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A periodic predator–prey-chain system with impulsive perturbation

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Abstract

A periodic predator–prey-chain system with impulsive effects is considered. By using the global results of Rabinowitz and standard techniques of bifurcation theory, the existence of its trivial, semi-trivial and nontrivial positive periodic solutions is obtained. It is shown that the nontrivial positive periodic solution for such a system may be bifurcated from an unstable semi-trivial periodic solution. Furthermore, the stability of these periodic solutions is studied.

Keywords: Predator–prey chain; Bifurcation; Impulsive perturbation; Positive periodic solution; Noncritical solution

1. Introduction

Predator–prey phenomena occur commonly in ecological systems, and they are always interesting topics of population dynamics. For autonomous predator–prey systems, i.e. all coefficients being constants, we usually pay much attention to the existence and stability of their equilibria, especially positive equilibria; but we investigate the existence and stability of periodic solutions for nonautonomous systems [1–3], whose coefficients are time dependent. When the seasonal effects, food supply, mating habits, etc., are considered, the nonautonomous systems are necessary.

The exploitation of the species and ecosystem interact, and both are affected by environment changes and human activities. In fact, the population is harvested from or stocked into the system in regular pulses. And many species are immigrated at fixed moments every year. Consequently, the population levels usually experience short-time abrupt changes at fixed moments after undergoing relatively long periods of smooth variation. Naturally, we consider combining nonautonomous differential dynamical systems and impulsive effects at fixed moments, and construct nonautonomous impulsive differential systems. The fundamental theory for these systems is described in Bainov and Simeonov's book [7], but there are some difficulties in applying it to the dynamical behaviors of population directly, such as the existence and stability of periodic solutions, persistence and extinction, and so on. Recently, some progress has been made for Lotka–Volterra systems. Two-species predator–prey periodic systems with impulsive effects at fixed moments were studied in [4–6]. To the author's knowledge, three-species periodic Lotka–Volterra impulsive systems have not been discussed in any generality, so we will consider the following T-periodic predator–prey-chain system with impulsive effects:

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$$\left.\begin{cases} \dot{N}_1(t) = N_1(b_1(t) - a_{11}(t)N_1(t) - a_{12}(t)N_2(t)) \\ \dot{N}_2(t) = N_2(-b_2(t) + a_{21}(t)N_1(t) - a_{22}(t)N_2(t) - a_{23}(t)N_3(t)) \\ \dot{N}_3(t) = N_3(-b_3(t) + a_{32}(t)N_2(t) - a_{33}(t)N_3(t)) \end{cases}\right\} \quad t \neq t_k, \\ \left.\begin{cases} N_1(t_k^+) = (1+h_k)N_1(t_k) \\ N_2(t_k^+) = (1+g_k)N_2(t_k) \\ N_3(t_k^+) = (1+f_k)N_3(t_k) \end{cases}\right\}, \tag{1.1}$$

where $b_i(t), a_{i2}(t)$ $(i = 1, 2, 3)$, $a_{i1}(t)$ $(i = 1, 2)$, $a_{i3}(t)$ $(i = 2, 3)$ are positive continuous T-periodic functions, and h_k, g_k, f_k $(k \in N)$ are constants satisfying $1 + h_k > 0$, $1 + g_k > 0$, $1 + f_k > 0$ for biological reasons. Moreover, we assume that there exists an integer $q > 0$ such that $h_{k+q} = h_k$, $g_{k+q} = g_k$, $f_{k+q} = f_k$, $t_{k+q} = t_k + T$ for all $k \in N$.

With system (1.1), we can take into account the potential exterior effects under which the population densities change rapidly. The existence and stability of trivial periodic solutions, including semi-trivial periodic solutions, are given by some lemmas and Floquet theory in the next section. In Section 3, nontrivial positive periodic solutions are obtained by bifurcation techniques.

