Asymptotic Control and Stabilization of Nonlinear Oscillators with Non-isolated Equilibria

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Received June 8, 2000; revised January 23, 2001

Let \(\Phi: H \rightarrow \mathbb{R}\) be a \(C^1\) function on a real Hilbert space \(H\) and let \(\gamma > 0\) be a positive (damping) parameter. For any control function \(c: \mathbb{R}_+ \rightarrow \mathbb{R}_+\), which tends to zero as \(t \rightarrow +\infty\), we study the asymptotic behavior of the trajectories of the damped nonlinear oscillator

\[
(HBFC) \quad x''(t) + \gamma x'(t) + \nabla \Phi(x(t)) + c(t) x(t) = 0.
\]

We show that if \(c(t)\) does not tend to zero too rapidly as \(t \rightarrow +\infty\), then the term \(c(t) x(t)\) asymptotically acts as a Tikhonov regularization, which forces the trajectories to converge to a particular equilibrium. Indeed, in the main result of this paper, it is established that, when \(\Phi\) is convex and \(S = \text{argmin} \Phi \neq \emptyset\), under the key assumption that \(c\) is a “slow” control, i.e., \(\int_{+}^{t} c(t) \, dt = +\infty\), then each trajectory of the (HBFC) system strongly converges, as \(t \rightarrow +\infty\), to the element of minimal norm of the closed convex set \(S\). As an application, we consider the damped wave equation with Neumann boundary condition

\[
\left\{ \begin{array}{ll}
u_{tt} + \gamma u_t - Au + c(t) \, u(t) = 0 & \text{in } \Omega \times \mathbb{R}_+, \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega \times \mathbb{R}_+.
\end{array} \right.
\]

Key Words: nonlinear oscillator; slow control; Tikhonov regularization; heavy ball with friction.

1. INTRODUCTION

(a) Let \(H\) be a real Hilbert space, with scalar product and corresponding norm respectively denoted by \(\langle \cdot, \cdot \rangle\) and \(|\cdot|\). Let \(\Phi: H \rightarrow \mathbb{R}\) be a given \(C^1\) real-valued function, called the potential function. The equilibria
are the solutions of the equation $\nabla \Phi(x) = 0$, where $\nabla \Phi$ is the gradient of $\Phi$. Among the equilibria, because of their deep physical or economical interpretation, the local or global minima of $\Phi$ are of special interest. In order to reach such optima, a powerful method consists in following the trajectories, as $t \to \infty$, of a corresponding dissipative dynamical gradient-like system.

In this paper, we are specially interested in the case where $\Phi$ has non isolated equilibria. This is a particular important situation which occurs for example when minimizing a convex function $\Phi$ which is not strictly convex (like linear programming or semi-coercive Neumann problems), and more generally when considering a local version of this type of situation.

The aim of this paper is to study these questions with the help of the following second order (in time) gradient-like system

\[(HBFC) \quad \ddot{x}(t) + c \dot{x}(t) + \nabla \Phi(x(t)) + \varepsilon(t)x(t) = 0,\]

where $c > 0$ is a positive damping parameter and $\varepsilon: \mathbb{R}_+ \to \mathbb{R}_+$ is a control function such that $\lim_{t \to +\infty} \varepsilon(t) = 0$.

(b) Let us first explain why considering second order in time gradient-like systems in this optimization context. Indeed, besides the classical steepest descent method

\[(SD) \quad \dot{x}(t) + \nabla \Phi(x(t)) = 0\]

which naturally appears in various domains like mechanics, differential geometry, economics..., it has been appearing (in the last two decades), with more and more evidence, that second order in time dissipative gradient-like systems also enjoy remarkable optimization properties. Among these, a particular important dynamical system is

\[(HBF) \quad \ddot{x}(t) + \gamma \dot{x}(t) + \nabla \Phi(x(t)) = 0\]

called, because of its mechanical interpretation, the Heavy Ball with Friction system. The (HBF) system is an asymptotic approximation of the equation describing the motion of a material point with positive mass, subjected to stay on the graph of $\Phi$, and which moves under the action of the gravity force, the reaction force, and the friction force ($\gamma > 0$ is the friction parameter). The (HBF) system is dissipative, and can be studied in the classical framework of the theory of dissipative dynamical systems (see, for example, Hale [15] and Haraux [16]).

It is worth pointing out that, in a series of recent papers, most of the convergence results known in the case of the steepest descent, have been proved to be also valid in the case of the (HBF) system. To quote only some of them, when $\Phi$ is convex, Bruck’s theorem [10] known for the
steepest descent, has been extended by Alvarez [1] in the case of the (HBF) system. When \( \Phi \) is real analytic on \( H = \mathbb{R}^n \), the Łojasiewicz theorem [19, 20] has been extended to the second order in time system by Jendoubi [18], Haraux and Jendoubi [17].

The introduction of the inertial term \( \ddot{x}(t) \) in the dynamical system permits to overcome some drawbacks of the steepest descent method. By contrast with (SD), the (HBF) system is not a descent method. It is the global energy (kinetic + potential) which decreases. So doing, by following the trajectories, one can go up and down along the graph of \( \Phi \) (“‘montagnes russes’” method) and explore the equilibria of \( \Phi \), see Attouch, Goudou, Redont [6]. Moreover it has been proved, see Goudou [14], that when \( \Phi \) is a Morse function, then generically with respect to the initial data, the trajectories converge to local minima of \( \Phi \).

(c) Let us now justify the introduction of a Tikhonov-like asymptotic regularization term \( \varepsilon(t) x(t) \) in the dynamics of the (HBFC) system. The idea of coupling approximation methods with the steepest descent has been considered in particular by Attouch and Cominetti [5]. To consider only a simple case of their paper, they proved that when \( \Phi \) is convex and \( \varepsilon : \mathbb{R}_+ \to \mathbb{R}_+ \) is a \( C^1 \) control function which tends to zero slowly, i.e., such that \( \int_0^{+\infty} \varepsilon(t) \, dt = +\infty \), then each trajectory of the system

\[
(SDC) \quad \ddot{x}(t) + \nabla \Phi(x(t)) + \varepsilon(t) x(t) = 0
\]

strongly converges to the point of minimal norm of the set \( S = \operatorname{argmin} \Phi \) (which is assumed to be nonempty). Roughly speaking, the condition \( \int_0^{+\infty} \varepsilon(t) \, dt = +\infty \) just expresses that \( \varepsilon(t) \) does not tend to zero too rapidly, which allows the Tikhonov regularization term \( \varepsilon(t) x(t) \) to be effective asymptotically.

This result can be viewed as an asymptotic selection property: by using such a slow control \( \varepsilon \), one can force all the trajectories to converge to the same equilibrium, which here is the equilibrium of minimal norm. This makes a sharp contrast with the non controlled situation (or fast control) where the limits of the trajectories are only weak limits, depend on the initial data, and are in general difficult to identify.

It is then a natural question to know if it is possible to extend the selection properties of the Tikhonov regularization to the second order gradient-like system (HBF). That is where the (HBFC) (Heavy Ball with Friction and Control) system comes into consideration. Let us notice that the (HBFC) system has a similar mechanical interpretation as the (HBF) system with an extra attraction force directed towards the origin (for example, with a spring of varying stiffness \( \varepsilon(t) \)). One can easily conceive that, if \( \varepsilon(t) \) does

\footnote{Precisely defined by \( \operatorname{argmin} \Phi = \{ z \in H \mid \forall z' \in H, \Phi(z) \leq \Phi(z') \} \).}
not tend to zero too rapidly, the mechanical system will select an equilibrium which is as close as possible to the origin.

(d) The main results of the paper are the convergence Theorems 2.3 and 2.4 where we restrict ourselves to the case of a convex function \( \Phi \). For convenience of the reader, we state these two results in a unifying way in the following statement.

**Theorem.** Assume that \( \Phi: H \to \mathbb{R} \) is a convex, \( \mathcal{C}^1 \) function, such that \( \nabla \Phi \) is Lipschitzian on the bounded sets, and such that \( S = \text{argmin}\ \Phi \neq \emptyset \). Let \( \gamma > 0 \) be a positive parameter. Consider a \( \mathcal{C}^1 \) function \( e: \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \lim_{t \to +\infty} e(t) = \lim_{t \to +\infty} \dot{e}(t) = 0 \) for every \( t \in \mathbb{R}_+ \). Then, for every \( (x_0, \dot{x}_0) \in H \times H \), there exists a unique solution \( x: [0, +\infty) \to H \) of the (HBFC) Cauchy problem:

\[
\begin{cases}
\ddot{x}(t) + \gamma \dot{x}(t) + \nabla \Phi(x(t)) + e(t) x(t) = 0, \\
x(0) = x_0, \ \dot{x}(0) = \dot{x}_0.
\end{cases}
\]

The solution \( x \) satisfies the following asymptotical behavior:

(i) There is \( x_\ast \in S \) such that \( x(t) \) weakly converges to \( x_\ast \) as \( t \to +\infty \).

(ii) (Slow parametrization) Additionally assume that \( \int_0^{+\infty} e(t) \, dt = +\infty \). Then \( x(t) \) strongly converges to \( \text{proj}_S(0) \) as \( t \to +\infty \).

The proof of (i) (fast parametrization) relies on Opial’s lemma and is not very different from Alvarez paper [1], which corresponds to the case \( \dot{e} = 0 \). On the opposite, the slow parametrization case is much more involved than in the case of the steepest descent considered in [5]. It does not seem possible to follow the method used in [5] in the case of the steepest descent, which consists in proving that the trajectories of the (SDC) system get close asymptotically to the trajectory of the Tikhonov approximation. We need to combine analytic and geometric arguments, taking into account the possibility for the trajectories to enter some particular subdomain of \( H \).

This result can be interpreted as the construction of a smooth time-varying feedback. For a survey on stabilization of nonlinear systems by nonautonomous feedbacks, we refer to Coron [12].

(e) The paper is organized as follows. In Section 2.1, we precisely state the global existence results (Theorems 2.1 and 2.2), which first are not reduced to the convex case, and secondly consider the case of a possibly increasing control \( \varepsilon \). In Section 2.2, we precisely state the asymptotic convergence results (Theorems 2.3 and 2.4). The results are proved in Section 3 (asymptotic convergence) and Section 5 (global existence). It would be more natural to first prove the global existence results, then to study the asymptotic control problem. However, to facilitate access to the proof of
the main results, which are the control results, we choose to do the oppo-
site. This is possible since the proofs of these two different aspects (global
existence and asymptotic control) are largely independent. So doing, the
major results are considered and proved from the very beginning of the
paper. In Section 4, we show that the method of control developed in this
paper can be applied to some infinite dimensional hyperbolic systems such
as the wave equation. Finally, in Section 6, we give some remarks and
questions, which may give directions for further research on the subject.

