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Quasi-invariance and integration by parts for determinantal and permanental processes

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Abstract

Determinantal and permanental processes are point processes with a correlation function given by a determinant or a permanent. Their atoms exhibit mutual attraction of repulsion, thus these processes are very far from the uncorrelated situation encountered in Poisson models. We establish a quasi-invariance result: we show that if atom locations are perturbed along a vector field, the resulting process is still a determinantal (respectively permanental) process, the law of which is absolutely continuous with respect to the original distribution. Based on this formula, following Bismut approach of Malliavin calculus, we then give an integration by parts formula.

Keywords: Determinantal processes; Integration by parts; Malliavin calculus; Point processes

1. Motivations

Point processes are widely used to model various phenomena, such as arrival times, arrangement of points in space, etc. It is thus necessary to know into details as large a catalog of point processes as possible. The Poisson process is one example which has been widely studied for a long time. Our motivation is to study point processes that generate a more complex correlation structure, such as a repulsion or attraction between points, but still remain simple enough so that their mathematical properties are analytically tractable. Determinantal and permanental

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point processes hopefully belong to this category. They were introduced in [21] in order to represent configurations of fermions and bosons. Elementary particles belong exclusively to one of these two classes. Fermions are particles like electrons or quarks; they obey the Pauli exclusion principle and hence the Fermi–Dirac statistics. The other sort of particles are particles like photons which obey the Bose–Einstein statistics. The interested reader can find in [26] an illuminating account of the determinantal (respectively permanental) structure of fermions (respectively bosons) ensemble. A mathematical unified presentation of determinantal/permanental point processes (DPPP for short) was for the first time, introduced in [23]. Let χ be the space of locally finite, simple configurations on a Polish space E and K a locally trace-class operator in $L^2(E)$ with a Radon measure λ. For $\alpha \in \mathfrak{A} = \{2/m,\ m \in \mathbf{N}\} \cup \{-1/m,\ m \in \mathbf{N}\}$, where $\mathbf{N}$ is the set of positive integers, for any positive, compactly supported f and $\xi = \sum_j \delta_{x_j} \in \chi$, the α-DPPP is the measure, $\mu_{\alpha,K,\lambda}$, on χ such that

$$\int_{\chi} e^{-\int f\,\mathrm{d}\xi}\,\mathrm{d}\mu_{\alpha,K,\lambda}(\xi) = \operatorname{Det}\bigl(\mathrm{I} + \alpha\sqrt{1-e^{-f}}\,K\sqrt{1-e^{-f}}\bigr)^{-\frac{1}{\alpha}}. \tag{1}$$

The values $\alpha = -1$ and $\alpha = 1$ correspond to determinantal and permanental point processes respectively. Starting from (1), existence of α-DPPP for any value of α is still a challenge as explained in [25]. Actually, existence is (not easily) proved for $\alpha = \pm 1$ and DPPP for other values of $\alpha \in \mathfrak{A}$ are constructed as superposition of these basic processes. DPPP recently regained interest because they have strong links with the spectral theory of random matrices [19,25]: for instance, eigenvalues of matrices in the Ginibre ensemble a.s. form a determinantal configuration. DPPP also appear in polynuclear growth [17,18], non-intersecting random walks, spanning trees, zero set of Gaussian analytic functions (see [16] and references therein), etc. Mathematically speaking, a few of their properties are known. The most complete references to date are, to the best of our knowledge, [16,23] and references therein. The overall impression seems to be that DPPP are rather hard to describe and analyze, their properties being highly dependent of the kernel and its eigenvalues.

Our aim is to investigate further some of the stochastic properties of α-DPPP. In the spirit of [28], we are interested in the differential calculus associated to these processes. We here address the problem within the point of view of Malliavin calculus. To date, Malliavin calculus for point processes has been developed namely for Poisson processes [2,6,7,10,13,22] and some of their extensions: Gibbs processes [3], marked processes [4], filtered Poisson processes [13], cluster processes [9] and Lévy processes [5,14]. There exist three approaches to construct a Malliavin calculus framework for point processes: one based on white noise analysis, one based on a difference operator and chaos decomposition and one which relies on quasi-invariance of the law of Poisson process with respect to some perturbations. This is the last track we follow here since neither the white noise framework nor the chaos decomposition exist so far.

We first show that the action of a diffeomorphism of E into itself onto the atoms of an α-DPPP yields another α-DPPP, the law of which is absolutely continuous with the distribution of the original process; a property usually known as quasi-invariance. Then, following the lines of proof of [2,8,9], we can derive an integration by parts formula for the differential gradient as usually constructed on configuration spaces. This gives another proof of the closability of the Dirichlet form canonically associated to an α-DPPP as in [28].

This paper is organized as follows. In Section 2, we give definitions concerning point processes and α-DPPP. In Section 3, we prove the quasi-invariance for α-DPPP. Then, in Section 4, we compute the integration by parts formula. We begin by determinantal point processes and then extend to α-determinantal point processes. Permanental processes are then analyzed on the same basis.

2. Preliminaries

2.1. Point processes

We remind here some properties of point processes we refer to [12,20] for more details. Let E be a Polish space and λ a Radon measure on $(E, \mathcal{B})$, the Borel σ-algebra on E. By χ we denote the space of all locally finite configurations on E:

$$\chi = \big\{\xi \subset E\colon\ |\xi \cap \Lambda| < \infty \text{ for any compact } \Lambda \subset E\big\},$$

where $|A|$ is the cardinality of a set A. Hereafter we identify a locally finite configuration ξ, defined as a set, and the atomic measure $\sum_{x\in\xi} \delta_x$. The space χ is then endowed with the vague topology of measures and $\mathcal{B}(\chi)$ denotes the corresponding Borel σ-algebra. For any measurable non-negative function f on E, we denote equivalently:

$$\langle f, \xi\rangle = \sum_{x\in\xi} f(x) = \int f\,\mathrm{d}\xi.$$

We also denote by $\chi_0 = \{\alpha \in \chi,\ |\alpha(E)| < \infty\}$ the set of all finite configurations in χ and χ_0 is equipped with the σ-algebra $\mathcal{B}(\chi_0)$. The restriction of a configuration ξ to a compact $\Lambda \subset E$, is denoted by ξ_Λ. We introduce the set $\chi_\Lambda = \{\xi \in \chi,\ \xi(E\backslash\Lambda) = 0\}$. Then for any integer n, we denote by $\chi_\Lambda^{(n)} = \{\xi \in \chi,\ \xi(\Lambda) = n\}$, the set of all configurations in with n points in Λ. Note that we have $\chi_\Lambda = \bigcup_{n=0}^\infty \chi_\Lambda^{(n)}$.

Definition 1. A random point process is a triplet $(\chi, \mathcal{B}(\chi), \mu)$, where μ is a probability measure on $(\chi, \mathcal{B}(\chi))$.

Every measure μ on the configuration space χ can be characterized by its Laplace function, that is to say for any measurable non-negative function f on E:

$$f \mapsto \mathbb{E}_\mu\big[e^{-\int f\,\mathrm{d}\xi}\big] = \int_\chi e^{-\int f\,\mathrm{d}\xi}\,\mathrm{d}\mu(\xi).$$

For instance, let π_σ denote the Poisson measure on $(\chi, \mathcal{B}(\chi))$ with intensity measure σ. Then its Laplace transform is, for any measurable non-negative function f:

$$\int_\chi e^{-\int f\,\mathrm{d}\xi}\,\mathrm{d}\pi_\sigma(\xi) = \exp\bigg(\int_E \big(1 - e^{-f(x)}\big)\,\mathrm{d}\sigma(x)\bigg).$$

Another way to describe the distribution of a point process is to give the probabilities $\mathbf{P}(|\xi_{\Lambda_k}| = n_k,\ 1 \leqslant k \leqslant n)$ for any n and any mutually disjoints Borel subsets of Λ, $\Lambda_1, \ldots, \Lambda_k$, $1 \leqslant k \leqslant n$. For instance, the Poisson measure π_σ with intensity measure σ can be defined in this way as:

$$\mathbf{P}\big(|\xi_{\Lambda_k}| = n_k,\ 1 \leqslant k \leqslant n\big) = \prod_{k=1}^{n} e^{-\sigma(\Lambda_k)} \frac{\sigma(\Lambda_k)^{n_k}}{n_k!}.$$

But in many cases, specifying the joint distribution of the $\xi(D)$'s is not that simple. It is then easier to describe the distribution of a point process by its correlation functions.

Definition 2. A locally integrable function $\rho_n : E^n \to \mathbf{R}_+$ is the n-point correlation function of μ if for any disjoint bounded Borel subsets $\Lambda_1, \ldots, \Lambda_m$ of E and $n_i \in \mathbf{N}$, $\sum_{i=1}^m n_i = n$:

$$\mathbb{E}_\mu\left[\prod_{i=1}^{m} \frac{|\xi_{\Lambda_i}|!}{(|\xi_{\Lambda_i}| - n_i)!}\right] = \int\limits_{\Lambda_1^{n_1} \times \cdots \times \Lambda_m^{n_m}} \rho_n(x_1, \ldots, x_n)\, \mathrm{d}\lambda(x_1) \ldots \mathrm{d}\lambda(x_n),$$

where $\mathbb{E}_\mu$ denotes the expectation relatively to μ.

For example, if $m = 1$ and $n_1 = n$, the formula becomes:

$$\begin{aligned}
\mathbb{E}_\mu\left[\frac{|\xi_\Lambda|!}{(|\xi_\Lambda| - n)!}\right] &= \mathbb{E}_\mu\big[|\xi_\Lambda|\big(|\xi_\Lambda| - 1\big) \ldots \big(|\xi_\Lambda| - n + 1\big)\big] \\
&= \int\limits_{\Lambda^n} \rho_n(x_1, \ldots, x_n)\, \mathrm{d}\lambda(x_1) \ldots \mathrm{d}\lambda(x_n).
\end{aligned}$$

We recognize here the n-th factorial moment of $|\xi_\Lambda|$. In particular:

$$\mathbb{E}_\mu\big[|\xi_\Lambda|\big] = \int\limits_{\Lambda} \rho_1(x)\, \mathrm{d}\lambda(x),$$

i.e., ρ_1 is the mean density of particles. More generally, the function ρ_n has the following interpretation: $\rho_n(x_1, \ldots, x_n)\, \mathrm{d}\lambda(x_1) \ldots \mathrm{d}\lambda(x_n)$ is approximately the probability to find a particle in each one of the $[x_i, x_i + \mathrm{d}\lambda(x_i)]$, $i = 1, \ldots, n$. A third way to define a point process proceeds via the Janossy densities. Denote by $\pi_{n,\Lambda}(x_1, \ldots, x_n)$ the density (assumed to exist) with respect to $\lambda^{\otimes n}$ of the joint distribution of $(x_1, \ldots, x_n)$ given that there are n points in Λ.

Definition 3. The density distributions or Janossy densities of a random process μ are the measurable functions j_Λ^n such that:

$$\begin{gathered}
j_\Lambda^n(x_1, \ldots, x_n) = n!\mu\big(\xi(\Lambda) = n\big)\pi_{n,\Lambda}(x_1, \ldots, x_n) \quad \text{for } n \in \mathbf{N}, \\
j_\Lambda^0(\emptyset) = \mu\big(\xi(\Lambda) = 0\big).
\end{gathered}$$

Hence, the Janossy density $j_\Lambda^n(x_1,\dots,x_n)$ appears as the probability density that there are exactly n points in Λ located around $x_1,\dots,x_n$, and no points anywhere else. For $n=0$, $j_\Lambda^0(\emptyset)$ is the probability that there is no point in Λ. For $n\geqslant 1$, the Janossy densities satisfy the following properties:

- Symmetry:

$$j_\Lambda^n(x_{\sigma(1)},\dots,x_{\sigma(n)}) = j_{n,\Lambda}(x_1,\dots,x_n),$$

  for every permutation σ of $\{1,\dots,n\}$.
- Normalization constraint. For each compact Λ:

$$\sum_{n=0}^{+\infty}\int_{\Lambda^n}\frac{1}{n!}j_\Lambda^n(x_1,\dots,x_n)\,\mathrm{d}\lambda(x_1)\dots\mathrm{d}\lambda(x_n)=1.$$

It is clear that the ρ_n's, j_n's, μ should satisfy some relationships. We will not dwell on that here (see the references cited above), we just mention the relation between μ and j_Λ^n, which is:

$$\int_{\mathfrak{X}} f(\xi)\,\mathrm{d}\mu(\xi)=\sum_{n=0}^{+\infty}\frac{1}{n!}\int_{\Lambda^n} f(x_1,\dots,x_n)j_\Lambda^n(x_1,\dots,x_n)\,\mathrm{d}\lambda(x_1)\dots\mathrm{d}\lambda(x_n). \tag{2}$$

2.2. *Fredholm determinants*

For details on this part, we refer to [15,24]. For any compact $\Lambda\subset E$, we denote by $L^2(\Lambda,\lambda)$ the set of functions square integrable with respect to the restriction of the measure λ to the set Λ. This becomes a Hilbert space when equipped with the usual norm:

$$\|f\|^2_{L^2(\lambda,\Lambda)}=\int_\Lambda\bigl|f(x)\bigr|^2\,\mathrm{d}\lambda(x).$$

For Λ a compact subset of E, P_Λ is the projection from $L^2(E)$ onto $L^2(\Lambda)$, i.e., $P_\Lambda f=f\mathbf{1}_\Lambda$. The operators we will deal with are special cases of the general category of continuous maps from $L^2(E,\lambda)$ into itself.

