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On certain remarkable curves of genus five

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ABSTRACT

The aim of this note is twofold. First to show the existence of genus five curves having exactly twenty four Weierstrass points, which constitute the set of fixed points of three distinct elliptic involutions on them. Second to characterize these curves, in fact we prove that all such curves are bielliptic double cover of Fermat's quartic.

1. INTRODUCTION

The plane quartic F defined by $x^4 + y^4 + z^4 = 0$, known as *Fermat's quartic*, and that defined by $x^4 + y^4 + z^4 + 3(x^2y^2 + y^2z^2 + x^2z^2) = 0$ are, up to isomorphisms, the only two curves of genus three having exactly twelve Weierstrass points, or, equivalently, the only non-hyperelliptic genus three curves whose Weierstrass points are all of maximal weight [12]. Between these two curves, the quartic of Fermat has the peculiar property of having its twelve Weierstrass points lying, by fours, onto the three coordinates axis and these sets of four points are the loci of fixed points of the three elliptic involutions on F [5]. Are there non-hyperelliptic curves X of genus $g > 3$ with the property $(*)$ of having their Weierstrass points all of maximal weight and each of them fixed point of some elliptic involutions? Since a curve with $g \geqslant 6$ may carry at most one elliptic involution, an easy computation shows that numbers

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fit well only if $g = 5$: when the maximal weight is 5 [11] and X carries exactly three elliptic involutions (see [4, Question 5.11]). Here, after some preliminaries, in Section 3 we prove the existence of curves of genus five satisfying $(*)$ by showing that the curve X_0 defined by the following equations

$$\begin{cases} x_1^2 + x_4^2 + x_5^2 = 0, \\ x_2^2 + x_4^2 - x_5^2 = 0, \\ x_3^2 + x_4 x_5 = 0, \end{cases}$$

where $[x_1, \dots, x_5]$ are projective coordinates in $\mathbb{P}^4$, is an explicit example of such curves. Then, in Section 4, we prove our main result:

Theorem 1.1. *If a curve of genus five satisfies property* $(*)$*, then it is a bielliptic double cover of Fermat's quartic.*

Then it follows that there are three non-isomorphic curves of genus five with the requested property $(*)$: the three (unbranched) double covers of F associated to the three half-periods on F which are lifting of the three nonzero half-periods on E.

Notations. As usual we denote by ω_C the canonical sheaf of the irreducible, smooth projective curve C. For each invertible sheaf $\mathfrak{F}$ on C we denote by $|\mathfrak{F}|$ the projectivization of $H^0(C, \mathfrak{F})$ and by $|\mathfrak{F}|^\vee$ its dual. If $a_1, \dots, a_k$ are elements of a certain group G, we denote $\langle a_1, \dots, a_k \rangle$ the subgroup of G generated by them. For all other notations we refer to [9].

2. PRELIMINARIES

In this section we shortly explain the geometry of bielliptic curves of genus five (see for instance [7] for details) and we recall some results that we need in the following sections.

Let X be an irreducible, smooth, projective curve of genus 5 defined over the complex field $\mathbb{C}$, and suppose X bielliptic, i.e., suppose that it admits a degree two morphism $\varphi : X \to E$ onto an elliptic curve; to such morphism corresponds an elliptic involution $\iota : X \to X$ so that $X/\langle \iota \rangle \simeq E$. To the cover φ one can associate its ramification locus B on E (the image under φ of the points fixed by ι) and a half $\mathfrak{H}$ of the divisor class of B. From the datum $(E; B, \mathfrak{H})$ one can reconstruct X up to isomorphisms. By Castelnuovo–Severi inequality (see for instance [1, p. 21]) X cannot be either hyperelliptic nor trigonal and its canonical model $\widetilde{X}$ is the complete intersection of a net of quadrics $\mathcal{N} = \{u_1 Q_1 + u_2 Q_2 + u_3 Q_3 = 0\}$ in the canonical $\mathbb{P}^4 = |\omega_X|^\vee$. The linear series of degree four and dimension one on $\widetilde{X}$ are cut out by the rulings of rank 4 quadrics of $\mathcal{N}$. Moreover $\widetilde{X}$ lies on the elliptic normal cone

$$\Gamma = \bigcup_{P \in \widetilde{X}} \overline{P, \iota P} \subset |\omega_X|^\vee,$$

where $\overline{P, \iota P}$ denotes the line joining P and its conjugated ιP under ι. From the natural decomposition

$$H^0(X, \omega_X) \simeq H^0(E, \omega_E) \oplus H^0(E, \mathfrak{H}),$$

it follows that $|\omega_E|^\vee$ and $|\mathfrak{H}|^\vee$ are the two linear subspaces of fixed points of the involutory homology of $\mathbb{P}^4 = |\omega_X|^\vee$ inducing ι on $\widetilde{X}$. So Γ has vertex $V = |\omega_E|^\vee$ and its section by $|\mathfrak{H}|^\vee$ is the elliptic normal curve $\widetilde{E}$ which is the embedding of E via the map associated with the linear series $|\mathfrak{H}|$.