For convenience, we define

$$\begin{aligned} PC_T &= \{x(t) \mid x(t) = x(t+T) \text{ for } \forall t \in R^+; \lim_{s\to t} x(s) = x(t) \text{ for } t \neq t_k \ (k \in N); \\ &\quad \lim_{t\to t_k^-} x(t) = x(t_k) \ (k \in N) \text{ and } \lim_{t \to t_k^+} x(t) \text{ exists}\}, \\ PC_T^n &= \{X(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \mid x_i(t) \in PC_T, i = 1, 2, \ldots n, n \in N\}. \end{aligned}$$

Under the supremum norm $\|X(t)\| = \sum_{i=1}^n \sup_{0\le t\le T} |x_i(t)|$, PC_T^n is a Banach space. At the same time, we define $X(t) = (x_1(t), \ldots, x_n(t)) > 0$, if $x_i(t) > 0$ $(i = 1, 2, \ldots, n)$, and $[f] = \frac{1}{T}\int_0^T f(x)\mathrm{d}x$, for $f(x) \in PC_T$.

It is obvious that $R_+^3 = \{(N_1, N_2, N_3) \mid N_i \geq 0 \ (i = 1, 2, 3)\}$ is a positive invariant set of system (1.1); in particular, if there exists some $i \in \{1, 2, 3\}$ such that $N_i(t_0) = 0$, then $N_i(t) \equiv 0$ for all $t \geq t_0$.

2. Trivial periodic solutions

First, we introduce a useful conclusion about the one-species Lotka–Volterra impulsive system.

Lemma 2.1 ([7]). *Consider the equation*

$$\begin{cases} \dot{x}(t) = r(t)\left(1 - \dfrac{x}{K(t)}\right)x & t \neq t_k, \\ \Delta x(t) = c_k x(t) & t = t_k, \end{cases} \tag{2.1}$$

where there exist $T > 0$ and $q \in N$ such that $r(t+T) = r(t)$, $K(t+T) = K(t)$ $(t \in R)$, $t_{k+q} = t_k + T$, $c_{k+q} = c_k$ $(k \in N)$ and $\inf_{t\in[0,T]} K(t) > 0$, and $r(t) > 0$, $1 + c_k > 0$ $(k \in N)$. If

$$\mu = \prod_{k=1}^q \frac{1}{1+c_k} \exp\left(-\int_0^T r(s)\mathrm{d}s\right) < 1,$$

i.e.

$$[r] > \frac{1}{T} \ln \prod_{k=1}^q \frac{1}{1+c_k}$$

holds, system (2.1) has a unique positive T-periodic solution.

Second, for the linear T-periodic impulsive equation

$$\begin{cases} \dot{X}(t) = A(t)X(t) & t \neq t_k, \\ \Delta X(t) = B_k X(t) & t = t_k, \end{cases} \tag{2.2}$$

where $A(t) \in PC(R, C^{n\times n})$, $A(t+T) = A(t)$, $B_k \in C^{n\times n}$, $\det(E + B_k) \neq 0$, and $t_k < t_{k+1}$ $(k \in N)$, and there exists $q \in N$ such that $B_{k+q} = B_k$, $t_{k+q} = t_k + T$, we have:

Lemma 2.2 ([7]). *The linear T-periodic impulsive equation (2.2) is asymptotically stable if and only if all its multipliers* μ_j $(j = 1, 2, \ldots, n)$ *satisfy* $\mu_j < 1$.

In order to discuss the local stability of the periodic solution $N^*(t) = (N_1^*(t), N_2^*(t), N_3^*(t))$ of (1.1), we set $N_i(t) = N_i^*(t) + x_i(t)$ $(i = 1, 2, 3)$. Then a linearized approximate system with respect to $(x_1(t), x_2(t), x_3(t))$ reads

$$\begin{cases} \left.\begin{aligned} &\dot{x}_1(t) = (b_1 - 2a_{11}N_1^* - a_{12}N_2^*)x_1 - a_{12}N_1^* x_2 \\ &\dot{x}_2(t) = a_{21}N_2^* x_1 + (-b_2 + a_{21}N_1^* - 2a_{22}N_2^* - a_{23}N_3^*)x_2 - a_{23}N_2^* x_3 \\ &\dot{x}_3(t) = a_{32}N_3^* x_2 + (-b_3 + a_{32}N_2^* - 2a_{33}N_3^*)x_3 \end{aligned}\right\} \quad t \neq t_k, \\ \left.\begin{aligned} &x_1(t_k^+) = (1 + h_k)x_1(t_k) \\ &x_2(t_k^+) = (1 + g_k)x_2(t_k) \\ &x_3(t_k^+) = (1 + f_k)x_3(t_k) \end{aligned}\right\}, \end{cases} \tag{2.3}$$