Definition 4. A map T from $L^2(E)$ into itself is said to be an integral operator whenever there exists a measurable function, we still denote by T, such that

$$Tf(x)=\int_E T(x,y)f(y)\,\mathrm{d}\lambda(y).$$

The function T is called the kernel of T.

Definition 5. Let T be a bounded map from $L^2(E,\lambda)$ into itself. The map T is said to be trace-class whenever for one complete orthonormal basis (CONB for short) $(h_n, n \geqslant 1)$ of $L^2(E,\lambda)$,

$$\sum_{n\geqslant 1} \left|(Th_n, h_n)_{L^2}\right| \quad \text{is finite.}$$

Then, the trace of T is defined by

$$\operatorname{trace}(T) = \sum_{n\geqslant 1} (Th_n, h_n)_{L^2}.$$

It is easily shown that the notion of trace does not depend on the choice of the CONB. Note that if T is trace-class then T^n also is trace-class for any $n \geqslant 2$.

Definition 6. Let T be a trace-class operator. The Fredholm determinant of $(\mathrm{I}+T)$ is defined by:

$$\operatorname{Det}(\mathrm{I}+T) = \exp\left(\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \operatorname{trace}\big(T^n\big)\right),$$

where I stands for the identity operator.

The practical computations of fractional power of Fredholm determinants involve the so-called α-determinants, which we introduce now.

Definition 7. For a square matrix $A = (a_{ij})_{i,j=1\dots n}$ of size $n \times n$, the α-determinant $\det_\alpha A$ is defined by:

$$\det_\alpha A = \sum_{\sigma\in\Sigma_n} \alpha^{n-\nu(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation is taken over the symmetric group Σ_n, the set of all permutations of $\{1,2,\dots,n\}$ and $\nu(\sigma)$ is the number of cycles in the permutation σ.

This is actually a generalization of the well-known determinant of a matrix. Indeed, when $\alpha = -1$, $\det_{-1} A$ is the usual determinant $\det A$. When $\alpha = 1$, $\det_1 A$ is the so-called permanent of A and for $\alpha = 0$, $\det_0 A = \prod_i a_{ii}$. We can then state the following useful theorem (see [23]):

Theorem 1. *For a trace-class integral operator T, if $\|\alpha T\| < 1$, we have*:

$$\operatorname{Det}(\mathrm{I}-\alpha T)^{-\frac{1}{\alpha}} = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{\Lambda^n} \det_\alpha\big(T(x_i,x_j)\big)_{1\leqslant i,j\leqslant n} \,\mathrm{d}\lambda(x_1)\dots\mathrm{d}\lambda(x_n).$$

If $\alpha \in \{-1/m;\ m \in \mathbf{N}\}$, this is true without the condition $\|\alpha T\| < 1$.

2.3. Determinantal–permanental point processes

The following set of hypothesis is of constant use.

Hypothesis 1. The Polish space E is equipped with a Radon measure λ. The map K is a Hilbert–Schmidt operator from $L^2(E, \lambda)$ into $L^2(E, \lambda)$ which satisfies the following conditions:

i) K is a bounded symmetric integral operator on $L^2(E, \lambda)$, with kernel $K(.,.)$, i.e., for any $x \in E$,

$$Kf(x) = \int_E K(x, y) f(y) \, \mathrm{d}\lambda(y).$$

ii) The spectrum of K is included $[0, 1[$.
iii) The map K is locally of trace class, i.e., for all compact $\Lambda \subset E$, the restriction $K_\Lambda = P_\Lambda K P_\Lambda$ of K to $L^2(\Lambda)$ is of trace class.

For a real $\alpha \in [-1, 1]$ and a compact subset $\Lambda \subset E$, the map $J_{\Lambda,\alpha}$ is defined by:

$$J_{\Lambda,\alpha} = (\mathrm{I} + \alpha K_\Lambda)^{-1} K_\Lambda,$$

so that we have:

$$(\mathrm{I} + \alpha K_\Lambda)(\mathrm{I} - \alpha J_{\Lambda,\alpha}) = \mathrm{I}.$$

For any compact Λ, the operator $J_{\Lambda,\alpha}$ is also a trace-class operator in $L^2(\Lambda, \lambda)$. In the following theorem, we define α-DPPP with the three equivalent characterizations: in terms of their Laplace transforms, Janossy densities and correlation functions. The theorem is also a theorem of existence, a problem which as said above is far from being trivial.

Theorem 2. *(See [23].) Assume Hypothesis* 1 *is satisfied. Let $\alpha \in \mathfrak{A}$. There exists a unique probability measure $\mu_{\alpha,K,\lambda}$ on the configuration space χ such that, for any non-negative bounded measurable function f on E with compact support, we have*:

$$\mathbb{E}_{\mu_{\alpha,K,\lambda}}\left[e^{-\int f \, \mathrm{d}\xi}\right] = \int_\chi e^{-\int f \, \mathrm{d}\xi} \, \mathrm{d}\mu_{\alpha,K,\lambda}(\xi) = \operatorname{Det}\big(\mathrm{I} + \alpha K\big[1 - e^{-f}\big]\big)^{-\frac{1}{\alpha}}, \tag{3}$$

where $K[1 - e^{-f}]$ is the bounded operator on $L^2(E)$ with kernel:

$$\big(K\big[1 - e^{-f}\big]\big)(x, y) = \sqrt{1 - \exp\big(-f(x)\big)} K(x, y) \sqrt{1 - \exp\big(-f(y)\big)}.$$

This means that for any integer n and any $(x_1, \ldots, x_n) \in E^n$, the correlation functions of $\mu_{\alpha,K,\lambda}$ are given by:

$$\rho_{n,\alpha,K}(x_1, \ldots, x_n) = \det{}_\alpha\big(K(x_i, x_j)\big)_{1 \leqslant i,j \leqslant n},$$

and for $n = 0$, $\rho_{0,\alpha,K}(\emptyset) = 1$. For any compact $\Lambda \subset E$, the operator $J_{\Lambda,\alpha}$ is a Hilbert–Schmidt, trace-class operator, whose spectrum is included in $[0, +\infty[$. For any $n \in \mathbf{N}$, any compact $\Lambda \subset E$, and any $(x_1, \ldots, x_n) \in \Lambda^n$ the n-th Janossy density is given by:

$$j^n_{\Lambda,\alpha,K_\Lambda}(x_1, \ldots, x_n) = \operatorname{Det}(\mathrm{I} + \alpha K_\Lambda)^{-1/\alpha} \det_\alpha\big(J_{\Lambda,\alpha}(x_i, x_j)\big)_{1 \leqslant i,j \leqslant n}. \tag{4}$$

For $n = 0$, we have $j^n_{\Lambda,\alpha,K_\Lambda}(\emptyset) = \operatorname{Det}(\mathrm{I} + \alpha K_\Lambda)^{-1/\alpha}$.

For $\alpha = -1$, such a process is called a determinantal process since we have, for any $n \geqslant 1$:

$$\rho_{n,-1,K}(x_1, \ldots, x_n) = \det\big(K(x_i, x_j)\big)_{1 \leqslant i,j \leqslant n}.$$

For $\alpha = 1$, such a process is called a permanental process, since we have, for any $n \geqslant 1$:

$$\rho_{n,1,K}(x_1, \ldots, x_n) = \sum_{\pi \in \Sigma} \prod_{i=1}^{n} K(x_i, x_{\pi(i)}) = \operatorname{per}\big(K(x_i, x_j)\big)_{1 \leqslant i,j \leqslant n}.$$

For any bounded function $g : E \to \mathbf{R}^+$, and any integral operator T of kernel $T(x, y)$, we denote by $T[g]$ the integral operator of kernel:

$$T[g](x, y) \to \sqrt{g(x)}T(x, y)\sqrt{g(y)}.$$

For calculations, it will be convenient to use the following lemma:

Lemma 1. *(See [23].) Let Λ be a compact subset of E and $f : E \to [0, +\infty)$, measurable with* $\operatorname{supp}(f) \in \Lambda$:

$$\operatorname{Det}\big(\mathrm{I} + \alpha K_\Lambda\big[1 - e^{-f}\big]\big)^{-1/\alpha} = \operatorname{Det}(\mathrm{I} + \alpha K_\Lambda)^{-1/\alpha} \operatorname{Det}\big(\mathrm{I} - \alpha J_{\Lambda,\alpha}\big[e^{-f}\big]\big)^{-1/\alpha}.$$

By differentiation into the Laplace transform, it is possible to compute moments of $\int f \,\mathrm{d}\xi$ for any deterministic f. We obtain, at the first order:

Theorem 3. *(See [23].) For any non-negative function f defined on E, we have*

$$\mathbb{E}\Bigg[\int_\Lambda f \,\mathrm{d}\xi\Bigg] = \int_\Lambda f(x)K(x, x)\,\mathrm{d}\lambda(x) = \operatorname{trace}\big(K_\Lambda[f]\big).$$

It is worth mentioning how the existence of α-DPPP is established. For $\alpha = -1$, there is a non-trivial work (see [23,25] and references therein) to show that the Janossy densities satisfy the positivity condition so that a point process with these densities does exist. For $\alpha = -1/m$, it is sufficient to remark from (3) that the superposition of m independent determinantal point processes of kernel K/m is an α-DPPP for kernel K. The point is that K/m satisfies Hypothesis 1, in particular that its spectrum is strictly bounded by $1/m < 1$, since $m > 1$. For $\alpha = 2$, a 2-permanental point process is in fact a Cox process based on a Gaussian random field. We know for sure that there exists X a centered Gaussian random field on E such that:

$$\mathbb{E}_{\mathbf{P}}\left[\int_\Lambda X^2(x)\,\mathrm{d}\lambda(x)\right] = \operatorname{trace}(K_\Lambda), \tag{5}$$

for any compact $\Lambda \subset E$ and

$$\mathbb{E}_{\mathbf{P}}\big[X(x)X(y)\big] = K(x,y)\ \lambda\otimes\lambda \quad \text{a.s.}, \tag{6}$$

where $\mathbf{P}$ is the probability measure on the probability space supporting X. Then the Cox process of random intensity $X^2(x)\,\mathrm{d}\lambda(x)$ has the same distribution as $\mu_{2,K,\lambda}$. Indeed, it follows from the formula:

$$\mathbb{E}_{\mathbf{P}}\left[\exp\left(-\int \big(1-e^{-f(x)}\big)X^2(x)\,\mathrm{d}\lambda(x)\right)\right] = \operatorname{Det}\big(\mathrm{I}+2\big(1-e^{-f}\big)K\big)^{-1/2}.$$

Thus, any $2/m$-permanental point process is the superposition of m independent 2-permanental point processes with kernel K/m.

Poisson process can be obtained formally as extreme case of 1-permanental process with a kernel K given by $K(x,y) = \mathbf{1}_{\{x=y\}}$. Of course, this kernel is likely to be null almost surely with respect to $\lambda\otimes\lambda$; nonetheless, it remains that replacing formally this expression in (3) yields the Laplace transform of a Poisson process of intensity λ. Another way to retrieve a Poisson process is to let α go to 0 in (3). With the above constructions, this means that a Poisson process can be viewed as an infinite superposition of determinantal or permanental point processes.

