A finite element, filtered eddy-viscosity method for the Navier–Stokes equations with large Reynolds number

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\textbf{A B S T R A C T}

The direct numerical simulation of the Navier–Stokes system in turbulent regimes is a formidable task due to the disparate scales that have to be resolved. Turbulence modeling attempts to mitigate this situation by somehow accounting for the effects of small-scale behavior on that at large-scales, without explicitly resolving the small scales. One such approach is to add viscosity to the problem; the Smagorinsky and Ladyzhenskaya models and other eddy-viscosity models are examples of this approach. Unfortunately, this approach usually results in over-dampening at the large scales, i.e., large-scale structures are unphysically smeared out. To overcome this fault of simple eddy-viscosity modeling, filtered eddy-viscosity methods that add artificial viscosity only to the high-frequency modes were developed in the context of spectral methods. We apply the filtered eddy-viscosity idea to finite element methods based on hierarchical basis functions. We prove the existence and uniqueness of the finite element approximation and its convergence to solutions of the Navier–Stokes system; we also derive error estimates for finite element approximations.

\section{1. Introduction}

It is generally accepted that incompressible fluid flows, even for high values of the Reynolds number, are faithfully modeled by the Navier–Stokes system. For small values of the Reynolds number, it can be shown that the motion is uniquely determined by the data, i.e., the solution of the Navier–Stokes system once the initial data, boundary data, and forcing function are specified. However, for values of the Reynolds number above a critical value and for all but sufficiently small data, it is not known if solutions of the Navier–Stokes system are globally unique. Also, for such values of the Reynolds number, flows become turbulent, i.e., they feature small eddies. Because of such small-scale behavior, the computational simulation of turbulence flows is a challenging task. In fact, the grid resolution needed to fully resolve eddies renders direct numerical simulation of the Navier–Stokes system infeasible, even using the most powerful computers available today or in the foreseeable future.

The detailed resolution of the small eddies is often not of practical interest. However, even though the small eddies do not contain much energy, they cannot be ignored because they have an appreciable effect on the large-scale structures of flows. This gives rise to the very active area of research known as turbulence modeling which can be defined as attempts to modify the Navier–Stokes system in such a way that the effects of small-scale behaviors on large-scale structures are accounted for without having to resolve the small scales.

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A turbulence model in common use was introduced by Smagorinsky [1] and a generalized version was independently developed by Ladyzhenskaya [2,3]. The Smagorinsky and Ladyzhenskaya models fall into the class of eddy-viscosity models for which the viscosity coefficient is modified so that, in regions where the gradient of the velocity is relatively large, additional dissipation is introduced. The Smagorinsky and Ladyzhenskaya models have been analyzed by various authors who take advantage of the additional viscosity to show that, e.g., under certain assumptions, the strong solutions of the Smagorinsky and Ladyzhenskaya models exist and are globally unique on a periodic domain [4]. For more detailed discussions and a survey of these models, see, e.g., [5,6] and the references therein.

Although the additional dissipation added in the Smagorinsky and Ladyzhenskaya models is effective in stabilizing the flow equations, in practice, these models are over diffusive, i.e., the large-scale structures are smeared out. This results because those models do not sufficiently discriminate between scales, i.e., additional viscosity is added at all scales. In this paper we study, in a finite element context, the idea of adding eddy viscosity only at the small scales, i.e., only to the high-frequency modes. This idea was used in the context of spectral [7–10], finite element [11,12], and wavelet [13] methods for hyperbolic conservation laws, of spectral methods for the Navier–Stokes equations [14–16], and of a computational implementation using finite element methods for the Navier–Stokes equations [17].

Related work is found in [18] where a subgrid modeling method based on a two-level decomposition of the approximation space was presented for stabilizing Galerkin finite element approximations of transport equations. That approach was extended to convection-dominated convection–diffusion equations in [19] by introducing a nonconforming, multiscale eddy viscosity method. The idea is equivalent to letting an artificial viscosity operator act only on the fluctuations in V u because it defines fluctuations through an elliptic projection into a coarse grid velocity finite element space. The development of this projection-based variational multiscale method was continued in, among many other papers, [20–23]. Here, we present a different approach by employing finite element basis functions having hierarchical structure.

In this paper, we pursue the idea of selectively applying artificial viscosity at only high frequencies in the context of finite element methods. The basic idea is that employed in the spectral eddy-viscosity method of [14–16] that are based on Fourier spectral basis functions. Unlike Fourier spectral basis functions, standard nodal finite element basis functions are all of the same frequency. Thus, to apply artificial viscosity in a selective manner, we turn to hierarchical finite element basis functions which can be clustered into the groups or levels that have different scales [24,25]. The multiscale nature of the hierarchical basis functions allows for the selective addition of eddy viscosity only at the small scales [11,12]. This paper is a step towards verifying that a finite element method based on hierarchical bases, when used to selectively apply an eddy viscosity, overcomes the drawback of severe smearing of large-scale structures occurring in simple eddy-viscosity turbulence models. There is already computational evidence to this effect in [17] where a two-level implementation is used.

The plan of the paper is as follows. In the remainder of this section, we present the model problem we study and, in Section 3, the finite element implementations of the modified Ladyzhenskaya and Smagorinsky models are introduced. In Section 4, we show the existence and uniqueness of a finite element approximation of the solution of modified models. Finally, in Section 5, we show that the approximate solution converges to a weak solution of Navier–Stokes equation. Future work will involve developing and applying codes in which the filtered-viscosity, hierarchical finite element method is implemented and tested using examples in the literature such as that given in [26].

2. High-pass filtered Ladyzhenskaya/Smagorinsky models

In this section, we present the high-pass filtered modifications of the Ladyzhenskaya and Smagorinsky models that we consider, show how they are related to the unfiltered versions of these models and to the Navier–Stokes equations, and briefly discuss their well posedness.

The models equations we consider have the form

\[
\begin{aligned}
\partial_t u - v \Delta u + u \cdot \nabla u + \nabla \pi - \epsilon Q \nabla \cdot (1 + |Q (\nabla u)|^{p-2}) Q (\nabla u) &= f, \\
\nabla \cdot u &= 0,
\end{aligned}
\]

where \( u \) and \( \pi \) denote the velocity and pressure fields, respectively, \( p \geq 2 \) a constant, \( f \) a given body force density, \( v \) the viscosity coefficient, and \( \epsilon \) a model parameter; after nondimensionalization, \( v \) the inverse of the Reynolds number. In (1), \( Q \) is a high-pass filter, i.e., it annihilates the low-frequency components of a function. An example of such a \( Q \) is the spectral filter \( Q_M = I - P_M \), where \( P_M \) denotes the projection operator

\[
P_M(g) = \sum_{|k| \leq M} \hat{g}(k)e^{ikx}
\]

with \( \hat{g} \) denoting the Fourier transform of the function \( g \). Another example is provided by the hierarchical finite element filter \( Q^N \) introduced in (5) which plays a crucial role in our algorithms and analyzes. If, in (1), \( Q = I \), e.g., if \( M = 0 \) in (2), then (1) reduces to one of the modified Navier–Stokes equations introduced by Ladyzhenskaya [2,3]: if, in addition, \( p = 3 \), we have the Smagorinsky model [1]; if \( \epsilon = 0 \), we end up with the Navier–Stokes equations.

Let \( \Omega \subset \mathbb{R}^3 \) denote a bounded domain with Lipschitz boundary \( \Gamma \) and \((0, T)\) be a bounded interval. Let \( L^p(\Omega) \) denote the Lebesgue space of all \( l \)-vector measurable functions with the norm \( \| v \|_{L^p(\Omega)} = (\int_{\Omega} |v|^p \, dx)^{1/p} \).
of $p$-th power of all components of $\mathbf{v}$. We omit $l$ if it is obvious in the context. Let $W^{m,p}(\Omega)$ denote the Sobolev space over $\Omega$ equipped with the norm $\| \cdot \|_{m,p,\Omega}$, where

$$\|u\|_{m,p,\Omega} = \left( \int_\Omega \sum_{|\alpha| \leq m} |\partial^\alpha u|^p \, dx \right)^{\frac{1}{p}}.$$

Let $H^m(\Omega) = W^{m,2}(\Omega)$ with the corresponding norm $\| \cdot \|_{m,\Omega}$. $H^0_0(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the space $H^1(\Omega)$. Throughout, $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ denote the $L^2$-norm and inner product, respectively. Let $X = H^1_0(\Omega)$ and $M = L^2_0(\Omega) \equiv \{ q \in L^2(\Omega) \mid \int_\Omega q \, dx = 0 \}$.

