Combined Effects of Singular and Superlinear Nonlinearities in Some Singular Boundary Value Problems\textsuperscript{1}

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This paper deals with a class of singular semilinear elliptic Dirichlet boundary value problems where the combined effects of a superlinear and a singular term allow us to establish some existence and multiplicity results.

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1. INTRODUCTION

In this paper we consider the following singular problem involving superlinear non-linearity (the super-linear problem)

\[
\begin{align*}
\Delta u + \lambda u^\beta + p(x) u^{-\gamma} &= 0, & \text{in } \Omega \\
u &= 0, & \text{in } \Omega \\
u &= 0, & \text{on } \partial \Omega,
\end{align*}
\]

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where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $p: \Omega \to \mathbb{R}$ is a given non-negative non-trivial function in $L^2(\Omega)$, $1 < \beta < 2^*-1$, $0 < \gamma < 1$ are two constants, $2^* = \frac{N \cdot 2}{N-2}$ is the limiting exponent in the Sobolev embedding, $N \geq 3$, and $\lambda > 0$ is a real parameter. To emphasize the dependence on $\lambda$, this problem is often referred to as problem $(1)_{\lambda}$ (the subscript $\lambda$ is omitted if no confusion arises). By a weak solution of $(1)_{\lambda}$ in $H^1_0(\Omega)$ we mean a function $u \in H^1_0(\Omega)$ such that $u(x) > 0, x \in \Omega$ and

$$
\int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \lambda \int_{\Omega} u^{\beta} \varphi \, dx + \int_{\Omega} p(x) u^{-\gamma} \varphi \, dx, \quad \forall \varphi \in H^1_0(\Omega).
$$

The existence of solutions to the elliptic equation

$$
\Delta u + p(x) u^{-\gamma} = 0, \quad \text{in } \Omega
$$
$$
u = 0, \quad \text{on } \partial \Omega
$$

on a smooth domain $\Omega \subseteq \mathbb{R}^N$ has been extensively studied (cf. [2, 4–6, 12–15] and their references). For bounded $\Omega$, in [4, Theorem 4] it is shown that problem (2) with $0 < \gamma < 1$ has a unique weak positive solution in $H^1_0(\Omega)$ if $p(x)$ is a nonnegative nontrivial function in $L^2(\Omega)$.

For the general problem $(1)_{\lambda}$, we have learned from M. M. Coclite and G. Palamieri [10] that there exists $\lambda^* \in (0, \infty)$ such that problem $(1)_{\lambda}$ has a solution if $\lambda < \lambda^*$ and has no solution if $\lambda > \lambda^*$, provided $p \equiv 1$ on $\Omega$. We are then interested in the question of whether this solution is unique or not. It is worth mentioning that, in [9, 11] the existence of a unique positive solution in the cases when $\beta = 1$ and $0 < \beta < 1$ (the sub-linear problem) has been proved.

Our goal in this paper is to show how variational methods can be used to establish some existence and multiplicity results for singular problems like $(1)_{\lambda}$. We work on the Sobolev space $H^1_0(\Omega)$ equipped with the norm $\|u\| = \int_{\Omega} |\nabla u|^2 \, dx$. For $u \in H^1_0(\Omega)$ we define $I_{\lambda}: H^1_0(\Omega) \to \mathbb{R}$ by

$$
I_{\lambda}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{\lambda}{\beta+1} \int_{\Omega} |u|^{\beta+1} \, dx - \frac{1}{1-\gamma} \int_{\Omega} p(x) |u|^{1-\gamma} \, dx.
$$

By investigating suitable minimization problems for the functional $I_{\lambda}$, we find the combined effects of singular and super-linear non-linearities change considerably the structure of the solution set. To be slightly more precise, we show that for suitable $p$'s, problem $(1)_{\lambda}$ possesses at least two weak positive solutions provided $\lambda > 0$ is small. Moreover, we provide some information about their location. It should be pointed out that since
$I_l$ fails to be Frechet differentiable in $\Omega$, Critical point theory could not be applied to obtain the existence of solutions. We mainly rely on the Ekeland’s variational principle (cf. [1]) and careful estimates inspired by Lair–Shaker [4] and Tarantello [8].

Concerning notations: $L^p(\Omega)$ denote Lebesgue spaces, the norm in $L^p$ is denoted by $\| \cdot \|_p$; $c, c_1, c_2, \ldots$ denote (possibly different) positive constants. We need the first eigenfunction $\varphi_1$ with $\Delta \varphi_1 + \lambda_1 \varphi_1 = 0$ in $\Omega$, $\varphi_1|_{\partial \Omega} = 0$, $0 < \varphi_1 \leq 1$ in $\Omega$.

Remark 1. When $N = 1$, the type of equations with a superlinear and a singular term has been studied by Agarwal–O’Regan [7] who proved the equation

$$y''(t) + \delta (y^{-\alpha}(t) + y^\beta(t) + 1) = 0, \quad 0 < t < 1$$

$$y(0) = y(1) = 0, \quad \delta > 0 \text{ a parameter}$$

with $0 \leq \alpha < 1$ and $\beta > 1$, has a nonnegative solution for all $\delta > 0$ small enough.

2. MULTIPLE POSITIVE SOLUTIONS FOR PROBLEM (1)$_l$

In this section we prove the existence of two weak positive solutions of (1)$_l$. Let us define

$$\Lambda = \left\{ u \in H^1_0(\Omega) : \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} p(x) |u|^{1-\gamma} \, dx - \lambda \int_{\Omega} |u|^{\beta+1} \, dx = 0 \right\}.$$

To start, notice that $u \in \Lambda$ if $u$ is a weak solution of (1)$_l$. This fact suggests to look at the following splitting for $\Lambda$:

$$\Lambda^+ = \left\{ u \in \Lambda : (1+\gamma) \int_{\Omega} |\nabla u|^2 \, dx - \lambda (\beta + \gamma) \int_{\Omega} |u|^{\beta+1} \, dx > 0 \right\}$$

$$\Lambda_0 = \left\{ u \in \Lambda : (1+\gamma) \int_{\Omega} |\nabla u|^2 \, dx - \lambda (\beta + \gamma) \int_{\Omega} |u|^{\beta+1} \, dx = 0 \right\}$$

$$\Lambda^- = \left\{ u \in \Lambda : (1+\gamma) \int_{\Omega} |\nabla u|^2 \, dx - \lambda (\beta + \gamma) \int_{\Omega} |u|^{\beta+1} \, dx < 0 \right\}$$
But first we state and prove five lemmas:

**Lemma 1.** Let

\[ \tilde{\lambda} = \left( \frac{1}{\|p\|_{2^{-1}}} \right)^{\frac{1}{2}} \left( \frac{1 + \gamma}{\beta + \gamma} \right) \left[ \left( \frac{\beta - 1}{\beta + \gamma} \right)^{\beta - 1} \right]^{\frac{1}{2}} \frac{1}{|\Omega|^{\frac{\alpha_p}{1 + \gamma}}} + \frac{1}{|\Omega|^{\frac{\alpha_p}{1 + \gamma}}} \right] \frac{\beta + \gamma}{|\Omega|^{\frac{\alpha_p}{1 + \gamma}}}, \]

where \( S \) is the best Sobolev constant, namely

\[ S = \inf \left\{ \frac{\|u\|^2}{\|u\|_{2^*}^2} : u \in H^1_0(\Omega), u \neq 0 \right\} \]

(it is well known that the infimum is never achieved and \( S > 0 \); cf. [8]) and \( \alpha = 2\left[ \frac{2 - (\beta + 1)}{\beta(\beta + 1)} \right] \). Then for all \( \lambda \in (0, \tilde{\lambda}) \) we have the following conclusions:

1. For every \( u \in A, u \neq 0, (1 + \gamma)\|u\|^2 - \lambda(\beta + \gamma)\|u\|_{\beta+1}^\gamma \neq 0 \) (i.e. \( \wedge_0 = 0 \));
2. \( A^\perp \) is closed in \( H^1_0(\Omega) \).