Remark 2.1. The fixed points of ι are the eight distinct points of $\widetilde{X} \cap |\mathfrak{H}|^\vee$.

Let N denote the projective plane, with homogeneous coordinates $[u_1, u_2, u_3]$, parametrizing the quadrics of $\mathcal{N}$. In N is defined the discriminant curve Δ of the net $\mathcal{N}$, i.e., the locus of points $P \in N$ corresponding to singular quadrics of $\mathcal{N}$. Clearly Δ is a quintic, possibly reducible. From [6, 6.1, 6.2 and Proposition 1.2], it follows

Proposition 2.2. *The discriminant curve Δ has at most ordinary double points as singularities. Moreover $P \in \Delta$ is singular if and only if it corresponds to a rank* 3 *quadric of $\mathcal{N}$.*

Any line contained in Δ corresponds to a pencil of singular quadrics through $\widetilde{X}$ with a common vertex [6, Lemma 6.8 and proof], say V. By projecting $\widetilde{X}$ from V, we obtain the complete intersection $\widetilde{E}$ of two quadrics in $\mathbb{P}^3$ (projection of any two quadrics of the pencil). Since $\widetilde{X}$ has degree eight and does not pass through V, it follows that $\widetilde{X}$ projects 2:1 onto the elliptic curve $\widetilde{E}$ and so X is bielliptic. By [3, p. 272] we have

Proposition 2.3. *There is a one-to-one correspondence between the lines contained in Δ and the bielliptic involutions on X.*

Suppose X carries exactly three elliptic involutions, then Δ contains three lines and a non-singular conic γ. In this case, see [14, p. 7], the points of γ correspond to a family of quadrics in $\mathcal{N}$ whose set of vertices is a line L, any plane through L contains exactly two points of $\widetilde{X}$ and the following holds:

Proposition 2.4. *The projection from L gives a degree two unramified morphism from $\widetilde{X}$ onto a non singular plane quartic curve Y.*

Notice that in this case we also have the following natural decomposition

$$H^0(X, \Omega_X) \simeq H^0(Y, \Omega_Y) \oplus H^0\big(Y, \Omega_Y(\sigma)\big),$$

where σ is the half-period on Y associated to τ and $L = |\Omega_Y(\sigma)|^\vee$ is the Prym-canonical space.

Since an elliptic involution on a curve of genus five has more then 4 fixed points by [13, Theorem 6] we have that they are all Weierstrass points. These points P have gap-sequences $(1,2,3,5,9)$ or $(1,2,3,5,7)$ according if $\varphi(P)$ is or not a point of order four on E (i.e., $4\varphi(P)=0$ in the group law on E) [4, Proposition 5.7]. In the first case the weight of P achieves the maximum possible for a non-hyperelliptic Weierstrass point on a curve of genus five. We like to remark the existence of bielliptic curves of genus five whose Weierstrass points are all of the second type and each of them is fixed point of an elliptic involution (see [4, 5.14]).

We end this section with a result concerning elliptic normal quartics in $\mathbb{P}^3$ (see [10, pp. 27–29]. Any such curve $\widetilde{E}$ is the complete intersection of a pencil of quadrics. In each pencil there are four quadric cones Σ_n, $n=1,\dots,4$, and the sixteen points p of order four on $\widetilde{E}$ are divided into four set of four points so that each set belongs to one, and only one, of such Σ_n. If $p\in\Sigma_n$, then the tangent line to $\widetilde{E}$ at p passes through the vertex of Σ_n so that the projection of $\widetilde{E}$ from that vertex gives a $2:1$ map onto $\mathbb{P}^1$. In particular we have:

Proposition 2.5. *If the tangent line at a point p of $\widetilde{E}$ passes through the vertex of a cone Σ_n, then p is a point of order four of $\widetilde{E}$.*