It is well known that there exists at least one weak solution $\mathbf{u}$ of the Navier–Stokes equations (the case $\varepsilon = 0$) such that $\mathbf{u} \in L^2(0, T; J) \cap L^\infty(0, T; L^2(\Omega))$, where $J = \{ \mathbf{u} \in H^1_0(\Omega) \mid \nabla \cdot \mathbf{u} = 0 \}$; however, the uniqueness of such solutions has not been demonstrated. On the other hand, it has been shown that if a solution $\mathbf{u}$ exists such that $\mathbf{u} \in L^\infty(0, T; J)$, then it is uniquely determined and the corresponding pressure $\pi$ belongs to $L^2(0, T; H^1(\Omega))$; however, the existence of such a solution $\mathbf{u}$ has not been demonstrated. See, e.g., [27,28], for details. In general, the existence and uniqueness of solutions of the Navier–Stokes equations has been demonstrated only if the viscosity and external force $f$ and other problem data satisfy very rigorous requirements [29].

The mathematical properties of solutions of the Ladyzhenskaya/Smagorinsky model (the case $Q = I$) were investigated by Ladyzhenskaya [2,3] as well as several others after that. It is known [4] that, for $p \geq 11/5$, a globally unique strong solution of the Ladyzhenskaya/Smagorinsky model exists on a periodic domain. In [15], the existence of a globally unique strong solution of the filtered Ladyzhenskaya/Smagorinsky model was also demonstrated for $p \geq 11/5$; moreover, it is shown there that weak solutions of the Ladyzhenskaya/Smagorinsky model converge to weak solutions of the Navier–Stokes equations as $\varepsilon \to 0$.

As mentioned in the introduction, direct discretization of the Navier–Stokes equations is a challenging task because of the presence of small-scale eddies. Thus, we define an approximation scheme based on applying a filtered hierarchical artificial viscosity scheme to the Ladyzhenskaya/Smagorinsky model.

3. Filtered, hierarchical finite element discretization

In [9], spectral viscosity method was introduced to damp the high-frequency modes in the approximation of the diffusion term in the periodic Burgers equation with Fourier basis functions; the spectral viscosity diffusion term compromises between not adding diffusion, which causes instability, and adding diffusion at all scales, which limits the convergence rate and smears out discontinuities in the solution. In the context of spectral viscosity method, multiscale basis functions are essential.

Hierarchical finite element basis functions fit into the framework of spectral viscosity methods due to the multiscale nature of the basis functions. The finite element, multiresolution viscosity method using hierarchical basis functions for hyperbolic conservation laws was established in [11] based on the above idea. Here, we apply the same idea to the eddy-viscosity term in the Ladyzhenskaya/Smagorinsky model ((1) with $Q = I$) so that the regularization only affects the high-frequency modes.

3.1. Hierarchical finite element bases

Let $T_k$ ($k = 0, \ldots, N$) denote the $k$-th level refinement of $\mathcal{T}$ into tetrahedral elements and let $h_k$ denote the smallest diameter of the tetrahedra in $T_k$, where elements in $T_k$ meet only along an entire common face or edge or vertex. Levels 0 and $N$ denote the coarsest and finest grids, respectively. We construct the set $T_{k+1}$ by partitioning each tetrahedron in $T_k$ into a number of subelements of equal volume such that

$$\frac{h_k}{h_{k+1}} = 2c_1 \quad \text{and} \quad \frac{h_0}{h_k} = c_2 2^k$$

for some $c_1, c_2 > 0$. Corresponding to $T_k$, we denote by $S_k$ the finite element space consisting of continuous functions that are piecewise polynomials on each element in $T_k$. Clearly, $S_l \subset S_k$ for $l < k$. $S_k$ is spanned by the nodal basis functions which are defined by $\phi_l \in S_k$ such that $\phi_l(x_j) = \delta_{ij}$ for all $x_j \in N_k$, where $N_k$ denotes the set of vertices of the tetrahedra in $T_k$. Then, the set of hierarchical basis functions $\hat{\phi}_l$ for $S_k$ is given by

$$\{ \phi_0^i \mid \forall i \text{ such that } x_i \in N_0 \} \bigcup_{l=1, \ldots, k} \{ \phi_l^i \mid \forall i \text{ such that } x_i \in N_l \setminus N_{l-1} \}.$$

That is, the hierarchical basis functions satisfy $\hat{\phi}_l = \phi_0^i$ for $x_i \in N_0$ and

$$\hat{\phi}_l = \phi_l^i, \quad \text{for } x_i \in N_l \setminus N_{l-1}.$$
Define the interpolation operator $I_k : S_N \to S_k$ by $(I_k u)(x) = u(x)$ for $x \in N_k$. For $u \in S_N$, because $u = I^0 u$, one has the splitting

$$u = I^0 u + \sum_{k=1}^{N} (I^k u - I^{k-1} u).$$

The subspace $\mathcal{V}_k$ of $S_k$ is defined as the span of the hierarchical basis functions $\hat{\phi}_i$, $x_i \in N_k \setminus N_{k-1}$. Then, $S_k$ is the direct sum

$$S_k = S_0 \oplus \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_k.$$

If $u \in S_N$ has the representation

$$u = u_0 + \sum_{k=1}^{N} v_k, \quad u_0 \in S_0, \ v_k \in \mathcal{V}_k,$$

then [30]

$$u_0 = I^0 u \quad \text{and} \quad v_k = I^k u - I^{k-1} u.$$

See [24,30,31] and the references cited therein for detailed discussions about hierarchical finite element spaces.

**Remark 3.1.** The implementation using the hierarchical basis can be handled easily by using the following relationship between a matrix $A$ corresponding standard to a nodal basis discretization and the matrix $\hat{A}$ corresponding to a hierarchical basis discretization [30,31]:

$$\hat{A} = S^T A S,$$

where the $(i, j)$ component of $S$ has the value $S_{ij} = \hat{\phi}_j(x_i)$. $S$ is a block unit lower triangular matrix if the nodes are numbered in hierarchical order, i.e., the nodes in level 0 are numbered first, the nodes in level 1 are numbered next, and so on. Even though $\hat{A}$ has a much more complicated structure and contains more nonzero elements compared to $A$, utilizing $\hat{A} \hat{v} = S^T (A(S \hat{v}))$, $\hat{A} \hat{v}$ can be evaluated with only little more effort than $A \hat{v}$.

Throughout this paper, we set $h = h_N$ and $c$ a generic constant unless it is explicitly specified.

Let $X^h \subset L^2(\Omega)$ denote a finite-dimensional subspace of continuous vector-valued functions whose components are in $S_N$ and $M^h \subset L^2(\Omega)$ denote a finite-dimensional subspace that consists of standard nodal basis. We assume the following.

**H1** Each function $v_h \in X^h$ satisfies

(i) \[ \int_{\partial T \cap \partial T'} (v_h|_T - v_h|_{T'}) \ ds = 0, \ \forall T, T' \in T_N. \]

(ii) \[ \int_{\partial T \cap \partial \Omega} v_h|_T \ ds = 0, \ \forall T \in T_N. \]

**H2** There exist operators $P^h \in L(H^1_0(\Omega) \cap H^2(\Omega); X^h)$ and $r^h \in L(H^1(\Omega); M^h)$ such that

(i) \[ \|\nabla(u - P^h u)\| \leq ch \|u\|_2, \ \forall u \in J \cap H^2(\Omega), \]

(ii) \[ \|\pi - r^h \pi\| \leq ch \|\pi\|_1, \ \forall \pi \in H^1(\Omega). \]

The above assumptions (H1) and (H2) are standard conditions satisfied by all the usual spaces of conforming and nonconforming elements (see [32]). Also we assume the classical inf–sup condition for mixed finite element methods:

**H3** for any $q_h \in M^h$, there exists a nonzero function $v_h \in X^h$ such that

$$\langle q_h, \nabla \cdot v_h \rangle \geq c \|\nabla v_h\| \|q_h\|.$$
3.2. Discretized problem

