Periodic Solutions for Second Order Systems with Not Uniformly Coercive Potential\textsuperscript{1}

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The existence and multiplicity of periodic solutions are obtained for the nonautonomous second order systems with locally coercive potential; that is, $F(t, x) \to +\infty$ as $|x| \to \infty$ for a.e. $t$ in some positive-measure subset of $[0, T]$, by using an analogy of Egorov’s Theorem, the properties of subadditive functions, the least action principle, and a three-critical-point theorem proposed by Brezis and Nirenberg.

Key Words: periodic solution; second order system; subadditivity; coercivity; Sobolev’s inequality; critical point.

1. INTRODUCTION AND MAIN RESULTS

Consider the second order systems

\[
\begin{align*}
\ddot{u}(t) &= \nabla F(t, u(t)) & \text{a.e. } t \in [0, T] \\
\dot{u}(0) - u(T) &= \dot{u}(0) - \dot{u}(T) = 0,
\end{align*}
\]

where $T > 0$ and $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in $t$ for every $x \in \mathbb{R}^N$ and continuously differentiable in $x$ for a.e. $t \in [0, T]$, and there exist $a \in C(R^+, R^+)$, $b \in L^1(0, T; R^+)$ such that

\[
|F(t, x)| \leq a(|x|)b(t), \quad |
\nabla F(t, x)| \leq a(|x|)b(t)
\]

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

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The corresponding functional $\varphi$ on $H^1_T$ given by

$$\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt + \int_0^T (F(t, u(t)) - F(t, 0)) \, dt$$

is continuously differentiable and weakly lower semicontinuous on $H^1_T$ (see [1]), where

$$H^1_T = \left\{ u : [0, T] \to \mathbb{R}^N \left| \begin{array}{l}
\text{ } u \text{ is absolutely continuous, } \\
\text{ } u(0) = u(T) \text{ and } \dot{u} \in L^2(0, T; \mathbb{R}^N)
\end{array} \right. \right\}$$

is a Hilbert space with the norm defined by

$$\|u\| = \left( \int_0^T |u(t)|^2 \, dt + \int_0^T |\dot{u}(t)|^2 \, dt \right)^{\frac{1}{2}}$$

for $u \in H^1_T$. It is well known that the solutions of problem (1) correspond to the critical points of $\varphi$.

It has been proved that problem (1) has at least one solution which minimizes $\varphi$ on $H^1_T$ by the least action principle (see [1–8]). Many solvability conditions are given, such as the coercivity condition (see [2]), the periodicity condition (see [3]), the convexity condition (see [4]), the boundedness condition (see [1]), the subadditive condition (see [5]), and the sublinear condition (see [6]). Specifically, under the condition that $F(t, x) \to +\infty$ as $|x| \to \infty$ uniformly for a.e. $t \in [0, T]$, Berger and Schechter [2] proved the existence of solutions for problem (1) (see Theorem 4.9 in [2]). On one hand, being based on [2], Mawhin and Willem [1] obtained the same result for the perturbation problem

$$\begin{cases}
\ddot{u}(t) = \nabla F(t, u(t)) + e(t) & \text{a.e. } t \in [0, T] \\
u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0
\end{cases}$$

in the case that $\tilde{e} \triangleq (1/T) \int_0^T e(t) \, dt = 0$. Recently Tang [8] generalized the results mentioned above. On the other hand, Brezis and Nirenberg [9] obtained three distinct solutions for problem (1) under some additional conditions, and Tang [10] made some extension.

In this paper, we replace the uniform coercivity by the local coercivity, that is, replacing $F(t, x) \to +\infty$ as $|x| \to \infty$ uniformly for a.e. $t \in [0, T]$ by $F(t, x) \to +\infty$ as $|x| \to \infty$ for a.e. $t$ in some positive-measure subset of $[0, T]$, and obtain some existence and multiplicity results of periodic solutions by using an analogy of Egorov's Theorem, the properties of subadditive functions, the least action principle, and a three-critical-point theorem proposed by Brezis and Nirenberg [9], which generalize some
well-known results in [1, 2, 8–10]. Our main results are the following theorems.

