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A slight improvement to Korenblum's constant

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Abstract

Let $A^2(D)$ be the Bergman space over the open unit disk D in the complex plane. Korenblum conjectured that there is an absolute constant $c \in (0, 1)$ such that whenever $|f(z)| \leqslant |g(z)|$ in the annulus $c < |z| < 1$, then $\|f(z)\| \leqslant \|g(z)\|$. This conjecture had been solved by Hayman [W.K. Hayman, On a conjecture of Korenblum, Analysis (Munich) 19 (1999) 195–205. [1]], but the constant c in that paper is not optimal. Since then, there are many papers dealing with improving the upper and lower bounds for the best constant c. For example, in 2004 C. Wang gave an upper bound on c, that is, $c < 0.67795$, and in 2006 A. Schuster gave a lower bound, $c > 0.21$. In this paper we slightly improve the upper bound for c.

Keywords: Bergman space; Korenblum's constant

1. Introduction

Let D be the open unit disk in the complex plane C. The Bergman space $A^2(D)$ consists of analytic functions f in D such that

$$\|f\| = \left[\int_D |f(z)|^2 \, dA(z)\right]^{1/2} < \infty,$$

where

$$dA(z) = \frac{1}{\pi}\,dx\,dy = \frac{1}{\pi} r\,dr\,d\theta, \quad z = x + iy = re^{i\theta},$$

is the normalized Lebesgue area measure on D. Korenblum conjectured that there is an absolute constant c, $0 < c < 1$, such that whenever $|f(z)| \leqslant |g(z)|$ in the annulus $c < |z| < 1$, then $\|f(z)\| \leqslant \|g(z)\|$. This conjecture is very natural and inspired many work. The answer to this conjecture was obtained by Hayman. He proved that the constant c exists and is greater than 0.04. But we can ask whether this bound is optimal, and many papers have been done to find the better upper and lower bounds for the constant c (see [2,3] and [4]). The best upper bound for c until now is 0.67795... in [4], and the best lower bound for this constant is 0.21 in [3]. In this paper we use Wang's example in a slightly more sophisticated way to improve the upper bound to 0.677905.

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2. Results

The following theorem shows that $c < 0.677905$.

Theorem 2.1. *Let*

$$f(z) = \frac{a + z^n}{2 - az^n}, \qquad g(z) = \frac{z(1 + az^n)}{2 - az^n},$$

where $a = 0.6666714$ and $n = 10$. Then $\|f(z)\| > \|g(z)\|$ and $|f(z)| \leqslant |g(z)|$ in $c < |z| < 1$, where $c = 0.6779049274\ldots$ is the real root of the equation $f(z) = g(z)$.

Proof. Let

$$h(r) = \max_{|z|=r}\left|\frac{f(z)}{g(z)}\right| = \frac{a + r^n}{r(1 + ar^n)}.$$

Then, $h(c) = h(1) = 1$. Since $f(z)/g(z)$ is analytic in $c \leqslant |z| \leqslant 1$, the maximum modulus theorem implies that $|f(z)| \leqslant |g(z)|$ in $c < |z| < 1$. Using Maple to solve the equation, we obtain that $c = 0.6779049274\ldots$. Next, a direct calculation shows that

$$\int_D \left|f(z)\right|^2 dA(z) - \int_D \left|g(z)\right|^2 dA(z)$$

$$= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \frac{a^2 + 2ar^n\cos n\theta + r^{2n}}{4 - 4ar^n\cos n\theta + a^2r^{2n}} r\,dr\,d\theta - \frac{1}{\pi}\int_0^{2\pi}\int_0^1 \frac{1 + 2ar^n\cos n\theta + a^2r^{2n}}{4 - 4ar^n\cos n\theta + a^2r^{2n}} r^3\,dr\,d\theta.$$

Let $r^n = \rho$ and $n\theta = \phi$, then

$$= \frac{1}{n\pi}\int_0^{2\pi}\int_0^1 \frac{a^2 + 2a\rho\cos\phi + \rho^2}{4 - 4a\rho\cos\phi + a^2\rho^2}\rho^{2/n-1}\,d\rho\,d\phi - \frac{1}{n\pi}\int_0^{2\pi}\int_0^1 \frac{1 + 2a\rho\cos\phi + a^2\rho^2}{4 - 4a\rho\cos\phi + a^2\rho^2}\rho^{4/n-1}\,d\rho\,d\phi.$$

Using Maple, we obtain that when $a = 0.6666714$, $n = 10$ and $c = 0.6779049274\ldots$, then

$$\int_D \left|f(z)\right|^2 dA(z) - \int_D \left|g(z)\right|^2 dA(z) \geqslant 0.22 \times 10^{-6}.$$

Hence,

$$\left\|f(z)\right\| > \left\|g(z)\right\|. \qquad \square$$

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