We seek an approximation of weak solutions of the Navier–Stokes equations. To do so, we discretize a corresponding weak formulation of the Ladyzhenskaya/Smagorinsky model in such a way that the eddy viscosity term is applied only to high-frequency modes; see (1). To this end, we define a high-pass filtering operator that eliminates the low-frequency modes. Let \( \{\phi_i\}_{k,i} \) denote a hierarchical basis for \( X^h \). Let \( Q^N : X^h \to X^h \) denote the high-pass filtering operator
\[
Q^N u_h = \sum_{k,i} Q_{k,i} \beta_{k,i} \phi_i^k, \quad \text{for } u_h = \sum_{k,i} \beta_{k,i} \phi_i^k \in X^h,
\]
where \( 0 \leq Q_{k,i} \leq 1 \) with
\[
Q_{k,i} = \begin{cases} 
0, & \text{for } k \leq m_h, \\
1, & \text{for } k > m_h.
\end{cases}
\]
Note that applying \( Q^N \) annihilates the low-frequency modes in \( u_h \), i.e., all the coefficients of \( Q^N u_h \) corresponding to the hierarchical basis functions at level \( m_h \) or lower vanish. Then, we seek an approximate solution of the weak formulation of the Ladyzhenskaya/Smagorinsky model, modified through the incorporation of a the high-pass filter \( Q^N \) in the eddy viscosity term: find \( (u_h, \tau_h) \in X^h \times M^h \) such that
\[
(\partial_t u_h, v_h) + \langle a(u_h, v_h) + \beta(u_h, u_h, v_h) - \langle \tau_h, \nabla \cdot v_h \rangle + \epsilon \| \nabla (Q^N u_h) \|^{p-2} \nabla (Q^N u_h), \nabla (Q^N v_h) \rangle = (f, v_h), \quad \forall v_h \in X^h,
\]
\[
(\nabla \cdot u_h, q_h) = 0, \quad \forall q_h \in M^h
\]
and \( u_h(0, x) = u_0^h(x) \), where
\[
a(u, v) = \langle \nabla u, \nabla v \rangle,
\]
\[
b(u, w, v) = \frac{1}{2} \langle u \cdot \nabla w, v \rangle - \frac{1}{2} \langle u \cdot \nabla v, w \rangle.
\]
If \( x, y, z \in H^1_0(\Omega) \) and \( \nabla \cdot x = 0 \), then \( b(x, y, z) = (x \cdot \nabla y, z) \). If the finite elements are nonconforming, then the inner product on the discrete space is defined as
\[
\langle u, v \rangle_T = \sum_{T \in \mathcal{T}_h} \int_T u \cdot v \, dx.
\]

4. Existence and uniqueness of the discrete solution

Define
\[
J^h = \{ v_h \in X^h \mid (\nabla \cdot v_h, q_h) = 0, \quad \forall q_h \in M^h \}
\]
so that (7) becomes: find \( u_h(t) \in J^h \) such that \( u_h(0) = u_0^h \) and, for \( t > 0 \),
\[
(\partial_t u_h, v_h) + \langle a(u_h, v_h) + b(u_h, u_h, v_h) + \epsilon \| \nabla (Q^N u_h) \|^{p-2} \nabla (Q^N u_h), \nabla (Q^N v_h) \rangle = (f, v_h)
\]
for all \( v_h \in J^h \).

**Theorem 4.1.** For given \( h \) and \( u_0^h \) chosen in \( J^h \), the problem (8) is uniquely solvable for all \( t > 0 \).

**Proof.** We substitute \( v_h = u_h \) into (8) to yield
\[
\frac{1}{2} \frac{d}{dt} \| u_h(t) \|^2 + \| \nabla u_h \|^2 + \epsilon \| Q^N u_h \|_0 \rho_\Omega^p = (f, u_h) \leq \| f \| \| u_h \|
\]
which implies \( \| u_h(t) \|^2 \leq \| u_0^h \|^2 + c \int_t^\infty \| f \|^2 \, dt \), for all \( t \geq 0 \). Therefore, the solution of (8) uniquely exists by the well-known theories of ordinary differential equations. \( \square \)

However, the uniqueness of the solution for (8) is ensured only for \( h \) fixed. In the following, we show that the finite element approximate solution is unique independently of \( h \). First, we introduce two lemmas which are useful in showing the \( h \)-independent uniqueness.
Lemma 4.2. (See [33].) Let \( p \geq 2 \). Then, for all \( a, b \in \mathbb{R}^m \) with \( m \) a nonnegative integer, we have
\[
\langle |a|^{p-2}a - |b|^{p-2}b, a - b \rangle \geq \gamma_0 |a - b|^p,
\]
where \( \gamma_0 \) depends only on \( p \) and \( m \).

Lemma 4.3. Let \( u_h \in X^h \) and \( T \in T_N \). Then,
\[
\|u_h\|_{0,3,T} \leq c\|u_h\|_{0,2,T}^\frac{1}{2}\|\nabla u_h\|_{0,2,T}^\frac{1}{2}.
\]

Proof. By the Hölder inequality, we have
\[
\left( \int_T |u_h|^2 \, dx \right)^{\frac{1}{2}} \leq \left( \int_T |u_h|^2 \frac{3}{2} \, dx \right)^{\frac{1}{2}} \left( \int_T |u_h|^{-\frac{3}{2}} \, dx \right)^{\frac{1}{2}}
\]
which implies
\[
\|u_h\|_{0,3,T} \leq c\|u_h\|_{0,2,T}^\frac{1}{2}\|u_h\|_{0,6,T}^\frac{1}{2}.
\]

Let \( L^h \) be an interpolation operator from \( X^h \) to the set of continuous functions which are piecewise linear on each element in \( T_N \). Interpolation error estimates, the linearity of \( L^h u_h \), (A.7) in [31], and the inverse inequality yield
\[
\|u_h\|_{0,0,T} \leq \|u_h - L^h u_h\|_{0,0,T} + \|L^h u_h\|_{0,0,T} \leq c\|\nabla u_h\|_{0,0,T} + c\|\nabla L^h u_h\|_{0,0,T} \leq c\|\nabla u_h\|_{0,0,T}
\]
\[
\leq c \cdot h^{-\frac{1}{4}} \|\nabla u_h\|_{0,2,T} = ch^{-\frac{1}{4}} \|\nabla u_h\|_{0,2,T}.
\]

From the Hölder inequality, we have
\[
\|u_h\|_{0,6,T} \leq \left( \int_T \|1\| \, dx \right)^{\frac{1}{6}} \|u_h\|_{0,0,T} \leq ch^\frac{1}{2} \|u_h\|_{0,0,T}.
\]

Therefore, by applying (11) and (12) to (10), the result holds. \( \square \)

Theorem 4.4. Choose \( p \geq \frac{5}{2} \) and \( m_h \leq \frac{N}{2} \) with \( m_h \) defined as in (6). The uniqueness of solution for (8) is independent on \( h \).

Proof. Suppose \( u_1 \) and \( u_2 \) are solutions of (8). Then \( w_h = u_1 - u_2 \) satisfies
\[
\langle \partial_t w_h, v_h \rangle + \nu \langle w_h, v_h \rangle + b(u_1, u_1, v_h) - b(u_2, u_2, v_h) + \varepsilon c(u_1, u_2, v_h) = 0,
\]
where
\[
c(u_1, u_2, v_h) = \|\nabla (Q^N u_1)\|^{p-2} \nabla (Q^N u_1) - \|\nabla (Q^N u_2)\|^{p-2} \nabla (Q^N u_2), \nabla (Q^N v_h)\|
\]
for all \( v_h \in X^h \). Setting \( v_h = w_h \) in the above equation and using Lemma 4.2, we have
\[
\frac{1}{2} \frac{d}{dt} \|w_h\|^2 + \nu \|\nabla w_h\|^2 + \varepsilon \|\nabla (Q^N w_h)\|^p_{0,p,Ω} \leq b(u_2, u_2, w_h) - b(u_1, u_1, w_h).
\]
Green's formula and the triangle inequality yield
\[
b(u_2, u_2, w_h) - b(u_1, u_1, w_h) = \frac{1}{2} \int_Ω (w_h \cdot \nabla u_2) \cdot w_h - (w_h \cdot \nabla w_h) \cdot u_2 \, dx
\]
\[
\leq c \int_Ω |w_h||\nabla w_h||u_2| \, dx
\]
\[
\leq c \int_Ω |w_h||\nabla w_h||Q^N u_2| + |w_h||\nabla w_h|||I - Q^N||u_2| \, dx.
\]
Consider the first term in (14):

\[
A = \int_{\Omega} |w_h| |\nabla w_h| |Q^N u_2| \, dx = \sum_{T \in \mathcal{T}_N} \int_T |w_h| |\nabla w_h| |Q^N u_2| \, dx
\]

\[
\leq \sum_{T \in \mathcal{T}_N} \|Q^N u_2\|_{0, \infty, T} \int_T |w_h| |\nabla w_h| \, dx
\]

\[
\leq \sum_{T \in \mathcal{T}_N} \|Q^N u_2\|_{0, \infty, T} \|w_h\|_{0, 2, T} \|\nabla w_h\|_{0, 2, T}.
\]