2. MAIN RESULTS

In the following, we will assume the following (rather standard) set of
hypotheses:

**Hypothesis 2.1.** Let $H$ be a real Hilbert space. Let us consider a map
$\Phi: H \rightarrow \mathbb{R}$ which satisfies the following conditions:

\[
\mathcal{H}_\Phi \begin{cases} 
(i) & \text{the map } \Phi \text{ is of class } C^1 \text{ on } H; \\
(ii) & \text{the map } \nabla \Phi \text{ is Lipschitzian on the bounded subsets of } H; \\
(iii) & \text{the map } \Phi \text{ is bounded from below on } H.
\end{cases}
\]

For $t_0 \in \mathbb{R}$, let $e: [t_0, +\infty) \rightarrow \mathbb{R}^+$ be a function of class $C^2$ such that

\[
\lim_{t \to +\infty} e(t) = 0, \quad \lim_{t \to +\infty} e'(t) = 0.
\]

Let $\gamma > 0$, $(x_0, \dot{x}_0) \in H \times H$, and the (HBFC) system is defined as

\[
\begin{cases}
\ddot{x}(t) + \gamma \dot{x}(t) + \nabla \Phi(x(t)) + e(t) \dot{x}(t) = 0, \\
x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0.
\end{cases}
\]

**2.1. Global Existence**

2.1.1. Non-increasing Control

The next theorem summarizes global existence and convergence properties
of solutions of the (HBFC) system, when the function $e$ is assumed to be
non-increasing. The results are quite similar to the results without a control
((HBF) system, see [6]).

**Theorem 2.1** (Global Existence with a Non-increasing Control). Assume
Hypothesis 2.1, together with:
(\(\mathcal{H}_{-\text{ii}}\)) the function \(\varepsilon\) is non-increasing, i.e., \(\dot{\varepsilon}(t) \leq 0\) for every \(t \in [t_0, +\infty)\).

- **Part (a).** Then,
  
  (i) there exists a unique maximal solution of the (HBFC) system 
  
  \[x : [t_0, +\infty) \to H, \text{ which is of class } \mathcal{C}^2.\]
  
  (ii) \(\dot{x}\) belongs to \(L^\infty([t_0, +\infty), H) \cap L^2([t_0, +\infty), H)\).

- **Part (b).** Additionally assume that:
  
  (\(\mathcal{H}_{-\text{i}v}\)) the map \(x\) belongs to \(L^\infty([t_0, +\infty), H)\).

  Then,
  
  (iii) \(\ddot{x}\) belongs to \(L^\infty([t_0, +\infty), H)\):
  
  (iv) \(\lim_{t \to +\infty} \dot{x}(t) = 0\) and \(\lim_{t \to +\infty} x(t) = 0;\)
  
  (v) \(\lim_{t \to +\infty} \nabla \Phi(x(t)) = 0.\)

The proof of Theorem 2.1 uses the energy function defined by

\[E(t) = \frac{1}{2} |\dot{x}(t)|^2 + \Phi(x(t)) + \frac{\varepsilon(t)}{2} |x(t)|^2,\]

and is given in Section 5.1.

**Remark 2.1.** The solution map \(x\) of the (HBFC) system may not be bounded in general, even when the map \(\Phi\) is assumed to be convex. For example, when \(\varepsilon = 0\), \(\Phi\) is minorized and does not attain its infimum, it is proved in Alvarez [1] that \(\lim_{t \to +\infty} \Phi(x(t)) = \inf \Phi\), which clearly forces \(\lim_{t \to +\infty} |x(t)| = +\infty.\)

Under the additional assumption that the map \(\Phi\) is coercive, we obtain that the solution \(x\) is bounded.

**Corollary 2.1.** Under the assumptions of Theorem 2.1 (Hypothesis 2.1 and (\(\mathcal{H}_{-\text{ii}}\))), additionally assume that:

(\(\mathcal{H}_{-\text{i}v}\)) the map \(\Phi\) is coercive, i.e., \(\lim_{|x| \to +\infty} \Phi(x) = +\infty.\)

Then the map \(x\) is in \(L^\infty([t_0, +\infty), H)\), hence satisfies all the conclusions of Theorem 2.1.

The proof of Corollary 2.1 is given in Section 5.1.1.

**2.1.2. Possibly Increasing Control**

In Theorem 2.1, we consider the case of a non-increasing control \(\varepsilon\). Having numerical applications in mind, it seems of importance to allow (small) errors on the control \(\varepsilon\). In particular, we want to consider the theoretical framework of a possibly increasing control. In fact, when the
function $\varepsilon$ is not assumed to be non-increasing, the global existence properties of solutions of the (HBFC) system still hold with some additional conditions that we precise now. When considering a possibly increasing control $\varepsilon$, the main difficulty is to show that the solution map $x$ is bounded. We now state the main result of Section 2.1, which also gives sufficient conditions for a solution of the (HBFC) system to be bounded.

**Theorem 2.2 (Global Existence with a Possibly Increasing Control).** Assume Hypothesis 2.1, together with:

$(\mathcal{H}_e - iii)$ the positive part of $\dot{\varepsilon}$ belongs to $L^1([t_0, +\infty), \mathbb{R}_+)$, i.e.,

$$\int_{t_0}^{+\infty} \dot{\varepsilon}(t)_+ \, dt < +\infty.$$

- **Part (a).** Let $x: [t_0, T_{\text{max}}) \to H$ be a maximal solution of the (HBFC) system (with $t_0 < T_{\text{max}} \leq +\infty$). Additionally assume that $x$ is bounded, i.e., $x \in L^\infty([t_0, T_{\text{max}}), H)$. Then,

  (i) $T_{\text{max}} = +\infty$;
  (ii) $\dot{x} \in L^\infty([t_0, +\infty), H)$ and $x \in L^2([t_0, +\infty), H)$;
  (iii) $\ddot{x} \in L^\infty([t_0, +\infty), H)$;
  (iv) $\lim_{t\to +\infty} \dot{x}(t) = 0$ and $\lim_{t\to +\infty} \ddot{x}(t) = 0$;
  (v) $\lim_{t\to +\infty} N_F(x(t)) = 0$.

- **Part (b).** Assume that the map $F$ is strongly coercive, i.e., there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $F(x) \geq \alpha |x|^2 - \beta$ for every $x \in H$. Then, every solution of the (HBFC) system is bounded.

- **Part (c) Only assume that the map $F$ is coercive, i.e., $\lim_{|x| \to +\infty} F(x) = +\infty$, and additionally assume that:

$$\int_{t_0}^{+\infty} \frac{\dot{\varepsilon}(t)_+}{\varepsilon(t)} \, dt < +\infty \quad \text{or} \quad \int_{t_0}^{+\infty} t^2 \dot{\varepsilon}(t)_+ \, dt < +\infty.$$  

Then, every solution of the (HBFC) system is bounded.

The proof of Theorem 2.2 uses Gronwall’s type arguments together with a majorization of the energy function, and is given in Section 5.2.

### 2.2. Convergence of the Trajectories in the Convex Case

Once the (global) existence is acquired, the main point in the study of a dissipative system is to investigate the convergence properties of the solution map. When the map $F$ is assumed to be convex, the main result

\[4\] Where the positive part of a real number $a \in \mathbb{R}$ is defined by $a_+ = \max\{0, a\}$. 


of Section 2.2, that we now state, shows the strong convergence of the solutions of the (HBFC) system, with a “slow” control, toward a specific point (i.e., the point of minimal norm in $S = \text{argmin } \Phi$).

**Theorem 2.3 (Slow Parametrization).** Under the assumptions of Theorem 2.1 (Hypothesis 2.1 and (H$_e$ − ii)), or under the assumptions of Theorem 2.2 (Hypothesis 2.1, (H$_e$ − iii), and $x$ bounded), additionally assume that the map $\Phi$ is convex, that $S = \text{argmin } \Phi$, and that:

($H_e$ − iv) the function $\varepsilon$ does not belong to $L^1([t_0, +\infty), \mathbb{R}_+)$, i.e.,

\[
\int_{t_0}^{+\infty} \varepsilon(t) \, dt = +\infty.
\]

Then, the map $x$ strongly converges to $\text{proj}_S(0)$, precisely $\lim_{t \to +\infty} |x(t) - \text{proj}_S(0)| = 0$ (hence $x \in L^\infty([t_0, +\infty), H)$).

**Remark 2.2.** From Theorem 2.3, for every $y \in H$, one easily deduces that the solution of the following system,

\[
\begin{align*}
\ddot{x}(t) + \gamma \dot{x}(t) + \nabla \Phi(x(t)) + \varepsilon(t)(x(t) - y) &= 0, \\
x(t_0) &= x_0, \quad \dot{x}(t_0) = \dot{x}_0,
\end{align*}
\]

converges to $\text{proj}_S(y)$ (under the assumptions of Theorem 2.3).

The next theorem shows that the solutions of the (HBFC) system weakly converge, with a “fast” control. As shown in Remarks 2.5 and 2.6, it does not seem to be possible to obtain a better result (in the sense of the specification of the limit point, or by obtaining strong convergence rather than weak convergence), without further assumptions.

**Theorem 2.4 (Fast Parametrization).** Under the assumptions of Theorem 2.1 (Hypothesis 2.1 and (H$_e$ − ii)), or under the assumptions of Theorem 2.2 (Hypothesis 2.1, (H$_e$ − iii), and $x$ bounded), additionally assume that the map $\Phi$ is convex, that $S = \text{argmin } \Phi \neq \emptyset$, and that:

($H_e$ − v) the function $\varepsilon$ belongs to $L^1([t_0, +\infty), \mathbb{R}_+)$, i.e., \(\int_{t_0}^{+\infty} \varepsilon(t) \, dt < +\infty\).

Then $x \in L^\infty([t_0, +\infty), \mathbb{R}_+)$ and there exists some $x_\infty \in S$ such that $x$ weakly converges to $x_\infty$, precisely $\lim_{t \to +\infty} x(t) = x_\infty$ and $\lim_{t \to +\infty} \Phi(x(t)) = \min \Phi$.

Theorems 2.3 and 2.4 are proved in Section 3.

**Remark 2.3.** From Theorems 2.3 and 2.4, one easily deduces the theorem stated in the Introduction.
Remark 2.4. The fast parametrization part can be also seen as a generalization of the result of Alvarez [1] who established that each trajectory of the (HBF) system (i.e., the (HBFC) system with an always zero control, precisely $e(t) = 0$ for every $t$), in the convex case, weakly converges to a global minimum of $\Phi$. Not surprisingly, our proof of Theorem 2.4 is greatly inspired from the proof of Alvarez Theorem given in [6]. Alvarez Theorem is itself an extension of the celebrated Bruck theorem [10] (first order steepest descent method) to the second order dissipative (HBF) system.

Remark 2.5. Under the assumptions of Theorem 2.4, the solution map $x$ does not strongly converge in general. See Baillon [7] and see [6] for a counterexample.

