Sharp Error Bounds for the Derivatives of Lidstone-Spline Interpolation II

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Abstract—In this paper, we shall derive explicit error estimates in $L_{\infty}$ norm between a given function $f(x) \in PC^{(n)}[a,b]$, $4 \leq n \leq 6$ and its quintic Lidstone-spline interpolate. The results obtained are then used to establish precise error bounds for the approximated and biquintic Lidstone-spline interpolates. We also include applications to integral equations and boundary value problems as well as sufficient numerical examples which dwell upon the sharpness of the obtained results.

Keywords—Lidstone-spline interpolation, Boundary value problems, Integral equations, Error bounds.

1. INTRODUCTION

In the year 1929, Lidstone [1] introduced a generalization of Taylor’s series. It approximates a given function in the neighborhood of two points instead of one. In terms of completely continuous functions it has been characterized in the work of Boas [2,3], Poritsky [4], Schoenberg [5], Whittaker [6,7], Widder [8,9], and others. In the field of approximation theory [10–14] the Lidstone interpolating polynomial of degree $(2m-1)$ satisfies the Lidstone conditions $P^{(2i)}(0) = A_i$, $P^{(2i)}(1) = B_i$, $0 \leq i \leq m-1$. Further, boundary value problems consisting of $2m$th order ordinary differential equation $f^{(2m)} = F(x, f, f', \ldots, f^{(2m-1)})$ and the Lidstone boundary conditions $f^{(2m)}(0) = A_i$, $f^{(2m)}(1) = B_i$, and several of its particular cases have been the subject matter of several recent investigations [11,15–23]. The motivation of this paper is in line with these related works, and the results obtained supplement those established earlier by the authors [11,24].

Let $-\infty < a < b < \infty$ and $-\infty < c < d < \infty$. For the intervals $[a, b]$ and $[c, d]$, we let $A : a = x_0 < x_1 < \cdots < x_{N+1} = b$, and $A' : c = y_0 < y_1 < \cdots < y_{M+1} = d$ denote uniform partitions of $[a, b]$ × $[c, d]$. Further, we let $p = A \times A'$ be a rectangular partition of $[a, b] \times [c, d]$. Let $PC^{n,\infty}[a, b]$ be the set of all real-valued functions $f(x)$ such that:

(i) $f(x)$ is $(n-1)$ times continuously differentiable on $[a, b]$,
(ii) there exist $t_i$, $0 \leq i \leq L + 1$ with $a = t_0 < t_1 < \cdots < t_{L+1} = b$ such that on each open subinterval $(t_i, t_{i+1})$, $0 \leq i \leq L$, $D^{n-1}f$ is continuously differentiable, and
(iii) the sup-norm of $D^n f$ is finite; i.e.,

$$\|D^n f\| = \max_{0 \leq i \leq L} \sup_{x \in (t_i, t_{i+1})} |D^n f(x)| < \infty.$$
For the functions $f(x, y)$ of two variables, the sets $PC^n,\infty([a, b] \times [c, d])$ and $PC^n,\infty([a, b] \times [c, d])$ are defined analogously.

The plan of the paper is as follows. In Section 2, we shall collect some results established earlier [10, 12, 24] which will be needed in later sections. In Section 3, for a given function $f(x) \in PC^n,\infty([a, b])$, $4 \leq n \leq 6$, we shall define its quintic Lidstone-spline interpolate $LS^5 f(x)$, which is different from that considered in [24]. Two explicit representations of $LS^5 f(x)$ are also given. We shall derive explicit upper estimates for $\|D^k(f - LS^5 f)\|$, $0 \leq k \leq n - 1$, $4 \leq n \leq 6$ in terms of $\|D^n f\|$ in Section 4. In Section 5, we shall construct approximated Lidstone splines $S_i f(x)$, $1 \leq i \leq 7$ under the conditions

(i) unknown $f^{(4)}_i$, $i = 0, N + 1$;
(ii) unknown $f^{(4)}_i$, $0 \leq i \leq N + 1$;
(iii) unknown $f^{(4)}_i$, $i = 0, N + 1$, $f^{(4)}_i$, $0 \leq i \leq N + 1$;
(iv) unknown $f^{(4)}_i$, $i = 0, N + 1$, $f''_i$, $1 \leq i \leq N$;
(v) unknown $f^{(4)}_i$, $i = 0, N + 1$, $f''_i$, $f_i$, $0 \leq i \leq N + 1$;
(vi) unknown $f^{(4)}_i$, $i = 0, N + 1$, $f''_i$, $f_i$, $1 \leq i \leq N$; and
(vii) unknown $f''_i$, $f_i$, $1 \leq i \leq N$.

Again explicit upper bounds for $\|D^k(f - S_i f)\|$, $0 \leq k \leq 5$, $1 \leq i \leq 7$ in terms of $\|D^5 f\|$ are established. These results directly conclude the stability properties of Lidstone spline. Section 6 deals with the treatment of two-dimensional Lidstone-spline interpolation. Once again we shall offer explicit error estimates in the two-dimensional case. Finally in Section 7, as applications of the results obtained in Sections 4 and 5, we shall construct sixth order $C^3$ approximate solutions of second order boundary value problems. We shall also approximate the solutions of Fredholm integral equations, and provide a priori as well as posteriori error bounds between the exact and approximate solutions. Throughout, sufficient numerical illustration, which dwells upon the sharpness and importance of the results obtained, is included.

2. PRELIMINARIES

DEFINITION 2.1. The unique polynomial $\Lambda_n(x)$ of degree $(2n + 1)$ defined by the relations

$$\begin{align*}
\Lambda_0(x) &= x, \\
\Lambda''_n(x) &= \Lambda_{n-1}(x), \\
\Lambda_n(0) &= \Lambda_n(1) = 0, & n \geq 1
\end{align*}$$

is called the Lidstone polynomial.

LEMMA 2.1. [10] The Lidstone polynomial $\Lambda_n(x)$ can be expressed as

$$\Lambda_n(x) = \int_0^1 g_n(x, t) \, dt, \quad n \geq 1,$$

where

$$g_1(x, t) = \begin{cases} (x - 1)t, & t \leq x, \\
(t - 1)x, & x \leq t, \end{cases}$$

$$g_n(x, t) = \int_0^1 g_1(x, t_1) g_{n-1}(t_1, t) \, dt_1, \quad n \geq 2.$$
Lemma 2.2. [10] The following equality holds:

\[ A_n(1 - x) = \int_0^1 g_n(x, t) (1 - t) \, dt, \quad n \geq 1. \]  

(2.5)

Let \( d_{2n,k} \) represent the numbers

\[
d_{2n,k} = \begin{cases} 
(-1)^{n-i} E_{2n-2i} & k = 2i, \quad 0 \leq i \leq n, \\
\frac{2(2n-2i-1)!}{(2n-2i)!} & k = 2i + 1, \quad 0 \leq i \leq n - 1, \\
2 & k = 2n + 1,
\end{cases}
\]  

(2.6)

where \( E_{2n} \) and \( B_{2n} \) are \( 2n \)th Euler and Bernoulli numbers, respectively. It is clear that \( d_{2n,k+2} = d_{2n-2,k}, 0 \leq k \leq 2n - 1. \)

Lemma 2.3. [10] The following holds:

\[
\int_0^1 (-1)^n g_n(x, t) \, dt = \int_0^1 |g_n(x, t)| \, dt = (-1)^n E_{2n}(x) \leq d_{2n,0},
\]  

(2.7)

where \( E_{2n}(x) \) is the Euler polynomial of degree \( 2n \).

Lemma 2.4. [10] The following holds:

\[
\int_0^1 |g_n(x, t)| \, dt = (-1)^n [2E_{2n}(x) + (1 - 2x)E_{2n-1}(x)] \leq d_{2n,1}.
\]  

(2.8)

For a fixed \( \Delta \), we define the set \( L_m(\Delta) = \{ h(\Delta) \in C[a, b] : h(\Delta) \) is a polynomial of degree at most \((2m - 1)\) in each subinterval \([x_i, x_{i+1}]\), \(0 \leq i \leq N\}. \) It is clear that \( L_m(\Delta) \) is of dimension \([2m(N + 1) - N]\).

Definition 2.2. For a given function \( f(x) \in C^{(2m-2)}[a, b], \) we say \( L_m^\Delta f(x) \) is the \( L_m(\Delta) \)-interpolate of \( f(x), \) also known as Lidstone interpolate of \( f(x), \) if \( L_m^\Delta f(x) \in L_m(\Delta) \) with \( D^k L_m^\Delta f(x_i) = f^{(2k)}(x_i), 0 \leq k \leq m - 1, 0 \leq i \leq N + 1. \)

In view of (2.1)-(2.5), it can be shown that (cf. [10]) for \( f(x) \in C^{(2m-2)}[a, b], \) \( L_m^\Delta f(x) \) uniquely exists and in the subinterval \([x_i, x_{i+1}]\) can be explicitly expressed as

\[
L_m^\Delta f(x) = \sum_{k=0}^{m-1} \left[ f_i^{(2k)} \Lambda_k \left( \frac{x_{i+1} - x}{h} \right) + f_{i+1}^{(2k)} \Lambda_k \left( \frac{x - x_i}{h} \right) \right] h^{2k}.
\]  

(2.9)

Therefore, it follows that

\[
L_m^\Delta f(x) = \sum_{i=0}^{N+1} \sum_{j=0}^{m-1} r_{m,i,j}(x) f_i^{(2j)} ,
\]  

(2.10)

where \( r_{m,i,j}(x); 0 \leq i \leq N + 1, 0 \leq j \leq m - 1 \) are the basic elements of \( L_m(\Delta) \) satisfying

\[
D^\nu r_{m,i,j}(x_{\mu}) = \delta_{i\mu} \delta_{\nu j}; \quad 0 \leq \nu \leq m - 1, \quad 0 \leq \mu \leq N + 1
\]  

(2.11)

and appear as

\[
r_{m,i,j}(x) = \Lambda_j \left( \frac{x_{i+1} - x}{h} \right) h^{2j}, \quad x_i \leq x \leq x_{i+1}, \quad 0 \leq i \leq N,
\]

\[
= \Lambda_j \left( \frac{x - x_{i-1}}{h} \right) h^{2j}, \quad x_{i-1} \leq x \leq x_i, \quad 1 \leq i \leq N + 1,
\]

\[
= 0, \quad \text{otherwise}.
\]  

(2.12)
It is clear that in the above Definition 2.2 as well as in the representation (2.10), the function \( f(x) \) need not be in \( C^{(2m-2)}[a, b] \); rather, it is sufficient (which we shall assume throughout) that for the function \( f(x) \), \( D^{2k}L_m f(x_i) = f^{(2k)}(x_i) = f_i^{(2k)} \); \( 0 \leq k \leq m - 1, 0 \leq i \leq N + 1 \) exist.

**Lemma 2.5.** [24] For \( 0 \leq i \leq N \) and \( 0 \leq k \leq 2j + 1, 0 \leq j \leq m - 1 \), the following equalities hold:

\[
\max_{x_i \leq x \leq x_{i+1}} \left[ |D^{k}r_{m,i,j}(x)| + |D^{k}r_{m,i+1,j}(x)| \right] = d_{2j,k}h^{2j-k}.
\]  

(2.13)

The first few \( d_{2j,k} \) are given in Table 2.1.

<table>
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<th>3</th>
<th>4</th>
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</table>

**Lemma 2.6.** [24] For \( 0 \leq i \leq N \) and \( 0 \leq k \leq 5 \), the following equalities hold:

\[
\max_{x_i \leq x \leq x_{i+1}} |D^k r_{3,i,j}(x)| = b_{2j,k}h^{2j-k}, \quad j = 1, 2,
\]  

(2.14)

where the constants \( b_{2j,k} \) are given in Table 2.2.

<table>
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<th>( j )</th>
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<td>\frac{1}{3} \sqrt{3}</td>
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**Lemma 2.7.** [10] Let \( F(x) \in PC^{2m,\infty}[a, b] \). Then,

\[
\|D^k(f - L_m^\Delta f)\| \leq d_{2m,k}h^{2m-k}\|D^{2m}f\|, \quad 0 \leq k \leq 2m - 1.
\]  

(2.15)

**Lemma 2.8.** [12] Let \( f(x) \in PC^{2m-2,\infty}[a, b] \). Then,

\[
\|D^k(f - L_m^\Delta f)\| \leq 2d_{2m-2,k}h^{2m-2-k}\|D^{2m-2}f\|, \quad 0 \leq k \leq 2m - 2.
\]  

(2.16)

**Lemma 2.9.** [12] Let \( f(x) \in PC^{2m-1,\infty}[a, b] \), \( 1 \leq m \leq 3 \). Then,

\[
\|D^k(f - L_m^\Delta f)\| \leq c_{2m-1,k}h^{2m-1-k}\|D^{2m-1}f\|, \quad 0 \leq k \leq 2m - 2,
\]  

(2.17)

where the constants \( c_{2m-1,k} \) are given in Table 2.3.
Results similar to that of Lemmas 2.7-2.9 for several other interpolating polynomials such as Hermite, Abel-Gontscharoff, special cases of Birkhoff, and their applications in the theory of ordinary differential equations are available in [11,25-27].

**Lemma 2.10.** [28] Let $A$ be a square matrix such that $\|A\| < 1$. Then, $(I \pm A)$ is nonsingular and

$$\|(I \pm A)^{-1}\| \leq (1 - \|A\|)^{-1},$$

where $I$ is the identity matrix.

### 3. Quintic Lidstone-Spline Interpolation

For a fixed $\Delta$, we define the spline space $S_m(\Delta) = \{s(x) \in C^{(2m-3)}([a, b]) : s(x)$ is a polynomial of degree at most $(2m - 1)$ in each subinterval $[x_i, x_{i+1}], 0 < i < N\}$. It is clear that $S_m(\Delta)$ is of dimension $2(N + m)$.

**Definition 3.1.** For a given $f(x) \in C^{(2m-2)}[a, b]$, we say $LS_m^f(x)$ is the Lidstone $S_m(\Delta)$-interpolate of $f(x)$, also known as Lidstone-spline interpolate of $f(x)$, if $LS_m^f(x) \in S_m(\Delta)$ with

$$D^{2k} LS_m^f(x_i) = f^{(2k)}(x_i) = f^{(2k)}_i, \quad 1 \leq i \leq N, \quad 0 \leq k \leq m - 2,$$

$$D^{2k} LS_m^f(x_i) = f^{(2k)}(x_i) = f^{(2k)}_i, \quad 0 = i = N + 1, \quad 0 \leq k \leq m - 1.$$  \hspace{1cm} (3.1)

Since $S_m(\Delta) \subseteq L_m(\Delta)$, we can represent $LS_m^f(x)$ in terms of the basic elements $r_{m,i,j}(x)$; $0 \leq i \leq N + 1$, $0 \leq j \leq m - 1$ of $L_m(\Delta)$ even though these functions may not belong to $S_m(\Delta)$. In fact, we have

$$LS_m^f(x) = \sum_{i=0}^{N+1} \sum_{j=0}^{m-1} r_{m,i,j}(x) D^{2j} LS_m^f(x_i).$$ \hspace{1cm} (3.2)

As earlier, here also we note that in the above Definition 3.1 as well as in the representation (3.2) for the function $f(x)$ it is sufficient to assume that $D^{2k} LS_m^f(x_i) = f^{(2k)}(x_i) = f^{(2k)}_i, \quad 0 \leq k \leq m - 1, \quad 0 \leq i \leq N + 1$ exist.