**Proof.** (1) Suppose, by contradiction that there exists some \( u_0 \in A, u_0 \neq 0 \) such that

\[ (1 + \gamma)\|u_0\|^2 - \lambda(\beta + \gamma)\|u_0\|_{\beta+1}^\gamma = 0. \]  (3)

Thus

\[ 0 = \|u_0\|^2 - \lambda\|u_0\|_{\beta+1}^\gamma - \int_\Omega p(x)|u_0|^{1-\gamma} \, dx \]

\[ = \left( \frac{\beta - 1}{\beta + \gamma} \right)\|u_0\|^2 - \int_\Omega p(x)|u_0|^{1-\gamma} \, dx. \]  (4)

From simple arguments we have

\[ \|u\|^2 > \frac{S}{|\Omega|^{\alpha}} \|u\|_{\beta+1}^\gamma, \quad \forall u \in H^1_0(\Omega), u \neq 0 \]  (5)

where \( \alpha = 2\left[ \frac{2 - (\beta + 1)}{\beta(\beta + 1)} \right] \), and \( S \) is the best Sobolev constant. Using this fact, we infer that
\[ c_2 := \left[ \frac{1 + \gamma}{\lambda(\beta + \gamma)} \right]^{(1+\gamma)/(\beta-1)} \left( \frac{\beta - 1}{\beta + \gamma} \right) \left[ \frac{\|u_0\|^2}{\|u_0\|_{\beta+1}^2} \right]^{1/(\beta-1)} \]

\[ - \int_B p(x) |u_0|^{1-\gamma} \, dx \]

\[ \forall \left[ \frac{1 + \gamma}{\lambda(\beta + \gamma)} \right]^{(1+\gamma)/(\beta-1)} \left( \frac{\beta - 1}{\beta + \gamma} \right) \left( \frac{S}{|\Omega|^s} \right)^{(\beta+\gamma)/(\beta-1)} \|u_0\|_{\beta+1}^{1-\gamma} \]

\[ - \int_B p(x) |u_0|^{1-\gamma} \, dx \]

\[ \forall \left[ \frac{1 + \gamma}{\lambda(\beta + \gamma)} \right]^{(1+\gamma)/(\beta-1)} \left( \frac{\beta - 1}{\beta + \gamma} \right) \left( \frac{S}{|\Omega|^s} \right)^{(\beta+\gamma)/(\beta-1)} \|u_0\|_{\beta+1}^{1-\gamma} \]

\[ - \|p\|_{L^1(|\Omega|^{(\beta+1)/2})} \|u_0\|_{\beta+1}^{1-\gamma}. \]

where in the last step we have used

\[ \int_B p(x) |u_0|^{1-\gamma} \, dx \leq \|p\|_2 \cdot |\Omega|^{\beta-1+2c \gamma} \cdot \|u_0\|_{\beta+1}^{-\gamma}. \]

Since \( \lambda < \bar{\lambda} \) it follows that \( c_2 > 0 \), which yields a contraction because from (3) and (4) we clearly have

\[ c_2 = \|u_0\|^2 \left( \frac{\beta - 1}{\beta + \gamma} \right) \left[ \left( \frac{1 + \gamma}{\lambda(\beta + \gamma)} \right) \left( \frac{\beta - 1}{\beta + \gamma} \right) \left[ \frac{\|u_0\|^2}{\|u_0\|_{\beta+1}^2} \right]^{1/(\beta-1)} - 1 \right] \]

\[ = 0. \]

Thus \( \Lambda_0 = \{0\} \). This completes the proof of (1);

(2) Let \( \{u_n\} \subset \wedge^- \) be a sequence such that \( u_n \to u_0 \) in \( H_0^1 \). Then \( u_n \to u_0 \) in \( L^{\beta+1}(\Omega) \) and \( u_0 \in \wedge^- \cup \Lambda_0 \). Now we prove \( u_0 \in \wedge^- \). Arguing by contradiction, we assume that \( u_0 \in \Lambda_0 \). Since \( \Lambda_0 = \{0\} \) (from the discussion in (1) we know \( \Lambda_0 = \{0\} \)) it follows that \( u_0 = 0 \), which is clearly impossible because

\[ \|u_0\|_{\beta+1} \geq \left[ \frac{1 + \gamma}{\lambda(\beta + \gamma)} \right] \left( \frac{S}{|\Omega|^s} \right)^{1/(\beta-1)} > 0 \]  

(6)

Indeed, using (5) we can easily obtain that

\[ \|u\|_{\beta+1} \geq \left[ \frac{1 + \gamma}{\lambda(\beta + \gamma)} \right] \left( \frac{S}{|\Omega|^s} \right)^{1/(\beta-1)} \), \forall u \in \wedge^-. \]  

(7)
Applying (7) with \( u = u_0 \) and passing to the limit as \( n \to \infty \), we then get (6). Hence \( u_0 \in \Lambda^- \). This completes the proof of Lemma 1.

**Lemma 2.** Suppose \( \lambda \in (0, 1] \). If \( p(x) \) is a nontrivial, nonnegative \( L^2(\Omega) \) function such that

\[
\|p\|_2 \leq a \left( \frac{S}{|\Omega|^\gamma} \right)^{(\beta + r)/(\beta - 1)},
\]

where

\[
a = \left( \frac{1}{\frac{\beta + 1}{\beta + \gamma}} \right) \left( \frac{\beta - 1}{\beta + \gamma} \right)^{(1 + \gamma)/(\beta - 1)} \quad \text{and} \quad \alpha = \frac{2^* - (\beta + 1)}{2(\beta + 1)} ,
\]

then for every \( u \in H^1_0 \), \( u \neq 0 \) there exists a unique \( t^+ = t^+(u) > 0 \) such that \( t^+ u \in \Lambda^- \).

The proofs of Lemma 2 and 3 are adaptations of that given by G. Tarantello [8] for a non-singular minimization problem. For the reader’s convenience, we still provide the details here.