Remark 2.6. In Theorem 2.4, the weak limit of the solution map $x$ is not equal to $\text{proj}_S(0)$ in general. It depends on the initial data, contrarily to the slow parametrization case (Theorem 2.3) where the limit is independent of the initial data. For example, consider the case where $\nabla\Phi = 0$ and $e = 0$, with the solution $x(t) = x(t_0) + \frac{1}{c} \dot{x}(t_0)(1 - e^{-c(t-t_0)})$.

It is worth completing Theorem 2.4 by strong convergence results. Let us first consider the case where the map $\Phi$ is additionally assumed to be strongly convex.

**Corollary 2.2.** Under the assumptions of Theorem 2.4, additionally assume that the map $\Phi$ is strongly convex, that is, for any $R > 0$, there exists a function $\beta_R: \mathbb{R}^+ \to \mathbb{R}^+$ such that, for every sequence $(t_n) \subset \mathbb{R}^+, \beta_R(t_n) \to 0 \Rightarrow t_n \to 0$, and

$$\forall (y, z) \in B(0, R) \times B(0, R), \quad \langle \nabla \Phi(y) - \nabla \Phi(z), y - z \rangle \geq \beta_R(|y-z|).$$

(1)

Then, each trajectory $x$ of the (HBFC) system strongly converges as $t$ goes to $+\infty$ to the unique global minimizer $\bar{x}$ of $\Phi$.

The proof of Corollary 2.2 is given in Section 3.3.

Strong convergence is also obtained when the map $\Phi$ is additionally assumed to be even. Note that $S$ is not reduced in general to a single element in that case, which makes this result quite subtle.

**Corollary 2.3.** Under the assumptions of Theorem 2.4, additionally assume that the map $\Phi$ is even, i.e., $\Phi(-z) = \Phi(z)$ for every $z$, then the solution map $x$ strongly converges to some $x_{\bar{e}} \in S$, precisely $\lim_{t \to +\infty} |x(t) - x_{\bar{e}}| = 0$.

The proof of Corollary 2.3 is given in Section 3.3.
3. PROOF OF THE CONVERGENCE RESULTS

In this section, we prove the convergence results stated in Section 2.2, i.e., Theorems 2.3 and 2.4. We now assume that the map $\Phi$ is convex and that the convex closed set $S = \text{argmin } \Phi = \{x \in H \mid \nabla \Phi(x) = 0\}$ is nonempty. For a matter of readability, we write the proofs under the assumptions of Theorem 2.1 (Hypothesis 2.1 and $(\mathcal{H}_i - ii)$). We let the reader check that the same proof holds under the assumptions of Theorem 2.2 (Hypothesis 2.1, $(\mathcal{H}_i - iii)$, and the solution map $x$ is bounded). Note indeed that a key assumption is that the map $\dot{x}$ belongs to $L^2([t_0, +\infty), H)$. We first recall the following classical result which is of importance in the following.

**Proposition 3.1.** Let $\Phi : H \to \mathbb{R}$ be convex and $C^1$, let $(x_n)$ be a sequence in $H$ and $z \in H$ such that $(x_n)$ converges weakly to $z$ and $(\nabla \Phi(x_n))$ strongly converges to 0. Then $z \in S = \text{argmin } \Phi$ and $\lim_{n \to +\infty} \Phi(x_n) = \min \Phi$.

**Proof of Proposition 3.1.** If $x_n \to z$ weakly, by using the graph closedness property of the maximal monotone operator $\nabla \Phi$ in $w-H \times s-H$, we conclude that $\nabla \Phi(z) = 0$. The result can be obtained more elementarily by noticing that

$$\forall \xi \in H \quad \Phi(\xi) \geq \Phi(x_n) + \langle \nabla \Phi(x_n), \xi - x_n \rangle.$$

By using the weak lower semicontinuity of the convex continuous function $\Phi$, and noticing that, in the duality bracket $\langle \nabla \Phi(x_n), \xi - x_n \rangle$, the two terms are respectively norm converging to zero and weakly convergent, hence bounded, we can pass to the lower limit to obtain

$$\forall \xi \in H, \quad \Phi(\xi) \geq \limsup_{n \to +\infty} \Phi(x_n) \geq \liminf_{n \to +\infty} \Phi(x_n) \geq \Phi(z),$$

that is, $z \in \text{argmin } \Phi = S$ and $\lim_{n \to +\infty} \Phi(x_n) = \min \Phi$.  

The proofs of Theorems 2.3 and 2.4 (the slow and the fast parametrization) rely on the study of the function $h_z$, that we now precisely define. Let $z \in S$, we define the function $h_z : [t_0, +\infty) \to \mathbb{R}_+$ by

$$h_z(t) = \frac{1}{2} |x(t) - z|^2.$$

Since $\dot{h}_z(t) = \langle x(t) - z, \dot{x}(t) \rangle$ and $\ddot{h}_z(t) = |\dot{x}(t)|^2 + \langle x(t) - z, \ddot{x}(t) \rangle$, we have

$$\ddot{h}_z(t) + \gamma \dot{h}_z(t) = |\dot{x}(t)|^2 + \langle x(t) - z, \ddot{x}(t) + \gamma \dddot{x}(t) \rangle.$$
Since the map $x$ is solution of the (HBFC) system, we have
\[ \ddot{x}(t) + \gamma \dot{x}(t) = -\nabla \Phi(x(t)) - \varepsilon(t) x(t). \]

Hence
\[ \ddot{h}_z(t) + \gamma \dot{h}_z(t) = |\dot{x}(t)|^2 + \langle x(t) - z, -\nabla \Phi(x(t)) \rangle - \varepsilon(t) \langle x(t) - z, x(t) \rangle. \]

Since $z \in S$, we have $\nabla \Phi(z) = 0$. From the monotonicity of $\nabla \Phi$, we have, for every $t \geq t_0$
\[ \langle x(t) - z, -\nabla \Phi(x(t)) \rangle = \langle x(t) - z, \nabla \Phi(z) - \nabla \Phi(x(t)) \rangle \leq 0. \]

Hence
\[ \ddot{h}_z(t) + \gamma \dot{h}_z(t) \leq |\dot{x}(t)|^2 - \varepsilon(t) \langle x(t) - z, x(t) \rangle. \tag{2} \]

3.1. Proof of Theorem 2.3: The Slow Parametrization

For simplicity of notation, we write $p = \text{proj}_S(0)$. Let us consider the function $h: [t_0, +\infty) \to \mathbb{R}_+$, defined by:
\[ h(t) = h_p(t) = \frac{1}{2} |x(t) - p|^2. \]

The proof of Theorem 2.3 consists in proving that $h$ converges to 0. From (2), we have, for every $t \geq t_0$
\[ \ddot{h}(t) + \gamma \dot{h}(t) \leq |\dot{x}(t)|^2 - \varepsilon(t) \langle x(t) - p, x(t) \rangle. \tag{3} \]

The main idea of the proof is to respectively distinguish the cases where $\langle x(t) - p, x(t) \rangle < 0$ and $\langle x(t) - p, x(t) \rangle \geq 0$. Precisely, noticing that
\[ \{x \in H : \langle x - p, x \rangle < 0\} = B \left( \frac{p}{2}, \frac{|p|}{2} \right), \]
we distinguish the three cases (illustrated in Fig. 1):

(a) $\exists T \geq t_0, \quad \forall t \geq T, \quad x(t) \notin B \left( \frac{p}{2}, \frac{|p|}{2} \right)$;

(b) $\exists T \geq t_0, \quad \forall t \geq T, \quad x(t) \in B \left( \frac{p}{2}, \frac{|p|}{2} \right)$;

(c) $\forall T \geq t_0, \quad \exists t \geq T, \quad x(t) \in B \left( \frac{p}{2}, \frac{|p|}{2} \right)$. 

Case (c) obviously contains Case (b), but the main points of the proof are made clearer with this distinction.

Case (a). It is illustrated in Fig. 2 on a numerical example, with two different trajectories.

We assume that there exists some $T \geq t_0$, such that, for every $t \geq T$, $x(t) \notin B(p/2, |p|/2)$. Equivalently, $\langle x(t) - p, x(t) \rangle \geq 0$, hence from (3), we deduce that, for every $t \geq T$

$$\dot{h}(t) + \gamma \dot{h}(t) \leq |\dot{x}(t)|^2.$$  

From Theorem 2.1, Part (a), we have $\dot{x} \in L^2([t_0, +\infty), H)$. In view of the following lemma from [6] we obtain that $h$ converges, hence that $x$ is bounded.

**Lemma 3.1** [6, Lemma 4.2]. Let $t_0 \in \mathbb{R}$ and $h \in C^2([t_0, +\infty), \mathbb{R}^+)$ satisfy the following differential inequality

$$\ddot{h}(t) + \gamma \dot{h}(t) \leq g(t)$$

with $g \in L^1([t_0, +\infty), \mathbb{R}^+)$. Then $(\dot{h})_+$, the positive part of $\dot{h}$, belongs to $L^1([t_0, +\infty), \mathbb{R})$ and, as a consequence, $\lim_{t \to +\infty} h(t)$ exists.

In order to prove that $\lim_{t \to +\infty} h(t) = 0$, we have to come back to Equation (3) without neglecting the term $\varepsilon(t) \langle x(t) - p, x(t) \rangle$. In fact, the idea of the proof is first to compare $\varepsilon(t) \langle x(t) - p, x(t) \rangle$ and $\varepsilon(t) h(t)$, and then to use the fact that $\int_{t_0}^{+\infty} \varepsilon(t) = +\infty$ to get a contradiction if $\lim_{t \to +\infty} h(t) \neq 0$.

Let us first prove that $\lim_{t \to +\infty} \langle x(t) - p, p \rangle > 0$. Indeed, let $(t_n) \subset \mathbb{R}_+$ be a sequence such that $\lim_{n \to +\infty} t_n = +\infty$ and $\lim_{n \to +\infty} \langle x(t_n) - p, p \rangle$ exists. Since, from above, the function $x$ is bounded, without any loss of

\footnote{Note that Case (a) is the only one to consider if we assume that $0 \notin S$. In that case, $p = \text{proj}_S(0) = 0$.}
generality, we may assume that there exists some $\bar{x} \in H$ such that $x(t_n)$ weakly converges to $\bar{x}$. From Theorem 2.1, Part (b)–(v), we have $\lim_{n \to +\infty} \langle x(t_n) - p, p \rangle = \langle \bar{x} - p, p \rangle$. Since $p = \text{proj}_S(0)$ and since $\bar{x} \in S$, we have the inequality $\langle \bar{x} - p, p - 0 \rangle \geq 0$, which clearly implies that $\lim_{n \to +\infty} \langle x(t_n) - p, p \rangle \geq 0$.