where $a_{ij} = a_{ij}(t)$, $b_i = b_i(t)$, and $N_i^* = N_i^*(t)$ $(i, j = 1, 2, 3)$, which are also valid in the following. We denote the monodromy matrix of (2.3) by M_{N^*}. On the basis of Lemmas 2.1 and 2.2, for the trivial periodic solutions of (1.1), we have

Theorem 2.1. *If the conditions*

$$[b_1] < \frac{1}{T} \ln \prod_{k=1}^{q} \frac{1}{1 + h_k}, \tag{2.4}$$

$$[b_2] > \frac{1}{T} \ln \prod_{k=1}^{q} (1 + g_k), \tag{2.5}$$

$$[b_3] > \frac{1}{T} \ln \prod_{k=1}^{q} (1 + f_k), \tag{2.6}$$

hold, then $N^0 = (0, 0, 0)$ *is a stable trivial periodic solution of (1.1), but if one of the above three inequalities is reversed, then* N^0 *is unstable.*

Theorem 2.2. *If* $[b_1] > \frac{1}{T} \ln \prod_{k=1}^{q} \frac{1}{1+h_k}$, *then there exists a semi-trivial periodic solution* $N^* = (N_1^*, 0, 0)$ *of (1.1), where* N_1^* *is a unique positive T-periodic solution of the one-species system*

$$\begin{cases} \dot{N}_1(t) = N_1(b_1(t) - a_{11}(t))N_1(t) & t \neq t_k, \\ N_1(t_k^+) = (1 + h_k)N_1(t_k). \end{cases}$$

And if

$$[b_2] > [a_{21}N_1^*] + \frac{1}{T} \ln \prod_{k=1}^{q} (1 + g_k) \tag{2.7}$$

and (2.6) are true, then N^* *is stable; if (2.6) or (2.7) is reversed,* N^* *is unstable.*

In fact, it is easily known that

$$M_{N^0} = \begin{bmatrix} \prod_{k=1}^{q} (1 + h_k) e^{T[b_1]} & 0 & 0 \\ 0 & \prod_{k=1}^{q} (1 + g_k) e^{-T[b_2]} & 0 \\ 0 & 0 & \prod_{k=1}^{q} (1 + f_k) e^{-T[b_3]} \end{bmatrix},$$

$$M_{N^*} = \begin{bmatrix} \mathrm{e}^{-T[a_{11}N_1^*]} & \# & 0 \\ 0 & \prod_{k=1}^{q}(1+g_k)\mathrm{e}^{T[-b_2+a_{21}N_1^*]} & 0 \\ 0 & 0 & \prod_{k=1}^{q}(1+f_k)\mathrm{e}^{-T[b_3]} \end{bmatrix},$$

where # does not need to be calculated. According to Lemma 2.2, we have the above two conclusions.

Theorem 2.3. *If* $[b_1] > \frac{1}{T}\ln\prod_{k=1}^{q}\frac{1}{1+h_k}$ *and* $[b_2] < [a_{21}N_1^*] + \frac{1}{T}\ln\prod_{k=1}^{q}(1+g_k)$ *hold, then there exists a small nonnegative constant* b_0 *such that for each* $b_2(t) \in C(R^+, R)$ *with* $b_2(t+T) = b_2(t)$, *if* $0 < [a_{21}N_1^*] + \frac{1}{T}\ln\prod_{k=1}^{q}(1+g_k) - b_0 \le [b_2] \le [a_{21}N_1^*] + \frac{1}{T}\ln\prod_{k=1}^{q}(1+g_k)$, *system (1.1) has another semi-trivial T-periodic solution* $N^{**} = (N_1^{**}, N_2^{**}, 0)$, *where* (N_1^{**}, N_2^{**}) *is the positive T-periodic solution of the two-species predator–prey system*