**Theorem 1.** Suppose that $F$ satisfies assumption (A) and there exists $\beta \in L^1(0, T)$ such that

$$F(x, t) \geq \beta(t)$$  \hspace{1cm} (2)

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. Assume that there exists a subset $E$ of $[0, T]$ with $\text{meas}(E) > 0$ such that

$$F(t, x) \to +\infty \quad \text{as} \quad |x| \to \infty$$  \hspace{1cm} (3)

for a.e. $t \in E$. Then problem (1) has at least one solution which minimizes $\varphi$ on $H^1_T$.

**Remark 1.** Theorem 1 generalizes Theorem 4.9 in 2. On one hand, if $F(t, x) \to +\infty$ as $|x| \to \infty$ for a.e. $t \in [0, T]$, there exists $M > 0$ such that $F(x, t) \geq 0$ for all $|x| \geq M$ and a.e. $t \in [0, T]$. Furthermore from assumption (A), one obtains (2) with

$$\beta(t) = -\left(\max_{0 \leq x \leq M} a(s)\right)b(t).$$

On the other hand, there are functions $F(t, x)$ satisfying our Theorem 1 and not satisfying Theorem 4.9 in 2 and other theorems in [1–8]. For example,

$$F(t, x) = t\left[2 + \sin(2\pi|x|^2)\right]\ln(1 + |x|^2)$$

is not convex in $x$, not periodic in $x$, not $\gamma$-subadditive in $x$, not convergent to $+\infty$ as $|x| \to \infty$ uniformly for a.e. $t \in [0, T]$, and $\nabla F(t, x)$ is not bounded by a function $g \in L^1(0, T)$ for all $x \in \mathbb{R}^N$.

**Theorem 2.** Suppose that $F$ satisfies assumption (A), (2), and (3). Assume that there exist $r > 0$ and an integer $k \geq 0$ such that

$$-\frac{1}{2}(k + 1)^2w^2|x|^2 \leq F(t, x) - F(t, 0) \leq -\frac{1}{2}k^2w^2|x|^2$$  \hspace{1cm} (4)

for all $|x| \leq r$ and a.e. $t \in [0, T]$, where $w = 2\pi/T$. Then problem (1) has at least three distinct solutions in $H^1_T$.

**Remark 2.** Theorem 2 generalizes Theorem 7 in [9] and Corollary 5 in [10]. There are functions $F(t, x)$ satisfying our Theorem 2 and not satisfying Theorem 7 in [9] and Corollary 5 in [10]. For example, let
\( F(t, x) \)
\[
\begin{cases}
  -\frac{1}{2}w^2|x|^2, & |x| \leq 1 \\
  t \left[ 2 + \sin(2\pi|x|^2) \right] \ln(1 + |x|^2) - \lambda \cos(2\pi|x|^2) - \mu \sin(2\pi|x|^2), & |x| \geq 1,
\end{cases}
\]

where
\[
\lambda = \frac{1}{2}w^2 + (2\ln 2)t \quad \text{and} \quad \mu = t \ln 2 + \frac{w^2 + 2t}{4\pi}
\]
are chosen such that \( F(t, x) \) is continuously differentiable in \( x \) for a.e. \( t \in [0, T] \).

We shall prove more general results than Theorems 1 and 2.

**THEOREM 3.** Suppose that \( F = F_1 + F_2 \), \( F_1 \), and \( F_2 \) satisfy assumption (A), and there exist a function \( \beta \in L^1(0, T) \) and a subset \( E \) of \( [0, T] \) with \( \text{meas}(E) > 0 \) such that
\[
F_1(x, t) \geq \beta(t) \tag{5}
\]
for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \), and
\[
F_1(t, x) \to +\infty \quad \text{as} \quad |x| \to \infty \tag{6}
\]
for a.e. \( t \in E \). Assume that there exist \( g \in L^1(0, T; \mathbb{R}^+) \) and \( C_0 \in \mathbb{R} \) such that
\[
|\nabla F_2(t, x)| \leq g(t) \tag{7}
\]
for all \( x \in \mathbb{R}^N \) and a.e. \( t \in [0, T] \), and
\[
\int_0^T F_2(t, x) \, dt \geq C_0 \tag{8}
\]
for all \( x \in \mathbb{R}^N \). Then problem (1) has at least one solution which minimizes \( \varphi \) on \( H^1_0 \).