3. THE EXAMPLE X_0

Let us put $Q_1^0 := x_1^2+x_4^2+x_5^2$, $Q_2^0 := x_2^2+x_4^2-x_5^2$ and $Q_3^0 := x_3^2+x_4x_5$. The discriminant locus of the net $\mathcal{N}^0=\{u_1Q_1^0+u_2Q_2^0+u_3Q_3^0=0\}$ is the quintic Δ defined by:

$$u_1u_2u_3\big(u_1^2-u_2^2-(u_3/2)^2\big)=0.$$

Clearly Δ is the union of the coordinate lines and an irreducible conic. Thus, according what we said in the previous section, X_0 carries three elliptic involutions ι_i, $i=1,2,3$. These involutions are induced by the following involutory homologies of $\mathbb{P}^4$ (that we also denote by ι_i):

$$x_i\to -x_i \quad\text{and}\quad x_j=x_j \quad \text{if } j\neq i;\quad \text{for } i=1,2,3.$$

The homology ι_i has center O_i, the point whose coordinates are all zero except the ith which is 1, and axis the hyperplane $H_i=\{x_i=0\}$. It is easy to determine the coordinates of all fixed points of X_0 under the involution ι_i, $i=1,\dots,3$, and we list them here below:

$$\iota_1:\begin{cases}\alpha_1:\ x_1=x_4+ix_5=0, & \alpha_1':\ x_1=x_4-ix_5=0,\\ A_{1,1}=(0,i\sqrt{2},\sqrt{-i},1,i), & A_{1,1}'=(0,i\sqrt{2},\sqrt{-i},1,-i),\\ A_{1,2}=(0,-i\sqrt{2},\sqrt{-i},1,i), & A_{1,2}'=(0,-i\sqrt{2},\sqrt{-i},1,-i),\\ A_{1,3}=(0,-i\sqrt{2},-\sqrt{-i},1,i), & A_{1,3}'=(0,-i\sqrt{2},-\sqrt{-i},1,-i),\\ A_{1,4}=(0,i\sqrt{2},-\sqrt{-i},1,i), & A_{1,4}'=(0,i\sqrt{2},-\sqrt{-i},1,-i),\end{cases}$$

$$\iota_2\colon \begin{cases} \alpha_2\colon x_2 = x_4 + x_5 = 0, & \alpha_2'\colon x_2 = x_4 - x_5 = 0,\\ A_{2,1} = (i\sqrt{2}, 0, 1, 1, -1), & A_{2,1}' = (i\sqrt{2}, 0, i, 1, 1),\\ A_{2,2} = (-i\sqrt{2}, 0, 1, 1, -1), & A_{2,2}' = (i\sqrt{2}, 0, -i, 1, 1),\\ A_{2,3} = (i\sqrt{2}, 0, -1, 1, -1), & A_{2,3}' = (-i\sqrt{2}, 0, i, 1, 1),\\ A_{2,4} = (i\sqrt{2}, 0, -1, 1, -1), & A_{2,4}' = (-i\sqrt{2}, 0, -i, 1, 1), \end{cases}$$

$$\iota_3\colon \begin{cases} \alpha_3\colon x_3 = x_4 = 0, & \alpha_3'\colon x_3 = x_5 = 0,\\ A_{3,1} = (i, 1, 0, 0, 1), & A_{3,1}' = (i, i, 0, 1, 0),\\ A_{3,2} = (i, -1, 0, 0, 1), & A_{3,2}' = (i, -i, 0, 1, 0),\\ A_{3,3} = (-i, 1, 0, 0, 1), & A_{3,3}' = (-i, i, 0, 1, 0),\\ A_{3,4} = (-i, -1, 0, 0, 1), & A_{3,4}' = (-i, -i, 0, 1, 0). \end{cases}$$

For convenience we have divided each set of eight fixed points in sets of four, according if they are contained in the plane α_i or α_i' (defined by the equations written beside).

