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Comparison of the Adomian decomposition method and the variational iteration method in solving the moving boundary problem

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ABSTRACT

In this paper, a comparison between two methods: the Adomian decomposition method and the variational iteration method, used for solving the moving boundary problem, is presented. Both of the methods consist in constructing the appropriate iterative or recurrence formulas, on the basis of the equation considered and additional conditions, enabling one to determine the successive elements of a series or sequence approximating the function sought. The precision and speed of convergence of the procedures compared are verified with an example.

1. Introduction

The moving boundary problem describes a group of processes, in which the region considered is bounded by the moving boundary. The solving of this problem consists in determining the function which satisfies the given differential equation with the appropriate boundary conditions in the region under discussion, and the function describing the position of the moving boundary. In the current paper, we present a comparison between solutions of the moving boundary problem obtained by using the Adomian decomposition method (ADM) and the variational iteration method (VIM).

The Adomian decomposition method, introduced by G. Adomian, consists in presenting the function sought in the form of function series and deriving the iterative formula which enables one to calculate the successive elements of the series, with the aid of the given initial and boundary conditions [1]. The convergence of Adomian's series to the exact solution is considered for example in [2]. Similarly, applying the variational iteration method, created by Ji-Huan He [3–6], consists in constructing the appropriate correction functional connected with the considered equation. The correction functional contains a Lagrange multiplier, the determination of which leads to a recurrence formula. Convergence of the VIM method is discussed by Tatari and Dehghan in [7]. Both of the methods examined have found application in determining the approximate solutions of different technical problems. Examples of these applications are considered in, among other work, [8–12]. Adaptation of the VIM method for solving the heat conduction problem is also discussed by Chun in [13], whereas application of the VIM method for solving direct and inverse Stefan problems is presented by Słota in [14].

2. Formulation of the problem

In the current paper, we consider the moving boundary problem (the one-phase Stefan problem) defined in the domain D presented in Fig. 1.

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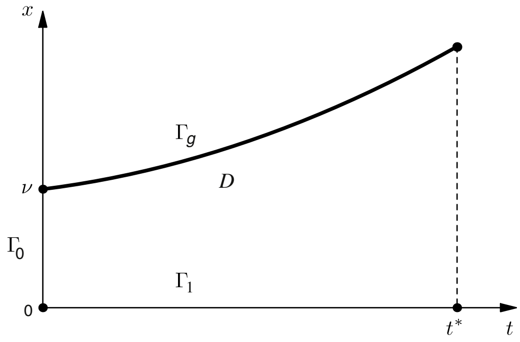

Fig. 1. Domain of the problem considered.

For solving this problem we need to determine the function $u(x, t)$, describing the temperature distribution in the domain D, and the function $\xi(t)$, denoting the moving boundary. The functions sought satisfy the following equations:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{a}\frac{\partial u(x,t)}{\partial t}, \quad \text{in } D, \tag{1}$$

$$u(x,0) = \varphi(x), \quad \text{on } \Gamma_0, \tag{2}$$

$$-\lambda \frac{\partial u(x,t)}{\partial x} = q(t), \quad \text{on } \Gamma_1, \tag{3}$$

$$u(\xi(t),t) = u^*, \quad \text{on } \Gamma_g, \tag{4}$$

$$-\lambda \frac{\partial u(x,t)}{\partial x} = \kappa \frac{\mathrm{d}\xi(t)}{\mathrm{d}t}, \quad \text{on } \Gamma_g, \tag{5}$$

where a is the thermal diffusivity, λ is the thermal conductivity, κ is the latent heat of fusion per unit volume, and u, t and x refer to temperature, time and spatial location, respectively.