**Remark 3.** Theorem 3 generalizes a result in [8]. There are functions \( F(t, x) \) satisfying our Theorem 1 and not satisfying the theorem and its corollaries in [8] and others in [1–7]. For example, let
\[
F(t, x) = t \left[ 2 + \sin(2\pi|x|^2) \right] \ln(1 + |x|^2) + (x, e(t)),
\]
where \( e \in L^1(0, T; \mathbb{R}^N) \) satisfies \( \int_0^T e(t) \, dt = 0 \). The reason is similar to that in Remark 1.
Theorem 4. Assume that (4) and the condition of Theorem 3 hold. Then problem (1) has at least three distinct solutions in $H^1_\gamma$.

2. PROOF OF THEOREMS

We first give analogies of Egorov’s Theorem: Lemmas 1 and 2. Lemma 1 treats the sequence case and Lemma 2 does the continuous variant case. They all deal with tending to $+\infty$.

Lemma 1. Suppose that $E$ is a Lebesgue measurable subset of $\mathbb{R}^1$ with $\text{meas} E < \infty$ and $f_n(t)$ is a sequence of Lebesgue measurable functions such that $f_n(t) \to +\infty$ as $n \to \infty$ for a.e. $t \in E$. Then there exists, for every $\delta > 0$, a subset $E_\delta$ with $\text{meas}(E \setminus E_\delta) < \delta$ such that $f_n(t) \to +\infty$ as $n \to \infty$ uniformly for all $t \in E_\delta$.

Proof. Without loss of generality, we may assume that $f_n(t) \to +\infty$ as $n \to \infty$ for all $t \in E$.

For every $M > 0$ and every positive integer $n$, define

$$E[n, M] = \bigcap_{k=n+1}^{\infty} \{ t \in E | f_n(t) > M \}.$$

Then $E[n, M]$ is measurable and

$$E[n, M] \subset E[k, M] \quad \text{if } n < k.$$

Hence we have

$$E = \bigcup_{n=1}^{\infty} E[n, M]$$

because that $f_n(t) \to +\infty$ as $n \to \infty$ for all $t \in E$. By the properties of Lebesgue’s measure one has

$$\text{meas } E = \lim_{n \to \infty} \text{meas } E[n, M],$$

which implies that

$$\lim_{n \to \infty} \text{meas}(E \setminus E[n, M]) = 0.$$ 

Hence for every $i$ there exists $n_i$ such that

$$\text{meas}(E \setminus E[n, M]) < \frac{\delta}{2^i}.$$
Set

\[ E_\delta = \bigcap_{i=1}^{\infty} E[n_i, i]. \]

Then one has

\[
\text{meas}(E \setminus E_\delta) = \text{meas}\left(\bigcap_{i=1}^{\infty} E[n_i, i]\right) \\
= \text{meas} \bigcup_{i=1}^{\infty} (E \setminus E[n_i, i]) \\
\leq \sum_{i=1}^{\infty} \text{meas}(E \setminus E[n_i, i]) \\
< \sum_{i=1}^{\infty} \frac{\delta}{2^i} = \delta.
\]

Furthermore, \( f_n(t) \to +\infty \) as \( n \to \infty \) uniformly for all \( t \in E_\delta \). In fact, for every \( M > 0 \), choose \( i_0 \geq M \). Then we have \( E_\delta \subset E[n_{i_0}, i_0] \), which implies that

\[ f_n(t) \geq i_0 \geq M \]

for all \( n \geq n_{i_0} \) and all \( t \in E_\delta \).

**Lemma 2.** Suppose that \( F \) satisfies assumption (A) and \( E \) is a measurable subset of \([0, T]\). Assume that

\[ F(x, t) \to +\infty \quad \text{as } |x| \to \infty \]

for a.e. \( t \in E \). Then for every \( \delta > 0 \) there exists subset \( E_\delta \) of \( E \) with \( \text{meas}(E \setminus E_\delta) < \delta \) such that

\[ F(x, t) \to +\infty \quad \text{as } |x| \to \infty \]

uniformly for all \( t \in E_\delta \).