(15)

Recall the interpolation operator $L_h$ defined in the proof of Lemma 4.3. Because $L_h Q^N u_2$ is linear on each $T \in \mathcal{T}_N$, the interpolation error estimates, (A.7) in [31], and the inverse inequality yield

\[
\|Q^N u_2\|_{0, \infty, T} \leq \|Q^N u_2 - L_h Q^N u_2\|_{0, \infty, T} + \|L_h Q^N u_2\|_{0, \infty, T}
\]

\[
\leq c h \|\nabla Q^N u_2\|_{0, \infty, T} + ch \|\nabla L_h Q^N u_2\|_{0, \infty, T}
\]

\[
\leq c h^{1-\frac{2}{p}} \|\nabla Q^N u_2\|_{0, p, T}.
\]

(16)

Also, (11) provides

\[
\|w_h\|_{0, 2, T} \leq c h^2 \|w_h\|_{0, \infty, T} \leq c h \|\nabla w_h\|_{0, 2, T}.
\]

(17)

Therefore, combining (15)–(17) leads us to

\[
A \leq \sum_{T \in \mathcal{T}_N} \|Q^N u_2\|_{0, \infty, T} \|w_h\|_{0, 2, T} \|\nabla w_h\|_{0, 2, T}
\]

\[
\leq \sum_{T \in \mathcal{T}_N} c h^{1-\frac{2}{p}} h^{1-\frac{2}{p}} \|\nabla Q^N u_2\|_{0, p, T} \|w_h\|_{0, 2, T} \|\nabla w_h\|_{0, 2, T}^2
\]

and Young’s inequality results in

\[
A \leq \sum_{T \in \mathcal{T}_N} \left( c h^{1-\frac{2}{p}} h^{1-\frac{2}{p}} \|\nabla Q^N u_2\|_{0, p, T} \|w_h\|_{0, 2, T}^2 + \frac{v}{4} \|\nabla w_h\|_{0, 2, T}^2 \right)
\]

\[
\leq \sum_{T \in \mathcal{T}_N} \left( c h^{1-\frac{2}{p}} h^{1-\frac{2}{p}} \|\nabla Q^N u_2\|_{0, p, T} \|w_h\|_{0, 2, T}^2 + \frac{v}{4} \|\nabla w_h\|_{0, 2, T}^2 \right)
\]

\[
= c h^{1-\frac{2}{p}} h^{1-\frac{2}{p}} \|\nabla Q^N u_2\|_{0, p, T} \|w_h\|_{0, 2, T}^2 + \frac{v}{4} \|\nabla w_h\|_{0, 2, T}^2.
\]

(18)

Next, consider the second term in (14). The Hölder inequality, Lemma 4.3, and Young’s inequality imply

\[
B = \int_{\Omega} |w_h| |\nabla w_h| |(I - Q^N) u_2| \, dx = \sum_{T \in \mathcal{T}_N} \int_T |w_h| |\nabla w_h| |(I - Q^N) u_2| \, dx
\]

\[
\leq c \sum_{T \in \mathcal{T}_N} h^{\frac{3}{2}} \|w_h\|_{0, 3, T} \|\nabla w_h\|_{0, 2, T} \|(I - Q^N) u_2\|_{0, \infty, T}
\]

\[
\leq c \sum_{T \in \mathcal{T}_N} h^{\frac{3}{2}} \|w_h\|_{0, 2, T}^\frac{3}{2} \|\nabla w_h\|_{0, 2, T}^\frac{3}{2} \|(I - Q^N) u_2\|_{0, \infty, T}
\]

\[
\leq \sum_{T \in \mathcal{T}_N} \left\{ \frac{c}{\nu^4} \left( h \|w_h\|_{0, 2, T}^\frac{3}{2} \|(I - Q^N) u_2\|_{0, \infty, T}^4 + \frac{v}{4} \|\nabla w_h\|_{0, 2, T}^\frac{3}{2} \right)^4 \right\}.
\]

(19)

In the first term in (19), for each $T$, let $T_{m_0} \in \mathcal{T}_{m_0}$ such that $T \subset T_{m_0}$. Then, the inverse inequality yields

\[
\|(I - Q^N) u_2\|_{0, \infty, T}^4 \leq \|(I - Q^N) u_2\|_{0, \infty, T_{m_0}}^4 \|(I - Q^N) u_2\|_{0, \infty, T}^2
\]

\[
\leq c h_{m_0}^{-3} \|(I - Q^N) u_2\|_{0, 2, T_{m_0}}^2 \|(I - Q^N) u_2\|_{0, \infty, T}^2.
\]

(20)
Recalling the definition of the filtering operator $Q^N$ and the representation (3)-(4) by hierarchical finite elements, we have

$$(I - Q^N)u_2 = \sum_{k=1}^{m_h} (I^k u_2 - I^{k-1} u_2) + t^0 u_2.$$ 

Again, we let $T_k \in T_k$ denote finite elements such that $T \subset T_k$ for each $k$, $0 \leq k \leq m_h$. By the triangle and inverse inequalities and interpolation error estimates, we have

$$\| (I - Q^N)u_2 \|_{0, \infty, T} \leq \sum_{k=1}^{m_h} \| I^k u_2 - I^{k-1} u_2 \|_{0, \infty, T} + \| I^0 u_2 \|_{0, \infty, \Omega}$$

$$\leq \sum_{k=1}^{m_h} \| I^k u_2 - I^{k-1} u_2 \|_{0, \infty, T} + \| I^0 u_2 \|_{0, \infty, T_0}$$

$$\leq \sum_{k=1}^{m_h} ch_k^{-2} \| I^k u_2 - I^{k-1} u_2 \|_{0, 2, T_k} + c H^{-\frac{3}{2}} \| I^0 u_2 \|_{0, 2, T_0}$$

$$\leq \sum_{k=1}^{m_h} ch_k^{-2} (\| I^k u_2 - u_2 \|_{0, 2, T_k} + \| u_2 - I^{k-1} u_2 \|_{0, 2, T_k}) + c H^{-\frac{3}{2}} \| u_2 \|_{0, 2, T_0}$$

$$\leq \sum_{k=1}^{m_h} ch_k^{-2} (\| I^k u_2 - u_2 \|_{0, 2, T_k} + \| u_2 - I^{k-1} u_2 \|_{0, 2, T_{k-1}}) + c H^{-\frac{3}{2}} \| u_2 \|_{0, 2, T_0}$$

$$\leq \sum_{k=1}^{m_h} ch_k^{-2} \| \nabla u_2 \|_{0, 2, \Omega} + c H^{-\frac{3}{2}} \| u_2 \|_{0, 2, \Omega}.$$  \hspace{1cm} (21)

where $H = h_0$ is the minimum mesh size of the initial refinement and $h_k = ch_k^{-1}/2$. Now, we observe some properties of $u_2$. Because $u_2 \in L^\infty (I; L^2 (\Omega))$, there exists a constant $M$ such that $\| u_2 \|_{0, 2, \Omega} \leq M$. Also, $I - Q^N$ is simply an interpolation operator with respect to the coarse grid, so that we have

$$\| (I - Q^N)u_2 \|_{0, 2, T_{m_h}} \leq \| (I - Q^N)u_2 \|_{0, 2, \Omega} \leq c \| u_2 \|_{0, 2, \Omega}. \hspace{1cm} (22)$$

Therefore, combining (20)-(22) results in

$$\| (I - Q^N)u_2 \|_{0, \infty, T}^4 \leq ch_m^{-3} \| (I - Q^N)u_2 \|_{0, 2, T_{m_h}}^2 (h_m^{-1} \| \nabla u_2 \|_{0, 2, \Omega}^2 + \| u_2 \|_{0, 2, \Omega}^2)$$

$$\leq ch_m^{-3} \| u_2 \|_{0, 2, \Omega}^2 (h_m^{-1} \| \nabla u_2 \|_{0, 2, \Omega}^2 + \| u_2 \|_{0, 2, \Omega}^2)$$

$$\leq ch_m^{-3} \| u_2 \|_{0, 2, \Omega}^2 + ch_m^{-3} M^4$$

$$\leq c(h_m^{-4} \| \nabla u_2 \|_{0, 2, \Omega}^2 + h_m^{-3}). \hspace{1cm} (23)$$