Theorem 4. *When α tends to* 0*, $\mu_{\alpha,K,\lambda}$ converges narrowly to a Poisson measure of intensity $K(x,x)\,\mathrm{d}\lambda(x)$.*

Proof. For any non-negative f, for any $n \geqslant 1$,

$$0 \leqslant \operatorname{trace}\big(\big(K_\Lambda\big[1-e^{-f}\big]\big)^n\big) \leqslant \operatorname{trace}\big(K_\Lambda\big[1-e^{-f}\big]\big),$$

hence,

$$\begin{aligned}
&\int_{\mathcal{X}} \exp\left(-\int f\,\mathrm{d}\xi\right)\mathrm{d}\mu_{\alpha,K_\Lambda,\lambda}(\xi) \\
&\quad = \operatorname{Det}\big(\mathrm{I}+\alpha K_\Lambda\big[1-e^{-f}\big]\big)^{-1/\alpha} \\
&\quad = \exp\left(-\frac{1}{\alpha}\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}\alpha^n \operatorname{trace}\big(\big(K_\Lambda\big[1-e^{-f}\big]\big)^n\big)\right) \\
&\quad \xrightarrow{\alpha\to 0} \exp\big(-\operatorname{trace}\big(K_\Lambda\big(1-e^{-f}\big)\big)\big) \quad = \int_E \big(1-e^{-f(x)}\big)K_\Lambda(x,x)\,\mathrm{d}\lambda(x).
\end{aligned} \tag{7}$$

Thus, when α goes to 0, the measure $\mu_{\alpha,K_\Lambda,\lambda}$ tends towards a measure that we call $\mu_{0,K_\Lambda,\lambda}$. According to (7), $\mu_{0,K_\Lambda,\lambda}$ is a Poisson process with intensity $K_\Lambda(x,x)\,\mathrm{d}\lambda(x)$. $\square$

3. Quasi-invariance

In this part we show the quasi-invariance property for any α-DPPP. Let $\mathrm{Diff}_0(E)$ be the set of all diffeomorphisms from E into itself with compact support, i.e., for any $\phi \in \mathrm{Diff}_0(E)$, there exists a compact Λ outside which ϕ is the identity map. For any $\xi \in \chi$, we still denote by ϕ the map:

$$\begin{aligned} \phi : \chi &\to \chi, \\ \sum_{x \in \xi} \delta_x &\mapsto \sum_{x \in \xi} \delta_{\phi(x)}. \end{aligned}$$

For any reference measure λ on E, λ_ϕ denotes the image measure of λ by ϕ. For $\phi \in \mathrm{Diff}_0(E)$ whose support is included in Λ, we introduce the isometry Φ,

$$\begin{aligned} \Phi : L^2(\lambda_\phi, \Lambda) &\to L^2(\lambda, \Lambda), \\ f &\mapsto f \circ \phi. \end{aligned}$$

Its inverse, which exists since ϕ is a diffeomorphism, is trivially defined by $f \circ \phi^{-1}$ and denoted by Φ^{-1}. Note that Φ and Φ^{-1} are isometries, i.e.,

$$\langle \Phi \psi_1,\ \Phi \psi_2 \rangle_{L^2(\lambda, \Lambda)} = \langle \psi_1,\ \psi_2 \rangle_{L^2(\lambda_\phi, \Lambda)},$$

for any ψ_1 and ψ_2 belonging to $L^2(\lambda, \Lambda)$. We also set:

$$K_\Lambda^\phi = \Phi^{-1} K_\Lambda \Phi \quad \text{and} \quad J_{\Lambda,\alpha}^\phi = \Phi^{-1} J_{\Lambda,\alpha} \Phi.$$

Lemma 2. *Let λ be a Radon measure on E and K a map satisfying Hypothesis* 1. *Let $\alpha \in \mathfrak{A}$. We have the following properties.*

a) K_Λ^ϕ *and* $J_{\Lambda,\alpha}^\phi$ *are continuous operators from* $L^2(\lambda_\phi, \Lambda)$ *into* $L^2(\lambda_\phi, \Lambda)$.
b) K_Λ^ϕ *is of trace class and* $\mathrm{trace}(K_\Lambda^\phi) = \mathrm{trace}(K_\Lambda)$.
c) $\mathrm{Det}(\mathrm{I} + \alpha K_\Lambda^\phi) = \mathrm{Det}(\mathrm{I} + \alpha K_\Lambda)$.

Proof. The first point is immediate according to the definition of an image measure. Since Φ^{-1} is an isometry, for any $(\psi_n, n \in \mathbf{N})$ a complete orthonormal basis of $L^2(\lambda, \Lambda)$, the family $(\Phi^{-1}\psi_n, n \in \mathbf{N})$ is a CONB of $L^2(\lambda_\phi, \Lambda)$. Moreover,

$$\begin{aligned} \sum_{n \geqslant 1} \left| \left\langle K_\Lambda^\phi \Phi^{-1} \psi_n, \Phi^{-1} \psi_n \right\rangle_{L^2(\lambda_\phi, \Lambda)} \right| &= \sum_{n \geqslant 1} \left| \left\langle \Phi^{-1} K \Phi \Phi^{-1} \psi_n, \Phi^{-1} \psi_n \right\rangle_{L^2(\lambda_\phi, \Lambda)} \right| \\ &= \sum_{n \geqslant 1} \left| \left\langle \Phi^{-1} K \psi_n, \Phi^{-1} \psi_n \right\rangle_{L^2(\lambda_\phi, \Lambda)} \right| \\ &= \sum_{n \geqslant 1} \left| \langle K \psi_n, \psi_n \rangle_{L^2(\lambda, \Lambda)} \right|. \end{aligned}$$

Hence, K_Λ^ϕ is of trace class and $\operatorname{trace}(K_\Lambda^\phi) = \operatorname{trace}(K_\Lambda)$. Along the same lines, we prove that $\operatorname{trace}((K_\Lambda^\phi)^n) = \operatorname{trace}(K_\Lambda^n)$ for any $n \geqslant 2$. According to Definition 6, the Fredholm determinant of K_Λ^ϕ is well defined and point c) follows. $\square$

Theorem 5. *Assume that K is a kernel operator. Then K_Λ^ϕ, as a map from $L^2(\lambda_\phi, \Lambda)$ into itself is a kernel operator whose kernel is given by $((x, y) \mapsto K_\Lambda(\phi^{-1}(x), \phi^{-1}(y)))$. An analog formula also holds for the operator $J_{\Lambda,\alpha}$.*

Proof. On the one hand, for any function f, the operator K_Λ^ϕ from $L^2(\Lambda, \lambda_\phi)$ into $L^2(\Lambda, \lambda_\phi)$ is given by:

$$K_\Lambda^\phi f(x) = \int_\Lambda K_\Lambda^\phi(x, z) f(z)\,\mathrm{d}\lambda_\phi(z).$$

On the other hand, using the definition $K_\Lambda^\phi = \Phi^{-1} K_\Lambda \Phi$

$$\begin{aligned} K_\Lambda^\phi f(x) &= \Phi^{-1} K_\Lambda \Phi f(x) \\ &= \int_\Lambda K_\Lambda\big(\phi^{-1}(x), y\big) f \circ \phi(y)\,\mathrm{d}\lambda(y) \\ &= \int_\Lambda K_\Lambda\big(\phi^{-1}(x), \phi^{-1}(z)\big) f(z)\,\mathrm{d}\lambda_\phi(z). \end{aligned}$$

The proof is thus complete. $\square$

Lemma 3. *Let $\rho : E \to \mathbf{R}$ be non-negative and assume that $\mathrm{d}\lambda = \rho\,\mathrm{d}m$ for some other Radon measure on E. Let K satisfy Hypothesis* 1. *Then, we have the following properties*:

(1) *The map $K[\rho]$ is continuous from $L^2(m)$ into itself.*
(2) *The map $K[\rho]$ is locally trace-class and* $\operatorname{trace}(K_\Lambda[\rho]) = \operatorname{trace}(K_\Lambda)$.
(3) *The measure $\mu_{\alpha,K,\lambda}$ is identical to the measure $\mu_{\alpha,K[\rho],m}$.*

That is to say, in some sense, we can "transfer" a part of the reference measure into the operator and vice versa.

Proof. Remember that

$$K[\rho](x, y) = \sqrt{\rho(x)} K(x, y) \sqrt{\rho(y)}.$$

Hence

$$K_\Lambda[\rho] f(x) = \sqrt{\rho(x)} \int_\Lambda K_\Lambda(x, y) \sqrt{\rho(y)}\,\mathrm{d}\lambda(y),$$

thus

$$\int_\Lambda \left|K_\Lambda[\rho]f\right|^2 \mathrm{d}m = \int_\Lambda |K_\Lambda f|^2 \,\mathrm{d}\lambda,$$

and the first point follows. Consider $(\psi_n, n \in \mathbf{N})$, a CONB of $L^2(\lambda)$. Then $(\psi_n\sqrt{\rho}, n \in \mathbf{N})$ is a CONB of $L^2(m)$. Furthermore, we have:

$$\begin{aligned}\sum_{n\geqslant 1}\left|\left\langle K_\Lambda[\rho]\psi_n, \psi_n\right\rangle_{L^2(\mathrm{d}m)}\right| &= \sum_{n\geqslant 1}\left|\langle K_\Lambda\sqrt{\rho}\psi_n, \sqrt{\rho}\psi_n\rangle_{L^2(\mathrm{d}m)}\right| \\ &= \sum_{n\geqslant 1}\left|\langle K_\Lambda\psi_n, \psi_n\rangle_{L^2(\lambda)}\right|.\end{aligned}$$

Therefore the operator $K_\Lambda[\rho]$ is of trace class and

$$\mathrm{trace}\big(K_\Lambda[\rho]\big) = \mathrm{trace}(K_\Lambda).$$

Similarly we can prove that for any $n \geqslant 2$, we have $\mathrm{trace}(K_\Lambda^n[\rho]) = \mathrm{trace}(K_\Lambda^n)$. Then, using the definition of a Fredholm determinant, we have:

$$\mathrm{Det}(\mathrm{I} + \alpha K_\Lambda) = \mathrm{Det}\big(\mathrm{I} + \alpha K_\Lambda[\rho]\big).$$

The third point then follows from the characterization of $\mu_{\alpha,K[\rho],m}$ by its Laplace transform. □

The expression $\det_\alpha J_{\Lambda,\alpha}(x_i, x_j)_{1\leqslant i,j\leqslant n}$ is now denoted $\det_\alpha J_{\Lambda,\alpha}(x_1, \dots, x_n)$. For any finite random configuration $\xi = (x_1, \dots, x_n)$, we call $J_{\Lambda,\alpha}(\xi)$ the matrix with terms $(J_{\Lambda,\alpha}(x_i, x_j),\ 1 \leqslant i, j \leqslant n)$. First, remind some results from [2] concerning Poisson measures. For any $\phi \in \mathrm{Diff}_0(E)$, we define $\phi^*\pi_\lambda$ as the image of the Poisson measure π_λ with intensity measure λ and λ_ϕ denotes the image measure of λ by ϕ.

Theorem 6. *(See [2].) For any $\phi \in \mathrm{Diff}_0(E)$, and a Poisson measure π_λ with intensity λ:*

$$\phi^*\pi_\lambda = \pi_{\lambda_\phi}.$$

That is to say, for any f non-negative and compactly supported on E:

$$\mathbb{E}_{\pi_\lambda}\left[e^{-\int f\circ\phi\,\mathrm{d}\xi}\right] = \exp\left(-\int 1 - e^{-f}\,\mathrm{d}\lambda_\phi\right). \tag{8}$$

We give the corresponding formula for α-determinantal measures. For any $\phi \in \mathrm{Diff}_0(E)$, we define $\phi^*\mu_{\alpha,K_\Lambda,\lambda}$ as the image of the measure $\mu_{\alpha,K_\Lambda,\lambda}$ under ϕ. We prove below that this image measure is an α-DPPP the parameters of which are explicitly known.

Theorem 7. *With the notations and hypothesis introduced above. For any $\phi \in \mathrm{Diff}_0(E)$, for any non-negative function f on E, for any compact $\Lambda \subset E$, we have:*

$$\mathbb{E}_{\mu_{\alpha,K_\Lambda,\lambda}}\big[e^{-\int f\circ\phi\,\mathrm{d}\xi}\big]=\mathbb{E}_{\mu_{\alpha,K^\phi_\Lambda,\lambda_\phi}}\big[e^{-\int f\,\mathrm{d}\xi}\big]=\operatorname{Det}\big(\mathrm{I}+\alpha K^\phi_\Lambda\big[1-e^{-f}\big]\big)^{-1/\alpha}. \tag{9}$$

That is to say the image measure of $\mu_{\alpha,K,\lambda}$ by ϕ is an α-determinantal process with operator K^ϕ and reference measure λ_ϕ.

