0],\mathbb{R}^3)\to\mathbb{R}$ is given by

$$f_i^c(\varphi_{1t},\varphi_{2t},\varphi_{3t})=f_i\big(\varphi_{1t}^c,\varphi_{2t}^c,\varphi_{3t}^c\big),\qquad \varphi_{it}^c(s)=\varphi_{it}(cs)=\varphi_i(t+cs),\quad s\in[-\tau,0],\ i=1,2,3,$$

where $(\varphi_{1-},\varphi_{2-},\varphi_{3-})$ and $(\varphi_{1+},\varphi_{2+},\varphi_{3+})$ are two equilibria of (2.1). Without loss of generality, we let $(\varphi_{1-},\varphi_{2-},\varphi_{3-})=(0,0,0)$ and $(\varphi_{1+},\varphi_{2+},\varphi_{3+})=(k_1,k_2,k_3)$. Then boundary conditions (2.2) become

$$\lim_{t\to-\infty}\varphi_i(t)=0,\qquad \lim_{t\to+\infty}\varphi_i(t)=k_i,\quad i=1,2,3. \tag{2.3}$$

Let

$$C_{[\mathbf{0},\mathbf{M}]}\big(\mathbb{R},\mathbb{R}^3\big)=\big\{(\varphi_1,\varphi_2,\varphi_3)\in C\big(\mathbb{R},\mathbb{R}^3\big):\ 0\leqslant\varphi_i(t)\leqslant M_i,\ i=1,2,3,\ t\in\mathbb{R}\big\},$$

where $k_i\leqslant M_i$, $i=1,2,3$.

For $(\varphi_1, \varphi_2, \varphi_3) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^3)$ and constants $\beta_i > 0$ $(i = 1, 2, 3)$, define $H = (H_1, H_2, H_3) : C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^3) \to C(\mathbb{R}, \mathbb{R}^3)$ by

$$H_i(\varphi_1, \varphi_2, \varphi_3)(t) = \int_{\mathbb{R}} J_i(y - t)\big[\varphi_i(y) - \varphi_i(t)\big]\,dy + f_i^c(\varphi_{1t}, \varphi_{2t}, \varphi_{3t}) + \beta_i \varphi_i(t), \quad i = 1, 2, 3. \tag{2.4}$$

Then (2.1) can be rewritten as

$$\begin{cases} c\varphi_1'(t) = -\beta_1\varphi_1(t) + H_1(\varphi_1, \varphi_2, \varphi_3)(t), \\ c\varphi_2'(t) = -\beta_2\varphi_2(t) + H_2(\varphi_1, \varphi_2, \varphi_3)(t), \\ c\varphi_3'(t) = -\beta_3\varphi_3(t) + H_3(\varphi_1, \varphi_2, \varphi_3)(t). \end{cases} \tag{2.5}$$

Define $F = (F_1, F_2, F_3) : C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^3) \to C(\mathbb{R}, \mathbb{R}^3)$ by

$$F_i(\varphi_1, \varphi_2, \varphi_3)(t) = \frac{1}{c} e^{-\frac{\beta_i}{c}t} \int_{-\infty}^{t} e^{\frac{\beta_i}{c}s} H_i(\varphi_1, \varphi_2, \varphi_3)(s)\,ds, \quad i = 1, 2, 3. \tag{2.6}$$

Then it is clear that the fixed point of F satisfies (2.5) is a traveling wave solution of (1.3) connecting $\mathbf{0} = (0, 0, 0)$ with $\mathbf{K} = (k_1, k_2, k_3)$ if it satisfies (2.3).

For $\mu \in (0, \min_{1\leqslant i\leqslant 3}\{\frac{\beta_i}{c}\})$, define

$$B_\mu\big(\mathbb{R}, \mathbb{R}^3\big) = \Big\{\Phi \in C\big(\mathbb{R}, \mathbb{R}^3\big):\ \sup_{t\in\mathbb{R}}\big|\Phi(t)\big|e^{-\mu|t|} < \infty\Big\}$$

and

$$|\Phi|_\mu = \sup_{t\in\mathbb{R}}\big|\Phi(t)\big|e^{-\mu|t|}.$$

Then it is easy to check that $(B_\mu(\mathbb{R}, \mathbb{R}^3), |\cdot|_\mu)$ is a Banach space.

For convenience, we give the following assumptions about f_i and J_i of (1.3) and all of them will be imposed throughout this paper.

(H1) $f_i(\widehat{\mathbf{0}}) = f_i(\widehat{\mathbf{K}}) = 0$, where $\widehat{\ }$ denotes the constant value function in $C([-\tau, 0], \mathbb{R}^3)$;

(H2) For any $\Phi, \Psi \in C([-\tau, 0], \mathbb{R}^3)$ satisfying $\mathbf{0} \leqslant \Phi(t), \Psi(t) \leqslant \mathbf{M} := (M_1, M_2, M_3)$ for each $t \in [-\tau, 0]$, there exist positive constants $L_i > 0$ such that

$$\big|f_i(\Phi) - f_i(\Psi)\big| \leqslant L_i\|\Phi - \Psi\|;$$

(H3) $J_i(x)$ is an even and non-negative function, $\int_{\mathbb{R}} J_i(x - t)u(x)\,dx \geqslant \int_{\mathbb{R}} J_i(x - t)v(x)\,dx$ for $t \in \mathbb{R}$ if $u \geqslant v$ satisfying that the two integrals above are convergent, and $\int_{\mathbb{R}} J_i(x)e^{\mu|x|}\,dx < \infty$ for any μ given above, $i = 1, 2, 3$.

3. Mixed quasimonotonicity case 1

In this section, we consider the nonlocal diffusion system (1.3) with the following mixed quasimonotonicity reaction terms:

MQM-1: There exist three positive constants β_1, β_2 and β_3 such that

$$\begin{cases} f_1(\varphi_{11}, \varphi_{21}, \varphi_{31}) - f_1(\varphi_{12}, \varphi_{22}, \varphi_{31}) + \Big(\beta_1 - \displaystyle\int_{\mathbb{R}} J_1(x)\,dx\Big)\big[\varphi_{11}(0) - \varphi_{12}(0)\big] \geqslant 0, \\ f_1(\varphi_{11}, \varphi_{21}, \varphi_{31}) - f_1(\varphi_{11}, \varphi_{21}, \varphi_{32}) \leqslant 0, \\ f_2(\varphi_{11}, \varphi_{21}, \varphi_{31}) - f_2(\varphi_{12}, \varphi_{22}, \varphi_{31}) + \Big(\beta_2 - \displaystyle\int_{\mathbb{R}} J_2(x)\,dx\Big)\big[\varphi_{21}(0) - \varphi_{22}(0)\big] \geqslant 0, \\ f_2(\varphi_{11}, \varphi_{21}, \varphi_{31}) - f_2(\varphi_{11}, \varphi_{21}, \varphi_{32}) \leqslant 0, \\ f_3(\varphi_{11}, \varphi_{21}, \varphi_{31}) - f_3(\varphi_{11}, \varphi_{21}, \varphi_{32}) + \Big(\beta_3 - \displaystyle\int_{\mathbb{R}} J_3(x)\,dx\Big)\big[\varphi_{31}(0) - \varphi_{32}(0)\big] \geqslant 0, \\ f_3(\varphi_{11}, \varphi_{21}, \varphi_{31}) - f_3(\varphi_{12}, \varphi_{21}, \varphi_{31}) \leqslant 0, \\ f_3(\varphi_{11}, \varphi_{21}, \varphi_{31}) - f_3(\varphi_{11}, \varphi_{22}, \varphi_{31}) \leqslant 0 \end{cases}$$

for $\Phi = (\varphi_{11}, \varphi_{21}, \varphi_{31}), \Psi = (\varphi_{12}, \varphi_{22}, \varphi_{32}) \in C([-\tau, 0], \mathbb{R}^3)$ with

$$\begin{cases} \text{(i)} \quad 0 \leqslant \varphi_{i2}(s) \leqslant \varphi_{i1}(s) \leqslant M_i, \quad s \in [-\tau, 0],\ i = 1, 2, 3; \\ \text{(ii)} \quad e^{\frac{\beta_l}{c}s}\big[\varphi_{l1}(s) - \varphi_{l2}(s)\big] \text{ is non-decreasing in } s \in [-\tau, 0] \text{ for } l \in \Theta, \end{cases}$$

where $\Theta \stackrel{\triangle}{=} \{i \mid \tau_{ii} > 0,\ i = 1, 2, 3\}$.