Let us observe that X_0 lies on the three elliptic normal cones defined respectively by the equations:

$$\Gamma_1\colon \begin{cases} x_2^2 + x_4^2 - x_5^2 = 0,\\ x_3^2 + x_4x_5 = 0, \end{cases}$$

$$\Gamma_2\colon \begin{cases} x_1^2 + x_4^2 + x_5^2 = 0,\\ x_3^2 + x_4x_5 = 0, \end{cases}$$

$$\Gamma_3\colon \begin{cases} x_2^2 + x_4^2 - x_5^2 = 0,\\ x_2^2 + x_4^2 - x_5^2 = 0, \end{cases}$$

with vertex $V_1 = (1, 0, 0, 0, 0)$, $V_2 = (0, 1, 0, 0, 0)$ and $V_3 = (0, 0, 1, 0, 0)$ respectively. The equations of Γ_i also define the elliptic normal curve $\widetilde{E}_i \subset H_i$. In the pencil of quadrics through $\widetilde{E}_i$ there are four quadric cones: those defined by the pairs of equations above, those defining the Γ_i's and the following pairs for each $\widetilde{E}_i$:

$$(\widetilde{E}_1) \quad \pm ix_2^2 + 2x_3^2 \pm ix_4^2 \mp ix_5^2 + 2x_4x_5 = 0,$$

with vertices $v_1 = (0, 0, 0, 1, -i)$ and $v_1' = (0, 0, 0, 1, i)$;

$$(\widetilde{E}_2) \quad \pm x_2^2 + 2x_3^2 \pm x_4^2 \mp x_5^2 + 2x_4x_5 = 0,$$

with vertices $v_2 = (0, 0, 0, 1, -1)$ and $v_2' = (0, 0, 0, 1, 1)$;

$$(\widetilde{E}_3) \quad x_1^2 + x_2^2 + 2x_4^2 = 0;\ -x_1^2 + x_2^2 - 2x_5^2 = 0,$$

with vertices $v_3 = (0, 0, 0, 0, 1)$ and $v_3' = (0, 0, 0, 1, 0)$.

One can easily verify that the tangent lines to $\widetilde{E}_i$ at $A_{i,1}, A_{i,2}, A_{i,3}$ and $A_{i,4}$ all pass through v_i, while the tangent lines to $\widetilde{E}_i$ at $A_{i,1}', A_{i,2}', A_{i,3}'$ and $A_{i,4}'$ all pass through v_i', $i = 1, 2, 3$. This, by Proposition 2.5, proves that all ramification points of the three elliptic involutions ι_i, $i = 1, 2, 3$, are Weierstrass points of weight 5.

Let us denote by ϕ the projection of $\mathbb{P}^4$ from the line $l = \{x_1 = x_2 = x_3 = 0\}$ onto the plane $\Pi = \{x_4 = x_5 = 0\}$, ϕ restricted to X_0 gives a $2:1$ map of X_0 onto the quartic curve C_0 defined by the equation:

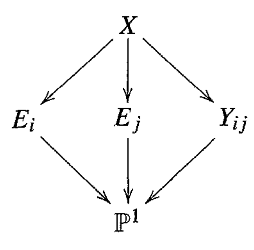
$$x_1^4 - x_2^4 - 4x_3^4 = 0,$$

that we get by eliminating the variables x_4 and x_5 from the equations of X_0. This curve C_0 is isomorphic to the quotient of X_0 by group generated by the involution (of genus three) induced by the homology of $\mathbb{P}^4$ that changes x_4 with $-x_4, x_5$ with $-x_5$ and leaves fixed the others (notice that this involution is exactly the composition of the three ι_i). Under ϕ, the planes α_i and α_i' are both mapped onto the line in Π defined by $x_i = 0$, this for $i = 1, 2, 3$. We also have $\phi(A_{i,j}) = \phi(A_{i,j}')$, $j = 1, 2, 3, 4$ and $i = 1, 2, 3$. Moreover for each $i = 1, 2, 3$, the hyperosculation planes $\Pi_{i,j}, \Pi_{i,j}'$ to $\widetilde{E}_i$ at the points $A_{i,j}$ and $A_{i,j}'$, are both projected onto the tangent line $l_{i,j}$ to C_0 at the $\phi(A_{i,j})$ for $j = 1, 2, 3, 4$, and all the four lines $l_{i,j}$, $j = 1, 2, 3, 4$, pass through the point V_i, $i = 1, 2, 3, 4$. Finally we see that $l_{i,j} \cdot C_0 = 4\phi(A_{i,j})$ for $j = 1, 2, 3, 4$ and $i = 1, 2, 3$, so all the twelve $\phi(A_{i,j})$ on C_0 are Weierstrass points of weight 2.

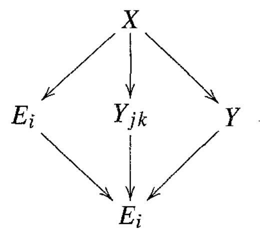
To our knowledge X_0 appeared for the first time in [15, p. 38] has an example of genus five curve having automorphisms group of order 192.