**Proof.** Set

\[ f_n(t) = \inf_{|x| \geq n} F(x, t) \]
for all $n$ and a.e. $t \in E$. By the continuity of $F(x, t)$ in $x$ for a.e. $t \in E$ one has
\[
    f_n(t) = \inf \left\{ F(x, t) \left| \begin{array}{c}
    |x| \geq n, x = (\xi_1, \xi_2, \ldots, \xi_N), \\
    \xi_i (i = 1, 2, \ldots, N) \text{ is rational number}
\end{array} \right. \right.
\]
for all $n$ and a.e. $t \in E$, which implies that $f_n(t)$ is measurable for all $n$.

Now the fact
\[
f_n(t) \to +\infty \quad \text{as } n \to \infty
\]
for a.e. $t \in E$ follows from the same property of $F(x, t)$. By Lemma 1 there exists, for every $\delta > 0$, a subset $E_\delta$ with $\text{meas}(E \setminus E_\delta) < \delta$ such that $f_n(t) \to +\infty$ as $n \to \infty$ uniformly for all $t \in E_\delta$, which implies the desired property of $F(x, t)$.

Next, we give a relation between the uniform coercivity and the subadditivity.

**Lemma 3.** Suppose that $F$ satisfies assumption (A) and $E$ is a measurable subset of $[0, T]$. Assume that
\[
    F(x, t) \to +\infty \quad \text{as } |x| \to \infty
\]
uniformly for all $t \in E$. Then there exist a real function $\gamma \in L^1(E)$, and $G \in C(R^N, R)$ which is subadditive, that is,
\[
    G(x + y) \leq G(x) + G(y)
\]
for all $x, y \in R^N$, and coercive, that is,
\[
    G(x) \to +\infty
\]
as $|x| \to \infty$, and satisfies that
\[
    G(x) \leq |x| + 4
\]
for all $x \in R^N$, such that
\[
    F(x, t) \geq G(x) + \gamma(t)
\]
for all $x \in R^N$ and a.e. $t \in E$.

**Proof.** Since $F(x, t) \to +\infty$ as $|x| \to \infty$ uniformly for all $t \in E$, there exists a sequence of positive integers $(n_k)$ with $n_{k+1} > 2n_k$ for all positive integers $k$ such that
\[
    F(x, t) \geq k
\]
for all $|x| \geq n_k$ and all $t \in E$. Let $n_0 = 0$ and define
\begin{equation}
G(x) = k + 2 + \frac{|x| - n_{k-1}}{n_k - n_{k-1}} \tag{14}
\end{equation}
for $n_{k-1} \leq |x| < n_k$, where $k \in N$.

By the definition of $G$ we have
\begin{equation}
k + 2 \leq G(x) \leq k + 3 \tag{15}
\end{equation}
for $n_{k-1} \leq |x| < n_k$. It follows that
\[F(x, t) \geq G(x) + \gamma(t)\]
for all $t \in \mathbb{R}^N$ and a.e. $t \in E$, where
\[\gamma(t) = -\left(\max_{0 \leq s \leq n_1} a(s)\right)b(t) - 4.\]

In fact, when $n_{k-1} \leq |x| < n_k$ for some $k \geq 2$, one has, by (13) and (15),
\[F(x, t) \geq k - 1 \geq G(x) - 4 \geq G(x) + \gamma(t)\]
for a.e. $t \in E$. When $|x| < n_1$, we have, by assumption (A) and (15),
\[F(x, t) \geq -\left(\max_{0 \leq s \leq n_1} a(s)\right)b(t) = 4 + \gamma(t) \geq G(x) + \gamma(t)\]
for a.e. $t \in E$.

It is obvious that $G$ is continuous and coercive. Moreover one has
\[G(x) \leq |x| + 4\]
for all $x \in \mathbb{R}^N$. In fact, for every $x \in \mathbb{R}^N$ there exists $k \in N$ such that
\[n_{k-1} \leq |x| < n_k\]
which implies that
\[G(x) \leq (k - 1) + 4 \leq n_{k-1} + 4 \leq |x| + 4\]
for all $x \in \mathbb{R}^N$ by (15) and the fact that $n_k \geq k$ for all integers $k \geq 0$.