We now prove that $\lim_{n \to +\infty} h(t) = 0$. Assume that it is not true. Then, there exists some $l > 0$ such that $\lim_{n \to +\infty} h(t) = l$. Hence there exists $t_1 \geq t_0$ such that, for every $t \geq t_1$, $|x(t) - p|^2 / 2 = h(t) > l/2$ and $\langle x(t) - p, p \rangle \geq -l/2$ (since $\lim \inf_{n \to +\infty} \langle x(t_n) - p, p \rangle \geq 0$). Hence $\langle x(t) - p, x(t) \rangle = |x(t) - p|^2 + \langle x(t) - p, p \rangle \geq l/2$. Together with Eq. (3), this implies that

$$\tilde{h}(t) + \gamma \tilde{h}(t) \leq |\dot{x}(t)|^2 - \varepsilon(t) \frac{l}{2}.$$ 

By multiplying each member of the above inequality by $e^{-\gamma t}$, and integrating between $t_1$ and $t$, we obtain

$$\tilde{h}(t) + \frac{l}{2} e^{-\gamma t} \int_{t_1}^{t} \varepsilon(s) e^\gamma ds \leq \tilde{h}(t_1) e^{\gamma (t_1 - t_0)} + e^{-\gamma t} \int_{t_1}^{t} e^\gamma |\dot{x}(s)|^2 ds.$$ 

Integrating again between $t_1$ and $t$, we obtain

$$\tilde{h}(t) - \tilde{h}(t_1) + \frac{l}{2} \int_{t_1}^{t} e^{-\gamma s} \int_{t_1}^{s} \varepsilon(u) e^{\gamma u} ds \ du$$

$$\leq \tilde{h}(t_1) \frac{1 - e^{\gamma (t_1 - t_0)}}{\gamma} + \int_{t_1}^{t} e^{-\gamma s} \int_{t_1}^{s} e^\gamma |\dot{x}(s)|^2 ds \ du.$$
In view of the following claim, letting $t \to +\infty$ in the above equation, we conclude that

$$l - h(t) + \frac{1}{2\gamma} \int_{t_n}^{+\infty} a(s) \, ds \leq \frac{1}{\gamma} h(t) + \frac{1}{\gamma} \int_{t_n}^{+\infty} |\dot{x}(s)|^2 \, ds,$$

which contradicts the facts that $\dot{x} \in L^2([t_0, +\infty), \mathbb{R}_+)$ and $e \notin L^1([t_0, +\infty), \mathbb{R}_+)$.

**Claim 3.1.** Let $t_1 \in \mathbb{R}$, $f: \mathbb{R} \to \mathbb{R}_+$ be a measurable function, then:

(i) $\forall t \geq t_1$, $\int_{t_1}^{t} e^{-\gamma u} \int_{t_1}^{u} e^{\gamma v} f(v) \, dv \, du \leq \frac{1}{\gamma} \int_{t_1}^{t} f(v) \, dv$;

(ii) $\int_{t_1}^{+\infty} e^{-\gamma u} \int_{t_1}^{u} e^{\gamma v} f(v) \, dv \, du = \frac{1}{\gamma} \int_{t_1}^{+\infty} f(v) \, dv$.

**Proof of Claim 3.1.** By Fubini's Theorem

$$\int_{t_1}^{t} e^{-\gamma u} \int_{t_1}^{u} e^{\gamma v} f(v) \, dv \, du = \int_{t_1}^{t} e^{-\gamma u} \int_{v_1}^{t} e^{\gamma v} f(v) \, dv \, du$$

$$= \int_{t_1}^{t} e^{\gamma v} f(v) \, \int_{v_1}^{t} e^{-\gamma u} \, du$$

$$= \frac{1}{\gamma} \int_{t_1}^{t} (1 - e^{-\gamma u}) f(v) \, dv \, du.$$

Hence we deduce (i) and (ii). 

**Case (b).** It is illustrated in Fig. 3 on a numerical example. We assume that there exists $T \geq t_0$, such that, for every $t \geq T$, $x(t) \in B(p/2, |p|/2)$ or, equivalently, $\langle x(t) - p, x(t) \rangle < 0$. Then the map $x$ is clearly bounded on $[T, +\infty)$. Since it is continuous, it is bounded on $[t_0, +\infty)$ and the function

**FIG. 3.** Illustration of Case (b), with $\Phi(x) = d_1(x)^2$, $S = B((2,0),1)$, $x(t) = 1/\sqrt{1+t}$, $x(0) = (0.1, -0.2)$, $\dot{x}(0) = (0, 0.6)$, $\gamma = 1.2$. 

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h is also bounded. Hence, in particular, from Theorem 2.1, Part (b)–(v), we have \( \lim_{t \to +\infty} \nabla \Phi(x(t)) = 0 \). We now prove that \( \lim_{n \to +\infty} h(t) = 0 \), i.e., \( \lim_{n \to +\infty} x(t) = p \). Consider a sequence \( (t_n) \subset \mathbb{R}^+ \) such that \( \lim_{n \to +\infty} t_n = +\infty \). From above, we have \( \lim_{t \to +\infty} \nabla \Phi(x(t)) = 0 \). Applying the following claim to the sequence \( x(t_n) \), we obtain that \( \lim_{n \to +\infty} x(t_n) = p \), which concludes the proof of Case (b).

**Claim 3.2.** Let \( (x_n) \) be a sequence in \( \bar{B}(p/2, |p|/2) \), such that \( \lim_{n \to +\infty} \nabla \Phi(x_n) = 0 \). Then

\[
\lim_{n \to +\infty} |x_n - p| = 0.
\]

**Proof of Claim 3.2.** We first prove that the sequence \( (x_n) \) weakly converges to \( p \). Let \( (x_{n(o)}) \) be a subsequence which weakly converges to \( x \in H \). Since \( \bar{B}(p/2, |p|/2) \) is closed and convex, it is weakly closed and \( x \in \bar{B}(p/2, |p|/2) \). Since \( \lim_{n \to +\infty} \nabla \Phi(x_{n(o)}) = 0 \), from Proposition 3.1, it follows that \( x \in S \). Since \( \bar{B}(p/2, |p|/2) \cap S = \{ p \} \), we obtain \( x = p \). Since \( \bar{B}(p/2, |p|/2) \) is weakly compact and \( p \) is the limit of every weakly convergent subsequence of \( (x_n) \), we conclude that the sequence \( (x_n) \) weakly converges to \( p \). Let us now prove that the sequence \( (x_n) \) strongly converges to \( p \). Since \( x_n \in \bar{B}(p/2, |p|/2) \), we have \( \langle x_n, x_n - p \rangle \leq 0 \) and \( |x_n - p|^2 = \langle x_n, x_n - p \rangle + \langle -p, x_n - p \rangle \leq \langle -p, x_n - p \rangle \). Since the sequence \( (x_n) \) weakly converges to \( p \), taking the limit when \( n \to +\infty \), we obtain \( \lim_{n \to +\infty} |x_n - p| = 0 \).

**Case (c).** It is illustrated in Fig. 4 on a numerical example. We now assume that, for every \( T \geq t_0 \), there exists some \( t \geq T \) such that \( x(t) \in \bar{B}(p/2, |p|/2) \). We first prove that \( h \) is bounded (hence that the map \( x \) is bounded). Let \( T \in \mathbb{R}^+ \) such that \( x(T) \in \bar{B}(p/2, |p|/2) \) and consider \( t \geq T \). If \( x(t) \in \bar{B}(p/2, |p|/2) \), then \( |x(t) - p| \leq |p| \). We now need a claim.

**Claim 3.3.** Let \( t \geq t_0 \) such that \( x(t) \notin \bar{B}(p/2, |p|/2) \), and let

\[
\tau(t) = \inf \left\{ u \in [t_0, t] \mid x([u, t]) \cap \bar{B} \left( \frac{p}{2}, \frac{|p|}{2} \right) = \emptyset \right\}.
\]

Then

\[
h(t) \leq h(\tau(t)) + \frac{1}{\gamma} \int_{\tau(t)}^{+\infty} |\dot{x}(u)|^2 \, du.
\]

\(^4\)This is also a consequence of a general result of Visintin involving weak convergence and convex extremality properties. See Valadier [24], Visintin [25].
Proof of Claim 3.3. For every $u \in [\tau(t), t]$, $x(u) \notin \mathcal{B}(p/2, |p|/2)$, that is, $\langle x(u) - p, x(u) \rangle \geq 0$. From (3), we deduce that

$$\dot{h}(u) + \gamma \dot{h}(u) \leq |\dot{x}(u)|^2.$$  

By multiplying each member of the above inequality by $e^{\gamma u}$, and integrating between $\tau(t)$ and $s$, we obtain

$$e^{\gamma s} \dot{h}(s) - e^{\gamma \tau(t)} \dot{h}(\tau(t)) \leq \int_{\tau(t)}^{s} e^{\gamma u} |\dot{x}(u)|^2 \, du.$$  

Hence

$$\dot{h}(s) \leq e^{\gamma (s - \tau(t))} \dot{h}(\tau(t)) + e^{-\gamma s} \int_{\tau(t)}^{s} e^{\gamma u} |\dot{x}(u)|^2 \, du.$$  

Integrating the above inequality between $\tau(t)$ and $t$, we obtain

$$h(t) \leq h(\tau(t)) + \dot{h}(\tau(t)) \frac{1 - e^{\gamma (t - \tau(t))}}{\gamma} + \int_{\tau(t)}^{t} e^{-\gamma s} \int_{\tau(t)}^{s} e^{\gamma u} |\dot{x}(u)|^2 \, du \, ds.$$  

From Claim 3.1, we have $\int_{\tau(t)}^{t} e^{-\gamma s} \int_{\tau(t)}^{s} e^{\gamma u} |\dot{x}(u)|^2 \, du \, ds \leq \frac{1}{\gamma} \int_{\tau(t)}^{t} |\dot{x}(u)|^2 \, du \leq \frac{1}{\gamma} \int_{\tau(t)}^{t} |\dot{x}(u)|^2 \, du$, which achieves the proof of the claim.  

We now come back to the proof of Case (c). If $x(t) \notin \mathcal{B}(p/2, |p|/2)$, clearly $T \leq \tau(t) < t$ and $x(\tau(t)) \in \mathcal{S}(p/2, |p|/2)$, which implies that $|x(\tau(t)) - p| \leq |p|$. Hence
\[ h(t) = \left| \frac{x(t(t)) - p}{2} \right| \leq \frac{|p|^2}{2}; \]

\[ \dot{h}(t) = \langle x(t(t)) - p, \dot{x}(t(t)) \rangle \leq |p| \| \dot{x} \|_{\infty}. \]

In view of (4) and Claim 3.3, we deduce that

\[ h(t) \leq \frac{|p|^2}{2} + \frac{1}{\gamma} |p| \| \dot{x} \|_{\infty} + \frac{1}{\gamma} \int_{T}^{+\infty} |\dot{x}(u)|^2 \, du. \]

This proves that \( h \) is bounded on \([T, +\infty)\). Since the function \( h \) is continuous, it is bounded on the interval \([t_0, +\infty)\). Hence the map \( x \) is bounded.