**Proof.** Set \( \varphi(t) = t^{1+\gamma} \|u\|^2 - \lambda t^{\beta + r} \|u\|_p^{\beta + 1} \). Easy computations show that \( \varphi \) achieves its maximum at

\[
t_{\text{max}} = \left[ \frac{1 + \gamma}{\lambda(\beta + r)} \right]^{1/(\beta - 1)} \left[ \frac{\|u\|^2}{\|u\|_p^{\beta + 1}} \right]^{1/(\beta - 1)} ,
\]

and

\[
\varphi(t_{\text{max}}) = \left[ \frac{1 + \gamma}{\lambda(\beta + r)} \right]^{(1 + \gamma)/(\beta - 1)} \left( \frac{\beta - 1}{\beta + r} \right)^{(1 + \gamma)/(\beta - 1)} \left[ \frac{\|u_0\|_p^{2(\beta + 1)}}{\|u_0\|_p^{(\beta + 1)/(\beta - 1)}} \right]^{1/(\beta - 1)} .
\]

Using assumption (\( \ast \)) and (5) and noting that \( 0 < \lambda \leq 1 \), we deduce that

\[
\int_\Omega p(x) |u|^{1-\gamma} \, dx < \varphi(t_{\text{max}}) .
\]

Consequently, there exists a unique \( t^+ > t_{\text{max}} \) such that

\[
\varphi(t^+) = \int_\Omega p(x) |u|^{1-\gamma} \, dx \quad \text{and} \quad \varphi'(t^+) < 0 .
\]

Equivalently \( t^+ u \in \Lambda^- \). This completes the proof of Lemma 2.

**Remark 2.** From Lemma 2 it follows that the set \( \Lambda^- \) is nonempty. In fact, it turns out that assumption (\( \ast \)) on \( p \) is only needed to guarantee
\[ \land^{-} \neq \emptyset \] and there are no other uses. Thus, assumption \((\ast)\) may be deleted if there are other methods to prove \(\land^{-} \neq \emptyset\).

**Lemma 3.** Given \(u \in \land^{-}\), then there exist \(\varepsilon > 0\) and a continuous function \(f = f(w) > 0\), \(w \in H^1_0\), \(\|w\| < \varepsilon\) satisfying that

\[
f(0) = 1, \quad f(w)(u+w) \in \land^{-}, \quad \forall w \in H^1_0(\Omega), \|w\| < \varepsilon.
\]

**Proof.** Define \(F: \mathbb{R} \times H^1_0(\Omega) \to \mathbb{R}\) as follows:

\[
F(t, w) = t^{1+\gamma} \int_{\Omega} |\nabla (u+w)|^2 \, dx - \lambda t^{\beta+\gamma} \int_{\Omega} |u+w|^{\beta+1} \, dx
\]

\[
- \int_{\Omega} p(x) |u+w|^{1-\gamma} \, dx.
\]

Since \(u \in \land^{-}(\subset \land)\), it follows that \(F(1, 0) = 0\) and

\[
F_t(1, 0) = (1+\gamma) \int_{\Omega} |\nabla u|^2 \, dx - \lambda (\beta+\gamma) \int_{\Omega} |u|^{\beta+1} \, dx < 0,
\]

then we can apply the implicit function theorem at the point \((1, 0)\) and obtain \(\varepsilon > 0\) and a continuous function \(f = f(w) > 0\), \(w \in H^1_0\), \(\|w\| < \varepsilon\) satisfying that

\[
f(0) = 1, \quad f(w)(u+w) \in \land^{-}, \quad \forall w \in H^1_0(\Omega), \|w\| < \varepsilon
\]

and hence, taking \(\varepsilon > 0\) possibly smaller \((\varepsilon < \bar{\varepsilon})\) we have

\[
f(w)(u+w) \in \land^{-}, \quad \forall w \in H^1_0(\Omega), \|w\| < \varepsilon
\]

This completes the proof of Lemma 3.

**Lemma 4.** Let

\[
\tilde{\lambda} = \left( \frac{1+\gamma}{\beta+\gamma} \right)^{(\beta+1)(1+\gamma)} \frac{(\beta-1)(1-\gamma)}{(\beta+1)(1+\gamma)} \left( \frac{1}{\|p\|_2} \right)^{(\beta-1)(1+\gamma)}
\]

Then for all \(\lambda \in (0, \tilde{\lambda}]\) the whole set \(\land^{-}\) lies at the nonnegative level, that is \(I_\lambda(u) \geq 0\), \(\forall u \in \land^{-}\).
Proof. We argue by contradiction. Suppose that there exists $u_0 \in \mathbb{R}^N$ such that

$$\frac{1}{2} \|u_0\|^2 - \frac{\lambda}{\beta + 1} \|u_0\|_{\beta+1}^{\beta+1} - \frac{1}{1-\gamma} \int_{\Omega} p(x) |u_0|^{1-\gamma} \, dx < 0.$$ 

Thus:

$$\lambda \left( \frac{1}{2} - \frac{1}{\beta + 1} \right) \|u_0\|_{\beta+1}^{\beta+1} - \left( \frac{1}{1-\gamma} - \frac{1}{2} \right) \int_{\Omega} p(x) |u_0|^{1-\gamma} \, dx < 0.$$ 

By Hölder’s inequality,

$$\int_{\Omega} p(x) |u_0|^{1-\gamma} \, dx \leq \|p\|_{2} \cdot |\Omega|^{(\frac{\beta-1+2\gamma}{\alpha(\beta+1)})} \cdot \|u_0\|_2^{\beta \gamma},$$

and hence

$$\|u_0\|_{\beta+1}^{\beta+1} < \frac{1}{\lambda} \cdot \frac{(\beta + 1)(1 + \gamma)}{(\beta - 1)(1 - \gamma)} \cdot |\Omega|^{(\frac{\beta-1+2\gamma}{\alpha(\beta+1)})} \cdot \|p\|_{2}.$$ 

This, together with (7) yields $\lambda > \tilde{\lambda}$, a contradiction. This completes the proof of Lemma 4.

Lemma 5. [Lazer–Mckenna [5]].

$$\int_{\Omega} \varphi'_i(x) \, dx < \infty$$

if and only if $r > -1$.

Theorem 1. Let $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$. $N \geq 3$. Let $0 < \gamma < 1 < \beta < 2^* - 1$. If $p$ is a nontrival, nonnegative $L^2(\Omega)$ function satisfying $(*)$, then there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ problem $(1)_1$ possesses at least two weak positive solutions $u_1(x), u_2(x) \in H^1_0(\Omega)$ in the sense that

$$\int_{\Omega} \nabla u_i \cdot \nabla \phi = \lambda u_i^\beta \cdot \phi - p(x) u_i^{\gamma-1} \cdot \phi \, dx = 0, \quad \forall \phi \in H^1_0(\Omega), i = 1, 2.$$ 

Moreover $u_1$ is a local minimizer of $I_\lambda$ in $H^1_0(\Omega)$ with $I_\lambda(u_1) < 0$; and $u_2$ is a minimizer of $I_\lambda$ on $\mathbb{R}^N$ with $I_\lambda(u_2) \geq 0$. 
Proof of Theorem 1. Using the Sobolev and Hölder inequalities we have:

\[ I_\lambda \geq \frac{1}{2} \|u\|^2 - \lambda c_1 \|u\|^p + c_2 \|u\|^{1-\gamma}, \quad \forall u \in H^1_0(\Omega). \]

From this we readily find that there exists \( \lambda^* > 0 \) such that for all \( \lambda \in (0, \lambda^*] \) there are \( r, a > 0 \) such that

(i) \( I_\lambda(u) \geq a \) for all \( \|u\| = r \);

(ii) \( I_\lambda \) is bounded on \( B_r = \{ u \in H^1_0 : \|u\| \leq r \} \).