Now we only need to prove the subadditivity of $G$. Let
\[n_{k-1} \leq |x| < n_k, \quad n_{j-1} \leq |y| < n_j\]
and $m = \max(k, j)$. Then we have
\[|x + y| \leq |x| + |y| < n_k + n_j \leq 2n_m < n_{m+1}\]
Hence we obtain, by (15),
\[ G(x + y) \leq m + 4 \leq k + 2 + j + 2 \leq G(x) + G(y), \]
which shows that \( G \) is subadditive.

At last, we prove our main results.

**Proof of Theorem 3.** By Lemma 2, for \( \delta = (1/2)\text{meas}(E) > 0 \), there exists subset \( E_{\delta} \) of \( E \) with \( \text{meas}(E \setminus E_{\delta}) < \delta \) such that
\[ F(x, t) \to +\infty \quad \text{as } |x| \to \infty \]
uniformly for all \( t \in E_{\delta} \), which implies that
\[ \text{meas } E_{\delta} = \text{meas } E - \text{meas}(E \setminus E_{\delta}) = \delta > 0. \]
Set \( \widetilde{u} = (1/T) \int_0^T u(t) \, dt \) and \( \tilde{u}(t) = u(t) - \widetilde{u} \). Then one has
\[ \|\tilde{u}\|_* \leq \left(\frac{T}{12}\right)^{\frac{1}{2}} \|\tilde{u}\|_{L^2}; \]
for all \( u \in H^1 \) (Sobolev's inequality). From (2), (12), and (9), one obtains
\[
\int_0^T F_1(t, u(t)) \, dt \\
\geq \int_{E_{\delta}} F_1(t, u(t)) \, dt + \int_{[0, T] \setminus E_{\delta}} B(t) \, dt \\
\geq \int_{E_{\delta}} G(u(t)) \, dt + \int_{E_{\delta}} \gamma(t) \, dt + \int_{[0, T] \setminus E_{\delta}} B(t) \, dt \\
\geq \int_{E_{\delta}} (G(\widetilde{u}) - G(-\tilde{u}(t))) \, dt + \int_{E_{\delta}} \gamma(t) \, dt + \int_{[0, T] \setminus E_{\delta}} B(t) \, dt \\
\geq G(\widetilde{u}) \text{meas } E_{\delta} - (\|\tilde{u}\|_* + 4) \text{meas } E_{\delta} \\
+ \int_{E_{\delta}} \gamma(t) \, dt + \int_{[0, T] \setminus E_{\delta}} B(t) \, dt
\]
for all \( u \in H^1 \). It follows from (7) that
\[
\int_0^T F_2(t, u(t)) \, dt = \int_0^T F_2(t, \widetilde{u}) \, dt + \int_0^T \int_0^1 (\nabla F_2(t, \widetilde{u} + s\tilde{u}(t)), \tilde{u}(t)) \, ds \, dt \\
\geq \int_0^T F_2(t, \widetilde{u}) \, dt - \|\tilde{u}\|_* \int_0^T g(t) \, dt
\]
for all \( u \in H^1_t \). Thus by Sobolev’s inequality we have

\[
\varphi(u) \geq \frac{1}{2} \int_0^T \dot{u}(t)^2 \, dt + G(\bar{u}) \text{meas } E_\delta - (\|\bar{u}\|_u + 4) \text{meas } E_\delta \\
+ \int_{E_\delta} \gamma(t) \, dt + \int_{[0,T] \setminus E_\delta} \beta(t) \, dt + \int_0^T F_2(t, \bar{u}) \, dt \\
- \|\bar{u}\| \int_0^T g(t) \, dt - \int_0^T F(t, 0) \, dt \\
\geq G(\bar{u}) \text{meas } E_\delta + C_0 + \frac{1}{2} \int_0^T \dot{u}(t)^2 \, dt \\
- \left( \text{meas } E_\delta + \int_0^T g(t) \, dt \right) \left( \frac{T}{12} \right)^{\frac{1}{2}} \left( \int_0^T \dot{u}(t)^2 \, dt \right)^{\frac{1}{2}} \\
+ \int_{E_\delta} \gamma(t) \, dt + \int_{[0,T] \setminus E_\delta} \beta(t) \, dt - \int_0^T F(t, 0) \, dt - 4 \text{meas } E_\delta
\]