3. ADM and VIM methods

According to the ADM method [1,2,8,9], we seek the approximate solution of the problem considered in the form of a function series:

$$u_n(x,t) = \sum_{i=0}^{n} g_i(x,t), \quad n \in \mathbb{N}. \tag{6}$$

After making some appropriate transformations and including the boundary conditions (3) and (4) we obtain the following recurrence formula (for details see [9]):

$$g_0(x,t) = \frac{1}{\lambda} q(t)\big(\xi(t) - x\big) + u^*,$$
$$g_n(x,t) = -\frac{1}{a}\int_{\xi(t)}^{x}\int_{x}^{0} \frac{\partial g_{n-1}(x,t)}{\partial t}\,\mathrm{d}x\,\mathrm{d}x, \quad n \ge 1. \tag{7}$$

The variational iteration method is useful for solving a wide range of nonlinear operator equations of the form

$$L(u(z)) + N(u(z)) = f(z), \tag{8}$$

where L is the linear operator, N is the nonlinear operator, f is some given function and u is the function sought. This method consists in constructing the correction functional, which for Eq. (1), describing the problem considered in this paper, has the form

$$u_n(x,t) = u_{n-1}(x,t) + \int_0^x \gamma\left(a\frac{\partial^2 u_{n-1}(s,t)}{\partial s^2} - \frac{\partial \tilde{u}_{n-1}(s,t)}{\partial t}\right)\mathrm{d}s, \tag{9}$$

where $\tilde{u}_{n-1}$ is the restricted variation [3,4], γ is the general Lagrange multiplier, which can be optimally identified with the aid of the variational theory [3], and $u_0(z)$ is the initial approximation. From the Eq. (9), the general Lagrange multiplier can be identified as the function $\gamma = s - x$, which gives the recurrence formula written below:

$$u_n(x,t) = u_{n-1}(x,t) + \int_0^x (s-x)\left(a\frac{\partial^2 u_{n-1}(s,t)}{\partial s^2} - \frac{\partial u_{n-1}(s,t)}{\partial t}\right)\mathrm{d}s. \tag{10}$$

Table 1
Values of the error in reconstruction of the moving interface position $\xi(t)$ and the temperature distribution $u(x,t)$.

	ADM			VIM		
	$m=2$	$m=3$	$m=4$	$m=2$	$m=3$	$m=4$
δ_ξ	0.01416	0.00897	0.00887	0.00067	0.00051	0.00050
Δ_ξ (%)	6.93836	4.39320	4.34508	2.32534	1.75462	1.75408
δ_u	0.01565	0.01223	0.01206	0.00137	0.00122	0.00121
Δ_u (%)	3.69274	2.88688	2.84560	0.13499	0.11950	0.11948

In order to determine the initial approximation we assume it to be of the form

$$u_0(x,t) = A + Bx, \tag{11}$$

where A and B are some parameters. For calculating those unknown parameters we require that the initial approximation $u_0(x,t)$ fulfills the Neumann boundary condition (3) and the condition of temperature continuity (4). Finally, the basic calculations lead to the following recurrence formula:

$$u_0(x,t) = u^* + \frac{1}{\lambda} q(t)\big(\xi(t) - x\big),$$
$$u_n(x,t) = u_{n-1}(x,t) + \int_0^x (s-x)\left(a\,\frac{\partial^2 u_{n-1}(s,t)}{\partial s^2} - \frac{\partial u_{n-1}(s,t)}{\partial t}\right) \mathrm{d}s, \quad n \geq 1, \tag{12}$$

determining the approximate solution of the problem considered, (1)–(5).

4. The functional

Application of both of the methods presented leads to the approximate solution u_n, having the form of the series (6) or determined by the formula (12). In both of those cases the recurrence formulas depend on the unknown function $\xi(t)$. We propose to derive this function in the form of a linear combination:

$$\xi(t) = \sum_{i=1}^{m} p_i\,\psi_i(t), \tag{13}$$

where $p_i \in \mathbb{R}$ and the base functions $\psi_i(t)$ are linearly independent. The coefficients p_i are selected in such a way as to obtain the minimal deviation of the approximated function $u_n(x,t)$ from the conditions (2) and (5) (considering the assumed measure). The measure of the error will be taken in the form of the following functional based on the least squares method:

$$J(p_1,\ldots,p_m) = \int_0^{\xi(0)} (u_n(x,0) - \varphi(x))^2\,\mathrm{d}x + \int_0^{t^*}\left(\lambda\,\frac{\partial u_n(x,t)}{\partial x}\bigg|_{x=\xi(t)} + \kappa\,\frac{\mathrm{d}\xi(t)}{\mathrm{d}t}\right)^2 \mathrm{d}t, \tag{14}$$

which has to be minimized. For minimizing the above functional we can use one of the gradient methods, since we are able to calculate the gradient of (14). In the course of minimizing the functional (14) the coefficients p_i are determined and, thereby, the approximated distribution of temperature $u(x,t)$ in the domain D and position of the moving interface $\xi(t)$ are obtained.