Letting \( \lambda_0 = \min\{\lambda^*, 1, \lambda\} \) where \( \lambda, \lambda^* \) are the values found in Lemma 1 and 4, henceforth we fix \( \lambda \in (0, \lambda_0) \) and drop the subscript \( \lambda \).

(Existence of \( u_1 \)) In view of Theorem 1.2 of [3] the infimum of \( I \) on \( B \) can be achieved at a point \( u_1 \in B \). Note that, since \( 1 - \gamma < 1 \), it follows that for every \( v > 0 \), \( I(tv) < 0 \) as \( t > 0 \) small and there exists \( v_1 \in B \) such that \( I(v_1) < 0 \), therefore \( I(u_1) = \inf_{B_1} I \leq I(v_1) < 0 \). This, together with (i) imply that \( u_1 \neq \partial B_1 \). Hence \( u_1 \) is a local minimizer of \( I \) in the \( H^1_0 \) topology.

Clearly, \( u_1 \neq 0 \). Moreover, since \( I(|u|) = I(u) \), we may assume that \( u_1 \neq 0 \) in \( \Omega \). Then, for any \( \varphi \in H^1_0, \varphi \geq 0 \),

\[
0 \leq I(u_1 + t\varphi) - I(u_1) = \frac{1}{2} \int_\Omega |\nabla(u_1 + t\varphi)|^2 dx - \frac{\lambda}{\beta + 1} \int_\Omega |u_1 + t\varphi|^{\beta + 1} dx
+ \int_\Omega p(x) |u_1 + t\varphi|^{1-\gamma} dx - \frac{1}{2} \int_\Omega |\nabla u_1|^2 dx
\]

provided \( t > 0 \) small enough. Dividing by \( t > 0 \) and passing to the limit as \( t \to 0 \), we derive

\[
\int_\Omega \nabla u_1 \cdot \nabla \varphi dx \geq 0, \quad \varphi \in H^1_0, \varphi \geq 0
\]

which means \( u_1 \in H^1_0 \) satisfies in a weak sense that

\[-Au_1 \geq 0 \quad \text{in} \quad \Omega,\]
since \( u_1 \geq 0, \ u_1 \neq 0 \), then the strong maximum principle yields
\[ u_1 > 0, \quad \text{in} \quad \Omega. \quad (9) \]

Moreover, from (8) we also have
\[
\frac{1}{1 - \gamma} \int_{\Omega} p(x)[(u_1 + t\varphi)^{1-\gamma} - u_1^{1-\gamma}] \, dx \\
\leq \frac{1}{2} \int_{\Omega} |\nabla (u_1 + t\varphi)|^2 \, dx - \frac{1}{2} \int_{\Omega} |\nabla u_1|^2 \, dx \\
- \frac{\lambda}{\beta + 1} \int_{\Omega} (u_1 + t\varphi)^{\beta+1} \, dx - \int_{\Omega} u_1^{\beta+1} \, dx,
\]
and therefore, dividing by \( t > 0 \) and passing to the limit, it follows that
\[
\frac{1}{1 - \gamma} \liminf_{t \to 0} \int_{\Omega} \frac{p(x)[(u_1 + t\varphi)^{1-\gamma} - u_1^{1-\gamma}]}{t} \, dx \\
\leq \int_{\Omega} \nabla u_1 \cdot \nabla \varphi \, dx - \lambda \int_{\Omega} u_1^\gamma \cdot \varphi \, dx.
\]

Observing
\[
\frac{1}{1 - \gamma} \int_{\Omega} \frac{p(x)[(u_1 + t\varphi)^{1-\gamma} - u_1^{1-\gamma}]}{t} \, dx = \int_{\Omega} p(x)(u_1 + \theta t\varphi)^{-\gamma} \varphi \, dx,
\]
where \( \theta \to 0^+ \) as \( t \to 0^+ \) and \( p(x)(u_1 + \theta t\varphi)^{-\gamma} \varphi \to p(x) u_1^{-\gamma} \varphi \) a.e. in \( \Omega \) as \( t \to 0^+ \). Since \( 0 \leq p(x)(u_1 + \theta t\varphi)^{-\gamma} \varphi, \forall x \in \Omega \), by Fatou’s Lemma of course, \( p(x) u_1^{-\gamma} \varphi \) is integrable and
\[
\int_{\Omega} p(x) u_1^{-\gamma} \varphi \, dx \leq \frac{1}{1 - \gamma} \liminf_{t \to 0^+} \int_{\Omega} \frac{p(x)[(u_1 + t\varphi)^{1-\gamma} - u_1^{1-\gamma}]}{t} \, dx.
\]

Putting together these relations we find that
\[
\int_{\Omega} \nabla u_1 \cdot \nabla \varphi - \lambda u_1^\gamma \varphi - p(x) u_1^{-\gamma} \varphi \, dx \geq 0, \ \varphi \in H_0^1, \ \varphi \geq 0. \quad (10)
\]

In particular, for \( u_i \) there is \( \eta_i \in (0, 1) \) such that \( u_i + tu_i \in B_i \) if \( |t| \leq \eta_i \). Then we define \( h_i: [-\eta_i, \eta_i] \to R \) by \( h_i(t) = I((1+t) u_i) \). Clearly, \( h_i(t) \) achieves its minimum at \( t = 0 \). Therefore,
\[
\frac{d}{dt} \bigg|_{t=0} h_i = \int_{\Omega} |\nabla u_i|^2 - \lambda u_i^\beta + 1 - p(x) u_i^{1-\gamma} \, dx = 0, \quad (11)
\]
which implies \( u_1 \in \mathcal{A} \). There remains only to show that (9) (10) and (11) imply that \( u_1 \) is a weak positive solution of (1). The proof is inspired by Lair–Shaker [4]. To this end, suppose \( \phi \in H^1_0(\Omega) \) and \( \varepsilon > 0 \), and define \( \Psi \in H^1_0 \), \( \Psi \geq 0 \) by

\[
\Psi \equiv (u_1 + \varepsilon \phi)^+.
\]