Proof. According to Theorem 2 and Lemma 1, we have for non-negative f:

$$\begin{aligned}\mathbb{E}_{\mu_{\alpha,K_\Lambda,\lambda}}\big[e^{-\int f\circ\phi\,\mathrm{d}\xi}\big]&=\operatorname{Det}\big(\mathrm{I}+\alpha K_\Lambda\big[1-e^{-f\circ\phi}\big]\big)^{-1/\alpha}\\&=\operatorname{Det}(\mathrm{I}+\alpha K_\Lambda)^{-1/\alpha}\operatorname{Det}\big(\mathrm{I}-\alpha J_{\Lambda,\alpha}\big[e^{-f\circ\phi}\big]\big)^{-1/\alpha}.\end{aligned}$$

According to Theorem 1, we get

$$\begin{aligned}&\operatorname{Det}\big(\mathrm{I}-\alpha J_{\Lambda,\alpha}\big[e^{-f\circ\phi}\big]\big)^{-1/\alpha}\\&\quad=\sum_{n=0}^{+\infty}\frac{1}{n!}\int_{\Lambda^n}\det{}_\alpha J_{\Lambda,\alpha}(x_1,\dots,x_n)e^{-\sum_{i=1}^n f(\phi(x_i))}\,\mathrm{d}\lambda(x_1)\dots\mathrm{d}\lambda(x_n)\\&\quad=\sum_{n=0}^{+\infty}\frac{1}{n!}\int_{\Lambda^n}\det{}_\alpha J^\phi_{\Lambda,\alpha}(x_1,\dots,x_n)e^{-\sum_{i=1}^n f(x_i)}\,\mathrm{d}\lambda_\phi(x_1)\dots\mathrm{d}\lambda_\phi(x_n)\\&\quad=\operatorname{Det}\big(\mathrm{I}-\alpha J^\phi_{\Lambda,\alpha}\big[e^{-f}\big]\big)^{-1/\alpha}.\end{aligned}$$

Since $\operatorname{Det}(\mathrm{I}+\alpha K_\Lambda)=\operatorname{Det}(\mathrm{I}+\alpha K^\phi_\Lambda)$, we have:

$$\mathbb{E}_{\mu_{\alpha,K_\Lambda,\lambda}}\big[e^{-\int f\circ\phi\,\mathrm{d}\xi}\big]=\operatorname{Det}\big(\mathrm{I}+\alpha K^\phi_\Lambda\big[1-e^{-f}\big]\big)^{-1/\alpha}=\mathbb{E}_{\mu_{\alpha,K^\phi_\Lambda,\lambda_\phi}}\big[e^{-\int f\,\mathrm{d}\xi}\big].$$

The proof is thus complete. □

For $\alpha=2$, Theorem 7 says that the image under ϕ of a Cox process is still a Cox process of parameters K^ϕ_Λ and λ_ϕ. Such a process can be constructed as follows: Let X be a centered Gaussian random field satisfying (5) and (6) and let $Y(x)=X(\phi^{-1}(x))$. Then, according to Lemma 2, we have: for any compact Λ,

$$\mathbb{E}_{\mathbf{P}}\bigg[\int_\Lambda Y^2(x)\,\mathrm{d}\lambda_\phi(x)\bigg]=\operatorname{trace}\big(K^\phi_\Lambda\big)$$

and

$$\mathbb{E}_{\mathbf{P}}\big[Y(x)Y(y)\big]=K^\phi(x,y)=K\big(\phi^{-1}(x),\phi^{-1}(y)\big),\quad \lambda_\phi\otimes\lambda_\phi,\ \text{a.s.}$$

From Theorem 6, by conditioning with respect to X, we also have:

$$\begin{aligned}\mathbb{E}_{\mu_{2,K,\lambda}}\left[e^{-\int f\circ\phi\,\mathrm{d}\xi}\right] &= \mathbb{E}_{\mathbf{P}}\left[\mathbb{E}\left[e^{-\int f\circ\phi\,\mathrm{d}\xi}\mid X\right]\right]\\ &= \mathbb{E}_{\mathbf{P}}\left[\exp\left(-\int\left(1-e^{-f\circ\phi}\right)X^2\,\mathrm{d}\lambda\right)\right]\\ &= \mathbb{E}_{\mathbf{P}}\left[\exp\left(-\int\left(1-e^{-f}\right)Y^2\,\mathrm{d}\lambda_\phi\right)\right].\end{aligned}$$

Thus the two approaches (fortunately) yield the same result.

We now want to prove that $\mu_{\alpha,K^\phi,\lambda_\phi}$ is absolutely continuous with respect to $\mu_{\alpha,K,\lambda}$ and compute the corresponding Radon–Nikodym derivative. For technical reasons, we need to assume that there exists a Jacobi formula (or change of variable formula) on the measured space (E,λ). This could be done in full generality for E a manifold; for the sake of simplicity, we assume hereafter that E is a domain of some $\mathbf{R}^d$. We denote by ∇^E the usual gradient on $\mathbf{R}^d$. We also introduce a new hypothesis.

Hypothesis 2. We suppose that the measure λ is absolutely continuous with respect to the Lebesgue measure m on E. We denote by ρ the Radon–Nikodym derivative of λ with respect to m. We furthermore assume that $\sqrt{\rho}$ is in $H^{1,2}_{loc}(K(x,x)\,\mathrm{d}m(x))$, i.e., ρ is weakly differentiable and for any compact Λ in E, we have:

$$\begin{aligned}\infty &> 2\int_\Lambda \left\|\nabla^E\sqrt{\rho(x)}\right\|^2 K(x,x)\,\mathrm{d}m(x)\\ &= \int_\Lambda \frac{\|\nabla^E\rho(x)\|^2}{\rho(x)}K(x,x)\,\mathrm{d}m(x)\\ &= \int_\Lambda \left(\frac{\|\nabla^E\rho(x)\|}{\rho(x)}\right)^2 K(x,x)\,\mathrm{d}\lambda(x).\end{aligned}$$

Then for any $\phi\in\mathrm{Diff}_0(E)$, λ_ϕ is absolutely continuous with respect to λ and

$$p^\lambda_\phi(x)=\frac{\mathrm{d}\lambda_\phi(x)}{\mathrm{d}\lambda(x)}=\frac{\rho(\phi^{-1}(x))}{\rho(x)}\,\mathrm{Jac}(\phi)(x),$$

where $\mathrm{Jac}(\phi)(x)$ is the Jacobian of ϕ at point x.

Lemma 4. *Assume* (E,K,λ) *satisfy Hypotheses* 1 *and* 2. *Let* $(u_n,\ n\geqslant 0)$ *be a sequence of non-negative real numbers such that for any* $x\in\mathbf{R}$,

$$\sum_{n\geqslant 0}\frac{u_n}{n!}\,|x|^n<+\infty. \tag{10}$$

For any compact $\Lambda\subset E$*, we have*:

$$\mathbb{E}_{\mu_{\alpha,K_\Lambda,\lambda}}\left[\frac{u_{|\xi|}}{\det_\alpha J_{\Lambda,\alpha}(\xi)}\right]<+\infty. \tag{11}$$

As a consequence, $\det_\alpha J_{\Lambda,\alpha}(\xi)$ *is* $\mu_{\alpha,K_\Lambda,\lambda}$ *almost-surely positive.*

Proof. According to Theorem 2, we have:

$$j^n_{\Lambda,\alpha,K_\Lambda}(x_1,\dots,x_n)=\operatorname{Det}(\mathrm{I}+\alpha K_\Lambda)^{-1/\alpha}\det_\alpha J_{\Lambda,\alpha}(x_1,\dots,x_n),$$

hence

$$\begin{aligned}\mathbb{E}\left[\frac{u_{|\xi|}}{\det_\alpha J_{\Lambda,\alpha}(\xi)}\right]&=\sum_{n=0}^{+\infty}\frac{1}{n!}\int_{\Lambda^n}\frac{u_n}{\det_\alpha J_{\Lambda,\alpha}(x_1,\dots,x_n)}\, j^n_{\Lambda,\alpha,K_\Lambda}(x_1,\dots,x_n)\otimes_{j=1}^n \mathrm{d}\lambda(x_j)\\&=\operatorname{Det}(\mathrm{I}+\alpha K_\Lambda)^{-1/\alpha}\sum_{n=0}^{+\infty}\frac{u_n}{n!}\lambda(\Lambda)^n<+\infty,\end{aligned}$$

because λ is assumed to be a Radon measure and Λ is compact. □

Theorem 8. *Assume* (E,K,λ) *satisfy Hypotheses* 1 *and* 2. *Then, the measure* $\mu_{\alpha,K,\lambda}$ *is quasi-invariant with respect to the group* $\operatorname{Diff}_0(E)$ *and for any* $\phi\in\operatorname{Diff}_0(E)$*, we have then*:

$$\frac{\mathrm{d}\phi^*\mu_{\alpha,K,\lambda}}{\mathrm{d}\mu_{\alpha,K,\lambda}}(\xi)=\left(\prod_{x\in\xi}p^\lambda_\phi(x)\right)\frac{\det_\alpha J^\phi_\alpha(\xi)}{\det_\alpha J_\alpha(\xi)}.$$

That is to say that for any measurable non-negative, compactly supported f *on* E:

$$\mathbb{E}_{\mu_{\alpha,K,\lambda}}\left[e^{-\int f\circ\phi\,\mathrm{d}\xi}\right]=\mathbb{E}_{\mu_{\alpha,K,\lambda}}\left[e^{-\int f\,\mathrm{d}\xi}e^{\int\ln(p^\lambda_\phi)\,\mathrm{d}\xi}\frac{\det_\alpha J^\phi_\alpha(\xi)}{\det_\alpha J_\alpha(\xi)}\right].\tag{12}$$

Proof. Since f is compactly supported and ϕ belongs to $\operatorname{Diff}_0(E)$, there exists a compact Λ which contains both the support of f and $f\circ\phi$. According to Theorem 7 and Lemma 5, we have:

$$\begin{aligned}\mathbb{E}_{\mu_{\alpha,K_\Lambda,\lambda}}\left[e^{-\int f\circ\phi\,\mathrm{d}\xi}\right]&=\mathbb{E}_{\mu_{\alpha,K^\phi_\Lambda,\lambda_\phi}}\left[e^{-\int f\,\mathrm{d}\xi}\right]\\&=\operatorname{Det}\big(\mathrm{I}+\alpha K^\phi_\Lambda\big)^{-1/\alpha}\left(\sum_{n=0}^{+\infty}\frac{1}{n!}A_n\right)\\&=\operatorname{Det}(\mathrm{I}+\alpha K_\Lambda)^{-1/\alpha}\left(\sum_{n=0}^{+\infty}\frac{1}{n!}A_n\right)\end{aligned}$$

where for any $n\in\mathbf{N}$, the A_n are the integrals:

$$\begin{aligned}A_n&=\int_{\Lambda^n}\det_\alpha J^\phi_{\Lambda,\alpha}(x_1,\dots,x_n)e^{-\sum_{i=1}^n f(x_i)}\,\mathrm{d}\lambda_\phi(x_1)\dots\mathrm{d}\lambda_\phi(x_n)\\&=\int_{\Lambda^n}\det_\alpha J^\phi_{\Lambda,\alpha}(x_1,\dots,x_n)e^{-\sum_{i=1}^n f(x_i)}\prod_{i=1}^n p^\lambda_\phi(x_i)\,\mathrm{d}\lambda(x_1)\dots\mathrm{d}\lambda(x_n)\end{aligned}$$

$$= \int_{\Lambda^n} \det_\alpha J_{\Lambda,\alpha}(x_1, \dots, x_n)\alpha_n(x_1, \dots, x_n)\, \mathrm{d}\lambda(x_1)\dots \mathrm{d}\lambda(x_n),$$

where

$$\alpha_n(x_1, \dots, x_n) = \frac{\det_\alpha J^\phi_{\Lambda,\alpha}(x_1, \dots, x_n)}{\det_\alpha J_{\Lambda,\alpha}(x_1, \dots, x_n)} e^{-\sum_i f(x_i)} \prod_{i=1}^n p^\lambda_\phi(x_i).$$

Hence according to (4), we can write:

$$\begin{aligned}
&\mathrm{Det}(\mathrm{I} + \alpha K_\Lambda)^{-1/\alpha} \sum_{n=0}^{+\infty} \frac{1}{n!} A_n \\
&\quad = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{\Lambda^n} j^n_{\Lambda,\alpha,K_\Lambda}(x_1, \dots, x_n)\alpha_n(x_1, \dots, x_n)\, \mathrm{d}\lambda(x_1)\dots \mathrm{d}\lambda(x_n).
\end{aligned}$$

Thus, we have (12). □

Should we consider Poisson process either as a 0-DPPP or as an α-DPPP with the singular kernel mentioned above, we see that the last fraction in (12) reduces to 1 and we find the well-known formula of quasi-invariance for Poisson processes (see [2]). In the following, we define:

$$L^\phi_{\mu_{\alpha,K,\lambda}}(\xi) = \left(\prod_{x\in\xi} p^\lambda_\phi(x)\right) \frac{\det_\alpha J^\phi_\alpha(\xi)}{\det_\alpha J_\alpha(\xi)}.$$

Then formula (12) can be rewritten as:

$$\mathbb{E}_{\mu_{\alpha,K,\lambda}}\left[e^{-\int f\circ\phi\, \mathrm{d}\xi}\right] = \mathbb{E}_{\mu_{\alpha,K,\lambda}}\left[e^{-\int f\, \mathrm{d}\xi}\ L^\phi_{\mu_{\alpha,K,\lambda}}(\xi)\right].$$

4. Integration by parts formula

In this section, we prove the integration by parts formula. The proof relies on a differentiation within (12). We thus need to put a manifold structure on χ. The tangent space $T_\xi \chi$ at some $\xi \in \chi$ is given as $L^2(\mathrm{d}\xi)$, i.e., the set of all maps V from E to $\mathbf{R}$ such that:

$$\int \left|V(x)\right|^2 \mathrm{d}\xi(x) < \infty.$$

Note that if $\xi \in \chi_0$ then $T_\xi \chi$ can be identified as $\mathbf{R}^{|\xi|}$ with the Euclidean scalar product.