For $m = 3$ we shall show that the unknown constants $D^{2m-2} LS_m^f(x_i), 1 \leq i \leq N$ are the solutions of a diagonally dominant system of linear algebraic equations, and hence can be obtained explicitly in terms of the known quantities. For this, we need the following results.
Lemma 3.1. Let $1 \leq i \leq N$ but fixed, and $p(x), q(x)$ be two quintic polynomials in $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$, respectively. Suppose $D^{2k}p(x_i) = D^{2k}q(x_i) = f_i^{(2k)}$, $0 \leq k \leq 2$; then $D^p(x_i) = D^q(x_i)$ if and only if
\[
\begin{align*}
&\frac{h^2}{6} \left[ D^2 p(x_{i-1}) + 4 f_i'' + D^2 q(x_{i+1}) \right] = p(x_{i-1}) - 2 f_i + q(x_{i+1}) \\
&+ \frac{h^4}{360} \left[ 7D^4 p(x_{i-1}) + 16 f_i^{(4)} + 7D^4 q(x_{i+1}) \right],
\end{align*}
\]
and $D^3p(x_i) = D^3q(x_i)$ if and only if
\[
D^2p(x_{i-1}) - 2 f_i'' + D^2 q(x_{i+1}) = \frac{h^2}{6} \left[ D^2 p(x_{i-1}) + 4 f_i^{(4)} + D^2 q(x_{i+1}) \right].
\]

Proof. By construction
\[
p(x) = \sum_{k=0}^{5} a_k (x - x_i)^k,
\]
where
\[
a_0 = f_i, \quad a_2 = \frac{1}{2} f_i'', \quad a_4 = \frac{1}{24} f_i^{(4)},
\]
a_1 = \frac{1}{h} \left[ f_i - p(x_{i-1}) + \frac{h^2}{3} f_i'' + \frac{h^2}{6} D^2 p(x_{i-1}) - \frac{h^4}{45} f_i^{(4)} - \frac{7h^4}{360} D^4 p(x_{i-1}) \right],
\]
a_3 = \frac{1}{6h} \left[ f_i'' - D^2 p(x_{i-1}) + \frac{h^2}{3} f_i^{(4)} + \frac{h^2}{6} D^2 q(x_{i-1}) \right],
\]
and hence, $D^p(x_i) = a_1$, $D^3p(x_i) = a_3$. Similarly,
\[
q(x) = \sum_{k=0}^{5} b_k (x - x_i)^k,
\]
where
\[
b_0 = f_i, \quad b_2 = \frac{1}{2} f_i'', \quad b_4 = \frac{1}{24} f_i^{(4)},
\]
b_1 = \frac{1}{h} \left[ q(x_{i+1}) - f_i - \frac{h^2}{3} f_i'' - \frac{h^2}{6} D^2 q(x_{i+1}) + \frac{h^4}{45} f_i^{(4)} + \frac{7h^4}{360} D^4 q(x_{i+1}) \right],
\]
b_3 = \frac{1}{6h} \left[ D^2 q(x_{i+1}) - f_i'' - \frac{h^2}{3} f_i^{(4)} - \frac{h^2}{6} D^2 q(x_{i+1}) \right],
\]
and hence, $D^q(x_i) = b_1$, $D^3q(x_i) = b_3$.

Therefore, $D^p(x_i) = D^q(x_i)$ if and only if $a_1 = b_1$, which is the same as (3.3), and $D^3p(x_i) = D^3q(x_i)$ if and only if $a_3 = b_3$, which is the same as (3.4).

Lemma 3.2. Let $h(x) \in L_3(\Delta)$ be a function for which, say, $c_i = h(x_i)$, $c_i'' = D^2 h(x_i)$ and $c_i^{(4)} = D^4 h(x_i)$, $0 \leq i \leq N + 1$ exist. Then, $h(x) \in S_3(\Delta)$ if and only if $c_i^{(4)}$, $1 \leq i \leq N$ satisfy the following relations:
\[
65c_1^{(4)} + 26c_2^{(4)} + c_3^{(4)} = \frac{120}{h^4} c_0'' - 18 c_0^{(4)}
\]
\[+ \frac{120}{h^4} (-2c_0 + 5c_1 - 4c_2 + c_3),
\]
\[
c_i^{(4)} + 26c_i^{(4)} + 66c_{i-1}^{(4)} + 26c_{i+1}^{(4)} + c_{i+2}^{(4)} = \frac{120}{h^4} (c_{i-2} - 4c_{i-1} + 6c_i - 4c_{i+1} + c_{i+2}),
\]
\[2 \leq i \leq N - 1,
\]
\[
c_N^{(4)} + 26c_N^{(4)} + 65c_N^{(4)} = \frac{120}{h^4} c_{N+1}'' - 18 c_N^{(4)}
\]
\[+ \frac{120}{h^4} (c_{N-2} - 4c_{N-1} + 5c_N - 2c_{N+1}).
\]
Moreover, from the system (3.5)-(3.7), the unknowns \( c_i^{(4)} \), \( 1 \leq i \leq N \) can be obtained uniquely in terms of \( c_i \), \( 0 \leq i \leq N+1 \), \( c_0' \), \( c_{N+1}' \), \( c_0^{(4)} \), and \( c_{N+1}^{(4)} \).

**PROOF.** By Lemma 3.1, the continuity of \( Dh(x) \) and \( D^3h(x) \) is equivalent to the equations (3.3) and (3.4), respectively, which are better written as

\[
P_i : c''_{i-1} + 4c_i'' + c_{i+1}'' = \frac{6}{h^2} (c_{i-1} - 2c_i + c_{i+1}) + \frac{h^2}{60} \left( 7c_{i-1}^{(4)} + 16c_i^{(4)} + 7c_{i+1}^{(4)} \right)
\]

and

\[
Q_i : c''_{i+1} - 2c_i'' + c_i'' = \frac{h^2}{6} \left( c_{i-1}^{(4)} + 4c_i^{(4)} + c_{i+1}^{(4)} \right).
\]

The operations \( 2P_1 - P_2 + 4Q_1 + Q_2, -P_{i-1} + 2P_i + Q_{i+1} + 4Q_{i+1}, Q_{i-1} + 4Q_i + Q_{i+1}, \) and \( -P_{N-1} + 2P_N + Q_{N-1} + 4Q_N \) give (3.5)-(3.7), respectively.

The system (3.5)-(3.7) in matrix form can be written as

\[
Be^4 = k,
\]

where \( e^4 = [c_i^{(4)}], \)

\[
B = [b_{ij}] = \begin{cases} 
26, & |i - j| = 1, \\
65, & i = j = 1, N, \\
66, & i = j = 2, \ldots, N - 1, \\
1, & |i - j| = 2, \\
0, & \text{otherwise,}
\end{cases}
\]

and \( k = [k_i] \), where

\[
k_i = \begin{cases} 
\frac{120}{h^2} c_0^{(4)} - 18c_0^{(4)} + \frac{120}{h^4} (-2c_0 + 5c_1 - 4c_2 + c_3), & i = 1, \\
-2c_0^{(4)} + \frac{120}{h^4} (c_0 - 4c_1 + 6c_2 - 4c_3 + c_4), & i = 2, \\
\frac{120}{h^4} (c_{i-2} - 4c_{i-1} + 6c_i - 4c_{i+1} + c_{i+2}), & 3 \leq i \leq N - 2, \\
-2c_{N+1}^{(4)} + \frac{120}{h^4} (c_{N-3} - 4c_{N-2} + 6c_{N-1} - 4c_N + c_{N+1}), & i = N - 1, \\
\frac{120}{h^2} c_{N+1}^{(4)} + 120c_{N+1}^{(4)} + \frac{120}{h^4} (c_{N-2} - 4c_{N-1} + 5c_N - 2c_{N+1}), & i = N.
\end{cases}
\]

**LEMMA 3.3.** For a given \( f(x) \in C^{(4)}[a, b] \), \( LS_3 f(x) \) exists and is unique.

**PROOF.** For a given \( g(x) \in C^{(4)}[a, b] \), \( LS_3 g(x) \) exists and is unique. Further, by Lemma 3.2 for the given set of numbers \( c_i = f_i, c_i'' = f_i'' \), \( 0 \leq i \leq N + 1 \), \( c_i^{(4)} = f_i^{(4)} \); \( i = 0, N + 1 \) there exist unique \( c_i^{(4)} \), \( 1 \leq i \leq N \) satisfying (3.5)-(3.7). Now, let \( g(x) \in C^{(4)}[a, b] \) be such that \( g^{(2k)}(x_i) = c_i^{(2k)} \), \( 0 \leq i \leq N + 1, 0 \leq k \leq 2 \). Then, again by Lemma 3.2, \( LS_3 g(x) \in S_3(\Delta) \). However, from the definition, this \( LS_3 g(x) \) is the same as \( LS_3 f(x) \).

**REMARK 3.1.** From Lemma 3.3 and (3.2), it is clear that \( LS_3 f(x) \) can be expressed as

\[
LS_3 f(x) = \sum_{i=0}^{N+1} \left[ r_{3,i,0}(x)f_i + r_{3,i,1}(x)f_i'' \right] + r_{3,0,2}(x)f_0^{(4)} + r_{3,N+1,2}(x)f_{N+1}^{(4)} + \sum_{i=1}^{N} r_{3,i,2}(x)c_i^{(4)},
\]

where \( c_i^{(4)} \), \( 1 \leq i \leq N \) satisfy (3.5)-(3.7).
REMARK 3.2. It is possible to describe a basis for $S_3(\Delta)$, namely the 'L-cardinal splines,' \( \{s_i(x)\}_{i=0}^{2N+5} \), defined by the following interpolation conditions:

\[
\begin{align*}
   s_j(x_i) &= \delta_{ij}, \quad D^2 s_j(x_i) = D^4 s_j(a) = D^4 s_j(b) = 0, \quad 0 \leq i, j \leq N + 1, \\
   s_j(x_i) &= 0, \quad D^2 s_j(x_i) = \delta_{ij-N-2}, \quad D^4 s_j(a) = D^4 s_j(b) = 0; \\
   N + 2 &\leq j \leq 2N + 3, \quad 0 \leq i \leq N + 1, \\
   s_j(x_i) &= D^2 s_j(x_i) = 0; \quad j = 2N + 4, 2N + 5, \quad 0 \leq i \leq N + 1, \\
   D^4 s_{2N+4}(a) &= 1, \quad D^4 s_{2N+4}(b) = 0, \\
   D^4 s_{2N+5}(a) &= 0, \quad D^4 s_{2N+5}(b) = 1.
\end{align*}
\]

Obviously, $L S_3^f(x)$ can be explicitly expressed as

\[
L S_3^f(x) = \sum_{i=0}^{N+1} \left[ s_i(x)f_i + s_{i+N+2}(x)f_i^{(4)} + s_{2N+4}(x)f_i^{(4)} + s_{2N+5}(x)f_i^{(4)} \right].
\] (3.12)

4. ERROR BOUNDS

Let $f(x) \in PC^{n,\infty}[a,b], 2m - 2 < n \leq 2m$. To obtain upper bounds for $\|D^k(f - L S_m^f)\|$, $0 < k < n - 1$ in terms of $\|D^n f\|$, we begin with the equality

\[
f(x) - L S_m^f(x) = (f(x) - L_m^f(x)) + (L_m^f(x) - L S_m^f(x)).
\] (4.1)

In (4.1) the term $(L_m^f(x) - L S_m^f(x))$ belongs to $L_m(\Delta)$, and

\[
D^{2k}(L_m^f(x_i) - L S_m^f(x_i)) = 0; \quad 1 \leq i \leq N, \quad 0 \leq k \leq m - 2, \\
i = 0, N + 1, \quad 0 \leq k \leq m - 1.
\]

Hence, from (2.10) and (3.2), it follows that

\[
L_m^f(x) - L S_m^f(x) = \sum_{i=1}^{N} r_{m,i,m-1}(x) e_{i,m}^{2m-2},
\] (4.2)

where $e_{i,m}^{2m-2} = f_i^{(2m-2)} - D^{2m-2} L S_m^f(x_i)$. Thus, on substituting (4.2) into (4.1) and differentiating the resulting relation $k$ times, $0 < k < n - 1$ at $x \in [x_i, x_{i+1}]$, $0 \leq i \leq N$, we obtain

\[
D^k \left( f(x) - L S_m^f(x) \right) = D^k \left( f(x) - L_m^f(x) \right) + \sum_{i=1}^{N} r_{m,i,m-1}(x) e_{i,m}^{2m-2}.
\] (4.3)

Let the vector $[e_{i,m}^{2m-2}]$ be denoted as $e_{m}^{2m-2}$. Then, from the triangle inequality, we find that

\[
\|D^k \left( f - L S_m^f \right)\| \leq \|D^k \left( f - L_m^f \right)\| + \|e_{m}^{2m-2}\| \\
\times \max_{0 \leq k \leq n} \max_{x_i \leq x \leq x_{i+1}} \left[ \left| r_{m,i,m-1}(x) \right| + \left| r_{m,i+1,m-1}(x) \right| \right], \quad 0 \leq k \leq n - 1,
\] (4.4)

where we have used the fact that $r_{m,i,m-1}(x)$ is nonzero only in the interval $(x_{i-1}, x_i) \cup (x_i, x_{i+1})$.

In the right side of (4.4), the equalities and the inequalities obtained in Lemmas 2.5 and 2.7-2.9 can be used, and hence we only need to compute an upper estimate for $\|e_{m}^{2m-2}\|$.

**LEMMA 4.1.** Let $f(x) \in PC^{n,\infty}[a,b], 4 \leq n \leq 6$. Then,

\[
\|e_3^4\| \leq \beta_n h^{n-4} \|D^n f\|,
\] (4.5)

where $\beta_4 = 20$, $\beta_6 = 0.25465679 + 1$ and $\beta_6 = 5/6$. 
PROOF. We shall prove (4.5) only for \( n = 6 \). The proof of the other cases is similar. Let \( r = [r_i(f)] \) be an \( N \times 1 \) vector defined by

\[
r = B e^3,
\]

where the matrix \( B \) is given in (3.9). Then, it follows that

\[
B f^4 = k + r,
\]

where \( k \) is defined in (3.10) and \( f^4 = [f_i^{(4)}] \) is an \( N \times 1 \) vector.