Inserting \( \Psi \) into (10) and (11), we infer that

\[
0 \leq \int_{\Omega} \nabla u_1 \cdot \nabla \Psi - \lambda u_1^\eta \cdot \Psi - p(x) u_1^{-\eta} \cdot \Psi \, dx
\]

\[
= \int_{[u_1 + \varepsilon \phi > 0]} \nabla u_1 \cdot \nabla (u_1 + \varepsilon \phi) - \lambda u_1^\eta \cdot (u_1 + \varepsilon \phi) - p(x) u_1^{-\eta} \cdot (u_1 + \varepsilon \phi) \, dx
\]

\[
= \left( \int_{\Omega} - \int_{[u_1 + \varepsilon \phi < 0]} \right) \nabla u_1 \cdot \nabla (u_1 + \varepsilon \phi)
\]

\[
- \lambda u_1^\eta \cdot (u_1 + \varepsilon \phi) - p(x) u_1^{-\eta} \cdot (u_1 + \varepsilon \phi) \, dx
\]

\[
= \|u_1\|^2 - \lambda \|u_1\|_{\eta+1}^{\eta+1} - \int_{\Omega} p(x) u_1^{-\eta} \, dx
\]

\[
+ \varepsilon \int_{\Omega} \nabla u_1 \cdot \nabla \phi - \lambda u_1^\eta \cdot \phi - p(x) u_1^{-\eta} \cdot \phi \, dx
\]

\[
- \int_{[u_1 + \varepsilon \phi < 0]} \nabla u_1 \cdot \nabla (u_1 + \varepsilon \phi) - \lambda u_1^\eta \cdot (u_1 + \varepsilon \phi) - p(x) u_1^{-\eta} \cdot (u_1 + \varepsilon \phi) \, dx
\]

\[
= \varepsilon \int_{\Omega} \nabla u_1 \cdot \nabla \phi - \lambda u_1^\eta \cdot \phi - p(x) u_1^{-\eta} \cdot \phi \, dx
\]

\[
- \int_{[u_1 + \varepsilon \phi < 0]} \nabla u_1 \cdot \nabla (u_1 + \varepsilon \phi) - \lambda u_1^\eta \cdot (u_1 + \varepsilon \phi) - p(x) u_1^{-\eta} \cdot (u_1 + \varepsilon \phi) \, dx
\]

\[
\leq \varepsilon \int_{\Omega} \nabla u_1 \cdot \nabla \phi - \lambda u_1^\eta \cdot \phi - p(x) u_1^{-\eta} \cdot \phi \, dx - \varepsilon \int_{[u_1 + \varepsilon \phi < 0]} \nabla u_1 \cdot \nabla \phi \, dx.
\]

Since the measure of the domain of integration \([u_1 + \varepsilon \phi < 0]\) tend to zero as \( \varepsilon \to 0 \), it follows that \( \int_{[u_1 + \varepsilon \phi < 0]} \nabla u_1 \cdot \nabla \phi \, dx \to 0 \) as \( \varepsilon \to 0 \). Dividing by \( \varepsilon \) and letting \( \varepsilon \to 0 \) therefore shows

\[
\int_{\Omega} \nabla u_1 \cdot \nabla \phi - \lambda u_1^\eta \cdot \phi - p(x) u_1^{-\eta} \cdot \phi \, dx \geq 0.
\]
Noting that \( \phi \) is arbitrary, this holds equally for \( -\phi \), it follows that \( u_1 \) is indeed a weak solution of (1). This completes the proof of the existence of \( u_1 \).

In the preceding part we have established the existence of a positive solution of (1), say \( u_1 \), which lies at the negative level (i.e. \( I(u_1) < 0 \)). Next, we prove the existence of a second positive solution of (1). In view of Lemma 4. It suffices to show that (1) possesses a weak positive solution in \( \wp^- \).

(Existence of \( u_2 \)). We start by showing that \( I \) is coercive on \( \wp^- \). Indeed, for \( u \in \wp^- \) we have:

\[
\|u\|^2 - \lambda \|u\|_{\beta+1} - \int_{\Omega} p(x) |u|^{1+\gamma} \, dx = 0.
\]

Thus:

\[
I(u) = \frac{1}{2} \|u\|^2 - \lambda \|u\|_{\beta+1} + \frac{1}{1-\gamma} \int_{\Omega} p(x) |u|^{1+\gamma} \, dx
\]

\[
= \left( \frac{1}{2} - \frac{1}{\beta+1} \right) \|u\|^2 - \left( \frac{1}{1-\gamma} - \frac{1}{\beta+1} \right) \int_{\Omega} p(x) |u|^{1+\gamma} \, dx
\]

\[
\geq \left( \frac{1}{2} - \frac{1}{\beta+1} \right) \|u\|^2 - c \left( \frac{1}{1-\gamma} - \frac{1}{\beta+1} \right) \|u|^{1+\gamma}.
\]

Noting that \( \wp^- \) is a closed set in \( H_0^1(\Omega) \), we apply Ekeland’s variational Principle to the minimization problem \( \inf_{\wp^-} I \). It gives a minimizing sequence \( \{w_n\} \subset \wp^- \) with the following properties:

(i) \( I(w_n) < \inf_{\wp^-} I + \frac{1}{n} \);

(ii) \( I(w) \geq I(w_n) - \frac{1}{n} \|w-w_n\|, \forall w \in \wp^- \).

Since \( I(|u|) = I(u) \), we may assume that \( w_n \geq 0 \) in \( \Omega \). By coerciveness, \( \{w_n\} \) is bounded in \( H_0^1(\Omega) \) (i.e. \( \|w_n\| \leq c_3, \, n = 1, 2, \ldots \) and hence, up to subsequences, converges to a function, say \( u_2 \geq 0 \), almost everywhere in \( \Omega \), strongly in \( L^{\beta+1} \) and weakly in \( H_0^1 \). From (7) it follows that \( u_2 \neq 0 \). In addition, for the minimizing sequence \( \{w_n\} \) there exists suitable constant \( c_4 > 0 \) such that

\[
(1+\gamma) \|w_n\|^2 - \lambda(\beta+\gamma) \|w_n\|_{\beta+1} \leq -c_4, \, n = 1, 2, \ldots
\]

(12)
Suppose, by contradiction, that for a subsequence, which is still denoted by \( w_n \), we have:

\[
(1 + \gamma) \| w_n \|^2 - \lambda (\beta + \gamma) \| w_n \|_{p+1} = o(1).
\]

Using (7) we infer

\[
I(w_n) = - \frac{1 + \gamma}{2(1 - \gamma)} \| w_n \|^2 + \frac{\beta + \gamma}{(1 - \gamma)(\beta + 1)} \| w_n \|_{p+1}^p
\]

\[
+ \frac{\lambda}{(1 - \gamma)} \left( \frac{1}{\beta + 1} - \frac{1}{\beta + p+1} \right) \| w_n \|_{p+1}^p
\]

\[
\leq - \frac{1}{2(1 - \gamma)} \left[ (1 + \gamma) \| w_n \|^2 - \lambda (\beta + \gamma) \| w_n \|_{p+1}^p \right] - c_\lambda,
\]

where \( c_\lambda > 0 \) is some constant independent of \( n \). Passing to the limit as \( n \to \infty \), we get:

\[
\lim_{n \to \infty} I(w_n) \leq - c_\lambda.
\]

This, together with condition (i) implies:

\[
\inf_{\nabla} I \leq - c_\lambda < 0.
\]

which is clearly impossible because from Lemma 4 it follows that \( \inf_{\nabla} \geq 0 \).