We now prove that \( h \) converges to 0. Take any sequence \((t_n) \subset \mathbb{R}_+\) such that \( \lim_{n \to +\infty} t_n = +\infty \). The proof of Case (c) will be complete if we prove that \( \lim_{n \to +\infty} h(t_n) = 0 \). First assume that there is a subsequence \((t'_{n})\) of \((t_n)\) such that \( x(t'_{n}) \in \bar{B}(p/2, |p|/2) \) and we prove that \( \lim_{n \to +\infty} h(t'_{n}) = 0 \). Since \( x(t'_{n}) \in \bar{B}(p/2, |p|/2) \), let \( \tau(t'_{n}) \) be defined by Claim 3.3. We first notice that \( \lim_{n \to +\infty} \tau(t'_{n}) = +\infty \) and \( x(t'_{n}) \in S(p/2, |p|/2) \) for \( n \) large enough. Indeed, let \( T \in \mathbb{R} \) such that \( x(T) \in \bar{B}(p/2, |p|/2) \) (by assumption of Case (c)). Let \( N \in \mathbb{N} \) such that \( t'_{n} > T \) for every \( n \geq N \) and consider \( n \geq N \). Then \( x(T) \in x([T, t'_{n}]) \cap \bar{B}(p/2, |p|/2) \), which implies that \( \tau(t'_{n}) \geq T \) and \( x(t'_{n}) \in S(p/2, |p|/2) \). Let \( n \) be large enough. Since \( x(t'_{n}) \in \bar{B}(p/2, |p|/2) \) (if \( \tau(t'_{n}) > t_0 \)), from Claim 3.2, we have \( \lim_{n \to +\infty} h(t'_{n}) = 0 \), i.e.,

\[ \lim_{n \to +\infty} h(t'_{n}) = 0. \]

Since \( \dot{h}(t'_{n}) = \langle x(t'_{n}) - p, \dot{x}(t'_{n}) \rangle \leq |x(t'_{n}) - p| \| \dot{x} \|_{\infty} \), we have

\[ \lim_{n \to +\infty} \dot{h}(t'_{n}) = 0. \]

Since \( \dot{x} \in L^2([t_0, +\infty), \mathbb{R}_+) \), we also have

\[ \lim_{n \to +\infty} \int_{t'_{n}}^{+\infty} |\dot{x}(u)|^2 \, du = 0. \]
3.2. Proof of Theorem 2.4: The Fast Parametrization

The proof of Theorem 2.4 consists first in proving the convergence of the function $h_z$ defined above and then to apply Opial’s lemma [21]. It goes along the same lines as the proof of Alvarez theorem given in [6] (Theorem 4.3), but it cannot be deduced from it because of the term $\varepsilon(t)$.

Recalling that, for $z \in S$, the function $h_z$ is defined by $h_z(t) := \frac{1}{2} |x(t) - z|^2$, we now prove that the function $h_z$ converges. First note that, for every $\xi \in H$, $\langle \xi - z, \xi \rangle \geq -\frac{1}{4} |z|^2$. Recalling that (see (2)) $\dot{h}_z(t) + \gamma \dot{h}_z(t) \leq |\dot{x}(t)|^2 - \varepsilon(t) \langle x(t) - z, x(t) \rangle$, we have the inequality

$$\dot{h}_z(t) + \gamma \dot{h}_z(t) \leq |\dot{x}(t)|^2 + \varepsilon(t).$$

Since $\dot{x} \in L^2([t_0, +\infty), H)$ and $\varepsilon \in L^1([t_0, +\infty), \mathbb{R}_+)$, in view of Lemma 3.1, we deduce that the function $h_z$ converges and that the function $x$ is bounded.

Since the function $x$ is bounded, from Theorem 2.1, Part (b) (v), it follows that, for every sequence $(t_n) \subset [t_0, +\infty)$ such that $t_n \to +\infty$ and $x(t_n) \rightharpoonup \bar{x}$ weakly in $H$, $\lim_{t \to +\infty} \Phi(x(t_n)) = 0$. Hence, in view of Proposition 3.1, we have $\bar{x} \in S$ and $\lim_{t \to +\infty} \Phi(x(t_n)) = \min \Phi$. Since, from above, $\lim_{t \to +\infty} |x(t) - z|$ exists for every $z \in S$, we deduce from Opial’s lemma (given below) that the map $x$ weakly converges to some element $\bar{x}$ of $S$.

**Lemma 3.2 (Opial [21]).** Let $H$ be a Hilbert space and $x: [t_0, +\infty) \to H$ be a function such that there exists a nonempty set $S \subset H$ which verifies:

(i) $\forall t_n \to +\infty$ with $x(t_n) \rightharpoonup \bar{x}$ weakly in $H$, we have $\bar{x} \in S$.

(ii) $\forall z \in S$, $\lim_{t \to +\infty} |x(t) - z|$ exists.

Then, $x(t)$ weakly converges as $t \to +\infty$ to some element $\bar{x}$ of $S$.

3.3. Proof of Corollaries 2.2 and 2.3

The proof of Corollary 2.2 goes along the same lines as the proof of Proposition 4.2 of [6] and is given below for the sake of completeness.
Proof of Corollary 2.2. Let us consider a trajectory $x$ of the (HBFC) system. From Theorem 2.4, the map $x$ is bounded. We now give a direct proof of Corollary 2.2. Since the map $x$ is bounded, there exists some $R > 0$ such that for all $t \in [t_0, +\infty)$, $|x(t)| \leq R$. Since $\Phi$ is strongly convex, it has a unique minimizer $\bar{x} = \text{argmin} \Phi$. Let us write the strong monotonicity property (1) at $\bar{x}$ and $x(t)$

$$\langle \nabla \Phi(\bar{x}) - \nabla \Phi(x(t)), \bar{x} - x(t) \rangle \geq \beta_R(|x(t) - \bar{x}|).$$

Since $\nabla \Phi(\bar{x}) = 0$ and $\nabla \Phi(x(t)) = -\bar{x}(t) - \gamma \bar{x}(t) - \epsilon(t) x(t)$, it follows that

$$\beta_R(|x(t) - \bar{x}|) \leq \langle \bar{x}(t) + \gamma \bar{x}(t) + \epsilon(t) x(t), \bar{x} - x(t) \rangle. \quad (5)$$

Since, from Theorem 2.1, Part (b), we have that $\lim_{t \to +\infty} \bar{x}(t) = \lim_{t \to +\infty} \beta_R(|x(t) - \bar{x}|) = 0$ and since the map $x$ is bounded, it follows from (5) that $\lim_{t \to +\infty} \beta_R(|x(t) - \bar{x}|) = 0$. From this we deduce that $x(t) \to \bar{x}$ strongly as $t \to +\infty$. \footnote{The proof of Corollary 2.3 is inspired by the proof of Theorem 2.2 of [1].}

Proof of Corollary 2.3. Consider a trajectory $x$ of the (HBFC) system. Let $T > t_0$, we define $g: [t_0, T] \to \mathbb{R}$ by

$$g(t) = |x(t)|^2 - |x(T)|^2 - \frac{1}{2} |x(t) - x(T)|^2.$$

Then $\dot{g}(t) = \langle \dot{x}(t), x(t) + x(T) \rangle$ and $\ddot{g}(t) = |\dot{x}(t)|^2 + \langle \ddot{x}(t), x(t) + x(T) \rangle$. Hence

$$\ddot{g}(t) + \gamma \dot{g}(t) = |\dot{x}(t)|^2 + \langle -\nabla \Phi(x(t)) - \epsilon(t) x(t), x(t) + x(T) \rangle.$$

Since the energy function (see Section 5.1) is decreasing, we have $E(t) \geq E(T)$, i.e., $\frac{1}{2} |\dot{x}(t)|^2 + \Phi(x(t)) + \frac{1}{2} \epsilon(t) |x(t)|^2 \geq |\dot{x}(T)|^2 + \Phi(x(T)) + \frac{1}{2} \epsilon(T) |x(T)|^2$. Since $\Phi(x(T)) = \Phi(-x(T))$, it follows that

$$\frac{1}{2} |\dot{x}(t)|^2 + \Phi(x(t)) + \frac{1}{2} \epsilon(t) |x(t)|^2 \geq \Phi(-x(T)). \quad (6)$$

By convexity of the map $\Phi$, we have that $\Phi(-x(T)) \geq \Phi(x(t)) + \langle \nabla \Phi(x(t)), -x(T) - x(t) \rangle$. Hence

$$\frac{1}{2} |\dot{x}(t)|^2 + \frac{1}{2} \epsilon(t) |x(t)|^2 \geq \langle \nabla \Phi(x(t)), -x(T) - x(t) \rangle.$$
Hence \( \ddot{g}(t) + c \dot{g}(t) \leq \frac{1}{2} |\dot{x}(t)|^2 + \varepsilon(t) |x(t)|^2 + \langle -\varepsilon(t) x(t), x(t) + x(T) \rangle \), which implies

\[
\ddot{g}(t) + \gamma g(t) \leq \frac{1}{2} |\dot{x}(t)|^2 - \varepsilon(t) \langle x(t), x(T) \rangle.
\]

Hence

\[
\ddot{g}(t) + \gamma g(t) \leq \frac{1}{2} |\dot{x}(t)|^2 + \varepsilon(t) \|x\|_\infty^2.
\]

(or note that \( \frac{1}{2} \varepsilon(t) |x(t)|^2 + \langle -\varepsilon(t) x(t), x(t) + x(T) \rangle = \langle -\varepsilon(t) x(t), \frac{1}{2} x(t) + x(T) \rangle = -2\varepsilon(t)(\frac{1}{2} x(t), \frac{1}{2} x(t) + x(T)) \leq 2\varepsilon(t)(|x(T)|^2/4) \)). After integrating the above inequality, we obtain

\[
g(T) - g(t) \leq \dot{g}(t_0) e^{\gamma t} \left( e^{-\gamma t_0} - e^{-\gamma T} \right)
\]

\[
\quad + \int_t^T e^{-\gamma s} \int_{t_0}^s e^{\gamma u} \left( \frac{3}{2} |\dot{x}(u)|^2 + \varepsilon(u) \|x\|_\infty^2 \right) du \, ds.
\]

Since \( g(T) = 0 \), we obtain that

\[
\frac{1}{2} |x(t) - x(T)|^2 \leq |x(t)|^2 - |x(T)|^2 + \dot{g}(t_0) e^{\gamma t_0} \left( e^{-\gamma t_0} - e^{-\gamma T} \right)
\]

\[
\quad + \int_t^T e^{-\gamma s} \int_{t_0}^s e^{\gamma u} \left( \frac{3}{2} |\dot{x}(u)|^2 + \varepsilon(u) \|x\|_\infty^2 \right) du \, ds.
\]