For \( i = 1 \), from (4.7), (3.9) and (3.10), we have

\[
\begin{align*}
  r_1(f) &= 18 f_0^{(4)} + 65 f_1^{(4)} + 26 f_2^{(4)} + f_3^{(4)} - \frac{120}{h^4} f_0'' + \frac{120}{h^4} (2 f_0 - 5 f_1 + 4 f_2 - f_3),
\end{align*}
\]

which in view of Peano's kernel theorem can be written as

\[
\begin{align*}
  r_1(f) &= \frac{1}{5!} \int_{x_0}^{x_3} (r_1)_x (x - t)_{+}^5 D^6 f(t) d t,
\end{align*}
\]

where

\[
(r_1)_x (x - t)_{+}^5 = 120 \left[ 18 (x_0 - t)_{+} + 65 (x_1 - t)_{+} + 26 (x_2 - t)_{+} + (x_3 - t)_{+} - \frac{2400}{h^2} (x_0 - t)^3 + \frac{120}{h^4} \left[ 2 (x_0 - t)^5 + 5 (x_1 - t)^5 + 4 (x_2 - t)^5 - (x_3 - t)^5 \right] 
\right]
\]

\[
\begin{align*}
  T - T^5 &= \alpha_1(T), & T &= \frac{(x_3 - t)}{h} \in [0, 1],
  \\
  22 T - 10 T^2 - 10 T^3 - 5 T^4 + 3 T^5 &= \alpha_2(T), & T &= \frac{(x_2 - t)}{h} \in [0, 1],
  \\
  32 T - 40 T^2 + 10 T^4 - 2 T^5 &= \alpha_3(T), & T &= \frac{(x_1 - t)}{h} \in [0, 1].
\end{align*}
\]

Therefore, from (4.8) it follows that

\[
|r_1(f)| \leq h^2 \| D^6 f \| \left[ \int_0^1 \alpha_1(T) d T + \int_0^1 \alpha_2(T) d T + \int_0^1 \alpha_3(T) d T \right] = \frac{28}{3} h^2 \| D^6 f \|. \quad (4.9)
\]

For \( 2 \leq i \leq N - 1 \), again from (4.7), (3.9) and (3.10), we have

\[
\begin{align*}
  r_i(f) &= f_i^{(4)} + 26 f_i^{(4)} + 66 f_i^{(4)} + 26 f_i^{(4)} + f_i^{(4)} + 120 \left( -f_{i-2} + 4 f_{i-1} - 6 f_i + 4 f_{i+1} - f_{i+2} \right),
  \\
  \text{and again by Peano's kernel theorem, we have}
  \\
  r_i(f) &= \frac{1}{5!} \int_{x_{i-2}}^{x_{i+2}} (r_i)_x (x - t)_{+}^5 D^6 f(t) d t.
\end{align*}
\]

Thus, as in the case \( i = 1 \), we find that

\[
|r_i(f)| \leq 10 h^2 \| D^6 f \|, \quad 2 \leq i \leq N - 1. \quad (4.10)
\]

Finally, for \( i = N \) we have

\[
\begin{align*}
  r_N(f) &= f_N^{(4)} + 26 f_N^{(4)} + 65 f_N^{(4)} + 18 f_{N+1}^{(4)} - \frac{120}{h^2} f_{N+1}'' \\
  & \quad - \frac{120}{h^2} (f_{N-2} - 4 f_{N-1} + 5 f_N - 2 f_{N+1}),
\end{align*}
\]
and hence as earlier in the case \( i = 1 \), we obtain

\[
|r_N(f)| \leq \frac{28}{3} h^2 \|D^6 f\|. \tag{4.11}
\]

Now we multiply both sides of (4.6) by the diagonal matrix \( \Box = [d_{ij}] \), where \( d_{ii} = 1/a, \ a \in \mathbb{R}^+ \), \( 1 \leq i \leq N \) to obtain \( \Box e_3 = \Box r \), which gives that

\[
\|e_3^i\| \leq \|\Box^{-1}\| \|\Box r\|. \tag{4.12}
\]

Writing \( \Box B = I + A \), where \( A \) is an \( N \times N \) matrix with the property that \( \|A\| < 1 \), it follows from (4.12) and Lemma 2.10 that

\[
\|e_3^i\| \leq \frac{1}{1 - \|A\|} \frac{1}{a} \max_{1 \leq i \leq N} |r_i(f)|. \tag{4.13}
\]

To obtain the smallest bound in (4.13), we need to maximize \((1 - \|A\|)\) over \( a \in \mathbb{R}^+ \). For this, from (3.9) we have

\[
\|A\| = \max \left\{ \frac{|65 - a| + 27}{a}, \frac{|66 - a| + 54}{a} \right\} = \max \left\{ \max_{0 < a \leq 65} \left\{ \frac{92}{a} - 1, \frac{120}{a} - 1 \right\}, \max_{65 \leq a \leq 66} \left\{ \frac{38}{a} - 1, \frac{120}{a} - 1 \right\}, \max_{a \geq 66} \left\{ \frac{120}{a} - 1, 1 - \frac{12}{a} \right\} \right\}.
\tag{4.14}
\]

Thus, the condition \( \|A\| < 1 \) is equivalent to \( 120/a - 1 < 1 \) \((1 - 12/a < 1 \text{ for all } a \geq 66)\), which gives that \( a > 60 \). Hence, from (4.14) we find that

\[
\max_{a > 60} (1 - \|A\|) a = \max \left\{ \max_{60 < a \leq 66} (2a - 120), \max_{a \geq 66} (12) \right\} = 12. \tag{4.15}
\]

Using (4.9)-(4.11) and (4.15) in (4.13), we obtain

\[
\|e_3^i\| \leq \frac{5}{6} h^2 \|D^6 f\|. \tag{4.16}
\]

**Theorem 4.2.** Let \( f(x) \in PC^{n,\infty}[a, b], 4 \leq n \leq 6 \). Then,

\[
\|D^k (f - LS_3 f)\| \leq \gamma_{n,k} h^{n-k} \|D^n f\|, \quad 0 \leq k \leq n - 1,
\tag{4.16}
\]

where the constants \( \gamma_{n,k} \) are given in Table 4.1.

**Proof.** Using Lemmas 2.5, 2.7–2.9, and 4.1 in (4.4), the inequalities (4.16) are immediate.

**Remark 4.1.** The sharpness of the inequalities (4.16) remains undecided. However, in Tables 4.2 and 4.3 we compute the actual values of \( \|D^k (f - LS_3 f)\| \) for some simple functions and compare these with the corresponding right side bounds in (4.16).
Table 4.1.

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</tbody>
</table>

Table 4.2. \(f(x) = (1 - x^2)e^x, \ x \in [-1, 1], \ n = 6\).

\[
\begin{array}{c|c|c|c}
N & 9 & 19 & 39 \\
\hline
\|D(f - LS_6^f)\| & 0.40074171 & 0.64615309 & 0.10246557 \\
Bound & 0.88957734 & 0.13899340 & 0.2177718 \\
\|D^2(f - LS_6^f)\| & 0.66801241 & 0.21571012 & 0.68472448 \\
Bound & 0.14207553 & 0.44398603 & 0.13875464 \\
\|D^3(f - LS_6^f)\| & 0.10345307 & 0.66902268 & 0.42500563 \\
Bound & 0.21406469 & 0.13379042 & 0.83619021 \\
\|D^4(f - LS_6^f)\| & 0.19318514 & 0.25007712 & 0.31793523 \\
Bound & 0.41861540 & 0.52326925 & 0.65408657 \\
\|D^5(f - LS_6^f)\| & 0.42994935 & 0.11824915 & 0.31022977 \\
Bound & 0.43764337 & 0.65408657 & 0.27352711 \\
\end{array}
\]

Table 4.3. \(f(x) = x(1 - x)e^x, \ x \in [0, 1], \ n = 6\).

\[
\begin{array}{c|c|c|c}
N & 9 & 19 & 39 \\
\hline
\|f - LS_6^f\| & 0.55345404 & 0.87801200 & 0.13819513 \\
Bound & 0.11913720 & 0.18615187 & 0.29086229 \\
\|D(f - LS_6^f)\| & 0.18475927 & 0.58667736 & 0.18476054 \\
Bound & 0.38055946 & 0.11892483 & 0.37164009 \\
\|D^2(f - LS_6^f)\| & 0.57301304 & 0.36416919 & 0.22945975 \\
Bound & 0.11467751 & 0.71673447 & 0.44795904 \\
\|D^3(f - LS_6^f)\| & 0.21418746 & 0.27242089 & 0.34332933 \\
Bound & 0.44851650 & 0.56064563 & 0.70080703 \\
\|D^4(f - LS_6^f)\| & 0.10115788 & 0.26564513 & 0.68073900 \\
Bound & 0.93780723 & 0.23445181 & 0.58612952 \\
\|D^5(f - LS_6^f)\| & 0.54815309 & 0.28694132 & 0.14681102 \\
Bound & 0.21202598 & 0.10601299 & 0.53006496 \\
\end{array}
\]
5. APPROXIMATED QUINTIC LIDSTONE-SPLINES

For a given function \( f(x) \in PC^{6,\infty}[a,b] \) and a fixed partition \( \Delta \), we shall construct approximates for the quintic Lidstone-spline interpolate \( LS_5^\Delta f(x) \) when

1. the values of \( f_i^{(4)}, \ i = 0, N + 1 \) are unknown;
2. the values of \( f_i^{(4)}, \ 0 \leq i \leq N + 1 \) are unknown;
3. the values of \( f_i^{(4)}, \ 0 \leq i \leq N + 1, f_i^{(4)}, \ i = 0, N + 1 \) are unknown;
4. the values of \( f_i^{(4)}, \ 1 \leq i \leq N, f_i^{(4)}, \ i = 0, N + 1 \) are unknown;
5. the values of \( f_i, f_i^{(4)}, 0 \leq i \leq N + 1, f_i^{(4)}, \ i = 0, N + 1 \) are unknown;
6. the values of \( f_i, f_i^{(4)}, 1 \leq i \leq N, f_i^{(4)}, \ i = 0, N + 1 \) are unknown; and
7. the values of \( f_i, f_i^{(4)}, 1 \leq i \leq N \) are unknown.

**CASE 1.** We approximate \( f_0^{(4)} \) and \( f_{N+1}^{(4)} \) by the formulae

\[ f_0^{(4)} \approx \tilde{f}_0^{(4)} = \frac{1}{h^4} \left( 3f_0 - 14f_1 + 26f_2 - 24f_3 + 11f_4 - 2f_5 \right), \tag{5.1} \]
\[ f_{N+1}^{(4)} \approx \tilde{f}_{N+1}^{(4)} = \frac{1}{h^4} \left( -2f_{N-4} + 11f_{N-3} - 24f_{N-2} + 26f_{N-1} - 14f_N + 3f_{N+1} \right), \tag{5.2} \]

which can easily be shown to have \( O(h^6) \) truncation error.

**DEFINITION 5.1.** We say \( S_1f(x) \) is an approximate for \( LS_5^\Delta f(x) \) if \( S_1f(x) \in S_0(\Delta) \) with \( S_1f(x_i) = f_i, D^2S_1f(x_i) = f_i'' \), \( 0 \leq i \leq N + 1 \) and \( D^4S_1f(x_i) = f_i^{(4)}, \ i = 0, N + 1 \).

We use (5.1) and (5.2) to replace \( f_i^{(4)}, \ i = 0, N + 1 \) in the system (3.5)-(3.7) and note that the resulting unknowns \( c_{i1}^{(4)}, 1 \leq i \leq N, \) say, can be obtained uniquely in terms of \( f_i, 0 \leq i \leq N + 1, f_i'' \) and \( f_{N+1}'' \). Further, by Remark 3.1, \( S_1f(x) \) can be explicitly expressed as

\[ S_1f(x) = \sum_{i=0}^{N+1} \left[ r_{3,0,i}(x)f_i + r_{3,1,i}(x)f_i'' \right] + \sum_{i=1}^{N} r_{3,2,i}(x)c_{i1}^{(4)}. \tag{5.3} \]

To obtain a priori bound for \( \|D^k(f - S_1f)\|, 0 \leq k \leq 5 \), we use the inequality

\[ \|D^k(f - S_1f)\| \leq \|D^k(f - LS_5^\Delta f)\| + \|D^k(LS_5^\Delta f - S_1f)\|, \tag{5.4} \]

in which the first term of the right side can be estimated by Theorem 4.2, whereas for the second term we proceed as follows: from Remark 3.1 and (5.3), we have

\[ (S_1f - LS_5^\Delta f) (x) = r_{3,0,2}(x)f_0^{(4)} + r_{3,N+1,2}(x)f_{N+1}^{(4)} + \sum_{i=1}^{N} r_{3,2,i}(x)c_{i1}^{(4)}, \tag{5.5} \]

where \( f_0^{(4)} = \tilde{f}_0^{(4)} - f_0^{(4)}, f_{N+1}^{(4)} = \tilde{f}_{N+1}^{(4)} - f_{N+1}^{(4)}, c_{i1}^{(4)} = \frac{1}{h^4} \left( \phi_{i1}^{(4)} - c_{i1}^{(4)} \right) \). Thus, for \( 0 \leq k \leq 5 \) it follows that

\[ \|D^k(S_1f - LS_5^\Delta f)\| \leq \left( \left| \phi_{01}^{(4)} \right| + \left| \phi_{N+1}^{(4)} \right| \right) \max_{0 \leq i \leq N} \max_{x_i \leq x \leq x_{i+1}} \|D^k r_{3,2,i}(x)\| \]
\[ + \left| \phi_{1}^{(4)} \right| \max_{0 \leq i \leq N} \max_{x_i \leq x \leq x_{i+1}} \left[ \|D^k r_{3,2,i}(x)\| + \|D^k r_{3,i+1,2}(x)\| \right], \tag{5.6} \]

where \( \phi^4 = [\phi_{i1}^{(4)}] \). In (5.6) we have used the fact that \( D^k r_{3,2,i}(x) \) is symmetrical about the line \( x = x_i \), and that \( D^k r_{3,i,2}(x) \) is nonzero only in the interval \( (x_{i-1}, x_i) \cup (x_i, x_{i+1}) \).
LEMMA 5.1. If \( f(x) \in PC^{6,\infty}[a,b] \), then

\[
\left| \phi_i^{(4)} \right| \leq \frac{17}{6} h^2 \|D^6 f\|; \quad i = 0, N+1.
\]  

(5.7)

PROOF. We shall prove (5.7) only for \( i = 0 \). Denoting \( \phi_0^{(4)} \) by \( r(f) \), we have

\[
r(f) = \frac{1}{h^4} (3f_0 - 14f_1 + 26f_2 - 24f_3 + 11f_4 - 2f_5),
\]

which in view of Peano’s kernel theorem can be written as

\[
r(f) = \frac{1}{5!} \int_{x_0}^{x_5} (r)_{x} (x-t)^5 D^6 f(t) dt,
\]

(5.8)

where

\[
(r)_x (x-t)^5 = \frac{1}{h^4} \left[3(x_0 - t)^5 - 14(x_1 - t)^5 + 26(x_2 - t)^5 - 24(x_3 - t)^5 + 11(x_4 - t)^5 - 2(x_5 - t)^5 - 120h^4 (x_0 - t)^5 \right].
\]

It is easy to see that \( (r)_x (x-t)^5 \) is nonpositive for all \( t \in [x_0, x_5] \). Thus, from (5.8) it follows that

\[
|r(f)| \leq \frac{1}{5!} \|D^6 f\| \int_{x_0}^{x_5} (r)_x (x-t)^5 dt = \frac{17}{6} h^2 \|D^6 f\|.
\]

LEMMA 5.2. If \( f(x) \in PC^{6,\infty}[a,b] \), then

\[
\| \phi^4 \| \leq \frac{17}{4} h^2 \|D^6 f\|.
\]  

(5.9)

PROOF. Letting \( c^4 = [c_i^{(4)}] \) and \( c_i^4 = [c_i^{(4)}] \), we have (3.8) and

\[
B c_i^4 = \bar{k},
\]

(5.10)

where \( \bar{k} \) is obtained from \( k \) by replacing \( f_i^{(4)} \) with \( f^{(4)}_i \), \( i = 0, N+1 \). Thus, from (3.8) and (5.10), it follows that

\[
B (c^4 - c_i^4) = k - \bar{k}.
\]

(5.11)

Multiplying both sides of (5.11) by the diagonal matrix \( \square = [d_{ij}] \), where \( d_{ii} = 1/a, a \in \mathbb{R}^+ \), \( 1 \leq i \leq N \), we get

\[
\square B (c^4 - c_i^4) = \square \left( k - \bar{k} \right),
\]

which implies that

\[
\| \phi^4 \| = \| c^4 - c_i^4 \| \leq \| \square (B^{-1}) \| \| \square \left( k - \bar{k} \right) \|.
\]

(5.12)

Now, as in Lemma 4.1, writing \( \square B = I + A \), where \( A \) is an \( N \times N \) matrix with the property that \( \| A \| < 1 \), it follows from (5.12) and Lemmas 2.10 and 5.1 that

\[
\| \phi^4 \| \leq \frac{1}{1 - \| A \|} \frac{1}{a} \max \left\{ 18 \left| \phi_i^{(4)} \right|, \left| \phi_0^{(4)} \right|, \left| \phi_i^{(4)} \right|, 18 \left| \phi_{i+1}^{(4)} \right| \right\}
\]

(5.13)

\[
\leq \frac{1}{1 - \| A \|} \frac{1}{a} 18 \left[ \frac{17}{6} h^2 \|D^6 f\| \right].
\]

Since in Lemma 4.1 we have noted that the maximum of \( (1 - \| A \|)a, a \in \mathbb{R}^+ \) subject to the condition \( \| A \| < 1 \) is 12, from (5.13) the inequality (5.9) is immediate.
Theorem 5.3. If \( f(x) \in PC^6,\infty[a,b] \), then
\[
\|D^k(f - S_1f)\| \leq \beta_{1,k} h^{6-k} \|D^6f\|, \quad 0 \leq k \leq 5,
\]
where \( \beta_{1,0} = 0.10447206 + 0, \beta_{1,1} = 0.34189815 + 0, \beta_{1,2} = 0.10119543 + 1, \beta_{1,3} = 0.44722222 + 1, \beta_{1,4} = 0.10875000 + 2, \) and \( \beta_{1,5} = 0.16333333 + 2. \)

Proof. We use Lemmas 2.5, 2.6, 5.1 and 5.2 in (5.6) to obtain an upper estimate for \( \|D^k(LS_3^\Delta f - S_1f)\| \). This estimate together with Theorem 4.2 in (5.4) then gives (5.14).