for all \( u \in H^1_t \). As \( \|u\| \to \infty \) if and only if \( (\|\bar{u}\| + \int_0^T \dot{u}(t)^2 \, dt)^{\frac{1}{2}} \to \infty \), it follows from (10) that \( \varphi \) is coercive. By Theorem 1.1 and Corollary 1.1 in [1] we complete our proof.

Now we prove Theorem 4. For convenience to quote we state a three-critical-point theorem proposed by Brezis and Nirenberg (see Theorem 4 in [9]).

**Lemma 4** [9]. Let \( X \) be a Banach space with a direct sum decomposition

\[
X = X_1 \oplus X_2
\]

with \( \dim X_2 < \infty \) and let \( \varphi \) be a \( C^1 \) function on \( X \) with \( \varphi(0) = 0 \), satisfying the (PS) condition. Assume that for some \( \delta_0 > 0 \)

\[
\varphi(v) \geq 0 \quad \text{for } v \in X_1 \text{ with } \|v\| \leq \delta_0
\]

and

\[
\varphi(v) \leq 0 \quad \text{for } v \in X_2 \text{ with } \|v\| \leq \delta_0.
\]

Assume also that \( \varphi \) is bounded from below and \( \inf_X \varphi < 0 \). Then \( \varphi \) has at least two nonzero critical points.

**Proof of Theorem 4.** From the proof of Theorem 3 we know that \( \varphi \) is coercive, which implies that \( \varphi \) satisfies the (PS) condition. Let \( X_2 \) be a finite-dimensional subspace of \( X = H^1_t \) given by

\[
X_2 = \left\{ \sum_{j=0}^k \left( a_j \cos jwt + b_j \sin jwt \right) \mid a_j, b_j \in \mathbb{R}^N, j = 0, \ldots, k \right\}
\]
and let $X_1 = X_2^+$. Then from (4) we obtain

$$\varphi(u) \leq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt - \frac{1}{2} k^2 w^2 \int_0^T |u(t)|^2 \, dt \leq 0$$

(16)

for all $u \in X_2$ with $\|u\| \leq C^{-1}r$, and

$$\varphi(u) \geq \frac{1}{2} \int_0^T |\dot{u}(t)|^2 \, dt - \frac{1}{2} (k + 1)^2 w^2 \int_0^T |u(t)|^2 \, dt \geq 0$$

for all $u \in X_1$ with $\|u\| \leq C^{-1}r$, where $C$ is a positive constant such that $\|u\|_\infty \leq C\|u\|$ for all $u \in H^1_T$. The existence of the constant $C$ follows from the inequality

$$\|u\|_\infty \leq |\bar{u}| + \|\bar{u}\|_\infty \leq \int_0^T |u(t)| \, dt + \|\bar{u}\|_\infty$$

$$\leq T^\frac{1}{2}\|u\|_{L^2} + \left( \frac{T}{12} \right)^{\frac{1}{2}} \|\dot{u}\|_{L^2} \leq 2T^{\frac{1}{2}}\|u\|$$

for all $u \in H^1_T$, where we have used the H"older inequality and the Sobolev inequality.

In the case that $\inf_X \varphi < 0$, Theorem 4 follows from Lemma 4.

In the case that $\inf_X \varphi \geq 0$, by (16) we have

$$\varphi(v) = \inf_X \varphi = 0$$

for all $v \in X_2$ with $\|v\| \leq C^{-1}r$, which implies that all $v \in X_2$ with $\|v\| \leq C^{-1}r$ are minimum points of $\varphi$. Hence all $v \in X_2$ with $\|v\| \leq C^{-1}r$ are solutions of problem (1), and problem (1) has infinite solutions in $H^1_T$. Therefore Theorem 4 is proved.

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