$$\begin{cases} \left.\begin{aligned}\dot N_1(t) = N_1(b_1(t) - a_{11}(t)N_1(t) - a_{12}(t)N_2(t)) \\ \dot N_2(t) = N_2(-b_2(t) + a_{21}(t)N_1(t) - a_{22}(t)N_2(t))\end{aligned}\right\} \quad t \ne t_k, \\ \left.\begin{aligned}N_1(t_k^+) = (1+h_k)N_1(t_k) \\ N_2(t_k^+) = (1+g_k)N_2(t_k)\end{aligned}\right\}. \end{cases} \tag{2.8}$$

Moreover, if $[b_3] < [a_{32}N_2^{**}] + \frac{1}{T}\ln\prod_{k=1}^{q}(1+f_k)$, *then* N^{**} *is unstable.*

For the existence of positive T-periodic solution (N_1^{**}, N_2^{**}) of (2.8), see [4]. With the stability of N^{**}, it is easily found that $\prod_{k=1}^{q}(1+f_k)\mathrm{e}^{T[-b_3+a_{32}N_2^{**}]}$ is a multiplier of monodromy matrix $M_{N^{**}}$.

3. Positive periodic solutions

Lemma 3.1 ([7]). *For the linear homogeneous T-periodic impulsive system (2.2), i.e.* $A(t)$ *and* B_k *satisfy all periodic conditions of Lemma 2.2, if* $\det(E - M) \ne 0$, *where E is the unit* $n \times n$ *matrix and M is the monodromy matrix of (2.2), then:*

(1) *System (2.2) has no T-periodic solution other than* $X \equiv 0$. *In this scenario, (2.2) is called noncritical.*

(2) *The linear nonhomogeneous T-periodic system*

$$\begin{cases} \dot X(t) = A(t)X(t) + g(t) & t \ne t_k, \\ \Delta X(t) = B_k X(t) + l_k & t = t_k, \end{cases} \tag{3.1}$$

where $g(t) \in PC(R, C^n)$, $l_k \in C^n$, $g(t+T) = g(t)$, *and* $l_{k+q} = l_k$, *has a unique T-periodic solution*

$$\tilde X(t) = \int_0^T G(t,s)g(s)\mathrm{d}s + \sum_{0 \le t_k < T} G(t, t_k^+)l_k,$$

and $G(t,s)$ *is a Green's function which is controlled by the linear system (2.2) corresponding to (3.1).* $\tilde X(t)$ *is called a noncritical periodic solution of system (3.1).*

In order to discuss the existence of a positive periodic solution of (1.1), we assume that (2.8) satisfies the following hypothesis.

$\mathcal{H}$: System (2.8) has a positive noncritical T-periodic solution $(N_1^{**}(t), N_2^{**}(t))$, i.e. all multipliers of the linearized system of (2.8) at $(N_1^{**}(t), N_2^{**}(t))$ are distinct from 1.

Theorem 3.1. *If* $\mathcal{H}$ *holds,* $\lambda^* = \frac{1}{T}\ln\prod_{k=1}^{q}(1+f_k) + [a_{32}N_2^{**}] \ne 0$, *and* $b_0(t)$ *is a given T-periodic function with* $[b_0] = 0$, *then there exists a continuum* $F = \{(N_1(t), N_2(t), N_3(t), \lambda) \in PC_T^3 \times R\}$ *with the following properties:*

(1) $(N_1, N_2, N_3, \lambda) \in F$ *implies that* $(N_1, N_2, N_3) \in PC_T^3$ *is a solution of (1.1) with* $b_3(t) = b_0(t) + \lambda$.

(2) $(N_1^{**}, N_2^{**}, 0, \lambda^*) \in F$.