We consider $V_0(E)$ the set of all C^∞-vector fields on E with compact support. For any $v \in V_0(E)$, we construct: $\phi^v_t : E \to E$, $t \in \mathbf{R}$, where the curve, for any $x \in E$

$$t \in \mathbf{R} \to \phi^v_t(x)$$

is defined as the solution to:

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t^v(x) = v\big(\phi_t^v(x)\big) \quad \text{and} \quad \phi_0^v(x) = x.$$

Because $v \in V_0(E)$, there is no explosion and ϕ_t^v is well defined for each $t \in \mathbf{R}$. The mappings $\{\phi_t^v, t \in \mathbf{R}\}$ form a one-parameter subgroup of diffeomorphisms with compact support, that is to say:

- $\forall t \in \mathbf{R}$, $\phi_t^v \in \mathrm{Diff}_0(E)$.
- $\forall t, s \in \mathbf{R}$, $\phi_t^v \circ \phi_s^v = \phi_{t+s}^v$. In particular, $(\phi_t^v)^{-1} = \phi_{-t}^v$.
- For any $T > 0$, there exists a compact K such that $\phi_t^v(x) = x$ for any $x \in K^c$, for any $|t| \leqslant T$.

In the following, we fix $v \in V_0(E)$. For any $\xi \in \chi$, we still denote by ϕ_t^v the map:

$$\begin{aligned}\phi_t^v : \chi &\to \chi, \\ \xi = \sum_{x\in\xi} \delta_{x_i} &\mapsto \sum_{x\in\xi} \delta_{\phi_t^v(x)} \in \chi.\end{aligned}$$

Definition 8. A function $F : \chi \to \mathbf{R}$ is said to be differentiable at $\xi \in \chi$ whenever for any vector field $v \in V_0(E)$, the directional derivative along the vector field v

$$\nabla_v F(\xi) = \frac{\mathrm{d}}{\mathrm{d}t} F\big(\phi_t^v(\xi)\big)\bigg|_{t=0}$$

is well defined.

Since ϕ_t^v does not change the number of atoms of ξ, if ξ belongs to χ_0, this notion of differentiability coincides with the usual one in $\mathbf{R}^{|\xi|}$ and

$$\nabla_v F(x_1, \dots, x_n) = \sum_{i=1}^{n} \partial_i F(x_1, \dots, x_n) v(x_i),$$

if $\xi = \{x_1, \dots, x_n\}$.

In the general case, a set of test functions is defined as is: Following the notations from [2], for a function $F : \chi \to \mathbb{R}$ we say that $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \chi)$ if:

$$F(\xi) = f\bigg(\int h_1\,\mathrm{d}\xi, \dots, \int h_N\,\mathrm{d}\xi\bigg),$$

for some $N \in \mathbf{N}$, $h_1, \dots, h_N \in \mathcal{D} = C^\infty(E)$, $f \in C_b^\infty(\mathbf{R}^N)$. Then for any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \chi)$, given $v \in V_0(E)$, we have:

$$F\big(\phi_t^v(\xi)\big) = f\bigg(\int h_1 \circ \phi_t^v\,\mathrm{d}\xi, \dots, \int h_N \circ \phi_t^v\,\mathrm{d}\xi\bigg).$$

It is then clear that the directional derivative of such F exists and that:

$$\nabla_v F(\xi) = \sum_{i=1}^{N} \partial_i f\left(\int h_1 \, \mathrm{d}\xi, \ldots, \int h_N \, \mathrm{d}\xi \right) \int \nabla_v^E h_i \, \mathrm{d}\xi.$$

The gradient ∇F of a differentiable function F is defined as a map from χ into $T\chi$ such that, for any $v \in V_0(E)$,

$$\int \nabla_x F(\xi) v(x) \, \mathrm{d}\xi(x) = \nabla_v F(\xi).$$

If $\xi \in \chi_0$ and F is differentiable at χ, then

$$\nabla_x F(\xi) = \sum_{i=1}^{|\xi|} \partial_i F\big(\{x_1, \ldots, x_{|\xi|}\}\big) \mathbf{1}_{\{x = x_i\}}.$$

If ξ belongs to χ, for any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \chi)$,

$$\nabla_x F(\xi) = \sum_{i=1}^{n} \partial_i f\left(\int h_1 \, \mathrm{d}\xi, \ldots, \int h_N \, \mathrm{d}\xi \right) \nabla^E h_i(x).$$

4.1. Determinantal point processes

In what follows, c and κ are positive constants which may vary from line to line.

In this part, we assume $\alpha = -1$ and that Hypotheses 1 and 2 hold. We denote by $\beta^\lambda(x)$ the logarithmic derivative of λ, given by: for any x in E,

$$\beta^\lambda(x) = \frac{\nabla \rho(x)}{\rho(x)} \quad \text{on } \big\{\rho(x) > 0\big\},$$

and $\beta^\lambda(x) = 0$ on $\{\rho(x) = 0\}$. Then, for any vector field v on E with compact support, we denote by B_v^λ the following function on χ:

$$\begin{aligned} B_v^\lambda &: \chi \to \mathbf{R}, \\ \xi &\mapsto B_v^\lambda(\xi) = \int_E \big(\beta^\lambda(x).v(x) + \operatorname{div}\big(v(x)\big)\big) \, \mathrm{d}\xi(x), \end{aligned}$$

where $x.y$ is the Euclidean scalar product of x and y in E. If $\lambda = m$,

$$B_v^m(\xi) = \int_E \operatorname{div}\big(v(x)\big) \, \mathrm{d}\xi(x)$$

and according to Theorem 3,

$$\mathbb{E}\big[\big|B_v^m(\xi)\big|\big] \leqslant \int\limits_E \big|\operatorname{div}\big(v(x)\big)\big| K(x,x)\,\mathrm{d}\lambda(x) \leqslant \|v\|_\infty \operatorname{trace}(K_\Lambda) < \infty,$$

where Λ is a compact containing the support of v. As in [28], we now define the potential energy of a finite configuration by

$$\begin{aligned} U : \chi_0 &\to \mathbf{R}, \\ \xi &\mapsto -\log\det J(\xi). \end{aligned}$$

Hypothesis 3. The functional U is differentiable at every configuration $\xi \in \chi_0$. Moreover, for any $v \in V_0(E)$, there exists $c > 0$ such that for any $\xi \in \chi_0$, we have

$$\big|\langle \nabla U(\xi), v\rangle_{L^2(\mathrm{d}\xi)}\big| \leqslant \frac{u_{|\xi|}}{\det J(\xi)}, \tag{13}$$

where $(u_n, n \geqslant 1)$ satisfy (10).

Theorem 9. *Assume that the kernel J is once differentiable with continuous derivative. Then, Hypothesis 3 is satisfied.*

Proof. Let $\xi = \{x_1, \dots, x_n\} \in \chi_0$ and let Λ be a compact subset of E whose interior contains ξ. Since $J(.,.)$ is differentiable

$$(y_1, \dots, y_n) \mapsto -\log\det\big(J(y_i, y_k),\ 1 \leqslant i, k \leqslant n\big)$$

is differentiable. The chain rule formula implies that

$$t \mapsto \log\det\big(J\big(\phi_t^v(x_i), \phi_t^v(x_k)\big),\ 1 \leqslant i, k \leqslant n\big)$$

is differentiable and its differential is equal to

$$\frac{1}{\det J(\phi_t^v(\xi))} \operatorname{trace}\bigg(\operatorname{Adj}\big(J\big(\phi_t^v(x_i), \phi_t^v(x_k)\big)\big)\bigg(E_t^v\bigg(\frac{\partial J(\xi)}{\partial x}\bigg)_t + \bigg(\frac{\partial J(\xi)}{\partial y}\bigg)_t E_t^v\bigg)\bigg),$$

where $(\frac{\partial J(\xi)}{\partial x})_t$ is the matrix with terms $(\frac{\partial J_\Lambda}{\partial x}(\phi_t^v(x_i), \phi_t^v(x_j)))_{x_i, x_j \in \xi}$, $(\frac{\partial J(\xi)}{\partial y})_t$ is the matrix with terms $(\frac{\partial J_\Lambda}{\partial y}(\phi_t^v(x_i), \phi_t^v(x_j)))_{x_i, x_j \in \xi}$, and E_t^v is the diagonal matrix with terms $(v(\phi_t^v(x_i)))_{x_i \in \xi}$. For $t = 0$, this reduces to

$$\big|\langle \nabla U(\xi), v\rangle_{L^2(\mathrm{d}\xi)}\big| = \frac{1}{\det J(\xi)} \operatorname{trace}\bigg(\operatorname{Adj}\big(J(\xi)\big)\bigg(E_0^v\bigg(\frac{\partial J(\xi)}{\partial x}\bigg)_0 + \bigg(\frac{\partial J(\xi)}{\partial y}\bigg)_0 E_0^v\bigg)\bigg).$$

Since J is continuous and Λ is compact,

$$\bigg\|\frac{\partial J}{\partial y}(\xi)\bigg\|_{HS} \leqslant |\xi| \|J\|_\infty \quad \text{and} \quad \big\|E_0^v(\xi)\big\|_{HS} \leqslant |\xi|^{1/2} \|v\|_\infty.$$

Hence, there exists c independent of ξ such that

$$\left|\langle \nabla U(\xi), v\rangle_{L^2(\mathrm{d}\xi)}\right| \leqslant c|\xi|^2 \frac{1}{\det J(\xi)} \left|\operatorname{trace}\bigl(\operatorname{Adj}\bigl(J(\xi)\bigr)\bigr)\right|.$$

From [15, p. 1021], we know that for any $n \times n$ matrix A, for any x and y in $\mathbf{R}^n$, we have

$$\left|(\operatorname{Adj} A)x.y\right| \leqslant \|y\| \|A\|_{HS}^{n-1} (n-1)^{-(n-1)/2}.$$

It follows that

$$\left|\operatorname{trace}\bigl(\operatorname{Adj} A\bigr)\right| = \left|\sum_{j=1}^n (\operatorname{Adj} A)e_j.e_j\right| \leqslant n\|A\|_{HS}^{n-1}(n-1)^{-(n-1)/2},$$

where $(e_j, j = 1, \dots, n)$ is the canonical basis of $\mathbf{R}^n$. Since J is bounded, $\|J(\xi)\|_{HS} \leqslant |\xi| \|J\|_\infty$, hence there exists c independent of ξ such that

$$\left|\langle \nabla U(\xi), v\rangle_{L^2(\mathrm{d}\xi)}\right| \leqslant \frac{c}{\det J(\xi)} |\xi|^{|\xi|/2}.$$

The proof is thus complete. □

Corollary 1. *Assume that Hypothesis* 3 *holds. For any $v \in V_0(E)$, for any $\xi \in \chi_0$, the function*

$$t \mapsto H_t(\xi) = \frac{\det J(\phi_t^v(\xi))}{\det J(\xi)}$$

is differentiable and

$$\sup_{|t| \leqslant T} \left|\frac{\mathrm{d}H_t(\xi)}{\mathrm{d}t}\right| \leqslant \frac{u_{|\xi|}}{\det J(\xi)},$$

where $(u_n = cn^{n/2}, n \geqslant 1)$ satisfy (10).