First, we give the *new* definition of a pair of upper and lower solutions of (2.1).

Definition 1. A pair of continuous functions $\overline{\Phi} = (\overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3)$, $\underline{\Phi} = (\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3)$ are called an upper solution and a lower solution of (2.1), respectively, if there exist constants T_i, $i = 1, \dots, m$, such that $\overline{\Phi}$ and $\underline{\Phi}$ are continuously differentiable in $\mathbb{R} \setminus \{T_i\colon i = 1, \dots, m\}$ and satisfy

$$\begin{cases} c\overline{\varphi}_1'(t) \geqslant \int_{\mathbb{R}} J_1(y-t)\big[\overline{\varphi}_1(y) - \overline{\varphi}_1(t)\big]\,dy + f_1^c(\overline{\varphi}_{1t}, \overline{\varphi}_{2t}, \underline{\varphi}_{3t}), \\ c\overline{\varphi}_2'(t) \geqslant \int_{\mathbb{R}} J_2(y-t)\big[\overline{\varphi}_2(y) - \overline{\varphi}_2(t)\big]\,dy + f_2^c(\overline{\varphi}_{1t}, \overline{\varphi}_{2t}, \underline{\varphi}_{3t}), \\ c\overline{\varphi}_3'(t) \geqslant \int_{\mathbb{R}} J_3(y-t)\big[\overline{\varphi}_3(y) - \overline{\varphi}_3(t)\big]\,dy + f_3^c(\underline{\varphi}_{1t}, \underline{\varphi}_{2t}, \overline{\varphi}_{3t}), \end{cases} \quad t \in \mathbb{R} \setminus \{T_i\colon i = 1, \dots, m\}, \tag{3.1}$$

and

$$\begin{cases} c\underline{\varphi}_1'(t) \leqslant \int_{\mathbb{R}} J_1(y-t)\big[\underline{\varphi}_1(y) - \underline{\varphi}_1(t)\big]\,dy + f_1^c(\underline{\varphi}_{1t}, \underline{\varphi}_{2t}, \overline{\varphi}_{3t}), \\ c\underline{\varphi}_2'(t) \leqslant \int_{\mathbb{R}} J_2(y-t)\big[\underline{\varphi}_2(y) - \underline{\varphi}_2(t)\big]\,dy + f_2^c(\underline{\varphi}_{1t}, \underline{\varphi}_{2t}, \overline{\varphi}_{3t}), \\ c\underline{\varphi}_3'(t) \leqslant \int_{\mathbb{R}} J_3(y-t)\big[\underline{\varphi}_3(y) - \underline{\varphi}_3(t)\big]\,dy + f_3^c(\overline{\varphi}_{1t}, \overline{\varphi}_{2t}, \underline{\varphi}_{3t}), \end{cases} \quad t \in \mathbb{R} \setminus \{T_i\colon i = 1, \dots, m\}. \tag{3.2}$$

We have some nice properties about operators H and F.

Lemma 3.1. *Assume that* MQM-1 *holds. Then*

$$H_1(\varphi_{11}, \varphi_{21}, \varphi_{32})(t) \geqslant H_1(\varphi_{12}, \varphi_{22}, \varphi_{31})(t),$$
$$H_2(\varphi_{11}, \varphi_{21}, \varphi_{32})(t) \geqslant H_2(\varphi_{12}, \varphi_{22}, \varphi_{31})(t),$$
$$H_3(\varphi_{11}, \varphi_{21}, \varphi_{32})(t) \leqslant H_3(\varphi_{12}, \varphi_{22}, \varphi_{31})(t)$$

for $\Phi = (\varphi_{11}, \varphi_{21}, \varphi_{31}), \Psi = (\varphi_{12}, \varphi_{22}, \varphi_{32}) \in C(\mathbb{R}, \mathbb{R}^3)$ *with*

$$\begin{cases} \text{(i)} \quad 0 \leqslant \varphi_{i2}(s) \leqslant \varphi_{i1}(s) \leqslant M_i, \quad s \in [-\tau, 0],\ i = 1, 2, 3; \\ \text{(ii)} \quad e^{\frac{\beta_l}{c}s}\big[\varphi_{l1}(s) - \varphi_{l2}(s)\big] \textit{ is non-decreasing in } s \in [-\tau, 0] \textit{ for } l \in \Theta. \end{cases}$$

Proof. By MQM-1 and the definition of operator H, we have

$$\begin{aligned} &H_1(\varphi_{11}, \varphi_{21}, \varphi_{32})(t) - H_1(\varphi_{12}, \varphi_{22}, \varphi_{31})(t) \\ &\quad = H_1(\varphi_{11}, \varphi_{21}, \varphi_{32})(t) - H_1(\varphi_{11}, \varphi_{21}, \varphi_{31})(t) + H_1(\varphi_{11}, \varphi_{21}, \varphi_{31})(t) - H_1(\varphi_{12}, \varphi_{22}, \varphi_{31})(t) \\ &\quad \geqslant \int_{\mathbb{R}} J_1(y-t)\big[\varphi_{11}(y) - \varphi_{12}(y)\big]\,dy + f_1^c(\varphi_{11t}, \varphi_{21t}, \varphi_{31t}) - f_1^c(\varphi_{12t}, \varphi_{22t}, \varphi_{31t}) \\ &\qquad + \Big(\beta_1 - \int_{\mathbb{R}} J_1(x)\,dx\Big)\big(\varphi_{11}(t) - \varphi_{12}(t)\big) \\ &\quad \geqslant 0, \end{aligned}$$

$$\begin{aligned}
&H_2(\varphi_{11},\varphi_{21},\varphi_{32})(t)-H_2(\varphi_{12},\varphi_{22},\varphi_{31})(t)\\
&\quad=H_2(\varphi_{11},\varphi_{21},\varphi_{32})(t)-H_2(\varphi_{11},\varphi_{21},\varphi_{31})(t)+H_2(\varphi_{11},\varphi_{21},\varphi_{31})(t)-H_2(\varphi_{12},\varphi_{22},\varphi_{31})(t)\\
&\quad\geqslant\int_{\mathbb{R}}J_2(y-t)\big[\varphi_{21}(y)-\varphi_{22}(y)\big]\,dy+f_2^c(\varphi_{11t},\varphi_{21t},\varphi_{31t})-f_2^c(\varphi_{12t},\varphi_{22t},\varphi_{31t})\\
&\qquad+\bigg(\beta_2-\int_{\mathbb{R}}J_2(x)\,dx\bigg)\big(\varphi_{21}(t)-\varphi_{22}(t)\big)\\
&\quad\geqslant 0,\\
&H_3(\varphi_{11},\varphi_{21},\varphi_{32})(t)-H_3(\varphi_{12},\varphi_{22},\varphi_{31})(t)\\
&\quad=H_3(\varphi_{11},\varphi_{21},\varphi_{32})(t)-H_3(\varphi_{12},\varphi_{21},\varphi_{31})(t)+H_3(\varphi_{12},\varphi_{21},\varphi_{31})(t)-H_3(\varphi_{12},\varphi_{22},\varphi_{31})(t)\\
&\quad=H_3(\varphi_{11},\varphi_{21},\varphi_{32})(t)-H_3(\varphi_{12},\varphi_{21},\varphi_{31})(t)+f_3^c(\varphi_{12t},\varphi_{21t},\varphi_{31t})-f_3^c(\varphi_{12t},\varphi_{22t},\varphi_{31t})\\
&\quad\leqslant H_3(\varphi_{11},\varphi_{21},\varphi_{32})(t)-H_3(\varphi_{11},\varphi_{21},\varphi_{31})(t)+H_3(\varphi_{11},\varphi_{21},\varphi_{31})(t)-H_3(\varphi_{12},\varphi_{21},\varphi_{31})(t)\\
&\quad=\int_{\mathbb{R}}J_3(y-t)\big[\varphi_{32}(y)-\varphi_{31}(y)\big]\,dy+f_3^c(\varphi_{11t},\varphi_{21t},\varphi_{32t})-f_3^c(\varphi_{11t},\varphi_{21t},\varphi_{31t})\\
&\qquad+\bigg(\beta_3-\int_{\mathbb{R}}J_3(x)\,dx\bigg)\big(\varphi_{32}(t)-\varphi_{31}(t)\big)+f_3^c(\varphi_{11t},\varphi_{21t},\varphi_{31t})-f_3^c(\varphi_{12t},\varphi_{21t},\varphi_{31t})\\
&\quad\leqslant 0.
\end{aligned}$$

The proof is complete. □

From the definition of F in (2.6), the following lemma is a direct consequence of Lemma 3.1.