We apply (23) to (19) to obtain

$$B \leq \sum_{T \in T_h} \left( \frac{c}{\epsilon^3} h^2 (h_m^{-4} \| \nabla u_2 \|_{0, 2, \Omega}^2 + h_m^{-2}) \| w_h \|_{0, 2, \Omega}^2 + \frac{\nu}{4} \| \nabla w_h \|_{0, 2, \Omega}^2 \right)$$

$$\leq c \nu^2 h^2 (h_m^{-4} \| \nabla u_2 \|_{0, 2, \Omega}^2 + h_m^{-2}) \| w_h \|_{0, 2, \Omega}^2 + \frac{\nu}{4} \| \nabla w_h \|_{0, 2, \Omega}^2.$$ 

Finally, gathering the bounds for $A$ and $B$ yields

$$\frac{1}{2} \frac{d}{dt} \| w_h \|^2 + \nu \| \nabla w_h \|^2 + \epsilon \| \nabla^2 w_h \|^p \geq c(v^{1-p} h^{2p-5} \| \nabla^2 w_h \|^p \| u_2 \|_{0, p, \Omega}^p + \nu^{-2} (h^{-4} \| \nabla u_2 \|_{0, 2}^2 + h^{-2} \| \nabla u_2 \|_{0, 2}^2)) \| w_h \|^2 + \nu \| \nabla w_h \|^2.$$ 

Hence, so long as we choose

$$p \geq \frac{5}{2} \text{ and } m_h \leq \frac{N}{2},$$
Gronwall’s lemma yields
\[
\left\| \mathbf{w}_h(\cdot, t) \right\| \leq \left\| \mathbf{w}_h(\cdot, 0) \right\|^2 e^{\int_0^t \beta(\tau) d\tau} = 0,
\]
where
\[
\beta(\tau) = \nu^1 p h^2 p - e^{-1} \varepsilon \left\| \nabla Q^N \mathbf{u}_2(\cdot, \tau) \right\|_{0, p, \Omega}^p + v^{-3} \left( h^2 - 4 \pi \right) \left\| \nabla \mathbf{u}_2(\cdot, \tau) \right\|^2 + h^2 - 2 \pi
\]
is bounded. Therefore, we have \( \mathbf{u}_1 = \mathbf{u}_2 \). \( \square \)

Once \( \mathbf{u}_h \in f^h \) is uniquely determined, we seek \( \pi_h \in M^h \) such that
\[
\langle \pi_h, \nabla \cdot \mathbf{v}_h \rangle = \langle \partial_t \mathbf{u}_h, \mathbf{v}_h \rangle + v a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{v}_h) + \varepsilon \left( \left\| \nabla Q^N \mathbf{u}_h \right\|_{2, \Omega}^p - 2 \left\| \nabla Q^N \mathbf{u}_h, \nabla Q^N \mathbf{v}_h \right\| - \langle f, \mathbf{v}_h \rangle \right) \tag{24}
\]
for all \( \mathbf{v}_h \in X^h \). It is easy to see that there exits a unique \( \pi_h \in M^h / \mathbb{R} \) satisfying (24). Here, the inf–sup condition (H3) ensures that the discrete pressure \( \pi_h \in M^h / \mathbb{R} \) depends continuously on the discrete velocity \( \mathbf{u}_h \in f^h \) uniformly in \( h \) [32].

In this section, we showed the existence and uniqueness of the solution to the discrete modified Ladyzhenskaya/ Smagorinsky model (7). For the approximate solution \( (\mathbf{u}_h, \pi_h) \), we next present basic convergence results.

5. Error estimates

The goal in this section is to verify that the approximation \( \mathbf{u}_h \), which was found by solving the problem (8) converges to a weak solution of Navier–Stokes equations. First we assume that the approximation \( \mathbf{u}_h^0 \) of the initial data \( \mathbf{u}_0 \) satisfies, uniformly for \( h \to 0 \),
\[
\left\| \mathbf{u}_0 - \mathbf{u}_h^0 \right\| \leq c h^l \left\| \mathbf{u}_0 \right\|_{l, 2, \Omega}, \quad l = 1, 2, \tag{25}
\]
provided that \( \mathbf{u}_0 \in H^k(\Omega) \). We also assume that \( p \geq 3 \). Then, we choose \( \varepsilon \) satisfying
\[
\varepsilon = c h^\theta \tag{26}
\]
for some positive constant \( \theta \). The following result is needed in the sequel.

**Lemma 5.1.** Let \( \mathbf{w}_h \in X^h \). Then,
\[
\left\| Q^N \mathbf{w}_h \right\| \leq c \left\| \mathbf{w}_h \right\|. \tag{27}
\]
\[
\left\| \nabla Q^N \mathbf{w}_h \right\| \leq c \left\| \nabla \mathbf{w}_h \right\|. \tag{28}
\]

**Proof.** Because \( Q^N \) is simply an interpolation operator on a fine grid, (27) is clearly true. To prove (28), we use an inverse inequality and recall the interpolation operator \( L^h \) and the inequalities (11) and (12) used in the proof of Lemma 4.3. Then, we have
\[
\left\| \nabla Q^N \mathbf{w}_h \right\|^2 \leq c h^{-2} \left\| Q^N \mathbf{w}_h \right\|^2 \leq c h^{-2} \left\| \mathbf{w}_h \right\|^2 = c h^{-2} \sum_{T \in T_h} \left\| \mathbf{w}_h \right\|^2_{0, 2, T}
\]
and
\[
\left\| \mathbf{w}_h \right\|_{0, 2, T} \leq \left\| \mathbf{w}_h - L^h \mathbf{w}_h \right\|_{0, 2, T} + \left\| L^h \mathbf{w}_h \right\|_{0, 2, T} \leq c h \left\| \nabla \mathbf{w}_h \right\|_{0, 2, T} + c h \left\| \nabla L^h \mathbf{w}_h \right\|_{0, 2, T} \leq c h \left\| \nabla \mathbf{w}_h \right\|_{0, 2, T}.
\]
The above two inequalities yield (28). \( \square \)

In order to achieve our goal, we follow the typical arguments used in [28] with some modifications.

**Theorem 5.2.** Let \( \mathbf{u}_0 \in J \) and \( f \in L^2(0, T; L^2(\Omega)) \). Let \( \mathbf{u} \in L^\infty(0, T; J) \) and \( \mathbf{u}_h \in L^\infty(0, T; f^h) \) denote a strong solution of the Navier–Stokes equations and the solution of (8), respectively. Assume that the discrete initial data approximation \( \mathbf{u}_h^0 \) for \( \mathbf{u}_h \) at \( t = 0 \) satisfies (25).

i) Then,
\[
\left\| \mathbf{u}(t) - \mathbf{u}_h(t) \right\|^2 + \int_0^t \left\| \nabla (\mathbf{u} - \mathbf{u}_h) \right\|^2 d\tau \leq c(t) h^\alpha, \tag{29}
\]
where \( \alpha = 2 \) if \( p \geq 4 \) and \( \alpha = 1 \) if \( 3 \leq p < 4 \), for \( t \in [0, T] \).
ii) Moreover, if the solution \( \mathbf{u} \) satisfies
\[
\partial_t \mathbf{u} \in L^2(0, T; H^2(\Omega)) \quad \text{and} \quad \partial_t \pi \in L^2(0, T; H^1(\Omega)),
\]
the data \( \mathbf{f} \) satisfies \( \mathbf{f} \in L^k(0, T; L^2(\Omega)), \ k > 4, \) and the parameter \( p \) in (8) satisfies \( p \geq 2 + k/2, \) then
\[
\left\| \nabla (\mathbf{u}(t) - \mathbf{u}_h(t)) \right\|^2 + \int_0^t \left\| \partial_t (\mathbf{u} - \mathbf{u}_h) \right\|^2 \, dt \leq c(t) h^{\min(2,(k-4)p/(2p-4))}.
\]