Proof. According to Hypothesis 3, the function $(t \mapsto U(\phi_t^v(\xi)))$ is differentiable and

$$\frac{\mathrm{d}U(\phi_t^v(\xi))}{\mathrm{d}t} = \bigl\langle \nabla U\bigl(\phi_t^v(\xi)\bigr), v\bigr\rangle_{L^2(\mathrm{d}\phi_t^v(\xi))}. \tag{14}$$

For any t, ϕ_t^v is a diffeomorphism hence, Theorem 8 applied to ϕ_t^v and ϕ_{-t}^v implies that $\mu_{-1,K^{\phi_t^v},\lambda_{\phi_t^v}}$ and $\mu_{-1,K,\lambda}$ are equivalent measures. According to Lemma 4, for any t, $\det J^{\phi_t^v}(\xi)$ is $\mu_{-1,K^{\phi_t^v},\lambda_{\phi_t^v}}$-a.s. positive hence it is also $\mu_{-1,K,\lambda}$-a.s. positive. Since for any $\xi \in \chi_0$,

$$t \mapsto \det J^{\phi_t^v}(\xi) = \exp\bigl(-U\bigl(\phi_t^v(\xi)\bigr)\bigr)$$

is continuous, it follows that there exists a set of full $\mu_{-1,K,\lambda}$ measure on which $\det J^{\phi_t^v}(\xi) > 0$ for any $|t| \leqslant T$, for any ξ. Furthermore,

$$\frac{\mathrm{d}H_t(\xi)}{\mathrm{d}t} = -\frac{\det J(\phi_t^v(\xi))}{\det J(\xi)} \frac{\mathrm{d}U(\phi_t^v(\xi))}{\mathrm{d}t}.$$

In view of (14) and of Hypothesis 3, this means that

$$\sup_{|t|\leqslant T}\left|\frac{\mathrm{d}H_t(\xi)}{\mathrm{d}t}\right| \leqslant \frac{\det J(\phi_t^v(\xi))}{\det J(\xi)} \frac{u_{|\xi|}}{\det J(\phi_t^v(\xi))} = \frac{u_{|\xi|}}{\det J(\xi)},$$

since $\phi_t^v(\xi)$ has the number of atoms as ξ. □

Lemma 5. *Assume that* $\lambda = m$ *and set*

$$P_t(\xi) = \prod_{x\in\xi} p_{\phi_t^v}(x) = \prod_{x\in\xi} \operatorname{Jac} \phi_t^v(x).$$

For any $v \in \mathrm{Diff}_0(E)$*, for any configuration* $\xi \in \chi$*,* P *is differentiable with respect to* t *and we have*

$$\frac{\mathrm{d}\log P_t}{\mathrm{d}t}(\xi) = \int \left(\operatorname{div} v - \int_0^t \nabla^E \operatorname{div} v \circ \eta_{r,t}.v(\eta_{r,t})\, \mathrm{d}r\right) \mathrm{d}\xi,$$

where for any $r \leqslant t$*,* $x \to \eta_{r,t}(x)$ *is the diffeomorphism of* E *which satisfies*:

$$\eta_{r,t}(x) = x - \int_r^t v\big(\eta_{s,t}(x)\big)\, \mathrm{d}s.$$

In particular for $t = 0$*, we have*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\prod_{x\in\xi} p^\lambda_{\phi_t^v}(x)\right)\Bigg|_{t=0} = B_v^m(\xi). \tag{15}$$

Moreover, there exist $c > 0$ *and* $\kappa > 0$ *such that for any* $\xi \in \chi_0$*,*

$$\sup_{t\leqslant T}\left|\frac{\mathrm{d}P_t}{\mathrm{d}t}(\xi)\right| \leqslant ce^{\kappa|\xi|}. \tag{16}$$

Proof. Introduce, for any $s \leqslant t$, $x \mapsto \eta_{s,t}(x)$, the diffeomorphism of E which satisfies:

$$\eta_{s,t}(x) = x - \int_s^t v\big(\eta_{r,t}(x)\big)\, \mathrm{d}r.$$

As a comparison, we remind that $\phi_t^v(x) = x + \int_0^t v(\phi_s^v(x))\,\mathrm{d}s$. It is well known that the diffeomorphism $x \mapsto \eta_{0,t}(x)$ is the inverse of $x \mapsto \phi_t^v(x)$. Then using [27], we have:

$$\operatorname{Jac}\phi_t^v(x) = \frac{\mathrm{d}(\phi_t^v)^* m(x)}{\mathrm{d}m(x)} = \exp\left(\int_0^t \operatorname{div} v \circ \eta_{r,t}(x)\,\mathrm{d}r\right), \tag{17}$$

and

$$\prod_{x\in\xi} \operatorname{Jac}\phi_t^v(x) = \exp\left(\sum_{x\in\xi} \int_0^t \operatorname{div} v \circ \eta_{r,t}(x)\,\mathrm{d}r\right).$$

Hence, we have:

$$\begin{aligned}\sum_{x\in\xi} \frac{\mathrm{d}}{\mathrm{d}t} \log \operatorname{Jac}\phi_t^v(x) &= \sum_{x\in\xi} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \operatorname{div} v \circ \eta_{r,t}(x)\,\mathrm{d}r \\ &= \sum_{x\in\xi} \operatorname{div} v(x) - \int_0^t \nabla^E \operatorname{div} v \circ \eta_{r,t}(x).v\big(\eta_{r,t}(x)\big)\,\mathrm{d}r.\end{aligned}$$

The first and second points follow easily. Now, v is assumed to have bounded derivatives of any order, hence for any $\xi \in \chi_0$,

$$\left|\frac{\mathrm{d}\log P_t}{\mathrm{d}t}(\xi)\right| \leqslant c|\xi|, \tag{18}$$

where c depends neither from t nor ξ. According to (17), there exists $\kappa > 0$ such that for any $\xi \in \chi_0$, we have:

$$\big|P_t(\xi)\big| \leqslant \exp\big(\kappa|\xi|\big). \tag{19}$$

Thus, combining (18) and (19), we get (16). $\square$

We are now in position to prove the main result of this section.

Theorem 10. *Assume* (E, K, λ) *satisfy Hypotheses* 1, 2 *and* 3, *let* $\alpha = -1$. *Let F and G belong to* $\mathcal{F}C_b^\infty$. *For any compact Λ, we have*:

$$\begin{aligned}&\int_{\chi_\Lambda} \nabla_v F(\xi) G(\xi)\,\mathrm{d}\mu_{-1,K_\Lambda,\lambda}(\xi) \\ &\quad= -\int_{\chi_\Lambda} F(\xi)\nabla_v G(\xi)\,\mathrm{d}\mu_{-1,K_\Lambda,\lambda}(\xi) + \int_{\chi_\Lambda} F(\xi)G(\xi)\big(B_v^\lambda(\xi) + \nabla_v U(\xi)\big)\,\mathrm{d}\mu_{-1,K_\Lambda,\lambda}(\xi).\end{aligned} \tag{20}$$

Proof. In view of Lemma 3, we can replace J by $J[\rho]$ and assume $\lambda = m$, i.e., λ is the Lebesgue measure. Note that

$$B_v^m(\xi) = \int \operatorname{div} v(x)\, \mathrm{d}\xi(x).$$

Let Λ be a fixed compact set in E, remember that $\chi_\Lambda \subset \chi_0$. Let M be an integer and $\chi^M = \{\xi \in \chi_0,\ |\xi| \leqslant M\}$. It is crucial to note that χ^M is invariant by any $\phi \in \mathrm{Diff}_0(E)$. On the one hand, by dominated convergence, we have:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\chi^M} F\big(\phi_t^v(\xi)\big)G(\xi)\,\mathrm{d}\mu_{-1,K_\Lambda[\rho],m}(\xi)\right)\bigg|_{t=0}\\
&\quad= \int_{\chi^M} \frac{\mathrm{d}}{\mathrm{d}t}\big(F\big(\phi_t^v(\xi)\big)\big)\bigg|_{t=0} G(\xi)\,\mathrm{d}\mu_{-1,K_\Lambda[\rho],m}(\xi)\\
&\quad= \int_{\chi^M} \nabla_v F(\xi) G(\xi)\,\mathrm{d}\mu_{-1,K_\Lambda[\rho],m}(\xi).
\end{aligned}$$

On the other hand, we know from (12) that

$$\begin{aligned}
&\int_{\chi^M} F\big(\phi_t^v(\xi)\big)G(\xi)\,\mathrm{d}\mu_{-1,K_\Lambda[\rho],m}(\xi)\\
&\quad= \int_{\chi_\Lambda} F\big(\phi_t^v(\xi)\big)G(\xi)\mathbf{1}_{\{|\xi|\leqslant M\}}\,\mathrm{d}\mu_{-1,K_\Lambda[\rho],m}(\xi)\\
&\quad= \int_{\chi_\Lambda} F(\xi)G\big(\phi_{-t}^v(\xi)\big)\mathbf{1}_{\{|\phi_{-t}^v(\xi)|\leqslant M\}}\,\mathrm{d}\mu_{-1,K_\Lambda^{\phi_t^v}[\rho],m_{\phi_t^v}}(\xi)\\
&\quad= \int_{\chi_\Lambda} F(\xi)G\big(\phi_{-t}^v(\xi)\big)\mathbf{1}_{\{|\xi|\leqslant M\}} L^{\phi_t^v}_{-1,K[\rho],\lambda}(\xi)\,\mathrm{d}\mu_{-1,K_\Lambda[\rho],m}(\xi).
\end{aligned}\tag{21}$$

According to Corollary 1 and Lemma 5, the function $(t \mapsto L^{\phi_t^v}_{-1,K[\rho],\lambda}(\xi))$ is differentiable and there exists c such that:

$$\sup_{t\leqslant T}\left|\frac{\mathrm{d}L^{\phi_t^v}_{-1,K[\rho],\lambda}}{\mathrm{d}t}(\xi)\right| \leqslant \frac{u_{|\xi|}}{\det J(\xi)},$$

where $(u_n, n \geqslant 0)$ satisfy (10).

Lemma 4 implies that the right-hand side of the last inequality is integrable with respect to $\mu_{-1,K_\Lambda,\lambda}$, thus, we can differentiate inside the expectations in (21) and we obtain:

$$
\begin{aligned}
&\int_{\chi_\Lambda} \nabla_v F(\xi) G(\xi) \mathbf{1}_{\{|\xi| \leqslant M\}} \, \mathrm{d}\mu_{-1,K_\Lambda,m}(\xi) \\
&\quad = \int_{\chi_\Lambda} F(\xi) \big(-\nabla_v G(\xi) + G(\xi) \big(B_v^m(\xi) + \nabla_v U(\xi) \big) \big) \mathbf{1}_{\{|\xi| \leqslant M\}} \, \mathrm{d}\mu_{-1,K_\Lambda,m}(\xi).
\end{aligned}
$$

According to Hypothesis 3 and Lemma 4, by dominated convergence, we have:

$$
\begin{aligned}
&\int_{\chi_\Lambda} \nabla_v F(\xi) G(\xi) \, \mathrm{d}\mu_{-1,K_\Lambda,m}(\xi) \\
&\quad = \int_{\chi_\Lambda} F(\xi) \big(-\nabla_v G(\xi) + G(\xi) \big(B_v^m(\xi) + \nabla_v U(\xi) \big) \big) \, \mathrm{d}\mu_{-1,K_\Lambda,m}(\xi).
\end{aligned}
$$

Now, we remark that

$$
\begin{aligned}
\nabla_v U[\rho](\xi) &= \nabla_v \log \det J[\rho](\xi) \\
&= \nabla_v \log \Big(\prod_{x \in \xi} \rho(x) \det J(\xi) \Big) \\
&= \nabla_v \int \log \rho(x) \, \mathrm{d}\xi(x) + \nabla_v U(\xi) \\
&= \int \frac{\nabla^E \rho(x)}{\rho(x)} . v(x) \, \mathrm{d}\xi(x) + \nabla_v U(\xi).
\end{aligned}
$$

Moreover, we have

$$
B_v^m(\xi) + \int_\Lambda \frac{\nabla^E \rho(x)}{\rho(x)} . v(x) \, \mathrm{d}\xi(x) = B_v^\lambda(\xi),
$$

and in view of Theorem 3,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big| \int_\Lambda \frac{\nabla^E \rho(x)}{\rho(x)} . v(x) \, \mathrm{d}\xi(x) \Big| \Big]^2 \\
&\quad \leqslant \mathbb{E}\Big[\int_\Lambda \Big(\frac{\|\nabla^E \rho(x)\|}{\rho(x)} \Big)^2 \mathrm{d}\xi(x) \Big] \mathbb{E}\Big[\int_\Lambda \big| v(x) \big|^2 \mathrm{d}\xi(x) \Big] \\
&\quad \leqslant \|v\|_\infty^2 \operatorname{trace}(K_\Lambda) \int_\Lambda \Big(\frac{\|\nabla^E \rho(x)\|}{\rho(x)} \Big)^2 K(x,x) \rho(x) \, \mathrm{d}m(x).
\end{aligned}
$$

Then, Hypothesis 2 implies that B_v^λ is integrable and we get (20) in the general case. □

4.2. α-Determinantal point processes

We now prove the integration by parts formula for α-determinantal point processes where $\alpha = -1/s$ for s integer greater than 2. In principle, we could follow the previous lines of proof modifying the definition of U as

$$U(\xi) = -\log \det_\alpha J_\alpha(\xi)$$

and assuming that Hypothesis 3 is still valid. Unfortunately, there is no (simple) analog of Theorem 9 since there is no rule to differentiate an α-determinant and control its derivative.