Case 2. We approximate \( f_i'' \), \( 0 \leq i \leq N + 1 \) by the following relations:
\[
f_0'' = f_0'' = \frac{1}{h^2} \left( \frac{15}{4} f_0 - \frac{77}{6} f_1 + \frac{107}{6} f_2 - 13 f_3 + \frac{61}{12} f_4 - \frac{5}{6} f_5 \right), \quad (5.15)
\]
\[
f_1'' = f_1'' = \frac{1}{h^2} \left( \frac{5}{6} f_0 - \frac{5}{4} f_1 - \frac{1}{3} f_2 + \frac{7}{6} f_3 - \frac{1}{2} f_4 + \frac{1}{12} f_5 \right), \quad (5.16)
\]
\[
f_i'' = f_i'' = \frac{1}{h^2} \left( -\frac{1}{12} f_{i-2} + \frac{4}{3} f_{i-1} - \frac{5}{2} f_i + \frac{4}{3} f_{i+1} - \frac{1}{12} f_{i+2} \right), \quad 2 \leq i \leq N - 1, \quad (5.17)
\]
\[
f_N'' = f_N'' = \frac{1}{h^2} \left( \frac{1}{12} f_{N-4} - \frac{1}{2} f_{N-3} + \frac{7}{6} f_{N-2} - \frac{1}{3} f_{N-1} - \frac{5}{4} f_N + \frac{5}{6} f_{N+1} \right), \quad (5.18)
\]
\[
f_{N+1}'' = f_{N+1}'' = \frac{1}{h^2} \left( -\frac{5}{6} f_{N-4} + \frac{61}{12} f_{N-3} - 13 f_{N-2} + \frac{107}{6} f_{N-1} - \frac{77}{6} f_N + \frac{15}{4} f_{N+1} \right). \quad (5.19)
\]

It is easy to see that each of these relations has \( O(h^6) \) truncation error.

Definition 5.2. We say \( S_2f(x) \) is an approximate for \( LS_3^\Delta f(x) \) if \( S_2f(x) \in S_3(\Delta) \) with \( S_2f(x_i) = f_i, D^2S_2f(x_i) = f_i'' \), \( 0 \leq i \leq N + 1 \), and \( D^4S_2f(x_i) = f_i^{(4)} \), \( i = 0, N + 1 \).

As in Case 1, we use (5.15) and (5.19) to replace \( f_i'' \), \( 0 \leq i \leq N + 1 \) in the system (3.5)—(3.7) and note that the resulting unknowns \( c_n^{(4)} \), \( 1 \leq i \leq N \), say, can be obtained uniquely in terms of \( f_i \), \( 0 \leq i \leq N + 1 \), \( f_0^{(4)} \) and \( f_{N+1}^{(4)} \). Further, by Remark 3.1, \( S_2f(x) \) can be explicitly expressed as
\[
S_2f(x) = \sum_{i=0}^{N+1} \left[ r_{3,i,0}(x) f_i + r_{3,i,1}(x) f_i'' \right] + r_{3,0,2}(x) f_0^{(4)} + r_{3,N+1,2}(x) f_{N+1}^{(4)} + \sum_{i=1}^{N} r_{3,i,2}(x) c_i^{(4)}. \quad (5.20)
\]

Thus, for this case, the relations corresponding to (5.4)—(5.6) are
\[
\|D^k(f - S_2f)\| \leq \|D^k(f - LS_3^\Delta f)\| + \|D^k(LS_3^\Delta f - S_2f)\|, \quad (5.21)
\]
\[
(S_2f - LS_3^\Delta f)(x) = \sum_{i=0}^{N+1} r_{3,i,1}(x) \theta_i'' + \sum_{i=1}^{N} r_{3,i,2}(x) \theta_i^{(4)}, \quad (5.22)
\]
\[
\|D^k(S_2f - LS_3^\Delta f)\| \leq \|\theta_i''\| \max_{0 \leq i \leq N} \max_{k \leq x \leq x_{i+1}} \left[ |D^k r_{3,i,1}(x)| + |D^k r_{3,i+1,1}(x)| \right] + \|\theta_i^{(4)}\| \max_{0 \leq i \leq N} \max_{k \leq x \leq x_{i+1}} \left[ |D^k r_{3,i,2}(x)| + |D^k r_{3,i+1,2}(x)| \right], \quad (5.23)
\]

where \( \theta_i'' = f_i'' - f_i'', \theta_i^{(4)} = c_i^{(4)} - c_i^{(4)} \), and \( \theta_0'' \) and \( \theta_0^{(4)} \) are the vectors \( [\theta_i''] \) and \( [\theta_i^{(4)}] \), respectively. As in Case 1, here in (5.23) also we have used the fact that \( D^k r_{3,i,1}(x) \) as well as \( D^k r_{3,i,2}(x) \) is nonzero only in the interval \( (x_{i-1}, x_i) \cup (x_i, x_{i+1}) \).

Further, the results corresponding to Lemmas 5.1 and 5.2 are as follows.

Lemma 5.4. If \( f(x) \in PC^6,\infty[a,b] \), then
\[
\|\theta_0''\| \leq \frac{137}{180} h^4 \|D^6f\|. \quad (5.24)
\]
PROOF. Using a similar technique as in Lemma 5.1, we find that

\[ |\theta_i''| \leq \begin{cases} 
\frac{137}{180} h^4 \|D^6 f\| ; & i = 0, N + 1, \\
\frac{13}{180} h^4 \|D^6 f\| ; & i = 1, N, \\
\frac{1}{90} h^4 \|D^6 f\|, & 2 \leq i \leq N - 1.
\end{cases} \]

Hence, (5.24) is immediate.

LEMMA 5.5. If \( f(x) \in PC^{6,\infty}[a, b] \), then

\[ \|\theta^4\| \leq \frac{137}{18} h^2 \|D^6 f\|. \]  

(5.25)

PROOF. The proof is similar to that of Lemma 5.2.

THEOREM 5.6. If \( f(x) \in PC^{6,\infty}[a, b] \), then

\[ \|D^k (f - S_2 f)\| \leq \beta_{2,k} h^{6-k} \|D^6 f\|, \quad 0 \leq k \leq 5, \]  

(5.26)

where \( \beta_{2,0} = 0.20641638 + 0, \beta_{2,1} = 0.73657407 + 0, \beta_{2,2} = 0.18296875 + 1, \beta_{2,3} = 0.57861111 + 1, \beta_{2,4} = 0.85694444 + 1, \) and \( \beta_{2,5} = 0.17388889 + 2. \)

PROOF. The proof is similar to that of Theorem 5.3.

CASE 3.

DEFINITION 5.3. We say \( S_3 f(x) \) is an approximate for \( LS_3 f(x) \) if \( S_3 f(x) \in S_3(\Delta) \) with \( S_3 f(x_i) = f_i, D^2 S_3 f(x_i) = \tilde{f}''_i, 0 \leq i \leq N + 1 \) and \( D^4 S_3 f(x_i) = \tilde{f}^{(4)}_i, i = 0, N + 1 \).

We use (5.15), (5.19), (5.1) and (5.2) to replace \( f''_i, f^{(4)}_i, i = 0, N + 1 \) in the system (3.5)-(3.7) and note that the resulting unknowns \( c^{(4)}_3, 1 \leq i \leq N, \) say, can be obtained uniquely in terms of \( f_i, 0 \leq i \leq N + 1 \). Further, by Remark 3.1, \( S_3 f(x) \) can be explicitly expressed as

\[ S_3 f(x) = \sum_{i=0}^{N+1} \left[ r_{3,i,0}(x)f_i + r_{3,i,1}(x)f''_i \right] + r_{3,0,2}(x)f''_0 + r_{3,N+1,2}(x)f''_{N+1} + \sum_{i=1}^{N} r_{3,i,2}(x)c^{(4)}_3. \]  

(5.27)

THEOREM 5.7. If \( f(x) \in PC^{6,\infty}[a, b] \), then

\[ \|D^k (S_3 f - LS_3^a f)\| \leq \beta_{3,k} h^{6-k} \|D^6 f\|, \quad 0 \leq k \leq 5, \]  

(5.28)

where \( \beta_{3,0} = 0.29871396 + 0, \beta_{3,1} = 0.10395833 + 1, \beta_{3,2} = 0.27244543 + 1, \beta_{3,3} = 0.98000000 + 1, \beta_{3,4} = 0.18486111 + 2, \) and \( \beta_{3,5} = 0.31555556 + 2. \)

PROOF. The proof is similar to that of Theorem 5.3. Here, the inequality corresponding to (5.6) is

\[ \|D^k (S_3 f - LS_3^a f)\| \leq \left\| \theta^2 \right\| \max_{0 \leq i \leq N} \max_{x_i \leq x \leq x_{i+1}} \left[ \|D^k \tilde{r}_{3,i,11}(x)\| + \|D^k \tilde{r}_{3,i+1,1}(x)\| \right] 
+ \left[ \|\phi^{(4)}_i\| + \|\phi^{(4)}_{N+1}\| \right] \max_{0 \leq i \leq N} \max_{x_i \leq x \leq x_{i+1}} \left[ \|D^k \tilde{r}_{3,i,2}(x)\| \right] 
+ \left[ \psi^4 \right] \max_{0 \leq i \leq N} \max_{x_i \leq x \leq x_{i+1}} \left[ \|D^k \tilde{r}_{3,i,2}(x)\| + \|D^k \tilde{r}_{3,i+1,2}(x)\| \right], \]  

(5.29)

where \( \psi^4 = [\psi^{(4)}_i, \psi^{(4)}_{i+1}] = c^{(4)}_3 - c^{(4)}_i \). By a technique similar to that in Lemma 5.2, we find that

\[ \|\psi^4\| \leq \frac{427}{36} h^2 \|D^6 f\|. \]

The rest of the proof is clear.
CASE 4.

DEFINITION 5.4. We say \( S_4f(x) \) is an approximate for \( LS_3^0f(x) \) if \( S_4f(x_i) = f_i, \ 0 \leq i \leq N + 1, \ D^2S_4f(x_i) = f_i'' \), \( 1 \leq i \leq N, \ D^3S_4f(x_i) = f_i''' \), \( i = 0, N + 1 \) and \( D^4S_4f(x_i) = f_i^{(4)} \), \( i = 0, N + 1 \).

We use (5.1) and (5.2) to replace \( f_i^{(4)} \), \( i = 0, N + 1 \) in the system (3.5)-(3.7) and note that the resulting unknowns \( c_i^{(4)} \), \( 1 \leq i \leq N \) (Case 1) can be obtained uniquely in terms of \( f_i \), \( 0 \leq i \leq N + 1, f_i'' \) and \( f_i^{(4)}_{N+1} \). Further, by Remark 3.1, \( S_4f(x) \) can be explicitly expressed as

\[
S_4f(x) = \sum_{i=0}^{N+1} r_{3,i,0}(x)f_i + r_{3,0,1}(x)f_i'' + r_{3,N+1,1}(x)f_{N+1}'' + \sum_{i=1}^{N} r_{3,i,1}(x)f_i''
\]

\[+ r_{3,0,2}(x)f_0'' + r_{3,N+1,2}(x)f_{N+1}'' + \sum_{i=1}^{N} r_{3,i,2}(x)c_i^{(4)}. \tag{5.30}\]

THEOREM 5.8. If \( f(x) \in PC_5^\infty[a,b] \), then

\[
\|D^k(f - S_4f)\| \leq \beta_{4,k} h^{6-k} \|D^6f\|, \quad 0 \leq k \leq 5, \tag{5.31}\]

where \( \beta_{4,0} = 0.11349984 + 0, \beta_{4,1} = 0.37800926 + 0, \beta_{4,2} = 0.10841766 + 1, \beta_{4,3} = 0.46166667 + 1, \beta_{4,4} = 0.10875000 + 2, \) and \( \beta_{4,5} = 0.16333333 + 2. \)

PROOF. We use the inequality

\[
\|D^k(f - S_4f)\| \leq \|D^k(f - S_1f)\| + \|D^k(S_1f - S_4f)\|, \quad 0 \leq k \leq 5,
\]

in which the first term of the right side can be estimated by Theorem 5.3, whereas for the second term we proceed as follows: from (5.3) and (5.30), we have

\[
(S_4f - S_1f)(x) = \sum_{i=1}^{N} r_{3,i,1}(x)\theta_i''.
\]

Hence, for \( 0 \leq k \leq 5 \) it follows that

\[
\|D^k(S_1f - S_4f)\| \leq \|\theta_2^2\| \max_{0 \leq i \leq N} \max_{x_i \leq x \leq x_{i+1}} \left[ |D^k r_{3,i,1}(x)| + |D^k r_{3,i+1,1}(x)| \right],
\]

where \( \theta_2^2 = [\theta_i^{(2)}]_{i=1}^{N} \). From the proof of Lemma 5.4, it is clear that

\[
\|\theta_2^2\| \leq \frac{13}{180} h^4 \|D^6f\|.
\]

The rest of the proof is obvious.

CASE 5.