Proof. i) Because we cannot use \( \mathbf{v}_h \in J^h \) as a test functions in the weak formulation of the Navier-Stokes equations, we start with the Navier-Stokes equations, i.e., with (1) with \( \varepsilon = 0, \) which we multiply by \( \mathbf{v}_h \in J^h \) and then integrate the result over \( \Omega \) to obtain
\[
\langle \partial_t \mathbf{u}_h, \mathbf{v}_h \rangle + \nu a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h) - (\pi, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}_h) + \Gamma^*(\mathbf{u}_h, \mathbf{u}_h),
\]
where
\[
\Gamma^*(\mathbf{u}_h, \mathbf{u}_h) = \sum_{T \in T_h} \int_{\partial T} \left( \nu (\mathbf{n} \cdot \nabla \mathbf{u}_h) \cdot \mathbf{v}_h + \frac{1}{2} (\mathbf{n} \cdot \mathbf{u}_h) (\mathbf{u}_h) - \pi (\mathbf{n} \cdot \mathbf{v}_h) \right) \, ds
\]
with \( \mathbf{n} \) the outward unit normal to \( \partial T, \) the boundary of \( T. \) We recall that \( \mathbf{u}_h \) satisfies (8). By subtracting (8) from (31) and letting \( \mathbf{e} = \mathbf{u} - \mathbf{u}_h, \) we have
\[
\langle \partial_t \mathbf{e}, \mathbf{w}_h \rangle + \nu a(\mathbf{e}, \mathbf{w}_h) + b(\mathbf{u}_h, \mathbf{u}_h) - b(\mathbf{u}, \mathbf{u}_h) + (\pi, \nabla \cdot \mathbf{w}_h)
\]
\[
+ \Gamma^*(\mathbf{u}_h, \mathbf{u}_h) + \nu (\nabla^2 \mathbf{u}_h) (\mathbf{w}_h) + \nu (\nabla \mathbf{u}_h) (\mathbf{w}_h).
\]
Let \( \Pi^h : L^2(\Omega) \to J^h \) denote an \( L^2 \)-projection operator, i.e., we have
\[
\langle \mathbf{v} - \Pi^h \mathbf{v}, \mathbf{w}_h \rangle = 0, \quad \forall \mathbf{w}_h \in J^h,
\]
such that
\[
\left\| \nabla \Pi^h \mathbf{v} \right\| \leq c \left\| \nabla \mathbf{v} \right\|, \quad \text{for} \ \mathbf{v} \in J \oplus J^h.
\]
\[
\left\| \mathbf{v} - \Pi^h \mathbf{v} \right\|_{L^2(\Omega)} \leq c h^{m-l} \left\| \mathbf{v} \right\|_{L^m(\Omega)}, \quad \text{for} \ l < m, \ \forall \mathbf{v} \in J \cap H^m(\Omega)
\]
and let \( \Theta^h : L^2(\Omega) \to M^h \) denote an \( L^2 \)-projection operator, i.e., we have
\[
\langle q - \Theta^h q, q_h \rangle = 0, \quad \forall q_h \in M^h,
\]
such that
\[
\left\| q - \Theta^h q \right\| \leq c h \left\| \nabla q \right\|, \quad \forall q \in H^1(\Omega).
\]
We introduce several inequalities which are useful in this proof:

- **Sobolev inequality:** \( \left\| \mathbf{v} \right\|_{0.6, K} \leq c \left\| \mathbf{v} \right\|_{1.2, K}, \quad \forall \mathbf{v} \in H^1(K). \)
- **Poincaré inequality:** \( \left\| \mathbf{v} \right\|_{1.2, K} \leq c \left\| \nabla \mathbf{v} \right\|_{0.2, K}, \quad \forall \mathbf{v} \in H^1_0(K). \)
- **a priori estimate:** \( \left\| \mathbf{v} \right\|_{2.2, K} \leq c \left\| \Delta \mathbf{v} \right\|_{0.2, K}, \quad \forall \mathbf{v} \in H^1_0(K) \cap H^2(K). \)
- **discrete Sobolev inequality:** \( \left\| \mathbf{v}_h \right\|_{0.6, K} \leq c \left\| \nabla \mathbf{v}_h \right\|_{0.2, K}. \quad \forall \mathbf{v}_h \in J^h. \)

The proof of discrete Sobolev inequality can be found in [32, Lemma 4.4]. From arguments used in [32,28], we have
\[
\Gamma^*(\mathbf{u}_h, \mathbf{u}_h) \leq c h \left\| \nabla \mathbf{v}_h \right\| \left( \left\| \Delta \mathbf{u} \right\| + \left\| \nabla \mathbf{u} \right\|^2 + \left\| \nabla \pi \right\| \right).
\]
Also, \( \mathbf{v}_h \) being an element of \( J^h \) and (36) yields
\[
\langle \pi, \nabla \cdot \mathbf{v}_h \rangle = \langle \pi - \Theta^h \pi, \nabla \cdot \mathbf{v}_h \rangle \leq c \left\| \pi - \Theta^h \pi \right\| \left\| \nabla \mathbf{v}_h \right\| \leq c h \left\| \nabla \pi \right\| \left\| \nabla \mathbf{v}_h \right\|.
\]
Set \( \mathbf{v}_h = \Pi^h \mathbf{e} \) in (32) to obtain
\[
\frac{1}{2} \frac{d}{dt} \|e\|^2 + v \|\nabla e\|^2 - \langle \dot{e}, e - \Pi^h e \rangle - v \langle \nabla e, \nabla (e - \Pi^h e) \rangle
\]

\[
= b(u_h, u_h, \Pi^h e) - b(u, u, \Pi^h e) + \langle \nabla \cdot e, \Pi^h e \rangle + \Gamma(u, u, \Pi^h e) + e \|\nabla Q^N u_h\|^{p-2} \nabla Q^N u_h \cdot \nabla Q^N \Pi^h e). \tag{43}
\]

Using the orthogonality of \(L^2\)-projections, (35), (39), and the Hölder and triangle inequalities, we obtain

\[
\langle \dot{e}, e - \Pi^h e \rangle + v \langle \nabla e, \nabla (e - \Pi^h e) \rangle \leq \frac{1}{2} \frac{d}{dt} \|e - \Pi^h e\|^2 + v \|\nabla e\| \|\nabla (e - \Pi^h e)\|
\]

\[
\leq \frac{1}{2} \frac{d}{dt} h^2 \|\nabla e\|^2 + \frac{v}{4} \|\nabla e\|^2 + c v h^2 \|u\|_2,\Omega. \tag{44}
\]

By (34), (41) and (42) yield

\[
\langle \nabla \cdot e, \Pi^h e \rangle + \Gamma(u, u, \Pi^h e) \leq c h \|\nabla e\| (\|\Delta u\| + \|\nabla u\|^3 + \|\nabla \pi\|)
\]

\[
\leq \frac{v}{4} \|\nabla e\|^2 + c h^2 (\|\Delta u\|^2 + \|\nabla u\|^6 + \|\nabla \pi\|^2). \tag{45}
\]

Now, consider \(b(u_h, u_h, \Pi^h e) - b(u, u, \Pi^h e)\) in (43):

\[
b(u_h, u_h, \Pi^h e) - b(u, u, \Pi^h e) = \frac{1}{2} \left( (e \cdot \nabla e, \Pi^h e) - (e \cdot \nabla e, \Pi^h e) - (e \cdot \nabla e, \Pi^h e) \right)
\]

\[
+ (e \cdot \nabla e, \Pi^h e) - (e \cdot \nabla \Pi^h e, \Pi^h e) + (e \cdot \nabla \Pi^h e, \Pi^h e).
\]

Because \(\Pi^h\) is an \(L^2\)-projection, we have \(\|\Pi^h e\| \leq \|e\|\). Applying the Hölder and triangle inequalities, (37)–(40), and (34) lead us to

\[
\|e \cdot \nabla e, \Pi^h e\| \leq \|\nabla e\|_{0.2, \Omega} \|e\|_{0.6, \Omega} \|\Pi^h e\|_{0.3, \Omega}
\]

\[
\leq \delta v \|\nabla e\|^2 + c \|\nabla u\|^2 \|\Pi^h e\|^{\beta} \|\Pi^h e\|_{0.6, \Omega}
\]

\[
\leq \delta v \|\nabla e\|^2 + c \delta v \|\Pi^h e\|^2_{0.6, \Omega} + c \|\nabla u\|^4 \|\Pi^h e\|^2
\]

\[
\leq \delta v \|\nabla e\|^2 + c \delta v \|\nabla e\|^2 + c \|\nabla u\|^4 \|e\|^2. \tag{46}
\]

In a similar manner, we have

\[
\|e \cdot \nabla e, \Pi^h e\| + \|e \cdot \nabla u, \Pi^h e\| - \|e \cdot \nabla \Pi^h e, e\| - \|e \cdot \nabla \Pi^h e, u\| \leq 7 \delta v \|\nabla e\|^2 + c \|\nabla u\|^4 \|e\|^2.
\]

Since \(\Pi^h u_h = u_h\) for \(u_h \in J^h\), adding, subtracting terms and (35) yield

\[
\|e \cdot \nabla e, \Pi^h e\| - \|e \cdot \nabla \Pi^h e, e\| - \|e \cdot \nabla \Pi^h e, u\|
\]

\[
\leq \|\nabla e\|_{0.3, \Omega} \|u - \Pi^h u\|_{0.6, \Omega} + \|\nabla (u - \Pi^h u)\| \|e\|_{0.3, \Omega} \|e\|_{0.6, \Omega}
\]

\[
\leq 2 \delta v \|\nabla e\|^2 + c \|\nabla u\|^4 \|e\|^2. \tag{47}
\]