Lemma 3.2. *Assume that* MQM-1 *holds. Then*

$$\begin{aligned}
&F_1(\varphi_{11},\varphi_{21},\varphi_{32})(t)\geqslant F_1(\varphi_{12},\varphi_{22},\varphi_{31})(t),\\
&F_2(\varphi_{11},\varphi_{21},\varphi_{32})(t)\geqslant F_2(\varphi_{12},\varphi_{22},\varphi_{31})(t),\\
&F_3(\varphi_{11},\varphi_{21},\varphi_{32})(t)\leqslant F_3(\varphi_{12},\varphi_{22},\varphi_{31})(t)
\end{aligned}$$

for $\Phi=(\varphi_{11},\varphi_{21},\varphi_{31})$, $\Psi=(\varphi_{12},\varphi_{22},\varphi_{32})\in C(\mathbb{R},\mathbb{R}^3)$ *with*

$$\begin{cases}\text{(i)}\quad 0\leqslant\varphi_{i2}(s)\leqslant\varphi_{i1}(s)\leqslant M_i,\quad s\in[-\tau,0],\ i=1,2,3;\\ \text{(ii)}\quad e^{\frac{\beta_l}{c}s}\big[\varphi_{l1}(s)-\varphi_{l2}(s)\big]\ \textit{is non-decreasing in } s\in[-\tau,0]\ \textit{for } l\in\Theta.\end{cases}$$

In what follows, we assume that there exist an upper solution $\overline{\Phi}=(\overline{\varphi}_1,\overline{\varphi}_2,\overline{\varphi}_3)$ and a lower solution $\underline{\Phi}=(\underline{\varphi}_1,\underline{\varphi}_2,\underline{\varphi}_3)$ of (2.1) satisfying (P1)–(P3):

(P1) $\mathbf{0}\leqslant(\underline{\varphi}_1,\underline{\varphi}_2,\underline{\varphi}_3)\leqslant(\overline{\varphi}_1,\overline{\varphi}_2,\overline{\varphi}_3)\leqslant\mathbf{M}=(M_1,M_2,M_3)$;

(P2) $\lim_{t\to-\infty}(\overline{\varphi}_1,\overline{\varphi}_2,\overline{\varphi}_3)=\mathbf{0}$, $\lim_{t\to+\infty}(\underline{\varphi}_1,\underline{\varphi}_2,\underline{\varphi}_3)=\lim_{t\to+\infty}(\overline{\varphi}_1,\overline{\varphi}_2,\overline{\varphi}_3)=\mathbf{K}=(k_1,k_2,k_3)$;

(P3) $e^{\frac{\beta_l}{c}t}[\overline{\varphi}_l(t)-\underline{\varphi}_l(t)]$, $l\in\Theta$ are non-decreasing for $t\in\mathbb{R}$.

Define the following profile set:

$$\Gamma=\left\{(\varphi_1,\varphi_2,\varphi_3)\in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R},\mathbb{R}^3):\ \begin{array}{l}\text{(i)}\quad(\underline{\varphi}_1,\underline{\varphi}_2,\underline{\varphi}_3)\leqslant(\varphi_1,\varphi_2,\varphi_3)\leqslant(\overline{\varphi}_1,\overline{\varphi}_2,\overline{\varphi}_3)\\ \text{(ii)}\quad e^{\frac{\beta_l}{c}t}[\overline{\varphi}_l(t)-\varphi_l(t)]\ \text{and}\ e^{\frac{\beta_l}{c}t}[\varphi_l(t)-\underline{\varphi}_l(t)]\\ \qquad l\in\Theta\ \text{are non-decreasing for}\ t\in\mathbb{R}\end{array}\right\}.$$

It is easy to see that Γ is non-empty. In fact, by (P1), we know that $(\overline{\varphi}_1,\overline{\varphi}_2,\overline{\varphi}_3)$ satisfies (i) of Γ. By (P3), we know that $e^{\frac{\beta_l}{c}t}[\overline{\varphi}_l(t)-\underline{\varphi}_l(t)]$, $l\in\Theta$ are non-decreasing in $t\in\mathbb{R}$ and $e^{\frac{\beta_l}{c}t}[\overline{\varphi}_l(t)-\overline{\varphi}_l(t)]=0$, $l\in\Theta$. Thus, $(\overline{\varphi}_1,\overline{\varphi}_2,\overline{\varphi}_3)$ satisfies (ii) of Γ. By (P3), we further know that Γ is a closed, bounded and convex subset of $B_\mu(\mathbb{R},\mathbb{R}^3)$, see Huang and Zou [7]. Moreover, the following result is obvious according to Pan et al. [12].

Lemma 3.3. $F=(F_1,F_2,F_3):C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R},\mathbb{R}^3)\to C(\mathbb{R},\mathbb{R}^3)$ *is continuous with respect to the norm* $|\cdot|_\mu$.

Lemma 3.4. *Assume that* MQM-1 *holds, then* $F\Gamma\subset\Gamma$.

Proof. For any $\Phi=(\varphi_1,\varphi_2,\varphi_3)\in\Gamma$, we first prove that $F(\Phi)$ satisfies (i) of Γ.

By Lemma 3.2, we have that

$$\begin{cases} F_1(\underline{\varphi}_1,\underline{\varphi}_2,\overline{\varphi}_3)\leqslant F_1(\varphi_1,\varphi_2,\varphi_3)\leqslant F_1(\overline{\varphi}_1,\overline{\varphi}_2,\underline{\varphi}_3),\\ F_2(\underline{\varphi}_1,\underline{\varphi}_2,\overline{\varphi}_3)\leqslant F_2(\varphi_1,\varphi_2,\varphi_3)\leqslant F_2(\overline{\varphi}_1,\overline{\varphi}_2,\underline{\varphi}_3),\\ F_3(\overline{\varphi}_1,\overline{\varphi}_2,\underline{\varphi}_3)\leqslant F_3(\varphi_1,\varphi_2,\varphi_3)\leqslant F_3(\underline{\varphi}_1,\underline{\varphi}_2,\overline{\varphi}_3).\end{cases}$$

Now, we only need to prove

$$\begin{cases} \underline{\varphi}_1\leqslant F_1(\underline{\varphi}_1,\underline{\varphi}_2,\overline{\varphi}_3)\leqslant F_1(\overline{\varphi}_1,\overline{\varphi}_2,\underline{\varphi}_3)\leqslant\overline{\varphi}_1,\\ \underline{\varphi}_2\leqslant F_2(\underline{\varphi}_1,\underline{\varphi}_2,\overline{\varphi}_3)\leqslant F_2(\overline{\varphi}_1,\overline{\varphi}_2,\underline{\varphi}_3)\leqslant\overline{\varphi}_2,\\ \underline{\varphi}_3\leqslant F_3(\overline{\varphi}_1,\overline{\varphi}_2,\underline{\varphi}_3)\leqslant F_3(\underline{\varphi}_1,\underline{\varphi}_2,\overline{\varphi}_3)\leqslant\overline{\varphi}_3.\end{cases}\tag{3.3}$$

According to the definitions of F and the upper-lower solutions, we have

$$\begin{aligned} F_1(\overline{\varphi}_1,\overline{\varphi}_2,\underline{\varphi}_3)(t)&=\frac{1}{c}\int_{-\infty}^{t}e^{-\frac{\beta_1}{c}(t-s)}H_1(\overline{\varphi}_1,\overline{\varphi}_2,\underline{\varphi}_3)(s)\,ds\\ &\leqslant\frac{1}{c}\left\{\left(\sum_{j=1}^{q-1}\int_{T_{j-1}}^{T_j}+\int_{T_{q-1}}^{t}\right)e^{-\frac{\beta_1}{c}(t-s)}\left[c\overline{\varphi}_1'(s)+\beta_1\overline{\varphi}_1(s)\right]ds\right\}\\ &=\overline{\varphi}_1(t)\end{aligned}$$

for $t\in(T_{q-1},T_q)$ with $q=1,\dots,m+1$ by letting $T_0=-\infty$, $T_{m+1}=+\infty$. Thus the continuity of $F_1(\overline{\varphi}_1,\overline{\varphi}_2,\underline{\varphi}_3)(t)$ and $\overline{\varphi}_1(t)$ implies that $\overline{\varphi}_1(t)\geqslant F_1(\overline{\varphi}_1,\overline{\varphi}_2,\underline{\varphi}_3)(t)$ for $t\in\mathbb{R}$. In a similar way, we can prove that (3.3) holds for $t\in\mathbb{R}$. This prove (i) of Γ.