DEFINITION 5.5. We say \( S_5f(x) \) is an approximate for \( LS_3^0f(x) \) if \( S_5f(x) \in S_3(\Delta) \) with \( S_5f(x_i) = g_i, \ 0 \leq i \leq N + 1, \) where the given \( g_i, 0 \leq i \leq N + 1 \) are such that

\[
\max_{0 \leq i \leq N+1} |f_i - g_i| = \xi,
\]

and \( D^2S_5f(x_i) = \tilde{f}_i'', \ 0 \leq i \leq N + 1, \) \( D^4S_5f(x_i) = \tilde{f}_i^{(4)} \), \( i = 0, N + 1, \) where \( \tilde{f}_i'' \) and \( \tilde{f}_i^{(4)} \) are obtained from \( f_i'' \) and \( f_i^{(4)} \) by replacing \( f_i \) with \( g_i, \ 0 \leq i \leq N + 1. \)
In system (3.5)-(3.7), we replace \( f_i \) by \( g_i \), \( 0 \leq i \leq N+1 \) and \( f_i^{(2k)} \) by \( f_i^{(2k)} \), \( k = 1, 2, i = 0, N+1 \), and note that the resulting unknowns \( c_{6i}^{(4)} \), \( 1 \leq i \leq N \), say, can be obtained uniquely in terms of \( g_i \), \( 0 \leq i \leq N+1 \). Further, by Remark 3.1, \( S_5f(x) \) can be explicitly expressed as

\[
S_5f(x) = \sum_{i=0}^{N+1} \left[ r_{3,i,0}(x) g_i + r_{3,i,1}(x) f_i^{(2)} + r_{3,0,2}(x) f_0^{(4)} + r_{3,N+1,2}(x) f_{N+1}^{(4)} + \sum_{i=1}^{N} r_{3,i,2}(x) c_{6i}^{(4)} \right].
\]

(5.32)

**Theorem 5.9.** If \( f(x) \in PC^{6,\infty}[a, b] \), then

\[
\| D^k (f - S_5f) \| \leq \beta_{3,k} h^{6-k} \| D^6f \| + \tau_k h^{-k} \xi, \quad 0 \leq k \leq 5,
\]

(5.33)

where \( \beta_{3,k} \) are defined in Theorem 5.7 and \( \tau_0 = 0.12529661 + 2, \tau_1 = 0.42444444 + 2, \tau_2 = 0.10026400 + 3, \tau_3 = 0.30666667 + 3, \tau_4 = 0.45333333 + 3, \) and \( \tau_5 = 0.74666667 + 3. \)

**Proof.** The relations corresponding to (5.4)-(5.6) are

\[
\| D^k (f - S_5f) \| \leq \| D^k (f - S_3f) \| + \| D^k (S_3f - S_5f) \|, \quad (5.34)
\]

\[
(S_3f - S_5f)(x) = \sum_{i=0}^{N+1} \left[ r_{3,i,0}(x) \xi_i + r_{3,i,1}(x) \gamma_i'' + r_{3,0,2}(x) \gamma_0^{(4)} + r_{3,N+1,2}(x) \gamma_0^{(4)} \right]
\]

\[
+ \sum_{i=1}^{N} r_{3,i,2}(x) \alpha_i^{(4)},
\]

and

\[
\| D^k (S_3f - S_5f) \| \leq \xi \max_{0 \leq i \leq N} \max_{x_i \leq x \leq x_{i+1}} \left[ |D^k r_{3,i,0}(x)| + |D^k r_{3,i+1,0}(x)| \right]
\]

\[
+ \frac{160}{3} h^{-2} \xi \max_{0 \leq i \leq N} \max_{x_i \leq x \leq x_{i+1}} \left[ |D^k r_{3,i,1}(x)| + |D^k r_{3,i+1,1}(x)| \right]
\]

\[
+ 2(80) h^{-4} \xi \max_{0 \leq i \leq N} \max_{x_i \leq x \leq x_{i+1}} \left[ |D^k r_{3,i,2}(x)| + |D^k r_{3,i+1,2}(x)| \right],
\]

(5.35)

where \( \xi_i = f_i - g_i, \gamma_i'' = \tilde{f}_i'' - \tilde{f}_i''', \gamma_i^{(4)} = \tilde{f}_i^{(4)} - \tilde{f}_i^{(4)}', \alpha_i^{(4)} = c_{6i}^{(4)} - c_{6i}^{(4)} \) and \( \alpha^4 = [\alpha_i^{(4)}] \).

Following a similar technique as in Lemma 5.2, we find that

\[
\| \alpha^4 \| \leq \frac{880}{3} h^{-4} \xi.
\]

Now using Lemmas 2.5 and 2.6 in (5.35) and then Theorem 5.7 in (5.34) yields (5.33) immediately.

**Case 6.**

**Definition 5.6.** We say \( S_6f(x) \) is an approximate for \( LS^6_S f(x) \) if \( S_6f(x) \in S_3(\Delta) \) with \( D^{2k} S_6f(x_i) = f_i^{(2k)} \), \( k = 0, 1, i = 0, N+1, S_6f(x_i) = g_i, 1 \leq i \leq N \), where the given \( g_i, 1 \leq i \leq N \) are such that

\[
\max_{1 \leq i \leq N} |f_i - g_i| = \xi^*,
\]

(5.36)

and \( D^2 S_6f(x_i) = \tilde{f}_i''', 1 \leq i \leq N, D^4 S_6f(x_i) = f_i^{(4)}, i = 0, N+1 \) where \( \tilde{f}_i''' \) and \( f_i^{(4)} \) are obtained from \( \tilde{f}_i' \) and \( f_i^{(4)} \) by replacing \( f_i \) with \( g_i, 1 \leq i \leq N \).

In system (3.5)-(3.7) we replace \( f_i \) by \( g_i, 1 \leq i \leq N \) and \( f_i^{(4)} \) by \( \tilde{f}_i^{(4)}, i = 0, N+1 \) and note that the resulting unknowns \( c_{6i}^{(4)}, 1 \leq i \leq N \), say, can be obtained uniquely in terms of \( f_i^{(2k)}, \)
\( k = 0, 1, \ i = 0, N + 1 \) and \( g_i, 1 \leq i \leq N \). Further, by Remark 3.1, \( S_0 f(x) \) can be explicitly expressed as

\[
S_0 f(x) = r_{3,0,0}(x)f_0 + r_{3,N+1,0}(x)f_{N+1} + \sum_{i=1}^{N} r_{3,1,i}(x)g_i + r_{3,0,1}(x)f''_0 + r_{3,N+1,1}(x)f''_{N+1}
\]

\[ + \sum_{i=1}^{N} r_{3,2,i}(x)f''_0 + r_{3,0,2}(x)f''_0 + r_{3,N+1,2}(x)f''_{N+1} + \sum_{i=1}^{N} r_{3,1,i}(x)c''_0 \]  

(5.37)

**Theorem 5.10.** If \( f(x) \in PC^{6,*}[a, b] \), then

\[
\| D^k (f - S_0 f) \| \leq \beta_{4,k} h^{6-k} \| D^6 f \| + \tau_{4}^k h^{-k} \| \epsilon \| = Mk, \quad 0 \leq k \leq 5,
\]

(5.38)

where \( \beta_{4,k} \) are defined in Theorem 5.8 and \( \tau_{4}^k = 0.54770726 + 1, \ \tau_{4}^k = 0.15068056 + 2, \ \tau_{4}^k = 0.42149938 + 2, \ \tau_{4}^k = 0.16975000 + 3, \ \tau_{4}^k = 0.36950000 + 3, \) and \( \tau_{4}^k = 0.58500000 + 3. \)

**Proof.** Here we use the triangle inequality

\[
\| D^k (f - S_0 f) \| \leq \| D^k (f - S_4 f) \| + \| D^k (S_4 f - S_0 f) \|, \quad 0 \leq k \leq 5,
\]

and apply Theorem 5.8 to the first term, whereas for the second term we note that

\[
(S_4 f - S_0 f)(x) = \sum_{i=1}^{N} [r_{3,1,0}(x) \xi_i + r_{3,1,1}(x) \eta_i'] + r_{3,0,2}(x) \eta_0'' + r_{3,N+1,2}(x) \eta_{N+1}''.
\]

(5.36)

where \( \eta''_i = f''_i - f''_0, \ \eta''_0 = f''_0 - f''_0 \) and \( \beta_i'' = c_i'' - c_0'' \). Denoting \( \beta^4 = [\beta_i''] \), for \( 0 \leq k \leq 5 \) it follows that

\[
\| D^k (S_4 f - S_0 f) \| \leq \xi^* \max_{0 \leq i \leq N} \max_{x_i \leq x \leq x_{i+1}} \left[ |D^k r_{3,1,0}(x)| + |D^k r_{3,1,1}(x)| \right]
\]

+ \( \frac{16}{3} h^{-2} \xi^* \max_{0 \leq i \leq N} \max_{x_i \leq x \leq x_{i+1}} \left[ |D^k r_{3,1,1}(x)| + |D^k r_{3,1,1}(x)| \right]
\]

\[
+ 2(77)h^{-4} \xi^* \max_{0 \leq i \leq N} \max_{x_i \leq x \leq x_{i+1}} |D^k r_{3,1,2}(x)|
\]

Following a similar technique as in Lemma 5.2, we find that

\[
\| \beta^4 \| \leq \frac{431}{2} h^{-4} \xi^*.
\]

The rest of the proof is clear.

**Case 7.**

**Definition 5.7.** We say \( S_{T} f(x) \) is an approximate for \( LS^2 f(x) \) if \( S_{T} f(x) \in S_3(\Delta) \) with \( D^{2k} S_{T} f(x_i) = f''^{(2k)}_i, \ 0 \leq k \leq 2, \ i = 0, N + 1, \ S_{T} f(x_i) = g_i, \ 1 \leq i \leq N, \) where the given \( g_i, \ 1 \leq i \leq N \) are such that (5.36) holds, and \( D^{2} S_{T} f(x_i) = f''^{(2k)}_i, \ 1 \leq i \leq N \) where \( f''^{(2k)}_i \) are obtained from \( f''_i \) by replacing \( f_i \) with \( g_i, \ 1 \leq i \leq N. \)

In system (3.5)--(3.7), we replace \( f_i \) by \( g_i, \ 1 \leq i \leq N \) and note that the resulting unknowns \( c''_i, \ 1 \leq i \leq N, \) say, can be obtained uniquely in terms of \( f''^{(2k)}_i, \ 0 \leq k \leq 2, \ i = 0, N + 1 \) and \( g_i, \ 1 \leq i \leq N. \)
\[ S_T f(x) = r_{3,0,0}(x)f_0 + r_{3,0,1}(x)f_0' + r_{3,0,2}(x)f_0'' + \sum_{i=1}^{N} r_{3,i,0}(x)g_i + r_{3,0,1}(x)f_N^{''} + r_{3,0,2}(x)f_N^{'''} + \sum_{i=1}^{N} r_{3,i,1}(x)f_i ' + r_{3,0,2}(x)f_0^{(4)} + r_{3,0,1}(x)f_N^{(4)} + \sum_{i=1}^{N} r_{3,i,2}(x)c_{4i}. \]

**Theorem 5.11.** If \( f(x) \in PC^6, \infty[a,b] \), then

\[
\|D^k (f - S_T f)\| \leq M_k + \beta_{k,h} h^{6-k} \|D^6 f\| + \tau_k h^{-k} \xi^*, \quad 0 \leq k \leq 5,
\]

where \( M_k \) are defined in Theorem 5.10, \( \beta_{7,0} = 0.92297585 - 1, \beta_{7,1} = 0.30300926 + 0, \beta_{7,2} = 0.89476684 + 0, \beta_{7,3} = 0.40138889 + 1, \beta_{7,4} = 0.99166667 + 1, \beta_{7,5} = 0.14166667 + 2, \tau_0 = 0.25083226 + 1, \tau_1 = 0.82347222 + 1, \tau_2 = 0.24316605 + 2, \tau_3 = 0.10908333 + 3, \tau_4 = 0.26950000 + 3, \)

and \( \tau_5 = 0.38500000 + 3. \)

**Proof.** Here we use the triangle inequality

\[
\|D^k (f - S_T f)\| \leq \|D^k (f - S_6 f)\| + \|D^k (S_6 f - S_T f)\|, \quad 0 \leq k \leq 5
\]

and apply Theorem 5.10 to the first term, whereas for the second term we note that

\[
(S_6 f - S_T f)(x) = r_{3,0,2}(x) \left( \frac{f_0^{(4)}}{f_0} - f_0^{(4)} \right) + r_{3,0,1}(x) \left( \frac{f_{N+1}^{(4)}}{f_{N+1}} - f_{N+1}^{(4)} \right) + \sum_{i=1}^{N} r_{3,i,2}(x) \zeta_i^{(4)}
\]

\[
= r_{3,0,2}(x) \left( \frac{f_0^{(4)}}{f_0} - f_0^{(4)} + \frac{f_0^{(4)}}{f_0} - f_0^{(4)} \right)
\]

\[
+ r_{3,0,1}(x) \left( \frac{f_{N+1}^{(4)}}{f_{N+1}} - f_{N+1}^{(4)} + \frac{f_{N+1}^{(4)}}{f_{N+1}} - f_{N+1}^{(4)} \right) + \sum_{i=1}^{N} r_{3,i,2}(x) \zeta_i^{(4)}
\]

\[
= r_{3,0,2}(x) \left( -\eta_0^{(4)} + \phi_0^{(4)} \right) + r_{3,0,1}(x) \left( -\eta_{N+1}^{(4)} + \phi_{N+1}^{(4)} \right) + \sum_{i=1}^{N} r_{3,i,2}(x) \zeta_i^{(4)},
\]

where \( \zeta_i^{(4)} = c_{4i}^{(4)} - c_{3i}^{(4)} \). Denoting \( \zeta^4 = [\zeta_i^{(4)}] \), for \( 0 \leq k \leq 5 \) it follows that

\[
\|D^k (S_6 f - S_T f)\| \leq 2 \left[ \|\eta_0^{(4)}\| + \|\phi_0^{(4)}\| \right] \max_{0 \leq i \leq N} \max_{x_i \leq x \leq x_{i+1}} |D^k r_{3,i,2}(x)|
\]

\[
+ \|\zeta^4\| \max_{0 \leq i \leq N} \max_{x_i \leq x \leq x_{i+1}} \left[ \|D^k r_{3,i,2}(x)\| + |D^k r_{3,i+1,2}(x)| \right].
\]

Following a similar technique as in Lemma 5.2, we find that

\[
\|\zeta^4\| \leq \frac{231}{2} h^{-4} \xi^* + \frac{17}{4} h^2 \|D^6 f\|.
\]

The rest of the proof is clear.