Since the map \( \Phi \) is even, we have \( \nabla \Phi(0) = -\nabla \Phi(-0) \), which implies \( 0 \in S \). As in the proof of Theorem 2.4, recalling that the function \( h_0 \) is defined by \( h_0(t) := \frac{1}{2} |x(t)|^2 \), we now prove that the function \( h_0 \) converges. Recalling that (see (2)) \( \dot{h}_0(t) + \gamma h_0(t) \leq |\dot{x}(t)|^2 - \varepsilon(t) \langle x(t), x(t) \rangle \), we obtain that \( \dot{h}_0(t) + \gamma h_0(t) \leq |\dot{x}(t)|^2 \). Since \( x \in L^1([t_0, +\infty), \mathbb{R}_+) \), in view of Lemma 3.1, we deduce that the function \( h_0 \) converges, i.e., that \( |x(t)| \) converges when \( t \to +\infty \). Recalling that \( \dot{x} \in L^2([t_0, +\infty)) \) and \( \varepsilon \in L^1([t_0, +\infty)) \), from Claim 3.1, we obtain

\[
\int_{t_0}^{+\infty} e^{-\gamma s} \int_{t_0}^s e^{\gamma u} \left( \frac{3}{2} |\dot{x}(u)|^2 + \varepsilon(u) \|x\|_\infty^2 \right) du \, ds
\]

\[
= \frac{1}{\gamma} \int_{t_0}^{+\infty} \left( \frac{3}{2} |\dot{x}(u)|^2 + \varepsilon(u) \|x\|_\infty^2 \right) du < +\infty.
\]
Hence the set \( \{x(t) | t \to +\infty \} \) is a Cauchy net, which implies that the trajectory \( x \) converges strongly when \( t \to +\infty \), and, from Theorem 2.4, its limit belongs to \( S \).

4. APPLICATION TO THE WAVE EQUATION

In this section, we show how the control techniques developed in this paper can be used in order to select particular solutions of some hyperbolic systems. Indeed, applications to PDE cannot be obtained directly by application of Theorem 2.3, because this would require considering \( \Phi \) lower semicontinuous (for example, \( \Phi(v) = \frac{1}{2} \|v\|^2 \) on \( H^1 \), \( +\infty \) on \( L^2 \setminus H^1 \), with \( A = \partial \Phi \) equal to \( -A \)). But as we show below the main ideas of the proof work. Detailed study goes beyond the scope of the present article. Let \( \Omega \) be a regular bounded domain in \( \mathbb{R}^n \), \( L^2(\Omega) \) stands for the usual Hilbert space endowed with the scalar product \( \langle u, v \rangle_{L^2} = \int_\Omega u(x) v(x) \, dx \) and the corresponding norm \( \|u\|_{L^2} = \sqrt{\langle u, u \rangle_{L^2}} \). For simplicity of notation, we write \( u(t) \) for the function \( x \to u(x,t) \). Given \( \gamma > 0 \), \( u_0 \in H^1(\Omega) \) and \( v_0 \in L^2(\Omega) \), consider the following hyperbolic problem:

\[
\begin{aligned}
&\frac{\partial u}{\partial t} + \gamma u_t - Au = 0 \quad \text{in} \quad \Omega \times ]0, \infty), \\
&\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega \times ]0, \infty), \\
&u(0) = u_0, \\
&u_t(0) = v_0.
\end{aligned}
\]

We refer to Alvarez and Attouch [2] and Zuazua [26] for a thorough study of convergence properties of the above system. In [26, Theorem 4.3], it is established that the unique solution \( u(x, t) \) of (7) satisfies:

(i) the map \( t \mapsto u_t(t) \) belongs to \( L^2([0, \infty); L^2(\Omega)) \);
(ii) \( u(t) \to u_\infty \) strongly in \( H^1(\Omega) \) as \( t \to \infty \), where \( u_\infty \) is the constant function given by

\[
u_\infty \equiv \frac{1}{|\Omega|} \int_\Omega \left[ u_0(x) + \frac{1}{\gamma} v_0(x) \right] \, dx,
\]

and \( |\Omega| \) is the Lebesgue measure of the domain \( \Omega \).

In fact, any constant function can be obtained as the limit of a solution of (7), with suitable initial conditions. By adding a control of the form
in the wave equation, one can force every solution of the controlled system to converge to a specific equilibrium. This selection property can be viewed as an asymptotic stabilization property. We now make this precise.

Let $\varepsilon : [0, +\infty) \to \mathbb{R}_+$ be a function of class $C^2$ such that $\lim_{t \to +\infty} \varepsilon(t) = 0$ and such that the function $\varepsilon$ is non-increasing, i.e., $\dot{\varepsilon}(t) \leq 0$ for every $t \in [0, +\infty)$. We consider the controlled system:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial t^2} + \Delta u + \varepsilon(t) u(t) &= 0 & \text{in } \Omega \times [0, +\infty[, \\
\frac{\partial u}{\partial n} &= 0 & \text{on } \partial \Omega \times [0, +\infty[, \\
u(0) &= u_0, \\
u_t(0) &= v_0.
\end{aligned}
$$

(8)

**Proposition 4.1 (Slow Parametrization).** Assume that $\int_0^\infty \varepsilon(t) \, dt = +\infty$. Then, the unique solution $u(x, t)$ of (8) satisfies:

(i) the map $t \mapsto u_t(t)$ belongs to $L^2([0, \infty); L^2(\Omega))$ and $\lim_{t \to +\infty} |u_t(t)|_{L^2} = 0$;

(ii) $u(t) \to 0$ strongly in $H^1(\Omega)$ as $t \to \infty$.

**Proof of Proposition 4.1.** The proof of (i) goes along the same lines as the proof of [2, Theorem 2.1].

**Proof of (ii).** Since $\lim_{t \to +\infty} |Vu(t)|_{L^2} = 0$, it is sufficient to prove that $\lim_{t \to +\infty} |u(t)|_{L^2} = 0$. Consider $h(t) = \frac{1}{2} |u(t)|_{L^2}^2$. Then $\dot{h}(t) = \langle u(t), u_t(t) \rangle_{L^2}$. Since $u_t(t) = \Delta u + \varepsilon(t) u(t)$, a standard argument yields $\dot{h}(t) = |u_t(t)|_{L^2}^2 + \langle u_t(t), u_t(t) \rangle_{L^2} - |Vu(t)|_{L^2}^2$. Hence

$$
\dot{h}(t) + \gamma h(t) + |Vu(t)|_{L^2}^2 + \varepsilon(t) |u(t)|_{L^2}^2 = |u_t(t)|_{L^2}^2 = |u_0(t)|_{L^2}^2.
$$

We deduce that

$$
\dot{h}(t) + \gamma h(t) + \varepsilon(t) |u(t)|_{L^2}^2 \leq |u_0(t)|_{L^2}^2.
$$

Hence

$$
\dot{h}(t) + \gamma h(t) \leq |u_0(t)|_{L^2}^2.
$$

Since the map $t \mapsto u_t(t)$ belongs to $L^2([0, \infty); L^2(\Omega))$, by using Lemma 3.1, we deduce that $\lim_{t \to +\infty} h(t)$ exists. If $l = \lim_{t \to +\infty} h(t) > 0$, and in view of (9), we have, for $t$ large enough

$$
\dot{h}(t) + \gamma h(t) + \varepsilon(t) l \leq |u_0(t)|_{L^2}^2.
$$
5. PROOFS OF THE GLOBAL EXISTENCE RESULTS

In this section, we prove the global existence results stated in Section 2.1.

5.1. Proof of Theorem 2.1

Proof of Part (a). Proof of (i). First note that the (HBFC) system can be written as a first order nonautonomous system in $H \times H$:

\[ Y = F(Y, t), \quad \text{with} \quad Y(t) = \left( \begin{array}{c} x(t) \\ \dot{x}(t) \end{array} \right), \]

and

\[ F(u, v, t) = \left( \begin{array}{c} v \\ -\gamma v - \nabla \Phi(u) - \delta(t) u \end{array} \right). \]

For $Y_0 = (\infty)$ given in $H \times H$, since the map $\nabla \Phi$ is locally Lipschitzian, the Cauchy–Lipschitz theorem ensures the existence of a unique local solution to the problem $\dot{Y} = F(Y, t)$, $Y(t_0) = Y_0$, hence of the (HBFC) system. Let $x$ be a maximal solution of the (HBFC) system, defined on $[t_0, T_{max})$ ($t_0 < T_{max} \leq +\infty$). The equation (HBFC) and the continuity of the map $\nabla \Phi$ automatically imply that the function $x$ is of class $C^2$ on $[t_0, T_{max})$. In order to prove that $T_{max} = +\infty$, let us show that the function $\dot{x}$ is bounded. We define the energy by

\[ E(t) = \frac{1}{2} |\dot{x}(t)|^2 + \Phi(x(t)) + \frac{1}{2} \delta(t) |x(t)|^2. \]

By differentiation of the energy function $E$, we obtain that

\[ \dot{E}(t) = \langle \dot{x}(t), \dot{x}(t) + \nabla \Phi(x(t)) + \delta(t) x(t) \rangle + \frac{1}{2} \dot{\delta}(t) |x(t)|^2. \]

Hence, in view of (HBFC), we infer

\[ \dot{E}(t) = -\gamma |\dot{x}(t)|^2 + \frac{1}{2} \dot{\delta}(t) |x(t)|^2. \]

Since $\dot{\delta}(t) \leq 0$, the function $E$ is decreasing, and, for all $t \in [t_0, T_{max})$, we have

\[ E(t) \leq E(t_0). \]
Equivalently, \( \frac{1}{2} |\dot{x}(t)|^2 + \Phi(x(t)) + \frac{1}{2} \varepsilon(t) |x(t)|^2 \leq \frac{1}{2} |\dot{x}_0|^2 + \Phi(x_0) + \frac{1}{2} \varepsilon(t_0) |x_0|^2 \),

which implies

\[
\frac{1}{2} |\dot{x}(t)|^2 + \Phi(x(t)) + \frac{1}{2} \varepsilon(t_0) |x_0|^2 = \frac{1}{2} |\dot{x}_0|^2 + \Phi(x_0) + \frac{1}{2} \varepsilon(t_0) |x_0|^2,
\]

(10)

Since \( \Phi \) is bounded from below, we obtain that

\[
\|x\|_\infty = \sup_{t \in [t_0, T_{\text{max}})} |\dot{x}(t)| < +\infty.
\]

It is a standard argument to derive from such estimation, that \( T_{\text{max}} = +\infty \); assume that \( T_{\text{max}} < +\infty \), since \( |x(t) - x(t')| \leq \|x\|_\infty |t - t'| \), then \( \lim_{t \rightarrow T_{\text{max}}} x(t) := x_\infty \) exists. By equation (HBFC), the map \( \dot{x} \) is also bounded on the interval \([t_0, T_{\text{max}})\) and \( \lim_{t \rightarrow T_{\text{max}}} \dot{x}(t) = \dot{x}_\infty \) exists. Applying the local existence theorem with initial data \((x_\infty, \dot{x}_\infty)\), we can extend the maximal solution to a strictly larger interval, a clear contradiction. Hence \( T_{\text{max}} = +\infty \), which completes the proof of (i).