5. An example

The comparison between the ADM and VIM methods will be presented with the aid of an example, in which: $a = 0.1$, $\lambda = 1$, $\varphi(x) = \mathrm{e}^{-x}$, $q(t) = \lambda\,\mathrm{e}^{a t}$, $\kappa = \lambda/a$, $u^* = 1$, $t^* = 1/2$. Under those assumptions, the exact solution of the problem considered is given by the following functions: $u(x,t) = \mathrm{e}^{a t - x}$ and $\xi(t) = a t$. With the known exact solution we will compare the approximate solutions obtained by using each of the methods considered. As the base functions for the linear combination (13) we take the monomials $\psi_i(t) = t^{i-1}$, for $i = 1,\ldots,m$.

Values of the absolute (δ_ξ and δ_u) and percentage relative errors (Δ_ξ and Δ_u) in the reconstruction of the temperature distribution $u(x,t)$ and position of the moving interface Γ_g (function $\xi(t)$) are compiled in Table 1. The errors are calculated for two elements in the sum or sequence describing the distribution of the temperature $u_n(x,t)$ ($n = 1$) and for a different number of basis functions $\psi_i(t)$ in the sum (13) ($m \in \{2,3,4\}$).

In Fig. 2 the positions of the moving interface reconstructed by using the ADM method (left figure) and by using the VIM method (right figure) for $n = 2$ and $m = 3$ are compared with the exact position. The errors obtained in this case are equal to: $\delta_\xi = 0.008203$, $\Delta_\xi = 4.0187$, $\delta_u = 0.011878$, $\Delta_u = 2.8031$ – for the ADM method; $\delta_\xi = 0.00045$, $\Delta_\xi = 1.56688$, $\delta_u = 0.00116$, $\Delta_u = 0.11450$ – for the VIM method.

6. Conclusion

As a conclusion we can note that the results obtained are satisfactory for both methods. The approximate solutions are convergent to the exact solution and the errors of approximation are small. However, the detailed analysis of the absolute

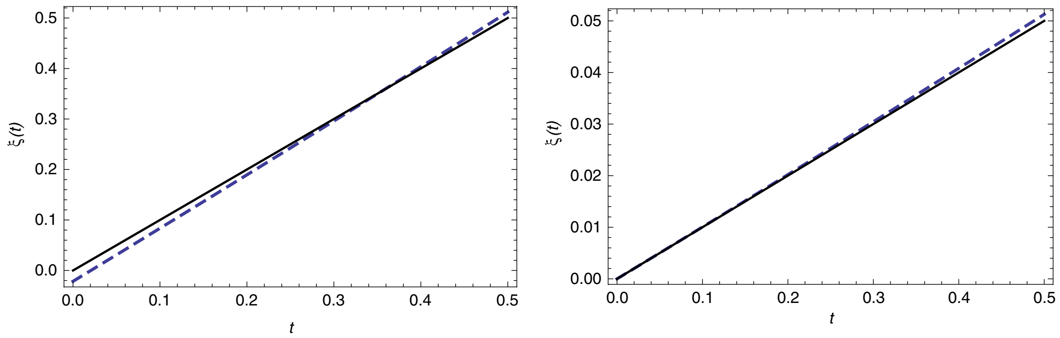

Fig. 2. Position of the moving interface reconstructed using the ADM (left figure) and using the VIM (right figure): solid line – exact position, dashed line – reconstructed position.

and relative errors suggests that the VIM method is slightly more effective for solving the moving boundary problem considered. The version of the VIM algorithm used in the current paper belongs to the group denoted as variational iteration algorithms I [15]. There are also alternative algorithms, i.e. variational iteration algorithms II and variational iteration algorithms III, whose application to the present problem is in progress.

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