Proof. Let $x_1 = N_1 - N_1^{**}$, $x_2 = N_2 - N_2^{**}$, $x_3 = N_3$ in (1.1); then

$$\begin{cases} \left.\begin{array}{l} \dot{x}_1 = (b_1 - 2a_{11}N_1^{**} - a_{12}N_2^{**})x_1 - a_{12}N_1^{**}x_2 + r_1(x_1, x_2, x_3) \\ \dot{x}_2 = a_{21}N_2^{**}x_1 + (-b_2 + a_{21}N_1^{**} - 2a_{22}N_2^{**})x_2 - a_{23}N_2^{**}x_3 + r_2(x_1, x_2, x_3) \\ \dot{x}_3 = (-b_3 + a_{32}N_2^{**})x_3 + r_3(x_1, x_2, x_3) \end{array}\right\} \quad t \neq t_k, \\ \left.\begin{array}{l} x_1(t_k^+) = (1 + h_k)x_1(t_k) \\ x_2(t_k^+) = (1 + g_k)x_2(t_k) \\ x_3(t_k^+) = (1 + f_k)x_3(t_k) \end{array}\right\}, \end{cases} \tag{3.2}$$

where $r_1(x_1, x_2, x_3) = -a_{11}x_1^2 - a_{12}x_1x_2$, $r_2(x_1, x_2, x_3) = a_{21}x_1x_2 - a_{22}x_2^2 - a_{23}x_2x_3$, and $r_3(x_1, x_2, x_3) = a_{32}x_2x_3 - a_{33}x_3^2$.

Define $b_3(t) = b_0(t) + \lambda$; then $\lambda = [b_3]$. At the same time, we set

$$A = \begin{bmatrix} b_1 - 2a_{11}N_1^{**} - a_{12}N_2^{**} & -a_{12}N_1^{**} \\ a_{21}N_2^{**} & -b_2 + a_{21}N_1^{**} - 2a_{22}N_2^{**} \end{bmatrix},$$

$$Z = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 \\ -a_{23}N_2^{**} \end{bmatrix}, \qquad R = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \qquad P_k = \begin{bmatrix} h_k & 0 \\ 0 & g_k \end{bmatrix},$$

and then (3.2) can be rewritten as

$$\begin{cases} \left.\begin{array}{l} \dot{Z} = AZ + x_3C + R \\ \dot{x}_3 = (a_{32}N_2^{**} - b_0(t))x_3 - \lambda x_3 + r_3(Z^\top, x_3) \end{array}\right\} \quad t \neq t_k, \\ Z(t_k^+) = (E + P_k)Z(t_k), \\ x_3(t_k^+) = (1 + f_k)x_3(t_k). \end{cases} \tag{3.3}$$

Here, $Z^\top$ denotes the transposed vector of Z. Let $G_1(t, s)$, $G_2(t, s)$ be Green's functions for the following linear equations:

$$\begin{cases} \dot{x}_3(t) = (a_{32}N_2^{**} - b_0(t))x_3 \quad t \neq t_k, \\ x_3(t_k^+) = (1 + f_k)x_3(t_k), \end{cases} \tag{3.3a}$$

and

$$\begin{cases} \dot{Z}(t) = AZ \quad t \neq t_k, \\ Z(t_k^+) = (E + P_k)Z(t_k), \end{cases} \tag{3.3b}$$

respectively; we know that $G_1(t, s)$ and $G_2(t, s)$ exist since $\prod_{k=1}^{q}(1+f_k)\exp\int_0^T(a_{32}N_2^{**} - b_0)\mathrm{d}t \neq 1$ and (N_1^{**}, N_2^{**}) is a noncritical solution of (2.8). Then these Green's functions may define compact linear operators $L_1 : PC \to PC$ and $L_2 : PC^2 \to PC^2$ by means of the integrals

$$L_1\xi = \int_0^T G_1(t, s)\xi(s)\mathrm{d}s \quad (\xi \in PC), \qquad L_2\eta = \int_0^T G_2(t, s)\eta(s)\mathrm{d}s \quad (\eta \in PC^2),$$

and system (3.3) is equivalent to the pair of operator equations

$$\begin{cases} Z = L_2(x_3C) + H_2(Z^\top, x_3), \\ x_3 = -\lambda L_1x_3 + H_1(Z^\top, x_3), \end{cases} \tag{3.4}$$