Next, we prove $F(\Phi)$ satisfies (ii) of Γ. For $t\in(T_{q-1},T_q)$ with $q=1,\dots,m+1$, we have

$$\begin{aligned} e^{\frac{\beta_i}{c}t}\left[\overline{\varphi}_i(t)-F_i(\varphi_1,\varphi_2,\varphi_3)(t)\right]&=\frac{e^{\frac{\beta_i t}{c}}}{c}\left\{\left(\sum_{j=1}^{q-1}\int_{T_{j-1}}^{T_j}+\int_{T_{q-1}}^{t}\right)e^{-\frac{\beta_i}{c}(t-s)}\left[c\overline{\varphi}_i'(s)+\beta_i\overline{\varphi}_i(s)\right]ds\right\}\\ &\quad-\frac{e^{\frac{\beta_i t}{c}}}{c}\left\{\left(\sum_{j=1}^{q-1}\int_{T_{j-1}}^{T_j}+\int_{T_{q-1}}^{t}\right)e^{-\frac{\beta_i}{c}(t-s)}H_i(\varphi_1,\varphi_2,\varphi_3)(s)\,ds\right\}\\ &=\frac{1}{c}\left(\sum_{j=1}^{q-1}\int_{T_{j-1}}^{T_j}+\int_{T_{q-1}}^{t}\right)e^{\frac{\beta_i s}{c}}\left(c\overline{\varphi}_i'(s)+\beta_i\overline{\varphi}_i(s)-H_i(\varphi_1,\varphi_2,\varphi_3)(s)\right)ds.\end{aligned}$$

From Lemma 3.2 and Definition 1, we know that

$$c\overline{\varphi}_i'(s)+\beta_i\overline{\varphi}_i(s)-H_i(\varphi_1,\varphi_2,\varphi_3)(s)\geqslant 0,\quad i=1,2,3.$$

Then the non-decreasing of $e^{\frac{\beta_l}{c}t}[\overline{\varphi}_l(t)-F_l(\Phi)(t)]$, $l\in\Theta$ is clear. In a similar way, we can prove that $e^{\frac{\beta_l}{c}t}[F_l(\Phi)(t)-\underline{\varphi}_l(t)]$ is non-decreasing in $t\in\mathbb{R}$, $l\in\Theta$.

Thus $F(\Phi)$ satisfies (ii) of Γ, and this completes the proof. □

Lemma 3.5. *Assume* MQM-1 *hold. Then* $F:\Gamma\to\Gamma$ *is compact.*

Remark 1. The proof of Lemma 3.5 is similar to those in Huang and Zou [7] and Pan et al. [12], so we omit it here.

Now, we are in a position to state and prove the following main theorem.

Theorem 3.6. *Assume that f satisfies* MQM-1 *condition and* (2.1) *has an upper and a lower solution $\Phi = (\overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3)$, $\Psi = (\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^3)$ satisfying* (P1)–(P3)*. Then* (1.3) *has a traveling wave solution satisfying* (2.3)*.*

Proof. From Lemmas 3.3–3.5, we know that $F\Gamma \subset \Gamma$ and F is compact. By Schauder's fixed point theorem there exists a fixed point $(\varphi_1^*, \varphi_2^*, \varphi_3^*) \in \Gamma$, which is a solution of (2.1), that is a traveling wave solution of (1.3).

Next, we verify the boundary conditions (2.3).

By (P2) and the inequality

$$0 \leqslant (\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3) \leqslant \left(\varphi_1^*, \varphi_2^*, \varphi_3^*\right) \leqslant (\overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3) \leqslant (M_1, M_2, M_3),$$

we see that

$$\lim_{t\to-\infty}\left(\varphi_1^*(t), \varphi_2^*(t), \varphi_3^*(t)\right) = (0,0,0), \qquad \lim_{t\to\infty}\left(\varphi_1^*(t), \varphi_2^*(t), \varphi_3^*(t)\right) = (k_1, k_2, k_3).$$

Therefore, the fixed point $(\varphi_1^*(t), \varphi_2^*(t), \varphi_3^*(t))$ satisfies the boundary conditions (2.3). The proof is complete. □

4. Mixed quasimonotonicity case 2 (type-K competition)

In this section, we propose another mixed quasimonotonicity condition.

MQM-2: There exist three positive constants β_1, β_2 and β_3 such that

$$\begin{cases} f_1(\varphi_{11}, \varphi_{21}, \varphi_{31}) - f_1(\varphi_{12}, \varphi_{21}, \varphi_{32}) + \left(\beta_1 - \int_{\mathbb{R}} J_1(x)\,dx\right)\left[\varphi_{11}(0) - \varphi_{12}(0)\right] \geqslant 0, \\ f_1(\varphi_{11}, \varphi_{21}, \varphi_{31}) - f_1(\varphi_{11}, \varphi_{22}, \varphi_{31}) \leqslant 0, \\ f_2(\varphi_{11}, \varphi_{21}, \varphi_{31}) - f_2(\varphi_{11}, \varphi_{22}, \varphi_{32}) + \left(\beta_2 - \int_{\mathbb{R}} J_2(x)\,dx\right)\left[\varphi_{21}(0) - \varphi_{22}(0)\right] \geqslant 0, \\ f_2(\varphi_{11}, \varphi_{21}, \varphi_{31}) - f_2(\varphi_{12}, \varphi_{21}, \varphi_{31}) \leqslant 0, \\ f_3(\varphi_{11}, \varphi_{21}, \varphi_{31}) - f_3(\varphi_{12}, \varphi_{22}, \varphi_{32}) + \left(\beta_3 - \int_{\mathbb{R}} J_3(x)\,dx\right)\left[\varphi_{31}(0) - \varphi_{32}(0)\right] \geqslant 0 \end{cases}$$

for $\Phi = (\varphi_{11}, \varphi_{21}, \varphi_{31})$, $\Psi = (\varphi_{12}, \varphi_{22}, \varphi_{32}) \in C([-\tau, 0], \mathbb{R}^3)$ with

$$\begin{cases} \text{(i)} \quad 0 \leqslant \varphi_{i2}(s) \leqslant \varphi_{i1}(s) \leqslant M_i, \quad s \in [-\tau, 0],\ i = 1,2,3; \\ \text{(ii)} \quad e^{\frac{\beta_l}{c}s}\left[\varphi_{l1}(s) - \varphi_{l2}(s)\right] \text{ is non-decreasing in } s \in [-\tau, 0] \text{ for } l \in \Theta, \end{cases}$$

where $\Theta \overset{\triangle}{=} \{i \mid \tau_{ii} > 0,\ i = 1,2,3\}$.

We give another definition of a pair of upper and lower solutions of (2.1).

Definition 2. A pair of continuous functions $\overline{\Phi} = (\overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3)$, $\underline{\Phi} = (\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3)$ are called an upper solution and a lower solution of (2.1), respectively, if there exist constants T_i, $i = 1, \ldots, m$, such that $\overline{\Phi}$ and $\underline{\Phi}$ are continuously differentiable in $\mathbb{R} \setminus \{T_i\colon i = 1, \ldots, m\}$ and satisfy

$$\begin{cases} c\overline{\varphi}_1'(t) \geqslant \int_{\mathbb{R}} J_1(y-t)\left[\overline{\varphi}_1(y) - \overline{\varphi}_1(t)\right]dy + f_1^c(\overline{\varphi}_{1t}, \underline{\varphi}_{2t}, \overline{\varphi}_{3t}), \\ c\overline{\varphi}_2'(t) \geqslant \int_{\mathbb{R}} J_2(y-t)\left[\overline{\varphi}_2(y) - \overline{\varphi}_2(t)\right]dy + f_2^c(\underline{\varphi}_{1t}, \overline{\varphi}_{2t}, \overline{\varphi}_{3t}), \qquad t \in \mathbb{R} \setminus \{T_i\colon i = 1, \ldots, m\}, \\ c\overline{\varphi}_3'(t) \geqslant \int_{\mathbb{R}} J_3(y-t)\left[\overline{\varphi}_3(y) - \overline{\varphi}_3(t)\right]dy + f_3^c(\overline{\varphi}_{1t}, \overline{\varphi}_{2t}, \overline{\varphi}_{3t}), \end{cases} \tag{4.1}$$

and

$$\begin{cases} c\underline{\varphi}_1'(t) \leqslant \int_{\mathbb{R}} J_1(y-t)\big[\underline{\varphi}_1(y)-\underline{\varphi}_1(t)\big]\,dy + f_1^c(\underline{\varphi}_{1t},\overline{\varphi}_{2t},\underline{\varphi}_{3t}), \\ c\underline{\varphi}_2'(t) \leqslant \int_{\mathbb{R}} J_2(y-t)\big[\underline{\varphi}_2(y)-\underline{\varphi}_2(t)\big]\,dy + f_2^c(\overline{\varphi}_{1t},\underline{\varphi}_{2t},\underline{\varphi}_{3t}), \qquad t\in\mathbb{R}\setminus\{T_i\colon i=1,\dots,m\}. \\ c\underline{\varphi}_3'(t) \leqslant \int_{\mathbb{R}} J_3(y-t)\big[\underline{\varphi}_3(y)-\underline{\varphi}_3(t)\big]\,dy + f_3^c(\underline{\varphi}_{1t},\underline{\varphi}_{2t},\underline{\varphi}_{3t}), \end{cases} \tag{4.2}$$