Applying Lemma 3 with \( u = w_n \) (\( n \) large enough so that \( \lambda(t - 0) < c_\lambda \)) and \( w = t\varphi, \ \varphi \in H_0^1, \ \varphi \geq 0, \ t > 0 \) small, we find \( f_n(t) := f_n(t\varphi) \) such that \( f_n(0) = 1 \) and \( f_n(t)(w_n + t\varphi) \in \nabla \). Note that, since

\[
0 = f_n^{1-\gamma}(t) \int_{\Omega} p(x)(w_n + t\varphi)^{1-\gamma} dx,
\]

\[
0 = \| w_n \|^2 - \lambda \| w_n \|_{p+1}^p - \int_{\Gamma} p(x) w_n^{1-\gamma} dx,
\]
then

\[
0 = \left[ f_n^2(t) - 1 \right] \| w_n + t \varphi \|^2 + (\| w_n + t \varphi \|^2 - \| w_n \|^2) \\
- \lambda \left[ f_n^{\beta+1}(t) - 1 \right] \| w_n + t \varphi \|_{\beta+1}^{\beta+1} - \lambda (\| w_n + t \varphi \|_{\beta+1}^{\beta+1} - \| w_n \|_{\beta+1}^{\beta+1}) \\
- \left[ f_n^{\gamma}(t) - 1 \right] \int_D p(x) (w_n + t \varphi)^{1-\gamma} dx \\
- \int_D p(x) [(w_n + t \varphi)^{1-\gamma} - w_n^{1-\gamma}] dx \\
\leq \left[ f_n^2(t) - 1 \right] \| w_n + t \varphi \|^2 + (\| w_n + t \varphi \|^2 - \| w_n \|^2) \\
- \lambda \left[ f_n^{\beta+1}(t) - 1 \right] \| w_n + t \varphi \|_{\beta+1}^{\beta+1} - \lambda (\| w_n + t \varphi \|_{\beta+1}^{\beta+1} - \| w_n \|_{\beta+1}^{\beta+1}) \\
- \left[ f_n^{\gamma}(t) - 1 \right] \int_D p(x) (w_n + t \varphi)^{1-\gamma} dx.
\]

Dividing by \( t > 0 \) and letting \( t \to 0 \), we infer that

\[
0 \leq 2 f_n^*(0) \| w_n \|^2 + 2 \int_D \nabla w_n \cdot \nabla \varphi \ dx - \lambda (\beta + 1) f_n^*(0) \| w_n \|_{\beta+1}^{\beta+1} \\
- \lambda (\beta + 1) \int_D w_n^\beta \cdot \varphi \ dx - (1-\gamma) f_n^*(0) \int_D p(x) w_n^{1-\gamma} \ dx \\
= f_n^*(0) \left[ 2 \| w_n \|^2 - \lambda (\beta + 1) \| w_n \|_{\beta+1}^{\beta+1} - (1-\gamma) \int_D p(x) w_n^{1-\gamma} \ dx \right] \\
+ 2 \int_D \nabla w_n \cdot \nabla \varphi \ dx - \lambda (\beta + 1) \int_D w_n^\beta \cdot \varphi \ dx \\
= f_n^*(0)[(1+\gamma) \| w_n \|^2 - \lambda (\beta + \gamma) \| w_n \|_{\beta+1}^{\beta+1}] \\
+ 2 \int_D \nabla w_n \cdot \nabla \varphi \ dx - \lambda (\beta + 1) \int_D w_n^\beta \cdot \varphi \ dx,
\]

(13)

where \( f_n^*(0) \in [\ -\infty, +\infty \] denotes the right derivate of \( f_n(t) \) at zero (for the sake of simplicity, we assume henceforth that the right derivate of \( f_n \) at \( t = 0 \) exists. Indeed, if it isn’t real, we let \( t_k \to 0 \) (instead of \( t \to 0 \)), \( t_k > 0 \) is chosen in such a way that \( q_n := \lim_{k \to -\infty} (f_n(t_k) - 1)/t_k \), where \( q_n \in [\ -\infty, +\infty \] , and then replace \( f_n^*(0) \) by \( q_n \). Since \( w_n \in W^- \) it follows that \( (1+\gamma) \| w_n \|^2 - \lambda (\beta + \gamma) \| w_n \|_{\beta+1}^{\beta+1} < 0 \), and thus from (13) we know immediately that \( f_n^*(0) \neq +\infty \). Now we show that \( |f_n^*(0)| < \infty \). Arguing
by contradiction, we assume that $f^*_n(0) = -\infty$, and so for $t > 0$ small there holds $f_n(t) < 1$. Then,

$$
\|f_n(t)(w_n + t\varphi) - w_n\| = \int_\Omega |(f_n(t) - 1)\nabla w_n + tf_n(t)\nabla \varphi|^2 \, dx^{1/2}
\leq [1 - f_n(t)] \|w_n\| + tf_n(t) \|\varphi\|
$$

provided $t > 0$ small. Thus from condition (ii) we have

$$
[1 - f_n(t)] \frac{\|w_n\|}{n} + tf_n(t) \frac{\|\varphi\|}{n}
\geq I(w_n) - I(f_n(t)(w_n + t\varphi))
= \frac{1 + \gamma}{2(1 - \gamma)} \|w_n + t\varphi\|^2 - \frac{1 + \gamma}{2(1 - \gamma)} [f_n^2(t) - 1] \|w_n + t\varphi\|^2
\leq -\lambda \frac{\beta + \gamma}{(1 - \gamma)(\beta + 1)} f_n^{\beta + 1}(t)(\|w_n + t\varphi\|^{\beta + 1} - \|w_n\|^{\beta + 1})
\leq -\lambda \frac{\beta + \gamma}{(1 - \gamma)(\beta + 1)} [f_n^{\beta + 1}(t) - 1] \|w_n\|^{\beta + 1},
$$

also dividing by $t > 0$ and passing to the limit as $t \to 0$, we derive that

$$
-f_n^*(0) \frac{\|w_n\|}{n} + \frac{\|\varphi\|}{n} \leq \frac{1 + \gamma}{1 - \gamma} \int_\Omega \nabla w_n \cdot \nabla \varphi \, dx + \frac{1 + \gamma}{1 - \gamma} f_n^*(0) \|w_n\|^2
\leq -\lambda \frac{\beta + \gamma}{1 - \gamma} f_n^*(0) \|w_n\|^{\beta + 1} - \lambda \frac{\beta + \gamma}{1 - \gamma} \int_\Omega w_n^{\beta} \varphi \, dx.
$$