in which $H_1 = L_1r_3 : PC^2 \times PC \to PC$ and $H_2 = L_2R : PC^2 \times PC \to PC^2$ are completely continuous operators of order higher than linear near $(Z^\top, x_3) = (0, 0)$. Eq. (3.4) is in turn equivalent to the equations

$$\begin{cases} Z = -\lambda L_2((L_1x_3)C) + H_3(Z^\top, x_3), \\ x_3 = -\lambda L_1x_3 + H_1(Z^\top, x_3), \end{cases} \tag{3.5}$$

where $H_3(Z^\top, x_3) = L_2(H_1(Z^\top, x_3))C + H_2(Z^\top, x_3)$ is also of order higher than linear near $(Z^\top, x_3) = (0, 0)$. In fact, (3.5) may be written in the concise form

$$w = \lambda Lw + H(w), \tag{3.6}$$

in which $L : PC^2 \times PC \to PC^2 \times PC$, $Lw = (-L_2(L_1x_3)C, -L_1x_3)$, and $H : PC^2 \times PC \to PC^2 \times PC$, $H(w) = (H_3(w), H_1(w))$, $w = (Z^\top, x_3) \in PC^2 \times PC$. The operator L is linear and compact and of order higher than linear near $w = (0, 0)$.

It is obvious that $w = 0$ is a trivial solution of (3.6) (corresponding to the solution $N^{**} = (N_1^{**}, N_2^{**}, 0)$ of (1.1)). But we are interested in nonzero solution $w \neq 0$ of (3.6) which can yield a solution $(N_1, N_2, N_3) = (x_1 + N_1^{**}, x_2 + N_2^{**}, x_3)$ of (1.1) for some $[b_3] = \lambda \in R$. In order to use the global bifurcation theorem of Rabinowitz [8], it must be shown that L has a characteristic value $\lambda = \lambda_0$ of odd multiplicity. The linear equation $w = \lambda Lw$ is equivalent to the linearized homogeneous T-periodic system

$$\begin{cases} \left.\begin{array}{l}\dot{Z} = AZ + x_3C \\ \dot{x}_3 = (a_{32}N_2^{**} - b_0(t) - \lambda)x_3\end{array}\right\} \quad t \neq t_k, \\ Z(t_k^+) = (E + P_k)Z(t_k), \\ x_3(t_k^+) = (1 + f_k)x_3(t_k). \end{cases} \tag{3.7}$$

For system (3.7), if $x_3 = 0$, then $w = (Z^\top, x_3) \equiv 0$ because (N_1^{**}, N_2^{**}) is a noncritical periodic solution of (2.8), so (3.7) has a nontrivial periodic solution if and only if $x_3 \neq 0$, which occurs if and only if $\lambda = \frac{1}{T} \ln \prod_{k=1}^{q} (1 + f_k) + [a_{32}N_2^{**}]$, and this shows that L has a unique nonzero characteristic value $\lambda = \lambda^*$.

Finally, we prove that $\lambda = \lambda^*$ is simple. Let $0 \neq w_0 = (Z_0^\top, x_{30}) \in PC^2 \times PC$ be a characteristic solution, i.e. $w_0 = \lambda^* L w_0$. Assume $w = (Z^\top, x_3) \in PC^2 \times PC$ satisfying $(I - \lambda^* L)^2 w = 0$, and let $(I - \lambda^* L)w = w^*$; then $(I - \lambda^* L)w^* = 0$, which implies that $w^* = m w_0$ for some real number m, i.e. $w^* = m\lambda^* L w_0$. Therefore we have $w = \lambda^* L(w + m w_0)$, which shows that $w = (Z^\top, x_3) \in PC^2 \times PC$ is a solution of nonhomogeneous linear system

$$\begin{cases} \left.\begin{array}{l}\dot{Z} = AZ + x_3C + \lambda^* m Z_0 \\ \dot{x}_3 = (a_{32}N_2^{**} - b_0(t) - \lambda^*)x_3 + \lambda^* m x_{30}\end{array}\right\} \quad t \neq t_k, \\ Z(t_k^+) = (E + P_k)Z(t_k), \\ x_3(t_k^+) = (1 + f_k)x_3(t_k). \end{cases}$$