In what follows, we assume that there exist an upper solution $\overline{\Phi}=(\overline{\varphi}_1,\overline{\varphi}_2,\overline{\varphi}_3)$ and a lower solution $\underline{\Phi}=(\underline{\varphi}_1,\underline{\varphi}_2,\underline{\varphi}_3)$ of (2.1) satisfying (P1)–(P3), which are defined in Section 3.

We choose the profile set as Γ, which is defined in Section 3.

Next, we get the nice properties for H and F in the MQM-2 case, and thus prove the important statement: $F\Gamma\subset\Gamma$.

Lemma 4.1. *Assume that* MQM-2 *holds. Then*

$$\begin{aligned} &H_1(\varphi_{11},\varphi_{22},\varphi_{31})(t)\geqslant H_1(\varphi_{12},\varphi_{21},\varphi_{32})(t), && F_1(\varphi_{11},\varphi_{22},\varphi_{31})(t)\geqslant F_1(\varphi_{12},\varphi_{21},\varphi_{32})(t),\\ &H_2(\varphi_{12},\varphi_{21},\varphi_{31})(t)\geqslant H_2(\varphi_{11},\varphi_{22},\varphi_{32})(t), && F_2(\varphi_{12},\varphi_{21},\varphi_{31})(t)\geqslant F_2(\varphi_{11},\varphi_{22},\varphi_{32})(t),\\ &H_3(\varphi_{11},\varphi_{21},\varphi_{31})(t)\geqslant H_3(\varphi_{12},\varphi_{22},\varphi_{32})(t), && F_3(\varphi_{11},\varphi_{21},\varphi_{31})(t)\geqslant F_3(\varphi_{12},\varphi_{22},\varphi_{32})(t) \end{aligned}$$

for $\Phi=(\varphi_{11},\varphi_{21},\varphi_{31})$, $\Psi=(\varphi_{12},\varphi_{22},\varphi_{32})\in C(\mathbb{R},\mathbb{R}^3)$ *with*

$$\begin{cases} \text{(i)} \quad 0\leqslant\varphi_{i2}(s)\leqslant\varphi_{i1}(s)\leqslant M_i, \quad s\in[-\tau,0],\ i=1,2,3; \\ \text{(ii)} \quad e^{\frac{\beta_l}{c}s}\big[\varphi_{l1}(s)-\varphi_{l2}(s)\big] \text{ is non-decreasing in } s\in[-\tau,0] \text{ for } l\in\Theta, \end{cases}$$

where $\Theta \stackrel{\triangle}{=} \{i\mid \tau_{ii}>0,\ i=1,2,3\}$.

Proof. By MQM-2 and the definition of operator H, we have

$$\begin{aligned} &H_1(\varphi_{11},\varphi_{22},\varphi_{31})(t)-H_1(\varphi_{12},\varphi_{21},\varphi_{32})(t)\\ &\quad = H_1(\varphi_{11},\varphi_{22},\varphi_{31})(t)-H_1(\varphi_{11},\varphi_{21},\varphi_{31})(t)+H_1(\varphi_{11},\varphi_{21},\varphi_{31})(t)-H_1(\varphi_{12},\varphi_{21},\varphi_{32})(t)\\ &\quad \geqslant f_1^c(\varphi_{11t},\varphi_{21t},\varphi_{31t})-f_1^c(\varphi_{12t},\varphi_{21t},\varphi_{32t})+\Big(\beta_1-\int_{\mathbb{R}}J_1(x)\,dx\Big)\big(\varphi_{11}(t)-\varphi_{12}(t)\big)\\ &\quad \geqslant 0,\\ &H_2(\varphi_{12},\varphi_{21},\varphi_{31})(t)-H_2(\varphi_{11},\varphi_{22},\varphi_{32})(t)\\ &\quad = H_2(\varphi_{12},\varphi_{21},\varphi_{31})(t)-H_2(\varphi_{11},\varphi_{21},\varphi_{31})(t)+H_2(\varphi_{11},\varphi_{21},\varphi_{31})(t)-H_2(\varphi_{11},\varphi_{22},\varphi_{32})(t)\\ &\quad \geqslant f_2^c(\varphi_{11t},\varphi_{21t},\varphi_{31t})-f_2^c(\varphi_{11t},\varphi_{22t},\varphi_{32t})+\Big(\beta_2-\int_{\mathbb{R}}J_2(x)\,dx\Big)\big(\varphi_{21}(t)-\varphi_{22}(t)\big)\\ &\quad \geqslant 0,\\ &H_3(\varphi_{11},\varphi_{21},\varphi_{31})(t)-H_3(\varphi_{12},\varphi_{22},\varphi_{32})(t)\\ &\quad = f_3^c(\varphi_{11t},\varphi_{21t},\varphi_{31t})-f_3^c(\varphi_{12t},\varphi_{22t},\varphi_{32t})+\Big(\beta_3-\int_{\mathbb{R}}J_3(x)\,dx\Big)\big(\varphi_{31}(t)-\varphi_{32}(t)\big)\\ &\quad \geqslant 0. \end{aligned}$$

From the definition of F in (2.6), we get the inequalities for F. The proof is complete. □

Lemma 4.2. *Assume that* MQM-2 *holds, then* $F\Gamma\subset\Gamma$.

Proof. For any $(\varphi_1,\varphi_2,\varphi_3)\in\Gamma$, by Lemma 4.1, we have that

$$\begin{cases} F_1(\underline{\varphi}_1, \overline{\varphi}_2, \underline{\varphi}_3) \leqslant F_1(\varphi_1, \varphi_2, \varphi_3) \leqslant F_1(\overline{\varphi}_1, \underline{\varphi}_2, \overline{\varphi}_3), \\ F_2(\overline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3) \leqslant F_2(\varphi_1, \varphi_2, \varphi_3) \leqslant F_2(\underline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3), \\ F_3(\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3) \leqslant F_3(\varphi_1, \varphi_2, \varphi_3) \leqslant F_3(\overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3). \end{cases}$$

Now, we only need to prove

$$\begin{cases} \underline{\varphi}_1 \leqslant F_1(\underline{\varphi}_1, \overline{\varphi}_2, \underline{\varphi}_3) \leqslant F_1(\overline{\varphi}_1, \underline{\varphi}_2, \overline{\varphi}_3) \leqslant \overline{\varphi}_1, \\ \underline{\varphi}_2 \leqslant F_2(\overline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3) \leqslant F_2(\underline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3) \leqslant \overline{\varphi}_2, \\ \underline{\varphi}_3 \leqslant F_3(\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3) \leqslant F_3(\overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3) \leqslant \overline{\varphi}_3. \end{cases} \tag{4.3}$$