**Proof of (ii).** Since the map \( \Phi \) is bounded from below, and since \( E(t) \geq \Phi(x(t)) \), the energy function \( E \) is also bounded from below. Since, from above, the function \( E \) is decreasing, it follows that \( E \) converges in \( \mathbb{R} \). Let \( E_\infty = \lim_{t \rightarrow +\infty} E(t) \). From (10), and since the map \( \Phi \) is bounded from below, we obtain that, for all \( t \geq t_0 \)

\[
\frac{1}{2} |\dot{x}(t)|^2 \leq \frac{1}{2} |\dot{x}_0|^2 + \Phi(x_0) + \frac{1}{2} \varepsilon(t_0) |x_0|^2 - \inf \Phi.
\]

Hence the map \( \dot{x} \) is bounded, i.e., \( \dot{x} \in L^\infty([t_0, +\infty), H) \). Since \( \delta(t) \leq 0 \), we have \( \dot{E}(t) = -\gamma |\dot{x}(t)|^2 + \frac{1}{2} \delta(t) |x(t)|^2 \leq -\gamma |\dot{x}(t)|^2 \). We deduce that, for all \( t_0 \leq t < +\infty \)

\[
\int_{t_0}^t |\dot{x}(s)|^2 ds \leq \frac{1}{\gamma} (E(t_0) - E(t)).
\]

Since \( E(t) \) decreases to \( E_\infty \) as \( t \) increases to \( +\infty \), we obtain

\[
\int_{t_0}^{+\infty} |\dot{x}(s)|^2 ds \leq \frac{1}{\gamma} (E(t_0) - E_\infty),
\]

and \( \dot{x} \in L^2([t_0, +\infty), H) \).

**Proof of Part (b).** **Proof of (iii).** We now assume that the map \( x \) is in \( L^\infty([t_0, +\infty), H) \). From (ii), we have \( \dot{x} \in L^\infty([t_0, +\infty), H) \). Equation (HBFC), and the fact that \( V \Phi \) is bounded (since it is Lipschitzian) on the bounded subsets of \( H \), imply that \( \dot{x} \in L^\infty([t_0, +\infty), H) \).
Proof of (iv) and (v). Let us now observe that the function $h(t) := \dot{x}(t)$ satisfies both

$$h \in L^2([t_0, +\infty), H) \quad \text{and} \quad \dot{h} \in L^\infty([t_0, +\infty), H).$$

Hence $\lim_{t \to +\infty} h(t) = 0$: assume that it is not true, since $h$ is Lipschitzian, then there is $\delta > 0$, and a sequence $(t_n)_n$ in $\mathbb{R}$ such that $\lim_{n \to +\infty} t_n = +\infty$ and $|h(t)| > \delta$ for $t \in [t_n - \delta, t_n + \delta]$, a contradiction with $h \in L^2([t_0, +\infty), H)$. Hence $\lim_{t \to +\infty} \dot{x}(t) = 0$.

Since $\lim_{t \to +\infty} \varepsilon(t) = 0$, $\lim_{t \to +\infty} x(t) = 0$, and since the map $x$ is bounded, it follows from Equation (HBFC) that

$$\lim_{t \to +\infty} \left[ \ddot{x}(t) + \nabla \Phi(x(t)) \right] = \lim_{t \to +\infty} \left[ \dot{x}(t) + \gamma \dot{x}(t) + \nabla \Phi(x(t)) + \varepsilon(t) x(t) \right] = 0.$$

If we prove that $\lim_{t \to +\infty} \ddot{x}(t) = 0$, then we infer that $\lim_{t \to +\infty} \nabla \Phi(x(t)) = 0$. If the map $\Phi$ is assumed to be $C^2$, the proof consists in differentiating the equation (HBFC). When $\Phi$ is not $C^2$, the proof is an adaptation of this argument. The idea is to replace the derivative $\ddot{x}$, which a priori makes no sense, by a differential quotient. For any $h > 0$, let us define

$$u_h(t) := \frac{1}{h} (\dot{x}(t+h) - \dot{x}(t)).$$

Let us write the equation (HBFC) at the points $t$ and $t+h$, let us make the difference and divide by $h$. We obtain

$$\dot{u}_h(t) + \gamma u_h(t) = f_h(t), \quad (11)$$

where

$$f_h(t) = \nabla \Phi(x(t+h)) - \nabla \Phi(x(t)) - \frac{\varepsilon(t+h) x(t+h) - \varepsilon(t) x(t)}{h}.$$

Multiplying Eq. (11) by $e^{\gamma t}$ and integrating between $t_0$ and $t$, we obtain

$$u_h(t) = e^{-\gamma(t-t_0)} u_h(t_0) + e^{-\gamma t} \int_{t_0}^t e^{\gamma s} f_h(s) \, ds. \quad (12)$$

We now prove that $f_h(t)$ converges to 0 as $t \to +\infty$, uniformly with respect to $h$. If $L$ denotes the Lipschitz constant of the map $\nabla \Phi$ on the (bounded) set $\overline{B}(0, \|x\|_\infty)$, we have, for every $h > 0$, and for every $t \geq t_0$

$$|f_h(t)| \leq L \frac{|x(t+h) - x(t)|}{h} + \|x\|_\infty \frac{|\varepsilon(t+h) - \varepsilon(t)|}{h} + \varepsilon(t) \frac{|x(t+h) - x(t)|}{h}.$$
Since
\[
\frac{|x(t+h) - x(t)|}{h} \leq \sup_{t \geq t} |\dot{x}(t')|,
\]
\[
\frac{|e(t+h) - e(t)|}{h} \leq \sup_{t \geq t} |\dot{e}(t')|,
\]
and since \( \lim_{t \to +\infty} \dot{x}(t) = 0 \), \( \lim_{t \to +\infty} \dot{e}(t) = 0 \), we deduce that there exists a function \( f: [t_0, +\infty) \to \mathbb{R}_+ \) such that
\[
\forall h > 0, \quad |f_h(t)| \leq f(t) \quad \text{and} \quad \lim_{t \to +\infty} f(t) = 0.
\]
Hence, Eq. (12), together with a Cesaro type argument, implies
\[
\lim_{t \to +\infty} (\sup_{h > 0} |u_h(t)|) = 0.
\]
Since, for all \( t \geq 0 \), the following inequality holds
\[
|\ddot{x}(t)| \leq \sup_{h > 0} |u_h(t)|
\]
we conclude that \( \lim_{t \to +\infty} \ddot{x}(t) = 0 \).

5.1.1. Proof of Corollary 2.1

It is enough to observe that the inequality (10) gives
\[
\Phi(x(t)) \leq \Phi(x_0) + \frac{1}{2} |x_0|^2 + \frac{1}{2} e(t_0) |x_0|^2.
\]
This majorization on \( \Phi(x(t)) \) and the coerciveness of \( \Phi \) imply that the trajectory \( x \) remains bounded, i.e., \( x \in L^\infty([t_0, +\infty), H) \).

5.2. Proof of Theorem 2.2

Proof of Part (a). We recall (see, for example, the proof of Theorem 2.1, Section 5.1) that the Cauchy–Lipschitz theorem ensures the existence of a unique local solution to the (HBFC) system. Let \( x \) be a maximal solution of the (HBFC) system, defined on the interval \([t_0, T_{\text{max}})\) \((t_0 < T_{\text{max}} \leq +\infty)\), and which is assumed to be bounded. The equation (HBFC) and the continuity of the map \( \nabla \Phi \) automatically imply that the function \( x \) is of class \( \mathcal{C}^2 \) on
In order to prove that \( T_{\text{max}} = +\infty \), let us show that the function \( \dot{x} \) is bounded. We recall that the energy function is defined by

\[
E(t) = \frac{1}{2} |\dot{x}(t)|^2 + \Phi(x(t)) + \frac{1}{2} \varepsilon(t) |x(t)|^2.
\]

By differentiation of the energy function \( E \), and in view of (HBFC), we deduce that

\[
\dot{E}(t) = -\gamma |\dot{x}(t)|^2 + \frac{1}{2} \dot{\varepsilon}(t) |x(t)|^2.
\]

Hence

\[
\dot{E}(t) \leq -\gamma |\dot{x}(t)|^2 + \frac{1}{2} \dot{\varepsilon}(t) |x(t)|^2. \tag{13}
\]

Since the solution map \( x \) is assumed to be bounded, this implies, for all \( t \in [t_0, T_{\text{max}}) \)

\[
E(t) \leq E(t_0) + \frac{1}{2} \|x\|^2 \int_{t_0}^{t} \dot{\varepsilon}(s) \, ds.
\]

From Assumption (\( \mathcal{H}_3 - iii \)), we deduce that the energy function is bounded, and since the map \( \Phi \) is bounded from below, we obtain

\[
\|\dot{x}\| = \sup_{t \in [t_0, T_{\text{max}})} |\dot{x}(t)| < +\infty.
\]

It is a standard argument to derive from such estimation, that \( T_{\text{max}} = +\infty \). Following the proof of Theorem 2.1, we then deduce all the conclusions of Part (a).

**Proof of Part (b).** We assume that the map \( \Phi \) is strongly coercive, i.e., there exist \( \alpha > 0 \) and \( \beta \in \mathbb{R}_+ \) such that \( \Phi(z) \geq \alpha |z|^2 - \beta \) for every \( z \in H \).

Let \( x \) be a maximal solution of the (HBFC) system, defined on the interval \( [t_0, T_{\text{max}}) \) \( (t_0 < T_{\text{max}} \leq +\infty) \). From (13), we obtain that, for every \( t \in [t_0, T_{\text{max}}) \)

\[
E(t) + \gamma \int_{t_0}^{t} |\dot{x}(s)|^2 \, ds \leq E(t_0) + \frac{1}{2} \int_{t_0}^{t} \dot{\varepsilon}(s) \, |x(s)|^2 \, ds. \tag{14}
\]

Since \( E(t) \geq \Phi(x(t)) \) and since the map \( \Phi \) is strongly coercive, we deduce that

\[
\alpha |x(t)|^2 + \gamma \int_{t_0}^{t} |\dot{x}(s)|^2 \, ds \leq \beta + E(t_0) + \frac{1}{2} \int_{t_0}^{t} \dot{\varepsilon}(s) \, |x(s)|^2 \, ds.
\]
Applying Gronwall’s lemma to the map $|x|^2$, we obtain, for every $t \in [t_0, T_{\text{max}})$

$$|x(t)|^2 \leq \frac{\beta + E(t_0)}{\alpha} \exp \left( \frac{1}{2\alpha} \int_{t_0}^t \dot{\varepsilon}(s) \, ds \right),$$

which implies that the solution map is bounded on the interval $[t_0, T_{\text{max}})$.