That is,

$$
\frac{\|\varphi\|}{n} \geq \frac{1}{1 - \gamma} \left[ (1 + \gamma) \|w_n\|^2 - \lambda (\beta + \gamma) \|w_n\|^{\beta + 1} + \frac{(1 - \gamma) \|w_n\|}{n} \right] f_n^*(0)
+ \frac{1 + \gamma}{1 - \gamma} \int_\Omega \nabla w_n \cdot \nabla \varphi \, dx - \lambda \frac{\beta + \gamma}{1 - \gamma} \int_\Omega w_n^{\beta} \varphi \, dx,
$$

which is clearly impossible if $f_n^*(0) = -\infty$ because from (12) it follows that

$$(1 + \gamma) \|w_n\|^2 - \lambda (\beta + \gamma) \|w_n\|^{\beta + 1} + \frac{(1 - \gamma) \|w_n\|}{n} \leq -c_3 + \frac{(1 - \gamma) \|w_n\|}{n} < 0.
$$

Hence $|f_n^*(0)| < +\infty$. Furthermore, estimate (12) with $\|w_n\| \leq c_3$, $\forall n$, and the two inequalities (13)(14) also imply that

$$
|f_n^*(0)| \leq c_3, \forall n = 1, 2, \ldots (c_3 > 0 \text{ suitable constant})
$$

(15)
Now we show that $u_2 \in \Lambda^-$ is a weak positive solution of (1). From condition (ii) we infer

$$\frac{1}{n} \left[ \| f_n(t) - 1 \| \| w_n \| + t f_n(t) \| \varphi \| \right]$$

$$\geq \frac{1}{n} \left[ f_n(t)(w_n + t \varphi) - w_n \right]$$

$$\geq I(w_n) - I(f_n(t)(w_n + t \varphi))$$

$$= - \left[ \frac{f_n^2(t) - 1}{2} \right] \| w_n \|^2 + \lambda \left[ \frac{f_n^{p+1}(t) - 1}{\beta + 1} \right] \| w_n + t \varphi \|^\beta_{p+1}$$

$$+ \frac{f_n^{p+1}(t) - 1}{1 - \gamma} \int_D p(x)(w_n + t \varphi)^{1 - \gamma} dx + \frac{f_n^2(t)}{2} \left( \| w_n \|^2 - \| w_n + t \varphi \|^2 \right)$$

$$+ \frac{\lambda}{\beta + 1} \left( \| w_n + t \varphi \|^\beta_{p+1} - \| w_n \|^\beta_{p+1} \right)$$

$$+ \frac{1}{1 - \gamma} \int_D p(x)[(w_n + t \varphi)^{1 - \gamma} - w_n^{1 - \gamma}] dx,$$

dividing by $t > 0$ and passing to the limit as $t \to 0$, this yields

$$\frac{1}{n} \left[ f_n'(0) \| w_n \| + \| \varphi \| \right]$$

$$\geq - f_n'(0) \| w_n \|^2 + \lambda f_n'(0) \| w_n \|^\beta_{p+1} + f_n'(0) \int_D p(x) w_n^{1 - \gamma} dx$$

$$- \int_D \nabla w_n \cdot \nabla \varphi dx + \lambda \int_D w_n^\beta \varphi dx$$

$$+ \lim \inf_{t \to 0^+} \frac{1}{1 - \gamma} \int_D p(x)[(w_n + t \varphi)^{1 - \gamma} - w_n^{1 - \gamma}] \frac{1}{t} dx$$

$$= - f_n'(0) \left[ \| w_n \|^2 - \lambda \| w_n \|^\beta_{p+1} - \int_D p(x) w_n^{1 - \gamma} dx \right]$$

$$- \int_D \nabla w_n \cdot \nabla \varphi dx + \lambda \int_D w_n^\beta \cdot \varphi dx$$

$$+ \lim \inf_{t \to 0^+} \frac{1}{1 - \gamma} \int_D p(x)[(w_n + t \varphi)^{1 - \gamma} - w_n^{1 - \gamma}] \frac{1}{t} dx$$

$$= - \int_D \nabla w_n \cdot \nabla \varphi dx + \lambda \int_D w_n^\beta \cdot \varphi dx$$

$$+ \lim \inf_{t \to 0^+} \frac{1}{1 - \gamma} \int_D p(x)[(w_n + t \varphi)^{1 - \gamma} - w_n^{1 - \gamma}] \frac{1}{t} dx. \quad \text{(16)}$$
Since \( p(x) [(w_n + t \varphi)^{1-\gamma} - w_n^{1-\gamma}] \geq 0 \), \( \forall x \in \Omega \), \( \forall t > 0 \), then by Fatou's Lemma we have

\[
\int_{\Omega} p(x) w_n^{-\gamma} \varphi \, dx \leq \liminf_{t \to 0^+} \frac{1}{t} \int_{\Omega} p(x) \left[ (w_n + t \varphi)^{1-\gamma} - w_n^{1-\gamma} \right] \, dx.
\]

Inserting this into (16) and using (15) we find

\[
\int_{\Omega} p(x) w_n^{-\gamma} \varphi \, dx \leq \frac{1}{n} (|f_*'(0)| \|w_n\| + \|\varphi\|) + \int_{\Omega} \nabla w_n \cdot \nabla \varphi \, dx - \lambda \int_{\Omega} w_n^\delta \cdot \varphi \, dx
\]

\[
\leq \frac{(c_3 + c_5 + \|\varphi\|)}{n} + \int_{\Omega} \nabla w_n \cdot \nabla \varphi \, dx - \lambda \int_{\Omega} w_n^\delta \cdot \varphi \, dx,
\]

as \( n \to \infty \) we are led to

\[
\liminf_{n \to \infty} \int_{\Omega} p(x) w_n^{-\gamma} \varphi \, dx \leq \int_{\Omega} \nabla u_2 \cdot \nabla \varphi \, dx - \lambda \int_{\Omega} u_2^\delta \cdot \varphi \, dx;
\]

then using once more Fatou's Lemma, we infer that

\[
\int_{\Omega} p(x) u_2^{-\gamma} \varphi \, dx \leq \int_{\Omega} \nabla u_2 \cdot \nabla \varphi \, dx - \lambda \int_{\Omega} u_2^\delta \cdot \varphi \, dx,
\]

Thus

\[
\int_{\Omega} \nabla u_2 \cdot \nabla \varphi - \lambda u_2^\delta \varphi - p(x) u_2^{-\gamma} \varphi \, dx \geq 0, \quad \varphi \in H^1_0, \; \varphi \geq 0 \tag{17}
\]

which means \( u_2 \) satisfies in the weak sense that

\[-Au_2 \geq 0 \quad \text{in} \quad \Omega, \]

since \( u_2 \geq 0 \) and \( u_2 \not\equiv 0 \) in \( \Omega \), then the strong maximum principle yields

\[ u_2 > 0 \quad \text{in} \quad \Omega. \tag{18} \]

In particular, using (17) with \( \varphi = u_2 \), we infer that

\[
\|u_2\|^2 \geq \lambda \|u_2\|^{\delta+1}_{\delta+1} + \int_{\Omega} p(x) u_2^{-\gamma} \, dx,
\]
on the other hand, by the weakly lower semi-continuity of the norm,

\[
\|u_2\|^2 \leq \liminf_{n \to \infty} \|w_n\|^2 \leq \limsup_{n \to \infty} \|w_n\|^2
\]

\[
= \lim_{n \to \infty} \left[ \lambda \|w_n\|_{\beta+1}^\beta + \int_{\Omega} p(x) w_n^{1-\gamma} \, dx \right]
\]

\[
= \lambda \|u_2\|_{\beta+1}^\beta + \int_{\Omega} p(x) u_2^{1-\gamma} \, dx
\]

Therefore

\[
\|u_2\|^2 = \lim_{n \to \infty} \|w_n\|^2 = \lambda \|u_2\|_{\beta+1}^\beta + \int_{\Omega} p(x) u_2^{1-\gamma} \, dx. \tag{19}
\]

Consequently \( w_n \to u_2 \) strongly in \( H_1^1(\Omega) \) and \( I(u_2) = \inf_{\lambda \in N} I \). Also from Lemma 1 follows that necessarily \( u_2 \in \lambda^- \). Then, following the same arguments as in proving the existence of \( u_1 \) and using (17)–(19), we obtain \( u_2 \in \lambda^- \) is a weak positive solution of (1). This completes the proof of Theorem 1.