Now, consider the term \(D \equiv \epsilon |\nabla Q^N u_h|^{p-2} \nabla Q^N \Pi^h e\) in (43). Simply by adding and subtracting terms, we have

\[
D = \epsilon (\tau \nabla Q^N u_h, \nabla Q^N \Pi^h e) - \epsilon (\tau \nabla Q^N u_h, \nabla Q^N u_h)
\]

\[
\leq \epsilon (\tau \nabla Q^N u_h, \nabla Q^N \Pi^h e)
\]

\[
= \epsilon (\tau \nabla Q^N \Pi^h u, \nabla Q^N \Pi^h u) - \epsilon (\tau \nabla Q^N \Pi^h e, \nabla Q^N \Pi^h e) \leq \epsilon (\tau \nabla Q^N \Pi^h e, \nabla Q^N \Pi^h e),
\]

where \(\tau \equiv |\nabla Q^N u_h|^{p-2}\), and therefore

\[
2D \leq \epsilon \|\nabla Q^N u_h\|^{p-2} \nabla Q^N \Pi^h u, \nabla Q^N \Pi^h u) \equiv D_1. \tag{48}
\]

Now, we separate into two cases: \(p > 4\) and \(3 < p < 4\).

1) \((p > 4)\) By the Hölder and inverse inequalities, we have

\[
D_1 \leq c \varepsilon (h^{\frac{5}{2} - \frac{2}{p-2}})^2 \|\nabla Q^N u_h\|_{0.0, \Omega} \|\nabla Q^N u_h\|^{p-4} \|\nabla Q^N \Pi^h u\|^{2(p-4)} \|\nabla Q^N \Pi^h u\|^{2(p-4)}
\]

\[
\leq c \varepsilon (h^{\frac{5}{2} - \frac{2}{p-2}})^2 \|\nabla Q^N u_h\|_{0.0, \Omega} \|u_h\|^2 \|\Pi^h e\|^2. \tag{49}
\]

Associated with a linear finite element space \(W^h\), let \(R^h : L^2(\Omega) \rightarrow W^h\) be an \(L^2\)-projection such that
\[ \| R^h \mathbf{v} \| \leq c ( \| \mathbf{v} \| + h \| \nabla \mathbf{v} \| ) \quad \text{and} \quad \| \nabla R^h \mathbf{v} \| \leq c \| \nabla \mathbf{v} \| \]

(50)

and

\[ \| \mathbf{v} - R^h \mathbf{v} \| \leq c h \| \mathbf{v} \|_{1,2,\Omega}, \quad \forall \mathbf{v} \in H^1(\Omega). \]

(51)

The properties of \( P^h \) and \( R^h \) imply

\[ \| \nabla P^h u \| \leq \| \nabla P^h u - \mathbf{v} \| + \| \mathbf{v} - R^h \mathbf{v} \| + \| R^h \mathbf{v} \| \leq c h \| u \|_{2,2,\Omega} + \| R^h \mathbf{v} \|. \]

where \( \| R^h \mathbf{v} \|^2 = \sum_{T \in T_h} \| R^h \mathbf{v} \|_{0,2,1}^2 \). Also, (11) and (50) yield

\[ \| R^h \mathbf{v} \|^2_{0,2,1} \leq ch^2 \| \nabla \mathbf{v} \|^2_{0,\infty,\Omega} \leq ch^2 h^2 \| \nabla \mathbf{v} \|^2_{0,\infty,\Omega} \leq ch^2 \| \nabla \mathbf{v} \|^2_{0,2,1} = ch^2 \| \nabla R^h \mathbf{v} \|^2_{0,2,1} \leq ch^2 \| \nabla^2 \mathbf{u} \|^2_{0,2,1}. \]

Therefore, from the above inequalities and a priori estimate, we obtain

\[ \| \nabla P^h u \| \leq ch \| \Delta u \|. \]

(52)

Thus, we substitute (52) into (49) to obtain

\[ D_1 \leq c h^{\frac{12}{p} - \frac{6}{p} \frac{1}{p}} \| \nabla Q^h u_h \|_{p,\Omega}^{p-4} (\| \nabla \mathbf{e} \|^2 + \| \nabla \mathbf{v} \|)^{\frac{2}{p}} \| \nabla \mathbf{u} \|^{\frac{2}{p}} - \frac{1}{p} \]

\[ \leq c h^{\frac{12}{p} - \frac{6}{p} \frac{1}{p}} (\| \nabla Q^h u_h \|_{p,\Omega}^{p-4} \| \nabla \mathbf{e} \|^2 + \| \nabla \mathbf{v} \|)^{\frac{2}{p}} \| \nabla \mathbf{u} \|^{\frac{2}{p}} - \frac{1}{p} \]

\[ \leq c h^{\frac{12}{p} - \frac{6}{p} \frac{1}{p}} \| \nabla \mathbf{u} \|^{\frac{2}{p}}. \]

(53)

We choose \( \theta = 2p - 5 \) in (26) so that (we choose this value because we have the condition \( h^{\frac{1}{p}} \leq \epsilon \) in the proof of uniqueness, so \( \epsilon \) has a lower bound \( h^{2p-5} \))

\[ \epsilon \geq c h^{\frac{12}{p} - \frac{6}{p} \frac{1}{p}} \]

\[ \epsilon \geq h^{2p-5} \leq c. \]

2) \( (3 \leq p < 4) \) Using (52), we have

\[ D_1 \leq \mathcal{E} \| \nabla Q^h u_h \|_{p,\Omega}^{p-2} \| \nabla P^h u \|_{p,\Omega}^{p-2} \| \nabla \mathbf{u} \|^2 \]

\[ \leq c h^{\frac{12}{p} - \frac{6}{p} \frac{1}{p}} \| \nabla Q^h u_h \|_{p,\Omega}^{p-2} \| \nabla \mathbf{e} \|^2 + \| \nabla \mathbf{v} \|^{\frac{2}{p}} \| \nabla \mathbf{u} \|^{\frac{2}{p}} - \frac{1}{p} \]

\[ \leq c h^{\frac{12}{p} - \frac{6}{p} \frac{1}{p}} \| \nabla \mathbf{u} \|^{\frac{2}{p}}. \]

Again choosing \( \theta = 2p - 5 \) in (26), we then obtain

\[ D_1 \leq ch (\| \mathbf{f} \|^2 + \| \Delta \mathbf{u} \|^2) \| \nabla \mathbf{u} \|^{\frac{2}{p}}. \]

(54)

Gathering (44)–(48), (53), and (54) and applying to (43), we have

\[ \frac{1}{2} \frac{d}{dt} \| \mathbf{e} \|^2 + \| \nabla \mathbf{e} \|^2 \leq ch^2 A(t) + ch^2 B(t) + cC(t) \| \mathbf{e} \|^2. \]

where \( A(t) = \| \nabla \mathbf{u} \|^2 + \| \Delta \mathbf{u} \|^2 + \| \mathbf{u} \|^6 + \| \nabla \mathbf{e} \|^2 \), \( B(t) = (\| \mathbf{f} \|^2 + \| \Delta \mathbf{u} \|^2)(\| \nabla \mathbf{u} \|^2 + \| \nabla \mathbf{u} \|^2 - \frac{4}{p}) \), and \( C(t) = \| \Delta \mathbf{u} \|^2 + \| \nabla \mathbf{u} \|^2 + \| \mathbf{f} \|^2 \) with \( \alpha = 2 \) for \( p \geq 4 \) and \( \alpha = 1 \) for \( 3 \leq p < 4 \). Therefore, (29) can be easily obtained by Gronwall’s inequality. The proof for i) is completed.

ii) Let the generalized Stokes projection \( S^h : J \oplus J^h \to J^h \) be defined by

\[ \langle \nabla (\mathbf{v} - S^h \mathbf{v}), \nabla \mathbf{v} \rangle = 0, \quad \forall \mathbf{v} \in J^h. \]

This projection satisfies (see [28])

\[ \| \nabla S^h \mathbf{v} \| \leq c \| \nabla \mathbf{v} \|, \quad \forall \mathbf{v} \in J \oplus J^h, \]