**Remark 5.1.** As in Remark 4.1, in Tables 5.1–5.4 we compute the actual values of \( \|D^k (f - S_i f)\| \);

\( 1 \leq i \leq 4, 0 \leq k \leq 5 \) for the function \( f(x) = x(1 - x)e^x \) in the interval \([0, 1]\) and compare these with the corresponding right side bounds.
<table>
<thead>
<tr>
<th>Table 5.1.</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>9</td>
<td>19</td>
<td>39</td>
</tr>
<tr>
<td>$|f - S_1 f|$</td>
<td>0.90438267 - 6</td>
<td>0.15982997 - 7</td>
<td>0.26573929 - 9</td>
</tr>
<tr>
<td>Bound</td>
<td>0.10223442 - 4</td>
<td>0.15074128 - 5</td>
<td>0.24959575 - 8</td>
</tr>
<tr>
<td>$|D (f - S_1 f)|$</td>
<td>0.32694290 - 4</td>
<td>0.11550035 - 5</td>
<td>0.38396435 - 7</td>
</tr>
<tr>
<td>Bound</td>
<td>0.33457519 - 3</td>
<td>0.10455475 - 4</td>
<td>0.32673558 - 6</td>
</tr>
<tr>
<td>$|D^2 (f - S_1 f)|$</td>
<td>0.94450261 - 3</td>
<td>0.66743885 - 4</td>
<td>0.44379648 - 5</td>
</tr>
<tr>
<td>Bound</td>
<td>0.99027971 - 2</td>
<td>0.61892482 - 3</td>
<td>0.38682601 - 4</td>
</tr>
<tr>
<td>$|D^3 (f - S_1 f)|$</td>
<td>0.58743372 - 1</td>
<td>0.82840159 - 2</td>
<td>0.11005459 - 2</td>
</tr>
<tr>
<td>Bound</td>
<td>0.43764337 + 0</td>
<td>0.54054222 - 1</td>
<td>0.68317777 - 2</td>
</tr>
<tr>
<td>$|D^4 (f - S_1 f)|$</td>
<td>0.21894543 + 1</td>
<td>0.61533896 + 0</td>
<td>0.16323186 + 0</td>
</tr>
<tr>
<td>Bound</td>
<td>0.16420733 + 2</td>
<td>0.26605183 + 1</td>
<td>0.66512958 + 0</td>
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</table>

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<thead>
<tr>
<th>Table 5.2.</th>
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<tr>
<td></td>
<td>9</td>
<td>19</td>
<td>39</td>
</tr>
<tr>
<td>$|f - S_2 f|$</td>
<td>0.25871670 - 5</td>
<td>0.46360347 - 7</td>
<td>0.77610564 - 9</td>
</tr>
<tr>
<td>Bound</td>
<td>0.20199524 - 4</td>
<td>0.31561757 - 6</td>
<td>0.49315245 - 8</td>
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<tr>
<td>$|D (f - S_2 f)|$</td>
<td>0.15560057 - 3</td>
<td>0.55892535 - 5</td>
<td>0.18717064 - 6</td>
</tr>
<tr>
<td>Bound</td>
<td>0.72079773 - 3</td>
<td>0.22524929 - 4</td>
<td>0.70390403 - 6</td>
</tr>
<tr>
<td>$|D^2 (f - S_2 f)|$</td>
<td>0.56242373 - 2</td>
<td>0.40428441 - 3</td>
<td>0.27106966 - 4</td>
</tr>
<tr>
<td>Bound</td>
<td>0.17904983 - 1</td>
<td>0.11190614 - 2</td>
<td>0.69413384 - 4</td>
</tr>
<tr>
<td>$|D^3 (f - S_2 f)|$</td>
<td>0.80658454 - 1</td>
<td>0.11634084 - 1</td>
<td>0.15627302 - 2</td>
</tr>
<tr>
<td>Bound</td>
<td>0.56218110 + 0</td>
<td>0.70777636 - 1</td>
<td>0.88471579 - 2</td>
</tr>
<tr>
<td>$|D^4 (f - S_2 f)|$</td>
<td>0.13778194 + 1</td>
<td>0.39346765 + 0</td>
<td>0.10521958 + 0</td>
</tr>
<tr>
<td>Bound</td>
<td>0.83858994 + 1</td>
<td>0.20964748 + 1</td>
<td>0.52411871 + 0</td>
</tr>
<tr>
<td>$|D^5 (f - S_2 f)|$</td>
<td>0.23427072 + 2</td>
<td>0.13287403 + 2</td>
<td>0.70837784 + 1</td>
</tr>
<tr>
<td>Bound</td>
<td>0.17016444 + 3</td>
<td>0.85082222 + 2</td>
<td>0.42541111 + 2</td>
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</table>

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<th>Table 5.3.</th>
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<tbody>
<tr>
<td></td>
<td>9</td>
<td>19</td>
<td>39</td>
</tr>
<tr>
<td>$|f - S_3 f|$</td>
<td>0.16817937 - 5</td>
<td>0.30451725 - 7</td>
<td>0.51232884 - 9</td>
</tr>
<tr>
<td>Bound</td>
<td>0.29231594 - 4</td>
<td>0.45674366 - 6</td>
<td>0.71366197 - 8</td>
</tr>
<tr>
<td>$|D (f - S_3 f)|$</td>
<td>0.12115769 - 3</td>
<td>0.43732541 - 5</td>
<td>0.14692660 - 6</td>
</tr>
<tr>
<td>Bound</td>
<td>0.10173169 - 2</td>
<td>0.31791154 - 4</td>
<td>0.99347358 - 6</td>
</tr>
<tr>
<td>$|D^2 (f - S_3 f)|$</td>
<td>0.56242737 - 2</td>
<td>0.40428441 - 3</td>
<td>0.27106966 - 4</td>
</tr>
<tr>
<td>Bound</td>
<td>0.26661005 - 1</td>
<td>0.16631228 - 2</td>
<td>0.10414455 - 3</td>
</tr>
<tr>
<td>$|D^3 (f - S_3 f)|$</td>
<td>0.14154370 + 0</td>
<td>0.20190521 - 1</td>
<td>0.29716919 - 2</td>
</tr>
<tr>
<td>Bound</td>
<td>0.95900983 + 0</td>
<td>0.11987623 + 0</td>
<td>0.14984529 - 1</td>
</tr>
<tr>
<td>$|D^4 (f - S_3 f)|$</td>
<td>0.21894543 + 1</td>
<td>0.61533896 + 0</td>
<td>0.16323186 + 0</td>
</tr>
<tr>
<td>Bound</td>
<td>0.18060165 + 2</td>
<td>0.45225141 + 1</td>
<td>0.11306353 + 1</td>
</tr>
<tr>
<td>$|D^5 (f - S_3 f)|$</td>
<td>0.20052269 + 2</td>
<td>0.19073837 + 2</td>
<td>0.56935680 + 1</td>
</tr>
<tr>
<td>Bound</td>
<td>0.30879682 + 3</td>
<td>0.15439841 + 3</td>
<td>0.77192052 + 2</td>
</tr>
</tbody>
</table>
6. TWO VARIABLE LIDSTONE-SPLINE INTERPOLATION

For a fixed $p$, we define the set $L_m(p)$ as follows:

$$L_m(p) = L_m(\Delta) \oplus L_m(\Delta')$$

(the tensor product)

$$= \text{Span} \{ r_{m,i,\mu}(x)r_{m,j,\nu}(y) \}_{i=0}^{N+1} \mu=0 j=0 \nu=0^{M+1}$$

$$= \left\{ h(x,y) \in C([a,b] \times [c,d]) : h(x,y) \text{ is a two-dimensional polynomial of degree at most } (2m-1) \text{ in each variable and in each subrectangle } [x_i,x_{i+1}] \times [y_j,y_{j+1}] ; 0 \leq i \leq N, 0 \leq j \leq M \right\}.$$  

Since $L_m(p)$ is the tensor product of $L_m(\Delta)$ and $L_m(\Delta')$ which are of dimensions $[2m(N+1) - N]$ and $[2m(M+1) - M]$, respectively, $L_m(p)$ is of dimension $[2m(N+1) - N] \times [2m(M+1) - M].$

**DEFINITION 6.1.** For a given $f(x,y) \in C^{(2m-2,2m-2)}([a,b] \times [c,d]),$ we say $L_m^p f(x,y)$ is the $L_m(p)$-interpolate of $f(x,y),$ also known as the two-dimensional Lidstone interpolate of $f(x,y),$ if $L_m^p f(x,y) \in L_m(p)$ with $D_x^p D_y^p L_m^p f(x_i,y_j) = f^{(2p,2\nu)}_{i,j}; 0 \leq i \leq N + 1, 0 \leq j \leq M + 1, 0 \leq \mu, \nu \leq m - 1.$

For $f(x,y) \in C^{(2m-2,2m-2)}([a,b] \times [c,d]),$ it is clear that $L_m^p f(x,y)$ uniquely exists and can be explicitly expressed as

$$L_m^p f(x,y) = \sum_{i=0}^{N+1} \sum_{\mu=0}^{m-1} \sum_{j=0}^{M+1} \sum_{\nu=0}^{m-1} r_{m,i,\mu}(x)r_{m,j,\nu}(y)f^{(2\mu,2\nu)}_{i,j}.$$  \hspace{1cm} (6.1)

The set $S_m(p)$ is defined as follows:

$$S_m(p) = S_m(\Delta) \oplus S_m(\Delta')$$

(the tensor product)

$$= \left\{ s(x,y) \in C^{(2m-3,2m-3)}([a,b] \times [c,d]) : s(x,y) \text{ is a two-dimensional polynomial of degree at most } (2m-1) \text{ in each variable and in each subrectangle } [x_i,x_{i+1}] \times [y_j,y_{j+1}] ; 0 \leq i \leq N, 0 \leq j \leq M \right\}.$$  

Since $S_m(p)$ is the tensor product of $S_m(\Delta)$ and $S_m(\Delta')$ which are of dimensions $2(N+m)$ and $2(M+m),$ respectively, $S_m(p)$ is of dimension $4(N+m)(M+m).$
DEFINITION 6.2. For a given \( f(x, y) \in C^{(2m-2, 2m-2)}([a, b] \times [c, d]) \), we say \( L^{p}_{m}(f) = L^{p}_{m}(f(x, y)) \) is the Lidstone \( S_{m}(p) \)-interpolate of \( f(x, y) \), also known as Lidstone-spline interpolate of \( f(x, y) \), if \( L^{p}_{m}(f) \in S_{m}(p) \) with \( D^{2\mu}_{x}D^{2\nu}_{y}L^{p}_{m}(f(x, y)) = f^{(2\mu, 2\nu)}_{i,j} \), where \( \mu, \nu, i \) and \( j \) satisfy the following:

1. If \( 0 \leq \mu, \nu \leq m - 2 \), then \( 0 \leq i \leq N + 1, 0 \leq j \leq M + 1 \);
2. If \( \mu = m - 1, 0 \leq \nu \leq m - 2 \), then \( i = 0, N + 1, 0 \leq j \leq M + 1 \);
3. If \( 0 \leq \mu \leq m - 2, \nu = m - 1 \), then \( 0 \leq i \leq N + 1, j = 0, M + 1 \); and
4. If \( \mu = \nu = m - 1 \), then \((i, j) = (0, 0), (0, M + 1), (N + 1, 0), (N + 1, M + 1)\).

Since \( S_{m}(p) \subset L_{m}(p) \) in view of (6.1), it is clear that \( L^{p}_{m}(f(x, y)) \) can be written as

\[
L^{p}_{m}(f(x, y)) = \sum_{i=0}^{N+1} \sum_{r_{i,j}=0}^{m-1} \sum_{t_{r_i,j}=0}^{m-1} \sum_{u_{r_i,j}=0}^{m-1} r_{i,j}(x) r_{m,j,v}(y) D^{2\mu}_{x} D^{2\nu}_{y} L^{p}_{m}(f(x, y)).
\]  

In (6.2), \( D^{2\mu}_{x} D^{2\nu}_{y} L^{p}_{m}(f(x, y)) \) where \( \mu, \nu, i \) and \( j \) do not fulfill Definition 6.2 exist uniquely. In fact, for \( m = 3 \) we shall show that these unknown constants are the solutions of diagonally dominant systems of algebraic equations.

LEMMA 6.1. Let \( h(x, y) \in L_{3}(p) \) be a function for which, say, \( c_{i,j}^{2\mu, 2\nu} = D^{2\mu}_{x} D^{2\nu}_{y} h(x, y) \), \( 0 \leq \mu, \nu \leq 2, 0 \leq i \leq N + 1, 0 \leq j \leq M + 1 \) exist. Then, the function \( h(x, y) \in S_{3}(p) \) if and only if \( c_{i,j}^{2\mu, 2\nu} \), where \( \mu, \nu, i \) and \( j \) such that

1. If \( \mu = 2, \nu = 0, 1 \), then \( 1 \leq i \leq N, 0 \leq j \leq M + 1 \);
2. If \( \mu = 0, 1, \nu = 2 \), then \( 0 \leq i \leq N + 1, 1 \leq j \leq M \); and
3. If \( \mu = \nu = 2 \), then \( 1 \leq i \leq N, 1 \leq j \leq M \), satisfy the following relations:

\[
\begin{align*}
65c_{i,j}^{2\mu, 2\nu} + 26c_{i+2,j}^{2\mu, 2\nu} + c_{i+3,j}^{2\mu, 2\nu} &= \frac{120}{h^{2}} c_{i,j}^{2\mu, 2\nu} - 18c_{i,j}^{2\mu, 2\nu} \\
&+ \frac{120}{h^{4}} \left(-2c_{i,j}^{2\mu, 2\nu} + 5c_{i+1,j}^{2\mu, 2\nu} - 4c_{i+2,j}^{2\mu, 2\nu} + c_{i+3,j}^{2\mu, 2\nu}\right), \quad (6.3)
\end{align*}
\]

\[
\begin{align*}
c_{i,j-2}^{2\mu, 2\nu} + 66c_{i,j}^{2\mu, 2\nu} + 26c_{i+1,j}^{2\mu, 2\nu} + c_{i+2,j}^{2\mu, 2\nu} &= \frac{120}{h^{4}} \left(c_{i-2,j}^{2\mu, 2\nu} - 4c_{i-1,j}^{2\mu, 2\nu} + 6c_{i,j}^{2\mu, 2\nu} - 4c_{i+1,j}^{2\mu, 2\nu} + c_{i+2,j}^{2\mu, 2\nu}\right), \\
&\quad 2 \leq i \leq N - 1, \quad (6.4)
\end{align*}
\]

\[
\begin{align*}
c_{N-2,j}^{2\mu, 2\nu} + 66c_{N-1,j}^{2\mu, 2\nu} + 26c_{N,j}^{2\mu, 2\nu} &= \frac{120}{h^{4}} c_{N-1,j}^{2\mu, 2\nu} - 18c_{N+1,j}^{2\mu, 2\nu} \\
&+ \frac{120}{h^{4}} \left(c_{N-2,j}^{2\mu, 2\nu} - 4c_{N-1,j}^{2\mu, 2\nu} + 5c_{N,j}^{2\mu, 2\nu} - 2c_{N+1,j}^{2\mu, 2\nu}\right). \quad (6.5)
\end{align*}
\]

In the system (6.3)-(6.5), if \( \nu = 0, 1 \) then in (6.4), \( 2 \leq i \leq N - 1, 0 \leq j \leq M + 1 \), and in (6.3) and (6.5), \( 0 \leq j \leq M + 1 \); also if \( \nu = 2 \), then in (6.4), \( 2 \leq i \leq N - 1, j = 0, M + 1 \), and in (6.3) and (6.5), \( j = 0, M + 1 \).

Further, we have

\[
\begin{align*}
65c_{i,j}^{2\mu, 2\nu} + 26c_{i+2,j}^{2\mu, 2\nu} + c_{i+3,j}^{2\mu, 2\nu} &= \frac{120}{h^{2}} c_{i,j}^{2\mu, 2\nu} - 18c_{i,j}^{2\mu, 2\nu} \\
&+ \frac{120}{h^{4}} \left(-2c_{i,j}^{2\mu, 2\nu} + 5c_{i+1,j}^{2\mu, 2\nu} - 4c_{i+2,j}^{2\mu, 2\nu} + c_{i+3,j}^{2\mu, 2\nu}\right), \quad (6.6)
\end{align*}
\]

\[
\begin{align*}
c_{i,j-2}^{2\mu, 2\nu} + 66c_{i,j}^{2\mu, 2\nu} + 26c_{i+1,j}^{2\mu, 2\nu} + c_{i+2,j}^{2\mu, 2\nu} &= \frac{120}{h^{4}} \left(c_{i-2,j}^{2\mu, 2\nu} - 4c_{i-1,j}^{2\mu, 2\nu} + 6c_{i,j}^{2\mu, 2\nu} - 4c_{i+1,j}^{2\mu, 2\nu} + c_{i+2,j}^{2\mu, 2\nu}\right), \\
&\quad 2 \leq j \leq M - 1, \quad (6.7)
\end{align*}
\]

\[
\begin{align*}
c_{i,j-2}^{2\mu, 2\nu} + 66c_{i,j}^{2\mu, 2\nu} + 26c_{i+1,j}^{2\mu, 2\nu} + c_{i+2,j}^{2\mu, 2\nu} &= \frac{120}{h^{4}} \left(c_{i-2,j}^{2\mu, 2\nu} - 4c_{i-1,j}^{2\mu, 2\nu} + 6c_{i,j}^{2\mu, 2\nu} - 4c_{i+1,j}^{2\mu, 2\nu} + c_{i+2,j}^{2\mu, 2\nu}\right), \\
&\quad 2 \leq i \leq N - 1, \quad (6.8)
\end{align*}
\]
For $0 \leq \mu \leq 2$ in equation (6.7), $2 \leq j \leq M - 1$, $0 \leq i \leq N + 1$, and in (6.6) and (6.8), $0 \leq i \leq N + 1$.