**Remark 3.** The solutions we have found are in \( H_1^1(\Omega) \). We don’t know whether they are classical solutions. In addition, it is worth pointing out that for singular elliptic equations, a classical solution in \( C^2(\Omega) \cap C(\overline{\Omega}) \) may not be a weak solution in \( H_1^1(\Omega) \). The reader can refer Lazer–Mckenna [5] for more details.

3. **EXISTENCE OF POSITIVE SOLUTIONS FOR ALL \( \lambda > 0 \)**

In this section we establish existence for all \( \lambda > 0 \). To make the essence of our result simple, we study below the special case of (1),

\[
Au + \lambda u^\beta + \sigma u^{1-\gamma} = 0, \quad \text{in} \ \Omega
\]

\[
u > 0, \quad \text{in} \ \Omega \tag{20}
\]

\[
u = 0, \quad \text{on} \ \partial\Omega,
\]

where \( \lambda, \sigma > 0 \) are parameters, \( \beta, \gamma \) are the same as in (1).

**Theorem 2.** Let \( 0 < \gamma < 1 < \beta < 2^*-1 \). Then for every \( \lambda > 0 \) there exists \( \sigma^* > 0 \) such that for all \( \sigma \in (0, \sigma^*] \) problem (20) possesses at least one weak positive solution \( \nu_3 \in H_1^1(\Omega) \). Moreover, \( \nu_3 \geq \varepsilon \phi_1 \) for some constant \( \varepsilon > 0 \) in \( \Omega \).
Proof. We fix \( \lambda \in (0, \infty) \). As is well known,
\[
\nabla \varphi_1(x) \neq 0, \quad \forall x \in \partial \Omega.
\]
Letting \( d_0 = \inf_{\partial \Omega} [l(1-l) |V \varphi_1|^2 + \lambda l \varphi_1^2] \), where \( l \in (0, 1) \) is a constant, then \( d_0 > 0 \). Since \( \beta > 1 \), we can find \( \sigma^* > 0 \) such that for all \( 0 < \sigma \leq \sigma^* \) there exists \( \eta = \eta(\lambda, \sigma) > 0 \) such that
\[
d_0 \eta \geq \lambda \eta^\beta + \sigma \eta^{-\gamma}.
\]
As a consequence, the function \( \eta \varphi_1 (0 \leq \varphi_1 \leq 1) \) verifies
\[
-\Delta (\eta \varphi_1) = \eta l(1-l) |\nabla \varphi_1|^2 \varphi_1^{l-2} + \eta \lambda l \varphi_1^l
\geq d_0 \eta \varphi_1^{l-2}
\geq (\lambda \eta^\beta + \sigma \eta^{-\gamma}) \varphi_1^{l-2}
\geq \lambda \eta^\beta \varphi_1^l + \sigma \eta^{-\gamma} \varphi_1^{l-\gamma},
\]
and hence it is a supersolution of (20). Moreover, any \( \varepsilon \varphi_1 \) is a subsolution of (20), provided
\[
\varepsilon \lambda \varphi_1 = -\Delta (\varepsilon \varphi_1) \leq \lambda (\varepsilon \varphi_1)^\beta + \sigma (\varepsilon \varphi_1)^{-\gamma},
\]
which is satisfied for all \( \varepsilon > 0 \) small enough and all \( \lambda, \sigma \). Taking \( \varepsilon \) possibly smaller, we also have
\[
-\Delta (\eta \varphi_1) \geq d_0 \eta \varphi_1^{l-2} \geq \lambda \varepsilon \varphi_1 = -\Delta (\varepsilon \varphi_1),
\]
then the strong maximum principle yields
\[
\varepsilon \varphi_1 < \eta \varphi_1.
\]
Define \( J : H_0^1 \to \mathbb{R} \) by
\[
J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{\lambda}{\beta+1} \int_\Omega |u|^{\beta+1} \, dx - \frac{\sigma}{1-\gamma} \int_\Omega |u|^{1-\gamma} \, dx,
\]
and
\[
K \equiv \{ u \in H_0^1(\Omega) \mid \varepsilon \varphi_1(x) \leq u(x) \leq \eta \varphi_1(x) \text{ in } \Omega \}.
\]
For every \( u \in K \), we have \( u(x) \leq \eta \varphi_1(x) \leq \eta, \forall x \in \Omega \). From this fact we infer that \( J(u) \geq \frac{1}{\sigma} \|u\|^2 - c_{\lambda, \sigma}, \forall u \in K \). Hence \( J \) is coercive on \( K \). Clearly \( K \)
is closed and convex (and weakly closed) so that $J$ has a global minimizer, say $u_3$ on $K$. Thus, for any $\Psi \in K$, the function

$$h: [0, 1] \to \mathbb{R}$$

defined by

$$h(t) \equiv J(t\Psi + (1-t)u_3)$$

has a minimum at $t = 0$. Note that, since

$$|[t\Psi + (1-t)u_3]^{-\gamma} (\Psi - u_3)| \leq (\exp_1)^{-\gamma} \eta \Phi \cdot \varphi ^{-\gamma+t}, \quad \forall x \in \Omega$$

by Lemma 5 we know that $\Phi ^{-\gamma+t}$ is integrable, then from the Dominated Convergence Theorem we deduce that

$$\lim_{t \to 0} \int_{\Omega} \frac{\sigma}{1-\gamma} \left[ t\Psi + (1-t)u_3 \right]^{1-\gamma} - u_3^{1-\gamma} \, dx = \int_{\Omega} u_3^{1-\gamma}(\Psi - u) \, dx.$$

Thus we get

$$\frac{d}{dt} \bigg|_{t=0} h = \int_{\Omega} \left[ \nabla (\Psi - u_3) \cdot \nabla u_3 - \lambda u_3^\gamma (\Psi - u_3) - \sigma u_3^{1-\gamma}(\Psi - u_3) \right] \geq 0.$$ 

The rest of the proof follows exactly as in [4, p. 381].

In conclusion, $u_3 \ (u_3 \geq \exp_1)$ is a weak positive solution of (20).

REFERENCES