(55)

\[ \| \mathbf{v} - S^h \mathbf{v} \| + h \| \nabla (\mathbf{v} - S^h \mathbf{v}) \| \leq ch^2 \| \mathbf{v} \|_{2,\Omega}, \quad \forall \mathbf{v} \in J \cap H^2(\Omega). \]

(56)

Now, substitute \( \mathbf{v}_h = S^h \hat{a}_h \mathbf{e} \) into (32). Because \( \langle \hat{a}_h, S^h \hat{a}_h \mathbf{e} \rangle = \langle \hat{a}_h, S^h \hat{a}_h \mathbf{e} - \hat{a}_h \mathbf{e} \rangle + \langle \hat{a}_h \mathbf{e}, \hat{a}_h \mathbf{e} \rangle \) and \( S^h \hat{a}_h \mathbf{e} - \hat{a}_h \mathbf{e} = S^h \hat{a}_h \mathbf{u} - S^h \hat{a}_h \mathbf{u}_h - \hat{a}_h \mathbf{u} + \hat{a}_h \mathbf{u}_h = S^h \hat{a}_h \mathbf{u} - \hat{a}_h \mathbf{u}, \) (32) becomes
\[
\|\partial_t e\|^2 + \nu \frac{1}{2} \frac{d}{dt} \|\nabla e\|^2 = \langle \partial_t e, \partial_t u - S^h \partial_t u + b(u_h, u_h, S^h \partial_t e) \\
- b(u, u, S^h \partial_t e) + \langle \pi, \nabla \cdot S^h \partial_t e \rangle + \Gamma(u, u, S^h \partial_t e) \\
+ \varepsilon \|\nabla Q^N u_h\|^{p-2} \nabla Q^N u_h, \nabla Q^N S^h \partial_t e \rangle.
\] (57)

The following bound for the terms in the first and second lines of (57) can be obtained in a similar manner as in the proof of i) and the details can be found in [28]:

\[
\langle \partial_t e, \partial_t u - S^h \partial_t u + b(u_h, u_h, S^h \partial_t e) - b(u, u, S^h \partial_t e) + \langle \pi, \nabla \cdot S^h \partial_t e \rangle + \Gamma(u, u, S^h \partial_t e) \\
\leq \frac{d}{dt} (c_0(t) \|\nabla e\|^2 + c_1(t) h^2) + c_2(t) \|\nabla e\|^2 + \frac{1}{4} \|\partial_t e\|^2,
\]

where \(c_0(t)\) depends on \(\|\Delta u\|\), \(c_1(t)\) depends on \(\|\Delta \partial_t u\|\) and \(\|\nabla \pi\|\), and \(c_2(t)\) depends on \(\|\Delta \partial_t u\|\) and \(\|\nabla \partial_t \pi\|\). Therefore, we focus on the term

\[
\varepsilon \|\nabla Q^N u_h\|^{p-2} \nabla Q^N u_h, \nabla Q^N S^h \partial_t e \rangle.
\] (58)

Because \(f \in L^p(0, T; L^2(\Omega))\), we return to the uniqueness proof in Section 4 to weaken the constraint on \(\varepsilon\) regarding \(h\) in (18). Recalling the inequalities in (18) and modifying them we have

\[
A \leq \sum_{T \in T_N} \|Q^N u_2\|_{0, \infty, T} \|w_h\|_{0, 2, T} \|\nabla w_h\|_{0, 2, T}
\leq \sum_{T \in T_N} c h^{1-\frac{\theta}{p} - \frac{1}{2}} \|Q^N u_2\|_{0, \delta, T} \|w_h\|_{0, 2, T} \|\nabla w_h\|_{0, 2, T}
\leq \sum_{T \in T_N} \left( c v^{1-\frac{\theta}{p} h^{k-\frac{1}{2}} - \frac{1}{2} - \frac{1}{2}} \|Q^N u_2\|_{0, \delta, T} \|w_h\|_{0, 2, T} + \frac{v}{4} \|\nabla w_h\|_{0, 2, T}^2 \right)
\leq \left( c v^{1-\frac{\theta}{p} h^{k-\frac{1}{2}} - \frac{1}{2} - \frac{1}{2}} \|Q^N u_2\|_{0, \delta, T} \|w_h\|_{0, 2, T} + \frac{v}{4} \|\nabla w_h\|_{0, 2, T}^2 \right)
\leq c v^{1-\frac{\theta}{p} h^{k-\frac{1}{2}} - \frac{1}{2} - \frac{1}{2}} \|Q^N u_2\|_{0, \delta, T} \|w_h\|_{0, 2, T} + \frac{v}{4} \|\nabla w_h\|_{0, 2, T}^2.
\] (59)

Therefore, the uniqueness holds so long as \(h^{k-\frac{1}{2} - \frac{1}{2}} - \frac{1}{2} \leq c\). Now, we choose \(\theta = 2p - 3 - \frac{4}{k}\) in (26) and turn our attention to (58) again. By the Hölder inequality and (28) we have

\[
E \equiv \varepsilon \|\nabla Q^N u_h\|^{p-2} \nabla Q^N u_h, \nabla Q^N S^h \partial_t e \rangle 
\leq c h^{\frac{\theta}{p} - 3} \|Q^N u_h\|_{0, 2, T} \|\nabla S^h \partial_t e\|.
\]

The triangle inequality, (34), (56), and the inverse inequality yield

\[
\|\nabla u_h\| \leq \|\nabla e\| + \|\nabla u\|,
\|\nabla S^h \partial_t e\| \leq \|\nabla S^h \partial_t e - \nabla T^h \partial_t e\| + \|\nabla T^h \partial_t e\|
= \|\nabla S^h \partial_t u - \nabla T^h \partial_t u\| + \|\nabla T^h \partial_t e\|
\leq \|\nabla (S^h \partial_t u - \partial_t u)\| + \|\nabla (T^h \partial_t u - \partial_t u)\| + \|\nabla T^h \partial_t e\|
\leq c h \|\Delta \partial_t u\| + c h^{-1} \|\nabla \partial_t e\| \leq c \|\Delta \partial_t u\| + c h^{-1} \|\partial_t e\|.
\]

Thus, we have

\[
E \leq c h^{\frac{\theta}{p} - 3} \|Q^N u_h\|_{0, 2, T} \|\nabla e\| + \|\nabla u\| \left( h \|\Delta \partial_t u\| + h^{-1} \|\partial_t e\| \right)
\leq c^2 \|\nabla e\|^2 + h^2 \|\Delta \partial_t u\|^2 + \alpha^2 h^{-2} \|\nabla u\|^2 + \frac{1}{16} \|\partial_t e\|^2
+ \alpha^2 \|\nabla u\|^2 + h^2 \|\Delta \partial_t u\|^2 + \alpha^2 h^{-2} \|\nabla u\|^2 + \frac{1}{16} \|\partial_t e\|^2.
\] (60)

where \(\alpha = c h^{\frac{\theta}{p} - 3} \|Q^N u_h\|_{0, 2, T} \|\nabla e\| + \|\nabla u\| \left( h \|\Delta \partial_t u\| + h^{-1} \|\partial_t e\| \right)\). Because the coefficients need to be \(\alpha^2 \sim O(1), \alpha h^{-1} \sim O(1), \alpha \sim O(h^\gamma)\) and \(\alpha h^{-1} \sim O(h^\gamma)\) for some \(\gamma > 0\), we only need to show the bound for the \(\alpha^2 h^{-2} \|\nabla u\|^2\)-term: by the inverse inequality and (9),
\[ a^2 h^{-2} \| \nabla u \|^2 = c e^{2 h^{-2} - \beta} \| \nabla^N u_h \|_{p}^{2p-4} \| \nabla u \|^2 \]
\[ \leq c e^{\beta} \| \nabla^N u_h \|_{p}^{2p-4} \| \nabla u \|^2 \]
\[ \leq c e^{4 \beta} h^{2} \| \nabla^N u_h \|_{p}^{2p-4} \beta \| \nabla u \|^2 \]
\[ \leq c h^{-3} \| \nabla u \|^2 \]
\[ \leq \epsilon \]
\[
(61)
\]
where \( \beta = (kp - 4p + 8)/p - 2 \) and \( p \geq 2 + k/2 \). Applying (61) to (60) and gathering all the bounds and applying them to (57), we have
\[ |\partial_t \epsilon|^2 \leq \frac{d}{dt} (c_0(t) \| \nabla \epsilon \|^2 + c_1(t) h^2) + c_2(t) \| \nabla \epsilon \|^2 + c_3(t) h^2 + c_4(t) h^{(4-4p)/(2p-4)}. \]
Gronwall's inequality yields (30). □

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