We already saw that such an α-DPPP can be obtained as the superposition of s determinantal processes of kernel K/s.

Let $(E_1, \lambda_1, K_1), \dots, (E_s, \lambda_s, K_s)$ be s Polish spaces each equipped with a Radon measure and s linear operators satisfying Hypothesis 1 on their respective space. We set

$$E = \bigcup_{i=1}^{s} \{i\} \times E_i,$$

that is to say E is the disjoint union of the E_i's, often denoted as $\bigsqcup_{i=1}^{s} E_i$. An element of E is thus a couple (i, x) where x belongs to E_i for any $i \in \{1, \dots, s\}$. On the Polish space E, we put the measure λ defined by

$$\int_E f(i, x)\, \mathrm{d}\lambda(i,\ x) = \int_E f(i, x)\, \mathrm{d}\lambda_i(x).$$

We also define K as

$$Kf(i, x) = \int_{E_i} K_i(x, y) f(y)\, \mathrm{d}\lambda_i(y).$$

A compact set in E is of the form $\Lambda = \bigcup_{i=1}^{s} \{i\} \times \Lambda_i$ where Λ_i is a compact set of E_i hence

$$K_\Lambda f(i, x) = \int_{\Lambda_i} K_i(x, y) f(y)\, \mathrm{d}\lambda_i(y).$$

This means that K is a kernel operator the kernel of which is given by:

$$K\big((i, x), (j, y)\big) = K_i(x, y) \mathbf{1}_{\{i=j\}}. \tag{22}$$

In particular, for $\xi = ((i_l, x_l),\ l = 1, \dots, n)$, we have

$$\det K(\xi) = \prod_{j=1}^{s} \det K(\xi_j)$$

where $\xi_j = \{x,\ (j, x) \in \xi\}$.

It is straightforward that K is symmetric and locally of trace class. Moreover, its spectrum is equal to the union of the spectra of the K_i's. For, if ψ is such that $K\psi = \alpha\psi$ then $\psi(i, .)$ is an eigenvector of K_i and thus α belongs to the spectrum of K_i. In the reverse direction, if ψ is an eigenvector of K_i associated to the eigenvalue α then the function

$$f(j, x) = \psi(x)\mathbf{1}_{\{i=j\}}$$

is square integrable with respect to λ and is an eigenvector of K for the eigenvalue α. If we assume furthermore that each of the E_i's is a subset of $\mathbf{R}^d$, we can define the gradient on E as

$$\nabla^E f(i, x) = \nabla^{E_i} f(i, x).$$

Now χ_E is the set of locally finite point measures of the form

$$\xi = \sum_j \delta_{(i_j, x_j)}.$$

With these notations, it is clear that Hypotheses 1, 2 and 3 are satisfied provided, they are satisfied for each index i. Thus (20) is satisfied.

Now take $E_1 = \cdots = E_s$, $\lambda_1 = \cdots = \lambda_s$ and $K_1 = \cdots = K_s$. We introduce the map Θ defined as:

$$\begin{aligned} \Theta : E &\to E_1, \\ (i, x) &\mapsto x. \end{aligned}$$

Consistently with earlier defined notations, we still denote by Θ the map

$$\begin{aligned} \Theta : \chi_E &\to \chi_{E_1}, \\ \xi &\mapsto \sum_{(j,x)\in\xi} \delta_x. \end{aligned}$$

Then, according to what has been said above, $\mu_{-1/s, sK_1, \lambda_1}$ is the image measure of $\mu_{-1,K,\lambda}$ by the map Θ. Set

$$\xi_n = \sum_{(i,x)\in\xi} \delta_x \mathbf{1}_{\{i=n\}}.$$

The reciprocal problem, interesting in its own sake and useful for the sequel, is to determine the conditional distribution of ξ_1 given $\Theta\xi$.

Theorem 11. *Let s be an integer strictly greater than 1, for F non-negative or bounded, for any Λ compact subset of E,*

$$\mathbb{E}\big[F(\xi_1) \bigm| \Theta\xi\big] = \sum_{\eta\subset\Theta\xi} F(\eta) \times \binom{|\Theta\xi|}{|\eta|} \frac{j_{\beta,(s-1)K_{1,\Lambda,\lambda_1}}(\Theta\xi\backslash\eta)\, j_{-1,K_{1,\Lambda},\lambda_1}(\eta)}{j_{\alpha,sK_{1,\Lambda},\lambda_1}(\Theta\xi)}, \tag{23}$$

where $\beta = -1/(s-1)$. *Note that* (23) *also holds for* $s = 1$ *with the convention that* $j_{\beta,0}(\eta) = 0$ *for* $\eta \neq \emptyset$ *and* $j_{\beta,0}(\emptyset) = 1$, *which is analog to the usual convention* $0^0 = 1$.

Proof. Let $\zeta = \xi_2 \cup \cdots \cup \xi_s$, we known that ζ is distributed as $\mu_{-\beta,-K_1/\beta,\lambda_1}$. Consider Ξ, the map

$$\begin{aligned}\Xi : \chi_{E_1} \times \chi_{E_1} &\to \chi_{E_1} \times \chi_{E_1},\\ (\eta_1, \eta_2) &\mapsto (\eta_1, \eta_1 \cup \eta_2).\end{aligned}$$

By construction, the joint distribution of $\Xi(\xi_1, \zeta)$ is the same as the distribution of $(\xi_1, \Theta\xi)$. For any $\eta \subset \Theta\xi \in \chi_0$, we set:

$$R(\eta, \Theta\xi) = \binom{|\Theta\xi|}{|\eta|} \frac{j_{\beta,(s-1)K_{1,\Lambda},\lambda_1}(\Theta\xi \backslash \eta) j_{-1,K_{1,\Lambda},\lambda_1}(\eta)}{j_{\alpha,sK_{1,\Lambda},\lambda_1}(\Theta\xi)}.$$

Hence, for any F and G bounded, we have

$$\begin{aligned}
&\mathbb{E}\big[F(\xi_1)G(\Theta\xi)\big] = \mathbb{E}\big[(F \otimes G) \circ \Xi(\xi_1, \zeta)\big]\\
&\quad= \sum_{j,k=0}^{\infty} \frac{1}{j!}\frac{1}{k!} \int_{\Lambda^j \times \Lambda^k} F\big(\{x_1, \ldots, x_j\}\big) G\big(\{x_1, \ldots, x_j\} \cup \{y_1, \ldots, y_k\}\big)\\
&\qquad \times j_{-1,K_{1,\Lambda},\lambda_1}(x_1, \ldots, x_j)\ j_{\beta,(s-1)K_{1,\Lambda},\lambda_1}(y_1, \ldots, y_k)\,\mathrm{d}\lambda_1(x_1)\ldots\mathrm{d}\lambda_1(y_k)\\
&\quad= \sum_{j,k=0}^{\infty} \frac{1}{(k+j)!} \int_{\Lambda^j \times \Lambda^k} F\big(\{x_1, \ldots, x_j\}\big)(GR)\big(\{x_1, \ldots, x_j\} \cup \{y_1, \ldots, y_k\}\big)\\
&\qquad \times j_{\alpha,sK_{1,\Lambda},\lambda_1}(x_1, \ldots, x_j, y_1, \ldots, y_k)\,\mathrm{d}\lambda_1(x_1)\ldots\mathrm{d}\lambda_1(y_k)\\
&\quad= \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\Lambda^m} \Big(\sum_{j \leqslant m} F\big(\{x_1, \ldots, x_j\}\big) R\big(\{x_1, \ldots, x_j\}, \{x_1, \ldots, x_m\}\big)\Big)\\
&\qquad \times G\big(\{x_1, \ldots, x_m\}\big) j_{\alpha,sK_{1,\Lambda},\lambda_1}(x_1, \ldots, x_m)\,\mathrm{d}\lambda_1(x_1)\ldots\mathrm{d}\lambda_1(x_m)\\
&\quad= \int_{\chi_{E_1}} \Big(\sum_{\eta \subset \omega} F(\eta) R(\eta, \omega)\Big) G(\omega)\,\mathrm{d}\mu_{\alpha,sK_1,\lambda_1}(\omega).
\end{aligned}$$

The proof is thus complete. □

This formula can be understood by looking at the extreme case of Poisson process. Assume that $\Theta\xi$ is distributed according to a Poisson process of intensity $\lambda\,\mathrm{d}m$. Then, ξ_1 is a Poisson process of intensity $s^{-1}\lambda\,\mathrm{d}m$ and ζ also is a Poisson process of intensity $(1 - s^{-1})\lambda\,\mathrm{d}m$. The couple $(\xi_1, \Theta\xi)$ can then be constructed by random thinning of $\Theta\xi$: Keep each point of $\Theta\xi$ independently of the others, with probability $1/s$; the remaining points will be distributed as ξ_1. The conditional expectation of a functional $F(\xi_1)$ given $\Theta\xi$ is then the sum of the values of F taken for each realization of a thinning multiplied by the probability of each thinned configuration. Since $|\Theta\xi|$ is assumed to be known, the atoms of $\Theta\xi$ are independent and identically dispatched along E, hence the probability to obtain a specific configuration is equal to the probability that a random variable binomially distributed of parameters $|\Theta\xi|$ and $1/s$, is equal to the cardinal of the configuration. This means that

$$\mathbb{E}\big[F(\xi_1)\bigm|\Theta\xi\big]=\sum_{\eta\subset\Theta\xi}F(\eta)\times\binom{|\Theta\xi|}{|\eta|}\left(\frac{1}{s}\right)^{|\eta|}\left(1-\frac{1}{s}\right)^{|\Theta\xi|-|\eta|}.$$

This corresponds to (23) for $\alpha=0$. As a consequence, (23) can be read as a generalization of this procedure where the points cannot be drawn independently and with equal probability because of the correlation structure.

For h any map from E_1 into E_1, we define $h^{\sqcup}$ by

$$\begin{aligned} h^{\sqcup}: E &\to E,\\ (i,x) &\mapsto \big(i,h(x)\big). \end{aligned}$$

With this notation at hand, for v in $V_0(E_1)$, $(\phi_t^v)^{\sqcup}$ is the solution of the equations:

$$\mathrm{d}\big(\phi_t^v\big)^{\sqcup}(i,x)=v^{\sqcup}\big(\big(\phi_t^v\big)^{\sqcup}(i,x)\big),\quad 1\leqslant i\leqslant m.$$

Note that we only consider a restricted set of perturbations of configurations in the sense that we move atoms on each "layers" without "crossing": By the action of $(\phi_t^v)^{\sqcup}$, an atom of the form (i,x) is moved into an atom of the form (i,y), leaving its first coordinate untouched.