According to the definition of F and the upper-lower solution (4.1), (4.2), we have

$$\begin{aligned} F_1(\underline{\varphi}_1, \overline{\varphi}_2, \underline{\varphi}_3)(t) &= \frac{1}{c} \int_{-\infty}^{t} e^{-\frac{\beta_1}{c}(t-s)} H_1(\underline{\varphi}_1, \overline{\varphi}_2, \underline{\varphi}_3)(s)\, ds \\ &\geqslant \frac{1}{c} \left\{ \left(\sum_{j=1}^{q-1} \int_{T_{j-1}}^{T_j} + \int_{T_{q-1}}^{t} \right) e^{-\frac{\beta_1}{c}(t-s)} \left[c\underline{\varphi}_1'(s) + \beta_1 \underline{\varphi}_1(s) \right] ds \right\} \\ &= \underline{\varphi}_1(t) \end{aligned}$$

for $t \in (T_{q-1}, T_q)$ with $q = 1, \ldots, m+1$ by letting $T_0 = -\infty$, $T_{m+1} = +\infty$. Thus the continuity of $F_1(\underline{\varphi}_1, \overline{\varphi}_2, \underline{\varphi}_3)(t)$ and $\underline{\varphi}_1(t)$ implies that $F_1(\underline{\varphi}_1, \overline{\varphi}_2, \underline{\varphi}_3)(t) \geqslant \underline{\varphi}_1(t)$ for $t \in \mathbb{R}$. In a similar way, we can prove that (4.3) holds for $t \in \mathbb{R}$. This prove that (i) of Γ holds. The proof of (ii) of Γ is similar to Lemma 3.4. Thus $F\Gamma \subset \Gamma$. This completes the proof. □

Similar to Lemmas 3.3 and 3.5, we have:

Lemma 4.3. *Assume* MQM-2 *hold, then* $F : \Gamma \to \Gamma$ *is continuous with respect to the norm* $|\cdot|_\mu$ *and compact.*

Now, we state the main theorem in this section, and the proof is similar to Theorem 3.6.

Theorem 4.4. *Assume that f satisfies* MQM-2 *condition and* (2.1) *has an upper and a lower solution* $\Phi = (\overline{\varphi}_1, \overline{\varphi}_2, \overline{\varphi}_3)$, $\Psi = (\underline{\varphi}_1, \underline{\varphi}_2, \underline{\varphi}_3) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^3)$ *satisfying* (4.1), (4.2) *and* (P1)–(P3). *Then* (1.3) *has a traveling wave solution satisfying* (2.3).

Remark 2. Our results in Sections 3 and 4 above include the delays "$\tau_{ii} = 0$, $i = 1, 2, 3$ $(\Theta = \emptyset)$" and "$\tau_{ii} > 0$, $i = 1, 2, 3$ $(\Theta = \{1, 2, 3\})$" as special cases. That is, we get the existence results not only for f with mixed quasimonotone condition or exponential mixed quasimonotone condition, but also for f with the intermediate cases.

5. Applications

As mentioned in the introduction, in this section, we employ our conclusions in Sections 3 and 4 to establish the existence of traveling wave solutions for three-dimensional delayed K-type Lotka–Volterra nonlocal diffusion systems.

Definition 3. (See [9].) Consider the system

$$\dot{x}_i = G_i(x) = x_i g_i(x), \quad 1 \leqslant i \leqslant n,\ x_i \geqslant 0,$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, $\mathbb{R}_+ := [0, +\infty)$, $G = (G_1, \ldots, G_n)$ and $g = (g_1, \ldots, g_n) : \mathbb{R}_+^n \to \mathbb{R}^n$ is a C^1 mapping. This system is called type-K monotone if the Jacobian $\mathrm{D}g(x)$ of g is type-K monotone, that is, if $\mathrm{D}g(x)$ has the form

$$\begin{pmatrix} A_1 & -A_2 \\ -A_3 & A_4 \end{pmatrix},$$

in which A_1 is a $k \times k$ matrix, A_2 is a $k \times (n-k)$ matrix, A_3 is an $(n-k) \times k$ matrix and A_4 is an $(n-k) \times (n-k)$ matrix; Each off-diagonal element of A_1 and A_4 is non-negative, and A_2 and A_3 are non-negative matrices. This system is called a type-K competitive if $-\mathrm{D}g(x)$ is type-K monotone.

Example 5.1. We consider the existence of the traveling wave solution for the three-dimensional delayed type-K monotone nonlocal diffusive Lotka–Volterra system (for simplicity, we let $\tau_{11} = \tau_{22} = \tau_{33} = 0$), that is,

$$\begin{cases} \frac{\partial u_1}{\partial t} = \int\limits_{\mathbb{R}} J_1(x-y)\big[u_1(y,t)-u_1(x,t)\big]\,dy + r_1u_1\big[1-a_{11}u_1(x,t)+a_{12}u_2(x,t-\tau_1)-a_{13}u_3(x,t-\tau_2)\big], \\ \frac{\partial u_2}{\partial t} = \int\limits_{\mathbb{R}} J_2(x-y)\big[u_2(y,t)-u_2(x,t)\big]\,dy + r_2u_2\big[1+a_{21}u_1(x,t-\tau_3)-a_{22}u_2(x,t)-a_{23}u_3(x,t-\tau_4)\big], \\ \frac{\partial u_3}{\partial t} = \int\limits_{\mathbb{R}} J_3(x-y)\big[u_3(y,t)-u_3(x,t)\big]\,dy + r_3u_3\big[1-a_{31}u_1(x,t-\tau_5)-a_{32}u_2(x,t-\tau_6)-a_{33}u_3(x,t)\big], \end{cases} \tag{5.1}$$

where J_i satisfies (H3), $r_i > 0$, $a_{ij} \geqslant 0$, $i, j = 1, 2, 3$.

It is easy to show that system (5.1) has a trivial steady state $E_0(0,0,0)$ and a steady state $E^*(k_1,k_2,k_3)$, where

$$k_1 = \frac{\begin{vmatrix} 1 & -a_{12} & a_{13} \\ 1 & a_{22} & a_{23} \\ 1 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & -a_{12} & a_{13} \\ -a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \qquad k_2 = \frac{\begin{vmatrix} a_{11} & 1 & a_{13} \\ -a_{21} & 1 & a_{23} \\ a_{31} & 1 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & -a_{12} & a_{13} \\ -a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}, \qquad k_3 = \frac{\begin{vmatrix} a_{11} & -a_{12} & 1 \\ -a_{21} & a_{22} & 1 \\ a_{31} & a_{32} & 1 \end{vmatrix}}{\begin{vmatrix} a_{11} & -a_{12} & a_{13} \\ -a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}.$$

We choose the coefficients a_{ij} such that $k_i > 0$, $i, j = 1, 2, 3$.

Assume that $c > 0$. Letting $u_1(x,t) = \varphi_1(x+ct)$, $u_2(x,t) = \varphi_2(x+ct)$, $u_3(x,t) = \varphi_3(x+ct)$, and denoting the traveling wave coordinate $x+ct$ still by t, then the corresponding wave system is

$$\begin{cases} c\varphi_1'(t) = \int\limits_{\mathbb{R}} J_1(y-t)\big[\varphi_1(y)-\varphi_1(t)\big]\,dy + r_1\varphi_1(t)\big[1-a_{11}\varphi_1(t)+a_{12}\varphi_2(t-c\tau_1)-a_{13}\varphi_3(t-c\tau_2)\big], \\ c\varphi_2'(t) = \int\limits_{\mathbb{R}} J_2(y-t)\big[\varphi_2(y)-\varphi_2(t)\big]\,dy + r_2\varphi_2(t)\big[1+a_{21}\varphi_1(t-c\tau_3)-a_{22}\varphi_2(t)-a_{23}\varphi_3(t-c\tau_4)\big], \\ c\varphi_3'(t) = \int\limits_{\mathbb{R}} J_3(y-t)\big[\varphi_3(y)-\varphi_3(t)\big]\,dy + r_3\varphi_3(t)\big[1-a_{31}\varphi_1(t-c\tau_5)-a_{32}\varphi_2(t-c\tau_6)-a_{33}\varphi_3(t)\big] \end{cases} \tag{5.2}$$

with the following asymptotic boundary conditions

$$\lim_{t\to-\infty}\big(\varphi_1(t),\varphi_2(t),\varphi_3(t)\big) = (0,0,0), \qquad \lim_{t\to+\infty}\big(\varphi_1(t),\varphi_2(t),\varphi_3(t)\big) = (k_1,k_2,k_3).$$

For $\varphi_1, \varphi_2, \varphi_3 \in C([-\tau,0],\mathbb{R})$, where $\tau = \max\{\tau_i,\ i = 1,\dots,6\}$, denote

$$\begin{aligned} f_1(\varphi_1,\varphi_2,\varphi_3) &= r_1\varphi_1(0)\big[1-a_{11}\varphi_1(0)+a_{12}\varphi_2(-\tau_1)-a_{13}\varphi_3(-\tau_2)\big], \\ f_2(\varphi_1,\varphi_2,\varphi_3) &= r_2\varphi_2(0)\big[1+a_{21}\varphi_1(-\tau_3)-a_{22}\varphi_2(0)-a_{23}\varphi_3(-\tau_4)\big], \\ f_3(\varphi_1,\varphi_2,\varphi_3) &= r_3\varphi_3(0)\big[1-a_{31}\varphi_1(-\tau_5)-a_{32}\varphi_2(-\tau_6)-a_{33}\varphi_3(0)\big]. \end{aligned}$$

Obviously, (H1) and (H2) are satisfied. We now verify that $f = (f_1, f_2, f_3)$ satisfies MQM-1.