The Fredholm alternative with respect to impulsive differential equations shows that $\lambda^* m x_{30}$ must be orthogonal to the adjoint solution $1/x_{30}$ on $[0, T]$, where $1/x_{30}$ solves the adjoint equation

$$\begin{cases} \dot{x} = -(a_{32}N_2^{**} - b_0(t) - \lambda^*)x \quad t \neq t_k, \\ x(t_k^+) = \dfrac{1}{1 + f_k} x(t_k), \end{cases}$$

and since $\lambda^* \neq 0$, $m = 0$; consequently, $w = \lambda^* L w$, which implies that w is a multiple of w_0. According to Rabinowitz's theorem, there exists a continuum $D = \{(x_1, x_2, x_3, \lambda)\} \subseteq PC_T^3 \times R$ of nontrivial solutions of (3.2) such that the closure $\bar{D}$ of D contains $(0, 0, 0, \lambda^*)$. D can yield another continuum $F = \{(N_1, N_2, N_3, \lambda)\} \subseteq PC_T^3 \times R$ of nontrivial solutions of (1.1) whose closure $\bar{F}$ contains the bifurcation point $(N_1^{**}, N_2^{**}, 0, \lambda^*)$. And the proof is completed. □

Remark. Theorem 3.1 guarantees the existence of global branches of T-periodic solutions of (1.1), but it doesn't assert the positivity of these solutions.

In order to investigate the properties of the continuum D near the bifurcation point $(0, 0, 0, \lambda^*)$, we give the Lyapunov–Schmidt small parameter expansion of the solution (x_1, x_2, x_3, λ), which is

$$\begin{cases} Z(t) = Z_1(t)\varepsilon + Z_2(t)\varepsilon^2 + Z_3(t, \varepsilon)\varepsilon^2, \\ x_3(t) = x_{31}(t)\varepsilon + x_{32}(t)\varepsilon^2 + x_{33}(t, \varepsilon)\varepsilon^2, \\ \lambda = \lambda^* + \lambda_1\varepsilon + \lambda_2\varepsilon^2 + \lambda_3(\varepsilon)\varepsilon^2, \end{cases} \tag{3.8}$$

where ε is a small parameter, $Z_1(t)^\top, Z_2(t)^\top \in PC \times PC$, $x_{31}(t), x_{32}(t) \in PC$, $\lambda_1, \lambda_2 \in R$, and $|Z_3(t, \varepsilon)|$, $|x_{33}(t, \varepsilon)|$, $\lambda_3(\varepsilon)$ are of higher order than ε. Substituting (3.8) into the differential Eq. (3.2) and equating the coefficients of ε and ε^2 respectively, we have

$$\begin{cases} \dot{Z}_1(t) = AZ_1 + x_{31}C \quad t \neq t_k, \\ Z_1(t_k^+) = (E + P_k)Z_1(t_k), \end{cases} \tag{3.9a}$$

and

$$\begin{cases} \dot{x}_{31}(t) = (a_{32}N_2^{**} - b_0 - \lambda^*)x_{31} \quad t \neq t_k, \\ x_{31}(t_k^+) = (1 + f_k)x_{31}(t_k). \end{cases} \tag{3.9b}$$

Consequently,

$$\begin{cases} Z_1(t) = \int_0^t G_2(t, s)x_{31}(s)C\mathrm{d}s \triangleq (z_{11}(t), z_{12}(t))^\top, \\ x_{31}(t) = \prod_{0 \le t_k < t} (1 + f_k) \exp \int_0^t (a_{32}N_2^{**}(s) - b_0(s) - \lambda^*)\mathrm{d}s > 0, \end{cases}$$

in which $G_2(t, s)$ is defined by (3.3b). Meanwhile, we can obtain

$$\begin{cases} \dot{x}_{32}(t) = x_{32}(a_{32}N_2^{**} - b_0 - \lambda^*) + x_{31}(-\lambda_1 - a_{33}x_{31} + a_{32}z_{12}) \quad t \neq t_k, \\ x_{32}(t_k^+) = (1 + f_k)x_{32}(t_k). \end{cases}$$