Moreover, from the systems (6.3)-(6.5) and (6.6)-(6.8), the unknowns $c_{ij}^{2\mu,2\nu}$ where $\mu$, $\nu$, $i$ and $j$ satisfy the conditions of Lemma 6.1 can be obtained uniquely in terms of $c_{ij}^{2\mu,2\nu}$ where $\mu$, $\nu$, $i$ and $j$ fulfill Definition 6.2 for $m = 3$.

**PROOF.** The proof follows from Lemma 3.2.

**LEMMA 6.2.** For a given $f(x,y) \in C^{(4,4)}([a,b] \times [c,d])$, $LS_m^0 f(x,y)$ exists and is unique.

**PROOF.** The proof is similar to that of Lemma 3.3.

**REMARK 6.1.** In view of Remark 3.2, $LS_m^0 f(x,y)$ in terms of L-cardinal splines $\{s_i(x)\}_{i=0}^{2N+5}$ and $\{s_j(y)\}_{j=0}^{2M+5}$ can be explicitly expressed as

$$LS_m^0 f(x,y) = \sum_{i=0}^{N+1} \sum_{j=0}^{M+1} \left[ f_{i,j} s_i(x) s_j(y) + f_{i,j}^{(0,2)} s_i(x) s_{j+M+2}(y) + f_{i,j}^{(2,0)} s_{i+N+2}(x) s_j(y) \right] + \sum_{i=0}^{N+1} \left[ \left[ f_{i,0}^{(0,4)} s_{2M+4}(y) + f_{i,M+1}^{(0,4)} s_{2M+5}(y) \right] s_i(x) + \sum_{j=0}^{M+1} \left[ f_{0,j}^{(4,0)} s_{2N+4}(x) + f_{j+1,j}^{(4,0)} s_{2N+5}(x) s_{j+M+2}(y) \right] \right]$$

The following result, whose proof for $m = 3$ is immediate from Remarks 3.2 and 6.1, provides a characterization of $LS_m^0 f(x,y)$ in terms of one-dimensional interpolation schemes.

**LEMMA 6.3.** If $f(x,y) \in C^{(2m-2,2m-2)}([a,b] \times [c,d])$, then

$$LS_m^0 f(x,y) = LS_m^0 \Delta f \Delta f(x,y) = LS_m^0 \Delta L^0 f(x,y). \quad (6.10)$$

Now let $f(x,y) \in C^{(2m-2,2m-2)}([a,b] \times [c,d])$ be an arbitrary function. From Lemma 6.3 we have

$$f - LS_m^0 f = (f - LS_m^0 \Delta f) + LS_m^0 \Delta f \quad (6.11)$$

$$= (f - LS_m^0 \Delta f) + \left[ LS_m^0 \Delta f - (f - LS_m^0 \Delta f) \right] + \left( f - LS_m^0 \Delta f \right) \quad (6.12)$$

$$= (f - LS_m^0 f) + \left[ LS_m^0 \Delta f - (f - LS_m^0 \Delta f) \right] + \left( f - LS_m^0 f \right). \quad (6.13)$$

**THEOREM 6.4.** Let $f(x,y) \in PC^{n,n,\infty}([a,b] \times [c,d])$, $4 \leq n \leq 6$. Then,

$$\left\| D_x^k (f - LS_m^0 f) \right\| \leq \gamma_{n,k} h^{n-k} \left\| D_x^n f \right\| + \gamma_{n,k} \gamma_{n,0} h^{n-k} \ell^n \left\| D_y^p D_x^n f \right\| + \gamma_{n,0} \ell^n \left\| D_y^p D_x^p f \right\|, \quad (6.14)$$

where the constants $\gamma_{n,k}$ are defined in Table 4.1.
Proof. Since as a function of $x$, $(f - LS^\alpha f) \in PC^{n, \infty}[a, b]$, from (6.12) and Theorem 4.2 it follows that

$$
\|D^k_x (f - LS^\alpha f)\| \leq \|D^k_x (f - LS^\alpha f)\| + \|D^k_x \left[ LS^\alpha \left( f - LS^\alpha f \right) - \left( f - LS^\alpha f \right) \right]\|
\leq \gamma_{n,k} h^{n-k} \|D^k_x f\| + \gamma_{n,k} h^{n-k} \|D^k_x \left( f - LS^\alpha f \right)\| + \|D^k_x \left( f - LS^\alpha f \right)\|,
$$

(6.15)

Further, since as a function of $y$, for each $0 \leq k \leq n$, $D^k_y f \in PC^{n, \infty}[c, d]$, and $D^k_y LS^\alpha f = LS^\alpha D^k_y f$, Theorem 4.2 can be used again to obtain

$$
\|D^k_x \left( f - LS^\alpha f \right)\| \leq \gamma_{n,0} \ell^n \|D^\alpha_y D^k_x f\|.
$$

(6.16)

Now using (6.16) in (6.15), we obtain (6.14).

Remark 6.2. In Theorem 6.4, the corresponding bound for $\|D^k_y (f - LS^\alpha f)\|$ can be obtained by replacing $x$ with $y$ and $h$ with $\ell$.

### Table 6.1.

<table>
<thead>
<tr>
<th>$N = M$</th>
<th>7</th>
<th>9</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|f - LS^\alpha f|$</td>
<td>0.40775426 - 7</td>
<td>0.10782875 - 7</td>
<td>0.65070416 - 9</td>
</tr>
<tr>
<td>Bound</td>
<td>0.10101592 - 5</td>
<td>0.26476449 - 6</td>
<td>0.15780343 - 7</td>
</tr>
<tr>
<td>$|D^k_x (f - LS^\alpha f)|$</td>
<td>0.10835931 - 5</td>
<td>0.35828258 - 6</td>
<td>0.34608229 - 7</td>
</tr>
<tr>
<td>Bound</td>
<td>0.23597874 - 4</td>
<td>0.76650009 - 5</td>
<td>0.72147984 - 6</td>
</tr>
<tr>
<td>$|D^k_y (f - LS^\alpha f)|$</td>
<td>0.21812558 - 6</td>
<td>0.70027897 - 7</td>
<td>0.60778504 - 8</td>
</tr>
<tr>
<td>Bound</td>
<td>0.86601004 - 5</td>
<td>0.24806215 - 5</td>
<td>0.16436364 - 6</td>
</tr>
<tr>
<td>$|D^k_x (f - LS^\alpha f)|$</td>
<td>0.26926892 - 4</td>
<td>0.11130159 - 4</td>
<td>0.17204645 - 5</td>
</tr>
<tr>
<td>Bound</td>
<td>0.55172640 - 3</td>
<td>0.22488499 - 3</td>
<td>0.34137234 - 4</td>
</tr>
<tr>
<td>$|D^k_y (f - LS^\alpha f)|$</td>
<td>0.38211612 - 5</td>
<td>0.15833998 - 5</td>
<td>0.24549792 - 6</td>
</tr>
<tr>
<td>Bound</td>
<td>0.10681608 - 3</td>
<td>0.39474909 - 4</td>
<td>0.53125102 - 5</td>
</tr>
<tr>
<td>$|D^k_x (f - LS^\alpha f)|$</td>
<td>0.80496274 - 3</td>
<td>0.41591423 - 3</td>
<td>0.10389444 - 3</td>
</tr>
<tr>
<td>Bound</td>
<td>0.17082952 - 1</td>
<td>0.87357252 - 2</td>
<td>0.21299025 - 2</td>
</tr>
<tr>
<td>$|D^k_y (f - LS^\alpha f)|$</td>
<td>0.11543671 - 3</td>
<td>0.59571421 - 4</td>
<td>0.14717418 - 4</td>
</tr>
<tr>
<td>Bound</td>
<td>0.25695339 - 2</td>
<td>0.81201392 - 2</td>
<td>0.30623933 - 3</td>
</tr>
<tr>
<td>$|D^k_x (f - LS^\alpha f)|$</td>
<td>0.29006252 - 1</td>
<td>0.18961704 - 1</td>
<td>0.76511248 - 2</td>
</tr>
<tr>
<td>Bound</td>
<td>0.28524277 + 0</td>
<td>0.18242824 + 0</td>
<td>0.71235164 - 1</td>
</tr>
<tr>
<td>$|D^k_y (f - LS^\alpha f)|$</td>
<td>0.41864413 - 2</td>
<td>0.27311622 - 2</td>
<td>0.10986818 - 2</td>
</tr>
<tr>
<td>Bound</td>
<td>0.41285986 - 1</td>
<td>0.26196199 - 1</td>
<td>0.10184128 - 1</td>
</tr>
<tr>
<td>$|D^k_x (f - LS^\alpha f)|$</td>
<td>0.12960882 + 1</td>
<td>0.10582996 + 1</td>
<td>0.68218153 + 0</td>
</tr>
<tr>
<td>Bound</td>
<td>0.51558670 + 1</td>
<td>0.41323043 + 1</td>
<td>0.53125102 - 1</td>
</tr>
<tr>
<td>$|D^k_y (f - LS^\alpha f)|$</td>
<td>0.18694945 + 0</td>
<td>0.15235647 + 0</td>
<td>0.97926411 - 1</td>
</tr>
<tr>
<td>Bound</td>
<td>0.73917046 + 0</td>
<td>0.58697194 + 0</td>
<td>0.36813700 + 0</td>
</tr>
</tbody>
</table>

Remark 6.3. As in Remarks 4.1 and 5.1 in Table 6.1, for $n = 6$ we compute the actual values of $\|D^k_y (f - LS^\alpha f)\|$ and $\|D^k_x (f - LS^\alpha f)\|$, $0 \leq k \leq 5$ for the function $f(x, y) = x(1 - e^{xy}) \in C^{(\infty, \infty)}([0, 1] \times [0, 1])$ and use Theorem 6.4 to compare these with the corresponding right side bounds.
7. SOME APPLICATIONS

To show further the sharpness and the importance of our results here, we shall present the following four interesting applications.

Sixth Order Boundary Value Problems

For the two-point sixth order boundary value problem

\[ f^{(6)} = F(x, f), \quad a \leq x \leq b, \]
\[ f(a) = A_0, \quad f''(a) = A_2, \quad f^{(4)}(a) = A_4, \]
\[ f(b) = B_0, \quad f''(b) = B_2, \quad f^{(4)}(b) = B_4, \]

a quintic Lidstone-spline solution \( S_f(x) \) (which is in the class \( C^3[a, b] \)) can be constructed as follows:

1. Define \( x_i = a + ih, 0 < i < N + 1, h = (b - a)/(N + 1), f(x_i) = f(x_i), \) where \( f(x) \) is the exact solution of (7.1).
2. Compute \( g_i, 1 < i < N \) the approximation of \( f_i, 1 < i < N \) by using the sixth order method of Twizell and Boutayeb [21]. This means that \( \max_{1 < i < N} |f_i - g_i| = \xi^* = O(h^6) \).
3. Construct \( S_f(x) \) with the known conditions \( D^{2k} S_f(x_0) = A_{2k}, D^{2k} S_f(x_{N+1}) = B_{2k}, 0 < k < 2 \) and \( S_f(x_i) = g_i, 1 < i < N \).

From Theorem 5.11 it is clear that this \( S_f(x) \) is a sixth order \( C^3 \) approximation to the solution \( f(x) \) of the problem (7.1); i.e., \( \| f - S_f \| = O(h^6) \).

Fourth Order Boundary Value Problems

For the two-point fourth order boundary value problem

\[ f^{(4)} = F(x, f), \quad a \leq x \leq b, \]
\[ f(a) = A_0, \quad f''(a) = A_2, \]
\[ f(b) = B_0, \quad f''(b) = B_2, \]

a sixth order quintic Lidstone-spline solution \( S_f(x) \) of class \( C^3 \) can be constructed as for (7.1), except that now we use some known sixth order discrete method to compute \( g_i, 1 < i < N, \) and instead of \( D^4 S_f(x_0) = A_4, D^4 S_f(x_{N+1}) = B_4, \) simply use \( D^4 S_f(x_0) = F(a, A_0), \)

Second Order Boundary Value Problems

For the two-point second order boundary value problem

\[ f'' = F(x, f), \quad a \leq x \leq b, \]
\[ f(a) = A_0, \quad f(b) = B_0, \]

a sixth order quintic Lidstone-spline solution \( S_f(x) \) of class \( C^3 \) can be constructed as follows:

1. Define \( x_i = a + ih, 0 < i < N + 1, h = (b - a)/(N + 1), f_i = f(x_i), \) where \( f(x) \) is the exact solution of (7.3).
2. Compute \( g_i, 1 < i < N \) the approximation of \( f_i, 1 < i < N \) by using the sixth order method of Chawala [29].
3. Construct \( S_f(x) \) with the conditions \( S_f(x_0) = A_0, S_f(x_{N+1}) = B_0, S_f(x_i) = g_i, 1 < i < N, D^2 S_f(x_0) = F(a, A_0), D^2 S_f(x_{N+1}) = F(b, B_0), D^2 S_f(x_i) = F(x_i, g_i), 1 < i < N, \) and \( D^4 S_f(x_i) = f_i^{(4)}, i = 0, N + 1 \) (see Definition 5.6).
We use \( f_i^{(4)} \) to replace \( f_i^{(1)}, \ i = 0, N + 1 \) and \( g_i \) to replace \( f_i, 1 \leq i \leq N \) in the system (3.5)–(3.7) and note that the resulting unknowns \( c_i^{(4)}, 1 \leq i \leq N \) (Case 6 of Section 5) can be obtained uniquely in terms of \( f_i^{(k)}, k = 0, 1, i = 0, N + 1 \) and \( g_i, 1 \leq i \leq N \). Further, by Remark 3.1, \( S_8f(x) \) can be explicitly expressed as

\[
S_8f(x) = r_{3,0,0}(x)f_0 + r_{3,N+1,0}(x)f_{N+1} + \sum_{i=1}^{N} r_{3,i,0}(x)g_i \\
+ r_{3,0,1}(x)F(x_0, f_0) + r_{3,N+1,1}(x)F(x_{N+1}, f_{N+1}) + \sum_{i=1}^{N} r_{3,i,1}(x)F(x_i, g_i) \tag{7.4}
\]

\[
+ r_{3,0,2}(x)f_0^{(4)} + r_{3,N+1,2}(x)f_{N+1}^{(4)} + \sum_{i=1}^{N} r_{3,i,2}(x)c_i^{(4)}.
\]

**Theorem 7.1.** If \( f(x) \in PC^{6,\infty}[a, b] \), then

\[
\|D^k(f - S_8f)\| \leq \beta_{1,k} h^{6-k} \left\| D^0 f \right\| + \tau_k, \quad 0 \leq k \leq 5, \tag{7.5}
\]

where \( \beta_{1,k} \) are defined in Theorem 5.1 and

\[
\tau_0 = (0.48104060 + 1)\xi^* + \frac{1}{8} h^2 \mu, \quad \tau_1 = (0.12401389 + 2)h^{-1}\xi^* + \frac{1}{2} h\mu, \\
\tau_2 = (0.36816605 + 2)h^{-2}\xi^* + \mu, \quad \tau_3 = (0.15908333 + 3)h^{-3}\xi^* + 2h^{-1}\mu, \\
\tau_4 = (0.36950000 + 3)h^{-4}\xi^*, \quad \tau_5 = (0.58500000 + 3)h^{-5}\xi^*.
\]

\( \xi^* \) is defined in (5.36), and \( \mu = \max_{1 \leq i \leq N} |F(x_i, f_i) - F(x_i, g_i)| \).