Lemma 5.2. *The function f satisfies MQM-1.*

Proof. For any $\Phi(s) = (\varphi_{11}(s), \varphi_{21}(s), \varphi_{31}(s))$, $\Psi(s) = (\varphi_{12}(s), \varphi_{22}(s), \varphi_{32}(s)) \in C([-\tau,0],\mathbb{R}^3)$, with $0 \leqslant \varphi_{i2}(s) \leqslant \varphi_{i1}(s) \leqslant M_i$, $s \in [-\tau,0]$, $i = 1,2,3$, we have

$$\begin{aligned} f_1(\varphi_{11},\varphi_{21},\varphi_{31}) - f_1(\varphi_{12},\varphi_{22},\varphi_{31}) &= r_1\varphi_{11}(0)\big[1-a_{11}\varphi_{11}(0)+a_{12}\varphi_{21}(-\tau_1)-a_{13}\varphi_{31}(-\tau_2)\big] \\ &\quad - r_1\varphi_{12}(0)\big[1-a_{11}\varphi_{12}(0)+a_{12}\varphi_{22}(-\tau_1)-a_{13}\varphi_{31}(-\tau_2)\big] \\ &\geqslant r_1\big[1-a_{11}\big(\varphi_{11}(0)+\varphi_{12}(0)\big)-a_{13}\varphi_{31}(-\tau_2)\big]\big(\varphi_{11}(0)-\varphi_{12}(0)\big) \\ &\geqslant r_1(1-2a_{11}M_1-a_{13}M_3)\big(\varphi_{11}(0)-\varphi_{12}(0)\big). \end{aligned}$$

Choosing appropriate constants and $J_1(x)$ such that $\beta_1 := r_1(2a_{11}M_1 - a_{13}M_3 - 1) + \int_{\mathbb{R}} J_1(x)\,dx > 0$, then

$$f_1(\varphi_{11},\varphi_{21},\varphi_{31}) - f_1(\varphi_{12},\varphi_{22},\varphi_{31}) + \Big(\beta_1 - \int\limits_{\mathbb{R}} J_1(x)\,dx\Big)\big[\varphi_{11}(0)-\varphi_{12}(0)\big] \geqslant 0.$$

Also

$$\begin{aligned} f_1(\varphi_{11},\varphi_{21},\varphi_{31}) - f_1(\varphi_{11},\varphi_{21},\varphi_{32}) &= r_1\varphi_{11}(0)\big[1 - a_{11}\varphi_{11}(0) + a_{12}\varphi_{21}(-\tau_1) - a_{13}\varphi_{31}(-\tau_2)\big] \\ &\quad - r_1\varphi_{11}(0)\big[1 - a_{11}\varphi_{11}(0) + a_{12}\varphi_{21}(-\tau_1) - a_{13}\varphi_{32}(-\tau_2)\big] \\ &= -a_{13}r_1\varphi_{11}(0)\big(\varphi_{31}(-\tau_2) - \varphi_{32}(-\tau_2)\big) \\ &\leqslant 0. \end{aligned}$$

In a similar argument, we can prove that f_2 and f_3 satisfy MQM-1. The proof is complete. □

In order to apply Theorem 3.6, we need to construct an upper solution and a lower solution for (5.2).
For $\lambda \geqslant 0$, $c \geqslant 0$, define

$$\Delta_1(\lambda,c) = \int_{\mathbb{R}} J_1(x)\big(e^{\lambda x} - 1\big)\,dx - c\lambda + r_1(1 + a_{12}M_2),$$

$$\Delta_2(\lambda,c) = \int_{\mathbb{R}} J_2(x)\big(e^{\lambda x} - 1\big)\,dx - c\lambda + r_2(1 + a_{21}M_1),$$

$$\Delta_3(\lambda,c) = \int_{\mathbb{R}} J_3(x)\big(e^{\lambda x} - 1\big)\,dx - c\lambda + r_3.$$

Then $\Delta_i(\lambda,c)$ $(i = 1,2,3)$ are well defined by (H3) and (H4). Further, we have the following lemma.

Lemma 5.3. *There exist* $c_i > 0$ $(i = 1,2,3)$ *such that the following hold*:

(i) *If* $c > c_1$, $\Delta_1(\lambda,c)$ *has two distinct positive roots* $\lambda_1(c) < \lambda_2(c)$, *and*

$$\Delta_1(\lambda,c) = \begin{cases} > 0, & \lambda \in (0,\lambda_1(c)) \cup (\lambda_2(c),\infty), \\ < 0, & \lambda \in (\lambda_1(c),\lambda_2(c)); \end{cases}$$

if $c < c_1$, *then* $\Delta_1(\lambda,c)$ *has no real root*;

(ii) *If* $c > c_2$, $\Delta_2(\lambda,c)$ *has two distinct positive roots* $\lambda_3(c) < \lambda_4(c)$, *and*

$$\Delta_2(\lambda,c) = \begin{cases} > 0, & \lambda \in (0,\lambda_3(c)) \cup (\lambda_4(c),\infty), \\ < 0, & \lambda \in (\lambda_3(c),\lambda_4(c)); \end{cases}$$

if $c < c_2$, *then* $\Delta_2(\lambda,c)$ *has no real root*;

(iii) *If* $c > c_3$, $\Delta_3(\lambda,c)$ *has two distinct positive roots* $\lambda_5(c) < \lambda_6(c)$, *and*

$$\Delta_3(\lambda,c) = \begin{cases} > 0, & \lambda \in (0,\lambda_5(c)) \cup (\lambda_6(c),\infty), \\ < 0, & \lambda \in (\lambda_5(c),\lambda_6(c)); \end{cases}$$

if $c < c_3$, *then* $\Delta_3(\lambda,c)$ *has no real root.*

Let $c^* = \max\{c_1,c_2,c_3\}$. For $c > c^*$, we define the continuous function $\overline{\Phi}(t) = (\overline{\varphi}_1(t),\overline{\varphi}_2(t),\overline{\varphi}_3(t))$ and $\underline{\Phi}(t) = (\underline{\varphi}_1(t),\underline{\varphi}_2(t),\underline{\varphi}_3(t))$ as follows

$$\overline{\varphi}_1(t) = \begin{cases} e^{\lambda_1(c)t}, & t \leqslant t_1, \\ k_1 + \varepsilon_1 e^{-\lambda t}, & t > t_1; \end{cases} \qquad \overline{\varphi}_2(t) = \begin{cases} e^{\lambda_3(c)t}, & t \leqslant t_3, \\ k_2 + \varepsilon_3 e^{-\lambda t}, & t > t_3; \end{cases} \qquad \overline{\varphi}_3(t) = \begin{cases} e^{\lambda_5(c)t}, & t \leqslant t_5, \\ k_3 + \varepsilon_5 e^{-\lambda t}, & t > t_5, \end{cases} \tag{5.3}$$

and

$$\underline{\varphi}_1(t) = \begin{cases} 0, & t \leqslant t_2, \\ k_1 - \varepsilon_2 e^{-\lambda t}, & t > t_2; \end{cases} \qquad \underline{\varphi}_2(t) = \begin{cases} 0, & t \leqslant t_4, \\ k_2 - \varepsilon_4 e^{-\lambda t}, & t > t_4; \end{cases}$$
$$\underline{\varphi}_3(t) = \begin{cases} 0, & t \leqslant t_6, \\ k_3 - \varepsilon_6 e^{-\lambda t}, & t > t_6, \end{cases} \tag{5.4}$$

where $\varepsilon_i > 0$ $(i = 1,\dots,6)$ satisfy

$$\begin{cases} a_{11}\varepsilon_1 - a_{12}\varepsilon_3 - a_{13}\varepsilon_6 > 0, \\ a_{22}\varepsilon_3 - a_{21}\varepsilon_1 - a_{23}\varepsilon_6 > 0, \\ a_{33}\varepsilon_5 - a_{31}\varepsilon_2 - a_{32}\varepsilon_4 > 0, \\ a_{11}\varepsilon_2 - a_{12}\varepsilon_4 - a_{13}\varepsilon_5 > 0, \\ a_{22}\varepsilon_4 - a_{21}\varepsilon_2 - a_{23}\varepsilon_5 > 0, \\ a_{33}\varepsilon_6 - a_{31}\varepsilon_1 - a_{32}\varepsilon_3 > 0, \end{cases}$$

$\lambda_i(c)$ $(i = 1, \dots, 6)$ are given above, and $\lambda > 0$ is a constant to be chosen appropriately, such that t_i $(i = 1, \dots, 6)$ satisfy the formula (to be given in the proof).