**Proof of Part (c).** In the remainder of this section, we only assume that the map $\Phi$ is coercive, i.e., $\lim_{|z| \to +\infty} \Phi(z) = +\infty$. We first assume that $\int_{t_0}^{+\infty} (\dot{\varepsilon}(t) + \varepsilon(t)) \, dt < +\infty$. Let $x$ be a maximal solution of the (HBFC) system, defined on $[t_0, T_{\text{max}})$ ($t_0 < T_{\text{max}} \leq +\infty$). Consider $t \in [t_0, T_{\text{max}})$. Since $E(t) \equiv \frac{1}{2} \dot{\varepsilon}(t) |x(t)|^2 - \min \Phi$, from (14) we deduce that

$$\frac{1}{2} \dot{\varepsilon}(t) |x(t)|^2 + \gamma \int_{t_0}^t |\dot{x}(s)|^2 \, ds \leq E(t_0) - \min \Phi + \frac{1}{2} \int_{t_0}^t \dot{\varepsilon}(s) |x(s)|^2 \, ds. \quad (15)$$

Let $a(t) = \int_{t_0}^t \dot{\varepsilon}(s) |x(s)|^2$, then $\dot{a}(t) = \dot{\varepsilon}(t) |x(t)|^2$ and, in view of (15), it follows that

$$\frac{\dot{a}(t)}{\dot{\varepsilon}(t)} \leq 2(E(t_0) - \min \Phi) + a(t).$$

Hence

$$\dot{a}(t) \leq 2(E(t_0) - \min \Phi) \frac{\dot{\varepsilon}(t)}{\varepsilon(t)} + \frac{\dot{\varepsilon}(t)}{\varepsilon(t)} a(t).$$

Since $a(t_0) = 0$, by integrating between $t_0$ and $t$, we deduce that

$$a(t) \leq 2(E(t_0) - \min \Phi) \int_{t_0}^t \frac{\dot{\varepsilon}(s)}{\varepsilon(s)} \, ds + \int_{t_0}^t \frac{\dot{\varepsilon}(s)}{\varepsilon(s)} a(s) \, ds.$$

Applying Gronwall’s lemma to the map $a$, it follows that, for every $t \in [t_0, T_{\text{max}})$

$$a(t) \leq 2(E(t_0) - \min \Phi) \left( \int_{t_0}^{+\infty} \frac{\dot{\varepsilon}(s)}{\varepsilon(s)} \, ds \right) \exp \left( \int_{t_0}^t \frac{\dot{\varepsilon}(s)}{\varepsilon(s)} \, ds \right).$$

This implies that the function $a$ is bounded on the set $[t_0, T_{\text{max}})$ by the constant

$$C = 2(E(t_0) - \min \Phi) \left( \int_{t_0}^{+\infty} \frac{\dot{\varepsilon}(s)}{\varepsilon(s)} \, ds \right) \exp \left( \int_{t_0}^t \frac{\dot{\varepsilon}(s)}{\varepsilon(s)} \, ds \right).$$

From (13), we deduce that for all $t \in [t_0, T_{\text{max}})$

$$E(t) \leq E(t_0) + \frac{C}{2}.$$
Hence, the energy function is bounded, which implies that the set 
\( \{ \Phi(t) \mid t \in [t_0, T_{\text{max}}] \} \) is bounded from above. Since the map \( \Phi \) is coercive, this implies that the solution map \( x \) is bounded.

We now assume that \( \int_{t_0}^{+\infty} i^2 \dot{\phi}(t) \, dt < +\infty \). From (13), we have, for every \( t \in [t_0, T_{\text{max}}) \)

\[
\dot{E}(t) \leq \frac{1}{2} \dot{\phi}(t) |x(t)|^2. \tag{16}
\]

By Cauchy–Schwarz inequality, for every \( t \in [t_0, T_{\text{max}}) \)

\[
|x(t)| = |x(t_0) + \int_{t_0}^{t} \dot{x}(s) \, ds| \\
\leq |x(t_0)| + \int_{t_0}^{t} |\dot{x}(s)| \, ds \\
\leq |x(t_0)| + \sqrt{t-t_0} \left( \int_{t_0}^{t} |\dot{x}(s)|^2 \, ds \right)^{1/2}.
\]

Hence

\[
|x(t)|^2 \leq 2 |x(t_0)|^2 + 2(t-t_0) \int_{t_0}^{t} |\dot{x}(s)|^2 \, ds. \tag{17}
\]

Since, for every \( t \in [t_0, T_{\text{max}}) \), and assuming without any loss of generality that \( \phi \geq 0, E(t) \geq \frac{1}{2} |\dot{x}(t)|^2 \), and in view of (16) and (17), we have

\[
E(t) \leq \frac{1}{2} \dot{\phi}(t) \left( 2 |x(t_0)|^2 + 2(t-t_0) \int_{t_0}^{t} 2E(s) \, ds \right).
\]

By integrating between \( t_0 \) and \( t \), we obtain

\[
E(t) \leq E(t_0) + |x(t_0)|^2 \int_{t_0}^{t} \dot{\phi}(s) \, ds + 2 \int_{t_0}^{t} (s-t_0) \dot{\phi}(s) \int_{t_0}^{s} E(u) \, du \, ds,
\]

which implies, in view of Fubini’s theorem, and letting \( C = E(t_0) + |x(t_0)|^2 \int_{t_0}^{+\infty} \dot{\phi}(s) \, ds \), that

\[
E(t) \leq C + 2 \int_{t_0}^{t} E(s) \int_{s}^{+\infty} (u-t_0) \dot{\phi}(u) \, du \, ds.
\]

Applying Gronwall’s lemma to the energy function \( E \), we obtain

\[
E(t) \leq C \exp \left( 2 \int_{t_0}^{+\infty} \int_{s}^{+\infty} (u-t_0) \dot{\phi}(u) \, du \, ds \right).
\]
Since $\int_{t_0}^{+\infty} \int_{-\infty}^{+\infty} (u-t_0) \dot{\theta}(u) du ds = \int_{t_0}^{+\infty} (u-t_0)^2 \dot{\theta}(u) du$, the energy function is bounded, which—as above—implies that the set $\{ \Phi(x(t)) \mid t \in [t_0, T_{\max}] \}$ is bounded. Hence the solution map $x$ is bounded (since the map $\Phi$ is coercive).

6. MORE REMARKS

6.1. Numerical Experiments

Consider a solution $x$ of the (HBFC) system, with a slow parametrization. In general, it is not an easy task to verify to which case (cases (a), (b) and (c) in the proof of Theorem 2.3) the solution $x$ belongs. Enlarging Figure 4 (see Fig. 5) may suggest that after many oscillations, a solution which does not belong to Case (a) behaves (slowly) as a Tikhonov regularization. It may be of interest to numerically study the asymptotic behavior and the rate of convergence of these solutions.
Remark 6.1. Note also that Cases (a) and (b) are definitely not empty. Consider the case where $0 \in S$. Then $p = 0$ and Case (a) always holds. To show that Case (b) may hold, consider the map $F: \mathbb{R} \to \mathbb{R}$ defined by $F(x) = \frac{1}{2}(x - 1)^2$. Then the map $x(t) = 1 - \frac{1}{t}$ is a solution map of the corresponding (HBFC) system:

$$\ddot{x}(t) + \dot{x}(t) + x - \varepsilon(t) x(t) = 0,$$

if we let $\varepsilon(t) = (t^2 - t + 2)/(t^3 - t^2)$. Then $\varepsilon(t) \geq 0$ and $\dot{\varepsilon}(t) \leq 0$ for every $t > 1$, and $\lim_{t \to +\infty} \varepsilon(t) = \lim_{t \to +\infty} \dot{\varepsilon}(t) = 0$, hence the assumption of Theorem 2.3 are satisfied for $t_0 > 1$, $p = 1$, and $x(t) \in (0, 1) = B(p/2, |p|/2)$ for every $t \geq t_0$.

6.2. Toward a More General Control, a More General Map $Φ$

This paper should be viewed as a step on the asymptotic control of (nonlinear) hyperbolic systems having nonunique equilibria. It clearly calls for some further extensions.

A first one would be to replace the control $ε(t) x(t)$ by a general control $ε(t, x)$. A first investigation shows that, in this general setting, it leads to non-intrinsic assumptions on the control. Among this class, one should particularly mention the controls of the type $ε(t) U(x(t))$, with a potential $U: H \to \mathbb{R}$, of class $C^1$. An interesting case is the one where the potential $U$ is not convex. This situation occurs when considering a coupled system with a repulsion potential. In that case, the techniques used in this paper do not directly give a result, but they indicate a possible direction. See [11] for a theoretical study of the asymptotic properties of the coupled system in $H \times H$

$$(HBFC^2) \quad \begin{cases} \ddot{x} + γx + ∇Φ(x) + ε(t) ∇V(x - y) = 0 \\ \ddot{y} + γy + ∇Φ(y) - ε(t) ∇V(x - y) = 0, \end{cases}$$

where $V: H \to \mathbb{R}^+$ is a repulsive potential of class $C^1$ and assuming Hypothesis 2.1 for $Φ$, $γ$, $ε$, with initial datas $(x_0, \dot{x}_0, y_0, \dot{y}_0)$ in $H^1$. See also [3] for a numerical study of the exploration of local minima by $N$ coupled (HBFC)-type systems.

Another natural extension should be to relax the convexity assumption on the map $Φ$. In view of applications, for example to Optimization, Mechanics and Economics, one is often interested with nonsmooth and/or nonconvex objective functions, possibly constrained. With this in mind, note that most of the convexity properties used in this paper are monotonicity properties. These studies raise nontrivial difficulties, since the
existence of constraints implies the possibility of shocks, with $\dot{x}$ being discontinuous and $\ddot{x}$ being a measure in the (HBF) system. We refer to [4] for the study of shock solutions of the following (HBF)-type system

$$\ddot{x}(t) + \gamma \dot{x}(t) + \partial \Phi(x(t)) \ni 0,$$

with $\gamma > 0$ and $\Phi : \mathbb{R}^N \to \mathbb{R} \cup \{\pm\infty\}$ is convex lower semicontinuous. Hopefully the control techniques used in this paper could be applied to the case of shocks.

Also, as the example of the wave equation is encouraging, we hope that our results would generalize to the domain of PDE.

REFERENCES

14. X. Goudou, Genericity of the convergence towards a local minimum of the heavy ball method, to appear.