Theorem 12. *Assume that* (E_1,K_1,λ_1) *satisfy Hypotheses* 1, 2 *and* 3. *Let* $s=-1/\alpha$ *be an integer greater than* 1. *For* F *and* G *cylindrical functions, for* $v\in V_0(E_1)$, *we have*:

$$\begin{aligned}
&\int_{\chi_\Lambda}\nabla_v F(\omega)G(\omega)\,\mathrm{d}\mu_{\alpha,sK_{1,\Lambda},\lambda_1}(\omega)\\
&\quad=-\int_{\chi_\Lambda}F(\omega)\nabla_v G(\omega)\,\mathrm{d}\mu_{\alpha,K_{1,\Lambda},\lambda_1}(\omega)\\
&\qquad+\frac{1}{|\alpha|}\int_{\chi_\Lambda}F(\omega)G(\omega)\left(\sum_{\eta\subset\omega}\big(B_v^{\lambda_1}(\eta)+\nabla_v U(\eta)\big)R(\eta,\omega)\right)\mathrm{d}\mu_{\alpha,sK_{1,\Lambda},\lambda_1}(\omega).
\end{aligned}$$

Proof. We first apply (20) to the process $\xi=(\xi_1,\dots,\xi_s)$. Remember that $\Theta\xi$ is equal to $\xi_1\cup\dots\cup\xi_s$. A cylindrical function of $\Theta\xi$ is a function of the form:

$$H(\Theta\xi)=f\left(\int h_1\,\mathrm{d}\Theta\,\xi,\dots,\int h_N\,\mathrm{d}\Theta\,\xi\right)$$

where $h_1,\dots,h_N\in\mathcal{D}=C^\infty(E_1)$, $f\in C_b^\infty(\mathbf{R}^N)$. Such a functional can be written as $F\circ\Theta(\xi)$ where F is a cylindrical function of ξ. Moreover, for $v\in V_0(E_1)$,

$$\begin{aligned}\nabla_v H(\Theta\xi) &= \lim_{t\to 0}\frac{1}{t}\big(H\big(\phi_t^v(\Theta\xi)\big) - H(\Theta\xi)\big)\\ &= \lim_{t\to 0}\frac{1}{t}\big(F\big(\Theta\big(\phi_t^v\big)^{\sqcup}\xi\big) - F(\Theta\xi)\big)\\ &= \nabla_{v^{\sqcup}}F(\Theta\xi).\end{aligned}\tag{24}$$

In view of (22),

$$U(\xi) = -\log\det J(\xi_1,\ldots,\xi_s) = \sum_{j=1}^{s} U(\xi_j).\tag{25}$$

Analyzing the proof of (20), we see that the intrinsic definition of B_v^λ is

$$B_v^\lambda(\xi) = \int \operatorname{div}_\lambda(v)\,\mathrm{d}\xi$$

where

$$\operatorname{div}_\lambda(v)(x) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}(\phi_t^v)^*\lambda}{\mathrm{d}\lambda}(x)\right)\bigg|_{t=0}.$$

In view of (24), we only need to consider flows on E associated to vector fields of the form $v^{\sqcup}$ for $v \in V_0(E_1)$. Hence,

$$B_{v^{\sqcup}}^\lambda(\xi) = \sum_{j=1}^{s} B_v^{\lambda_j}(\xi_j).\tag{26}$$

It follows from the previous considerations that:

$$\begin{aligned}&\int_{\chi_{\Lambda^{\sqcup}}} \nabla_{v^{\sqcup}}F(\Theta\xi)G(\Theta\xi)\,\mathrm{d}\mu_{-1,K_\Lambda,\lambda}(\xi)\\ &\quad= -\int_{\chi_{\Lambda^{\sqcup}}} F(\Theta\xi)\nabla_{v^{\sqcup}}G(\Theta\xi)\,\mathrm{d}\mu_{-1,K_\Lambda,\lambda}(\xi)\\ &\qquad+\int_{\chi_{\Lambda^{\sqcup}}} F(\Theta\xi)G(\Theta\xi)\big(B_{v^{\sqcup}}^\lambda(\xi) + \nabla_{v^{\sqcup}}U(\xi)\big)\,\mathrm{d}\mu_{-1,K_\Lambda,\lambda}(\xi)\end{aligned}$$

where $\Lambda^{\sqcup} = \bigcup_{j=1}^{s}\{i\}\times\Lambda$. Since the ξ_j's are independent and identically distributed, according to (25) and (26), we have

$$\begin{aligned}\mathbb{E}\big[B_{v^{\sqcup}}^\lambda(\xi) + \nabla_{v^{\sqcup}}U(\xi)\bigm|\Theta\xi\big] &= s\mathbb{E}\big[B_v^{\lambda_1}(\xi_1) + \nabla_v U(\xi_1)\bigm|\Theta\xi\big]\\ &= -\frac{1}{\alpha}\sum_{\eta\subset\Theta\xi}\big(B_v^{\lambda_1}(\eta) + \nabla_v U(\eta)\big)R(\eta,\Theta\xi).\end{aligned}$$

Thus, we obtain:

$$\begin{aligned}&\int_{\chi_\Lambda} \nabla_v F(\omega) G(\omega)\, \mathrm{d}\mu_{\alpha, sK_{1,\Lambda},\lambda_1}(\omega)\\&\quad = -\int_{\chi_\Lambda} F(\omega)\nabla_v G(\omega)\, \mathrm{d}\mu_{\alpha, K_{1,\Lambda},\lambda_1}(\omega)\\&\qquad - \frac{1}{\alpha}\int_{\chi_\Lambda} F(\omega)G(\omega)\bigg(\sum_{\eta\subset\omega}\big(B_v^{\lambda_1}(\eta) + \nabla_v U(\eta)\big)R(\eta,\omega)\bigg)\, \mathrm{d}\mu_{\alpha, sK_{1,\Lambda},\lambda_1}(\omega).\end{aligned}$$

The proof is thus complete. □

4.3. α-Permanental point processes

For permanental point processes, we begin with the situation where $\alpha = 1$. In this case,

$$j_{1,K_\Lambda,\lambda}\big(\{x_1,\dots,x_n\}\big) = \operatorname{Det}(\mathrm{I}+K_\Lambda)^{-1}\operatorname{per}\big(J(x_i,x_j),\ 1 \leqslant i, j \leqslant n\big).$$

We aim to follow the lines of proof of Theorem 10, for, we need some preliminary considerations.

For any integer n, let $D[n]$ be the set of partitions of $\{1,\dots,n\}$. The cardinal of $D[n]$ is known to be the n-th Bell number (see [1]), denoted by $\mathfrak{B}_n$ and which can be computed by their exponential generating function: for any real x,

$$\sum_{n=0}^{\infty}\mathfrak{B}_n\frac{x^n}{n!} = e^{e^x} - 1. \tag{27}$$

For an $n\times n$ matrix $A = (a_{ij},\ 1 \leqslant i, j \leqslant n)$ and for τ a subset of $\{1,\dots,n\}$, we denote by $A[\tau]$ the matrix $(a_{ij},\ i\in\tau, j\in\tau)$. For a partition σ of $\{1,\dots,n\}$, $\iota(\sigma)$ is the number of non-empty parts of σ. This means that $\sigma = (\tau_1,\dots,\tau_{\iota(\sigma)})$, where the τ_i's are disjoint subsets of $\{1,\dots,n\}$ whose union is exactly $\{1,\dots,n\}$. Then, we set

$$\det A[\sigma] = \prod_{j=1}^{\iota(\sigma)}\det J[\tau_j].$$

It is proved in [11, Corollary 1.7] that

$$\operatorname{per} A = \sum_{\sigma\in D[n]}(-1)^{n+\iota(\sigma)}\det A[\sigma]. \tag{28}$$

We slightly change the definition of the potential energy of a finite configuration as

$$\begin{aligned}U : \chi_0 &\to \mathbf{R},\\ \xi &\mapsto -\log\operatorname{per} J(\xi).\end{aligned}$$

A new hypothesis then arises:

Hypothesis 4. The functional U is differentiable at every configuration $\xi \in \chi_0$. Moreover, for any $v \in V_0(E)$, there exists $(u_n,\ n \geqslant 1)$ a sequence of non-negative real as in Lemma 4 such that for any $\xi \in \chi_0$, we have

$$\left|\langle \nabla U(\xi), v\rangle_{L^2(\mathrm{d}\xi)}\right| \leqslant \frac{u_{|\xi|}}{\operatorname{per} J(\xi)}. \tag{29}$$

An analog of Theorem 9 now becomes.

Theorem 13. *Assume that K is of finite rank N and that the kernel J is once differentiable with continuous derivative. Then, Hypothesis* 4 *is satisfied.*

Proof. Since K is of finite rank N there are at most N points in any configuration. It is clear from (28) that $(t \mapsto U(\phi_t^v(\xi)))$ is differentiable. Since $|\det J(\xi)[\tau]| \leqslant c|\tau|^{|\tau/2|}$ where $|\tau|$ is the cardinal of $\tau \in D[|\xi|]$, we get

$$\left|\langle \nabla U(\xi), v\rangle_{L^2(\mathrm{d}\xi)}\right| \leqslant c\frac{\mathfrak{B}_{|\xi|}|\xi|^{|\xi|/2}}{\operatorname{per} J(\xi)}\mathbf{1}_{\{|\xi| \leqslant N\}}.$$

Hence the result. $\square$

Remark 1. The finite rank condition is rather restrictive but the sequence $(\mathfrak{B}_n n^{n/2},\ n \geqslant 1)$ has not a finite exponential generating function thus we can't avoid it. In order to circumvent this difficulty one would have to improve known upper-bounds on permanents.

We can then state the main result for this subsection.

Theorem 14. *Assume that (E, K, λ) satisfy Hypotheses* 1, 2 *and* 4. *Let F and G belong to $\mathcal{F}C_b^\infty$. For any compact Λ, we have*:

$$\begin{aligned}
&\int_{\chi_\Lambda} \nabla_v F(\xi) G(\xi)\, \mathrm{d}\mu_{1,K_\Lambda,\lambda}(\xi) \\
&\quad = -\int_{\chi_\Lambda} F(\xi)\nabla_v G(\xi)\, \mathrm{d}\mu_{1,K_\Lambda,\lambda}(\xi) + \int_{\chi_\Lambda} F(\xi)G(\xi)\big(B_v^\lambda(\xi) + \nabla_v U(\xi)\big)\, \mathrm{d}\mu_{1,K_\Lambda,\lambda}(\xi).
\end{aligned}$$

Proof. Same as the proof of Theorem 10. $\square$

Now then, we can work as in Section 4.2 and we obtain the integration by parts formula for α-permanental point processes.

Corollary 2. *Assume that (E_1, K_1, λ_1) satisfy Hypotheses* 1, 2 *and* 4. *Let $s = 1/\alpha$ be an integer greater than* 1. *For F and G cylindrical functions, for $v \in V_0(E_1)$, we have*:

$$\int_{\chi_\Lambda} \nabla_v F(\omega) G(\omega)\, \mathrm{d}\mu_{\alpha, sK_{1,\Lambda},\lambda_1}(\omega)$$

$$= -\int_{\chi_\Lambda} F(\omega)\nabla_v G(\omega)\, \mathrm{d}\mu_{\alpha,K_{1,\Lambda},\lambda_1}(\omega)$$

$$+ \frac{1}{\alpha}\int_{\chi_\Lambda} F(\omega)G(\omega)\Bigl(\sum_{\eta\subset\omega}\bigl(B_v^{\lambda_1}(\eta) + \nabla_v U(\eta)\bigr)R(\eta,\omega)\Bigr)\, \mathrm{d}\mu_{\alpha,sK_{1,\Lambda},\lambda_1}(\omega).$$

5. Conclusion

We showed that for any $\alpha \in \mathfrak{A}$, a stochastic integration by parts formula holds. A first well-known consequence of such a formula is the closability of ∇. We define the norm $\|.\|_{2,1}$ on $\mathcal{F}C_b^\infty(\mathcal{D},\chi)$ by:

$$\|F\|_{2,1}^2 = \|F\|_{L^2(\mu)}^2 + \mathbb{E}\bigl[\|\nabla F\|^2\bigr] = \mathbb{E}\bigl[F^2\bigr] + \mathbb{E}\Bigl[\int |\nabla_x F|^2\, \mathrm{d}\xi(x)\Bigr]$$

and we call $\mathcal{D}_{2,1}$ the closure of $\mathcal{F}C_b^\infty(\mathcal{D},\chi)$ for the norm $\|.\|_{2,1}$. A classical consequence of the previous results is then that, for any α-DPPP known to exist, the operator ∇ is closable and can thus be extended to $\mathcal{D}_{2,1}$. Moreover, the integration by parts remains valid as is for F and G in $\mathcal{D}_{2,1}$. With the same lines of proof we retrieve the result of [29], which says that the Dirichlet form: $\mathcal{E}(F,F) = \mathbb{E}[\langle\nabla F, \nabla F\rangle]$ is closable.

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