It is easy to know that $M_1 = \sup_{t\in\mathbb{R}} \overline{\varphi}_1 > k_1$, $M_2 = \sup_{t\in\mathbb{R}} \overline{\varphi}_2 > k_2$, $M_3 = \sup_{t\in\mathbb{R}} \overline{\varphi}_3 > k_3$, $\overline{\Phi}(t)$ and $\underline{\Phi}(t)$ satisfy (P1)–(P3). We now prove that $\overline{\Phi}(t)$ and $\underline{\Phi}(t)$ are an upper solution and a lower solution of (5.2), respectively.

Lemma 5.4. *$\overline{\Phi}(t) = (\overline{\varphi}_1(t), \overline{\varphi}_2(t), \overline{\varphi}_3(t))$ is an upper solution of* (5.2).

Proof. If $t \leqslant t_1$, $\overline{\varphi}_1(t) = e^{\lambda_1(c)t}$, we have

$$\begin{aligned} &\int_{\mathbb{R}} J_1(y-t)\big[\overline{\varphi}_1(y) - \overline{\varphi}_1(t)\big]\,dy - c\overline{\varphi}_1'(t) + r_1\overline{\varphi}_1(t)\big[1 - a_{11}\overline{\varphi}_1(t) + a_{12}\overline{\varphi}_2(t - c\tau_1) - a_{13}\underline{\varphi}_3(t - c\tau_2)\big] \\ &\quad \leqslant \int_{\mathbb{R}} J_1(y-t)\big[\overline{\varphi}_1(y) - \overline{\varphi}_1(t)\big]\,dy - c\overline{\varphi}_1'(t) + r_1(1 + a_{12}M_2)\overline{\varphi}_1(t) \\ &\quad = \Big[\int_{\mathbb{R}} J_1(x)\big(e^{\lambda_1(c)x} - 1\big)\,dx - c\lambda_1(c) + r_1(1 + a_{12}M_2)\Big]\overline{\varphi}_1(t) = 0. \end{aligned}$$

If $t > t_1$, $\overline{\varphi}_1(t) = k_1 + \varepsilon_1 e^{-\lambda t}$, let $t_1 - c\tau_2 \geqslant t_6$, then

$$\overline{\varphi}_2(t - c\tau_1) \leqslant k_2 + \varepsilon_3 e^{-\lambda(t - c\tau_1)}, \qquad \underline{\varphi}_3(t - c\tau_2) = k_3 - \varepsilon_6 e^{-\lambda(t - c\tau_2)},$$

and

$$\begin{aligned} &\int_{\mathbb{R}} J_1(y-t)\big[\overline{\varphi}_1(y) - \overline{\varphi}_1(t)\big]\,dy - c\overline{\varphi}_1'(t) + r_1\overline{\varphi}_1(t)\big[1 - a_{11}\overline{\varphi}_1(t) + a_{12}\overline{\varphi}_2(t - c\tau_1) - a_{13}\underline{\varphi}_3(t - c\tau_2)\big] \\ &\quad \leqslant \Big[\varepsilon_1 \int_{\mathbb{R}} J_1(x)\big(e^{-\lambda x} - 1\big)\,dx + c\varepsilon_1\lambda\Big]e^{-\lambda t} \\ &\qquad + r_1\overline{\varphi}_1(t)\big[1 - a_{11}\big(k_1 + \varepsilon_1 e^{-\lambda t}\big) + a_{12}\big(k_2 + \varepsilon_3 e^{-\lambda(t - c\tau_1)}\big) - a_{13}\big(k_3 - \varepsilon_6 e^{-\lambda(t - c\tau_2)}\big)\big] \\ &\quad = e^{-\lambda t}\Big[\varepsilon_1 \int_{\mathbb{R}} J_1(x)\big(e^{-\lambda x} - 1\big)\,dx + c\varepsilon_1\lambda + r_1\big(k_1 + \varepsilon_1 e^{-\lambda t}\big)\big(a_{12}\varepsilon_3 e^{\lambda c\tau_1} + a_{13}\varepsilon_6 e^{\lambda c\tau_2} - a_{11}\varepsilon_1\big)\Big]. \end{aligned}$$

Let

$$I_1(\lambda) = \varepsilon_1 \int_{\mathbb{R}} J_1(x)\big(e^{-\lambda x} - 1\big)\,dx + c\varepsilon_1\lambda + r_1\big(k_1 + \varepsilon_1 e^{-\lambda t}\big)\big(a_{12}\varepsilon_3 e^{\lambda c\tau_1} + a_{13}\varepsilon_6 e^{\lambda c\tau_2} - a_{11}\varepsilon_1\big).$$

Then, $a_{11}\varepsilon_1 - a_{12}\varepsilon_3 - a_{13}\varepsilon_6 > 0$ implies that $I_1(0) = r_1(k_1 + \varepsilon_1)(a_{12}\varepsilon_3 + a_{13}\varepsilon_6 - a_{11}\varepsilon_1) < 0$, and there exists a $\lambda_1^* > 0$, such that $I_1(\lambda) < 0$ for $\lambda \in (0, \lambda_1^*)$. Thus, we have

$$c\overline{\varphi}_1'(t) \geqslant \int_{\mathbb{R}} J_1(y-t)\big[\overline{\varphi}_1(y) - \overline{\varphi}_1(t)\big]\,dy + r_1\overline{\varphi}_1(t)\big[1 - a_{11}\overline{\varphi}_1(t) + a_{12}\overline{\varphi}_2(t - c\tau_1) - a_{13}\underline{\varphi}_3(t - c\tau_2)\big].$$

Similarly, there exist a $\lambda_2^* > 0$ such that for $\lambda \in (0, \lambda_2^*)$ we have

$$c\overline{\varphi}_2'(t) \geqslant \int_{\mathbb{R}} J_2(y-t)\big[\overline{\varphi}_2(y) - \overline{\varphi}_2(t)\big]\,dy + r_2\overline{\varphi}_2(t)\big[1 + a_{21}\overline{\varphi}_1(t - c\tau_3) - a_{22}\overline{\varphi}_2(t) - a_{23}\underline{\varphi}_3(t - c\tau_4)\big]$$

and a $\lambda_3^* > 0$ such that for $\lambda \in (0, \lambda_3^*)$ we have

$$c\overline{\varphi}_3'(t) \geqslant \int_{\mathbb{R}} J_3(y-t)\big[\overline{\varphi}_3(y) - \overline{\varphi}_3(t)\big]\,dy + r_3\overline{\varphi}_3(t)\big[1 - a_{31}\underline{\varphi}_1(t-c\tau_5) - a_{32}\underline{\varphi}_2(t-c\tau_6) - a_{33}\overline{\varphi}_3(t)\big].$$

Taking $\lambda \in (0, \min(\lambda_1^*, \lambda_2^*, \lambda_3^*))$, we see that $\overline{\Phi}(t)$ is an upper solution of (5.2). □

Similar to Lemma 5.4 in [13], we have the following lemma by repeating the above procedure.

Lemma 5.5. *$\underline{\Phi}(t) = (\underline{\varphi}_1(t), \underline{\varphi}_2(t), \underline{\varphi}_3(t))$ is a lower solution of* (5.2)*.*

By Theorem 3.6 and Remark 2, we have the following result.

Theorem 5.6. *For every $c > c^*$, system* (5.1) *always has a traveling wave solution with speed c connecting the trivial steady state E_0 and the positive steady state E^*.*

Remark 3. For the delayed type-K competitive nonlocal diffusive Lotka–Volterra system, we can easily verify that the reaction term satisfies the MQM-2 condition. By constructing similar upper solution and lower solution as (5.3) and (5.4), respectively, we can also get the existence result of the traveling wave solution.

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