4. PROOF OF THEOREM 1.1

By hypothesis X carries exactly three elliptic involutions ι_i, $i = 1, 2, 3$. We denote $\iota_{ij} = \iota_i \circ \iota_j$ the composition of two of them, since ι_i, ι_j always commute (e.g., [8] or [1, Lemma 5.13]) and do not have common couples, we have, by [2, p. 56] and [1, Theorem 5.9] three commutative diagrams of curves and morphisms of degree two of the following type:



where $Y_{ij} := X/\langle \iota_{ij} \rangle$ is a curve of genus three, $\mathbb{P}^1 \simeq X/\langle \iota_1, \iota_2 \rangle$ (we stress the fact that each Y_{ij} is hyperelliptic). Let us remark that the composition $\tau = \iota_1 \circ \iota_2 \circ \iota_3$ is also an involution without fixed points, so the quotient $Y := X/\langle \tau \rangle$ is a curve of genus three. Since the involutions ι_i, ι_{jk} and τ are pair-wise commuting [2, p. 56] and [1, Theorem 5.9] again yield the following commutative diagram of curves and morphisms of degree two:



In particular: Y_{jk} and Y are bielliptic and the latter one, as we will see, is also non-hyperelliptic. Let denote the morphisms: $\pi_{jk}: X \to Y_{jk}$, $\varepsilon_{jk}: Y_{jk} \to E_i$, $\varepsilon_i: Y \to E_i$. We notice that the unramified map $E_i \to E_i$ is associated to a nonzero half-period η_i (or 2-torsion point) on E_i so that the maps π_{jk} and τ are associated to the half-periods $\varepsilon^*_{jk}(\eta_i)$ and $\varepsilon^*(\eta_i)$ on Y_{jk} and Y respectively.

By recalling what we said in section two, it follows that Δ contains three lines l_i, $i = 1, 2, 3$, and that we can choose, without loss of generality, homogeneous coordinates (u_1, u_2, u_3) in N so that $l_i = \{u_i = 0\}$, $i = 1, 2, 3$. Then we can suppose Q_i given by:

$$x_i^2 + f_i(x_4, x_5) = 0,$$

where $f_i(x_4, x_5) = a_i x_4^2 + b_i x_4 x_5 + c_i x_5^2$, for $i = 1, 2, 3$.

The involutory homology of $\mathbb{P}^4$ that changes $x_i \to -x_i$ and leaves the others coordinates unchanged, induces on $X = \bigcap_{i=1,2,3} Q_i$ the elliptic involutions ι_i, this for each $i = 1, 2, 3$. A simple computation yields to the following equation for γ:

$$4(a_1u_1 + a_2u_2 + a_3u_3)(c_1u_1 + c_2u_2 + c_3u_3) - (b_1u_1 + b_2u_2 + b_3u_3)^2 = 0.$$

Clearly for general f_i's γ is nonsingular. Moreover it is not difficult to see that the set of vertices of the singular quadrics of $\mathcal{N}$ corresponding to the points of γ is the line $L = \{x_1 = x_2 = x_3 = 0\}$. Now let us consider the three pencils of quadrics:

$$\mathcal{F}_i := \{\lambda Q_j + Q_k = 0\}, \quad i \neq j \neq k,$$

in $\{x_i = 0\}$, $i = 1, 2, 3$. Each pencil $\mathcal{F}_i$ contains four quadric cones: Q_j, Q_k and the other two corresponding to values of $\lambda \neq 0, \infty$. These two latter cones, say Λ_i and Λ'_i, have vertex on L. The line L and the plane $\Pi := \{x_4 = x_5 = 0\}$ are the linear subspaces of $\mathbb{P}^4$ fixed by the involutory homology that changes $x_i \to -x_i$ for $i = 4, 5$ and fixes the other coordinates. This homology induces the involution ι_{123} on X.

Now suppose that all Weierstrass points of X are of weight 5, then the branch points of ι_i are fourth order points on E_i, $i = 1, 2, 3$. Recalling Proposition 2.4, this implies that in the projection from L the curve X is mapped $2:1$ onto a nonsingular quartic curve C whose Weierstrass points are all of weight 2 and lie, by fours, onto the three lines $x_i = 0$ of Π. Thus, by what we said in the introduction, C is isomorphic to F and this ends the proof.

From above we get immediately the following:

Corollary 4.1. *There are three non-isomorphic curves of genus five with the requested property* $(*)$: *the three (unbranched) double covers of F corresponding to the three half-periods on F which are lifting of the three nonzero half-periods on E.*

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