**Proof.** We use the inequality

\[
\|D^k(f - S_8f)\| \leq \|D^k(f - S_1f)\| + \|D^k(S_1f - S_8f)\|, \quad 0 \leq k \leq 5 \tag{7.6}
\]

and apply Theorem 5.1 to the first term, whereas for the second term we note that

\[
(S_1f - S_8f)(x) = \sum_{i=1}^{N} r_{3,i,0}(x)(f_i - g_i) + \sum_{i=1}^{N} r_{3,i,1}(x)[F(x_i, f_i) - F(x_i, g_i)] \\
+ r_{3,0,2}(x)\left(f_0^{(4)} - f_0^{(4)}\right) + r_{3,N+1,2}(x)\left(f_{N+1}^{(4)} - f_{N+1}^{(4)}\right) \tag{7.7}
\]

\[
+ \sum_{i=1}^{N} r_{3,i,2}(x)c_i^{(4)},
\]

where \( c_i^{(4)} = c_i^{(4)} - c_i^{(4)} \). Denoting \( \delta^4 = [c_i^{(4)}] \), for \( 0 \leq k \leq 5 \) it follows that

\[
\|D^k(S_1f - S_8f)\| \leq \xi^* \max_{0 \leq i \leq N, x_i \leq x \leq x_{i+1}} \left[\|D^k r_{3,i,0}(x)\| + \|D^k r_{3,i+1,0}(x)\|\right] \\
+ \mu \max_{0 \leq i \leq N, x_i \leq x \leq x_{i+1}} \left[\|D^k r_{3,i,1}(x)\| + \|D^k r_{3,i+1,1}(x)\|\right] \\
+ 2(77h^{-4}\xi^*) \max_{0 \leq i \leq N, x_i \leq x \leq x_{i+1}} |D^k r_{3,i,2}(x)| \\
+ \|\delta^4\| \max_{0 \leq i \leq N, x_i \leq x \leq x_{i+1}} \left[\|D^k r_{3,i,2}(x)\| + \|D^k r_{3,i+1,2}(x)\|\right]. \tag{7.8}
\]

By a technique similar to that in Lemma 5.2, we find that

\[
\|\delta^4\| \leq \frac{431}{2} h^{-4} \xi^*.
\]

The rest of the proof is clear.
Lidstone-Spline Interpolation

From Theorem 7.1, it is clear that this $S_8f(x)$ is a sixth order $C^3$ approximation to the solution $f(x)$ of the problem (7.3).

**Remark 7.1.** The above construction of $S_7f(x)$ and $S_8f(x)$ for the problems (7.2) and (7.3), respectively, is in contrast with Usmani and Warsi [22,23] and others, where the smooth solutions of the boundary value problems are obtained by using only the consistency relations for the spline functions such as the system (3.5)–(3.7). Although, this results into relatively less computation, then only lower order approximations are achieved.

In particular, for the problem

$$
f'' = \frac{(1-x)f + 1}{(1+x)^2}, \quad 0 \leq x \leq 1,
$$

$$
f(0) = 1, \quad f(1) = \frac{1}{2},
$$

for which the exact solution is $f(x) = 1/(1+x)$, we use the above procedure to construct the $C^3$ approximate solution $S_8f(x)$. In Table 7.1, we illustrate $\xi^*$ which is computed by using the known exact solution, the actual values of $\|D^k(f - S_8f)\|$, $0 \leq k \leq 5$ and compare these with the corresponding right side bounds in (7.5).

<table>
<thead>
<tr>
<th>$N$</th>
<th>9</th>
<th>19</th>
<th>39</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi^*$</td>
<td>0.45080060 - 8</td>
<td>0.73603471 - 10</td>
<td>0.11709800 - 11</td>
</tr>
<tr>
<td>$|f - S_8f|$</td>
<td>0.28696423 - 5</td>
<td>0.73948247 - 7</td>
<td>0.15304626 - 8</td>
</tr>
<tr>
<td>Bound</td>
<td>0.75241572 - 4</td>
<td>0.11756648 - 5</td>
<td>0.18359862 - 7</td>
</tr>
<tr>
<td>$|D^2(f - S_8f)|$</td>
<td>0.10424566 - 3</td>
<td>0.53553269 - 5</td>
<td>0.22136649 - 6</td>
</tr>
<tr>
<td>Bound</td>
<td>0.24622259 - 2</td>
<td>0.76945341 - 4</td>
<td>0.24045522 - 5</td>
</tr>
<tr>
<td>$|D^3(f - S_8f)|$</td>
<td>0.30084398 - 2</td>
<td>0.30930368 - 3</td>
<td>0.25579334 - 4</td>
</tr>
<tr>
<td>Bound</td>
<td>0.72877309 - 3</td>
<td>0.45548768 - 2</td>
<td>0.2848113 - 3</td>
</tr>
<tr>
<td>$|D^4(f - S_8f)|$</td>
<td>0.19090558 + 0</td>
<td>0.38736775 - 1</td>
<td>0.63700038 - 2</td>
</tr>
<tr>
<td>Bound</td>
<td>0.32207172 + 1</td>
<td>0.40259367 + 0</td>
<td>0.50324422 - 1</td>
</tr>
<tr>
<td>$|D^5(f - S_8f)|$</td>
<td>0.73500031 + 1</td>
<td>0.29196317 + 1</td>
<td>0.95126454 + 0</td>
</tr>
<tr>
<td>Bound</td>
<td>0.73166575 + 2</td>
<td>0.19579351 + 2</td>
<td>0.48948577 + 1</td>
</tr>
<tr>
<td>$|D^5(f - S_8f)|$</td>
<td>0.12173437 + 3</td>
<td>0.90619104 + 2</td>
<td>0.57347295 + 2</td>
</tr>
<tr>
<td>Bound</td>
<td>0.11762637 + 4</td>
<td>0.58813777 + 3</td>
<td>0.29407014 + 3</td>
</tr>
</tbody>
</table>

**Linear Integral Equations**

Consider the linear Fredholm integral equation of the second kind

$$
\phi(x) = \int_c^d f(x,y) \phi(y) dy + \psi(x), \quad x \in [a, b],
$$

where $f(x, y) \in C^{(4,4)}([a, b] \times [c, d])$ and $\psi(x) \in C[a, b]$.

In (7.10), we can approximate $f(x, y)$ by its biquintic Lidstone-spline interpolate, $LS^6_5 f(x, y)$, which in view of (6.9) is in degenerate form, i.e., (for the simplicity of notation) can be written as

$$
LS^6_5 f(x, y) = \sum_{\mu=1}^{\xi} \sum_{\nu=1}^{\eta} \beta_{\mu \nu} A_{\mu}(x) B_{\nu}(y).
$$
With this approximation, the resulting integral equation appears as

\[
\tilde{\phi}(x) = \int_c^d L S_2^o f(x, y) \tilde{\phi}(y) \, dy + \psi(x), \quad x \in [a, b],
\]

(7.11)

which determines an approximate solution \(\tilde{\phi}(x)\).

Equation (7.11) is the same as

\[
\tilde{\phi}(x) = \sum_{\mu=1}^\xi A_\mu(x) \beta_\mu + \psi(x), \quad x \in [a, b],
\]

(7.12)

where

\[
\beta_\mu = \int_c^d \sum_{\nu=1}^\eta \beta_{\mu\nu} B_\nu(y) \tilde{\phi}(y) \, dy, \quad 1 \leq \mu \leq \xi.
\]

(7.13)

From (7.12), the approximate solution \(\tilde{\phi}(x)\) can be obtained if we can determine the \(\xi \times 1\) vector \(\beta = [\beta_\mu]\). For this, we substitute (7.12) in (7.13), to obtain

\[
\beta_\mu = \sum_{\nu=1}^\eta \int_c^d \beta_{\mu\nu} B_\nu(y) \tilde{\phi}(y) \, dy + \sum_{\nu=1}^\eta \int_c^d \beta_{\mu\nu} B_\nu(y) \psi(y) \, dy, \quad 1 \leq \mu \leq \xi,
\]

(7.14)

which in system form can be written as \(\beta = P\beta + q\), or

\[
(I - P)\beta = q,
\]

(7.15)

where \(P = [p_{ij}]\) is an \(\xi \times \xi\) matrix

\[
p_{ij} = \sum_{\nu=1}^\eta \int_c^d \beta_{\nu j} B_\nu(y) A_i(y) \, dy, \quad 1 \leq i, j \leq \xi
\]

(7.16)

and \(q = [q_i]\) is an \(\xi \times 1\) vector

\[
q_i = \sum_{\nu=1}^\eta \int_c^d \beta_{\nu i} B_\nu(y) \psi(y) \, dy, \quad 1 \leq i \leq \xi
\]

(7.17)

It is clear from (7.12) and (7.15) that a unique \(\tilde{\phi}(x)\) exists if and only if the matrix \((I - P)\) is nonsingular. To provide sufficient conditions for the existence of a unique \(\tilde{\phi}(x)\), we introduce the operators \(R\) and \(S\) on \(C[a, b]\) as follows:

\[
R[\phi] = \int_c^d f(x, y) \phi(y) \, dy
\]

and

\[
S[\phi] = \int_c^d L S_2^o f(x, y) \phi(y) \, dy,
\]

so that (7.10) and (7.11) in operator form can be written as

\[
(I - R)[\phi] = \psi
\]

(7.18)

and

\[
(I - S)[\tilde{\phi}] = \psi,
\]

(7.19)

respectively.

**Definition 7.1.** Let \(T : C[a, b] \to C[a, b]\) be an operator defined by \(T[\phi] = \psi\). We say that the operator \(T\) is invertible if \(T[\phi] = \psi\) has a unique solution \(\phi \in C[a, b]\) for each \(\psi \in C[a, b]\).
LEMMA 7.2. If \((I - R)\) is invertible and
\[
\sigma = (d - c) \| f - LS^\text{opt} f \| \| (I - R)^{-1} \| < 1, \tag{7.20}
\]
then \((I - S)\) is invertible; i.e., \((7.11)\) has a unique solution \(\tilde{\phi}(x)\).

PROOF. The proof is similar to that of Lemma 3.1 in [30].

THEOREM 7.3. If \((I - R)\) is invertible and \((7.20)\) holds, then
\[
\| \phi - \tilde{\phi} \| \leq \frac{\sigma}{1 - \sigma} \| \phi \| \tag{7.21}
\]
and
\[
\| \phi - \tilde{\phi} \| \leq \sigma \| \tilde{\phi} \|. \tag{7.22}
\]
(Inequality \((7.21)\) gives a priori error bound, whereas \((7.22)\) provides a posteriori error bound.)

PROOF. The proof is similar to that of [30, Theorem 3.2].

From Theorem 6.4, the following corollary of Theorem 7.3 is immediate.

COROLLARY 7.4. If \((I - R)\) is invertible, \(f(x, y) \in PC^{4,n,\infty}([a, b] \times [c, d]), 4 \leq n \leq 6 \) and \(\rho\) is such that
\[
\sigma_n = (\gamma_{n,0} h^n \| D_y^n f \| + \gamma_{n,0} h^n \| D_y^n D_x^n f \| + \gamma_{n,0} \| D_y^n f \|) \times (d - c) \| (I - R)^{-1} \| < 1,
\]
then \((I - S)\) is invertible. Moreover, \((7.21)\) and \((7.22)\) hold.

In particular, we consider the integral equation
\[
\phi(x) = \int_0^1 x (1 - e^{xy}) \phi(y) dy + e^x - x, \quad x \in [0, 1] \tag{7.23}
\]
whose exact solution is known to be \(\phi(x) \equiv 1\). In this equation, the operator \(R\) is given by
\[
R[\phi] = \int_0^1 x (1 - e^{xy}) \phi(y) dy,
\]
and hence,
\[
\| R[\phi] \| \leq \max_{0 \leq x \leq 1} \left| \int_0^1 x (1 - e^{xy}) \phi(y) \right| = \max_{0 \leq x \leq 1} (e^x - x - 1) \| \phi \| = (e - 2) \| \phi \|,
\]
which gives that \(\| R \| \leq (e - 2) < 1\).

From the extension of Lemma 2.10 for bounded linear operators in Banach spaces, the term \(\| (I - R)^{-1} \|\) in \(\sigma_n\) (Corollary 7.4) can be replaced by \((3 - e)^{-1}\). With this modification, the two error bounds \((7.21)\) and \((7.22)\) will be larger. We shall find \(\tilde{\phi}(x)\) and the actual value of \(\| \phi - \tilde{\phi} \|\), and compare this with the modified error bounds.

Choose \(N\) and \(M\) such that in Corollary 7.4, \(\sigma_n < 1\). We first obtain the biquintic Lidstone-spline interpolate of the kernel \(f(x, y) = x(1 - e^{xy}) \in C^{4,\infty,\infty}([0, 1] \times [0, 1])\). For this, in view of Remark 6.1, we need only to construct the cardinal Lidstone-splines \(s_i(x), 0 \leq i \leq 2N + 5\) and \(s_j(y), 0 \leq j \leq 2M + 5\). From (3.11), we note that only the values of \(c_{4i}^i, 1 \leq i \leq N\) need to be computed for each cardinal spline as the explicit expressions of the functions \(r_{3,4,i,j}(x)\) are known. Thus, we need to solve the system \((3.5)-(3.7)\). To find \(\tilde{\phi}(x)\) we need to solve the system \((7.15)\) to obtain the vector \(\beta\); then from \((7.12)\) it follows that
\[
\tilde{\phi}(x) = \sum_{\mu=0}^{2N+5} s_\mu(x) \beta_\mu + e^x - x, \quad x \in [0, 1].
\]

In Table 7.2 we present the actual value of \(\| \phi - \tilde{\phi} \|\) and the two modified error bounds.
Table 7.2.

<table>
<thead>
<tr>
<th>$N = M$</th>
<th>7</th>
<th>9</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\left| \phi - \hat{\phi} \right|$</td>
<td>0.38805225 - 8</td>
<td>0.10835548 - 8</td>
<td>0.69038500 - 10</td>
</tr>
<tr>
<td>$\frac{a_6}{1 - a_6} \left| \phi \right|$</td>
<td>0.35857211 - 5</td>
<td>0.93982129 - 6</td>
<td>0.56014648 - 7</td>
</tr>
<tr>
<td>$a_6 \left| \phi \right|$</td>
<td>0.35857082 - 5</td>
<td>0.93982041 - 6</td>
<td>0.56014645 - 7</td>
</tr>
</tbody>
</table>

REFERENCES