This equation is a nonhomogeneous version of (3.9b) and consequently the nonhomogeneous term must be orthogonal to the solution $1/x_{31}(t)$ of the adjoint equation of (3.9b),

$$\int_0^T \frac{1}{x_{31}(t)} x_{31}(t)(-\lambda_1 - a_{33}(t)x_{31}(t) + a_{32}(t)z_{12}(t))\mathrm{d}t = 0,$$

so $\lambda_1 = [a_{32}z_{12} - a_{33}x_{31}]$. Since the sign of $z_{12}(t)$ is indefinite, we have the following conclusions.

Theorem 3.2. *If $\lambda_1 < 0$ and $[a_{32}N_2^{**}] + \frac{1}{T} \ln \prod_{k=1}^q (1 + f_k) > 0$, then there exists a small nonnegative constant m such that for each $b_3(t) \in C(R^+, R)$ with $[a_{32}N_2^{**}] + \frac{1}{T} \ln \prod_{k=1}^q (1 + f_k) - m \le [b_3] < [a_{32}N_2^{**}] + \frac{1}{T} \ln \prod_{k=1}^q (1 + f_k)$, system (1.1) has a solution $(N_1(t), N_2(t), N_3(t)) \in PC_T^3$, $N_i(t) > 0$ $(i = 1, 2, 3)$ for all $t \in R^+$.*

Proof. From the third equality of (3.8), we know that $\lambda - \lambda^* = \lambda_1\varepsilon + o(\varepsilon)$; therefore for each $b_3(t) \in C(R^+, R)$ satisfying $[a_{32}N_2^{**}] + \frac{1}{T} \ln \prod_{k=1}^q (1 + f_k) - m \le [b_3] < [a_{32}N_2^{**}] + \frac{1}{T} \ln \prod_{k=1}^q (1 + f_k)$, we have $\varepsilon > 0$. With $x_3(t) = x_{31}(t)\varepsilon + x_{32}(t)\varepsilon^2 + o(\varepsilon^2)$ and $x_{31}(t) > 0$ for all $t > 0$, clearly, $N_3(t) = x_3(t) > 0$ is true for all $t > 0$ under the conditions of Theorem 3.2. Now, we need only show that $(N_1(t), N_2(t)) = (x_1(t) + N_1^{**}(t), x_2(t) + N_2^{**}(t)) = Z(t) + (N_1^{**}(t), N_2^{**}(t)) > 0$. This can be easily proved. In fact, if m is small enough, then $[b_3] = \lambda$ is close enough to λ^*; therefore $(N_1(t), N_2(t))$ is near $(N_1^{**}(t), N_2^{**}(t))$ in the supremum norm of PC^2, and consequently $(N_1(t), N_2(t)) > 0$ holds since $(N_1^{**}(t), N_2^{**}(t)) > 0$ is bounded away from zero. □

Theorem 3.3. *If $\lambda_1 > 0$ and $[a_{32}N_2^{**}] + \frac{1}{T} \ln \prod_{k=1}^q (1 + f_k) > 0$, then there exists a small constant η such that for each $b_3(t) \in C(R^+, R)$ satisfying $[a_{32}N_2^{**}] + \frac{1}{T} \ln \prod_{k=1}^q (1 + f_k) \le [b_3] < [a_{32}N_2^{**}] + \frac{1}{T} \ln \prod_{k=1}^q (1 + f_k) + \eta$, system (1.1) has a positive solution $(N_1(t), N_2(t), N_3(t)) \in PC_T^3$.*

The proof is omitted since it is similar to that of Theorem 3.2.

In short, this paper gives some theoretical conditions that guarantee the existence of positive periodic solutions for periodic predator–prey-chain systems with impulsive perturbation. We hope our results can supply some useful suggestions for ecological protection.

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