Coefficient inequality for perturbed harmonic mappings

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ABSTRACT

The aim of this work is to give the coefficient inequality for the perturbed harmonic mappings in the open unit disc $\mathbb{D}$.

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1. Introduction

Let $\Omega$ be the family of functions $\phi(z)$ which are regular in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Next, denote by $\mathcal{P}$ the family of functions $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ regular in $\mathbb{D}$ such that $p(z)$ in $\mathcal{P}$ if and only if $p(z) = \frac{1 + \phi(z)}{1 - \phi(z)}$ for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$. Then we have $p(0) = 1$ and $\Re p(z) > 0$ whenever $p(z) \in \mathcal{P}$.

Let $\mathcal{S}_H$ denote the class of all harmonic, complex valued, orientation-preserving univalent mappings $f$ defined on the open unit disc $\mathbb{D}$ which are normalized at the origin by $f(0) = 0$ and $f_z(0) = 1$. Such functions admit the representation $f = h + \bar{g}$, where $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$ are analytic in $\mathbb{D}$.

One shows easily that the orientation-preserving property implies that $|b_1| < 1$.

The function $w = g'/h'$ is called the second dilatation of the harmonic mapping $f = h + \bar{g}$, and we denote the class of such functions by $W$. Note that $|w(z)| < 1$ and $w(0) \neq 0$ for all $z \in \mathbb{D}$. Therefore $w(z)$ is not in $\Omega$, but $zw(z)$ is.

Let $f$ be an element of $\mathcal{S}_H$. We define a new subclass of $\mathcal{S}_H$ for which the analytic part is a univalent function in $\mathbb{D}$. The family of such functions will be denoted by $\mathcal{S}^c_{\mathcal{S}_H}(\rho)$.

2. Result

Now we give the coefficient inequality for the perturbed harmonic mappings.

Theorem 2.1. If $f = h + \bar{g}$ is an element of $\mathcal{S}^c_{\mathcal{S}_H}(\rho)$, then

$$|b_n| \leq \frac{1}{n} \{2^n 6 - n^2 - 4n - 6\}$$

for all $n \in \mathbb{N}$.

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Proof. Let \( p(z) \in \mathcal{P} \). Then we have
\[
p(z) = \frac{1 + \phi(z)}{1 - \phi(z)} = \frac{1 + z w(z)}{1 - z w(z)} = \frac{h'(z) + zg'(z)}{\bar{h}'(z) - z g'(z)} \Rightarrow
\]
\[
p(z)(\bar{h}'(z) - z g'(z)) = h'(z) + zg'(z), \tag{2.2}
\]
where \( w(z) \in W \) and \( \phi(z) \in \Omega \) for all \( z \in D \). Equality (2.2) can be written by using the Taylor expansion of \( p(z) \), \( h(z) \) and \( g(z) \) in the form
\[
\left(1 + \sum_{n=1}^{\infty} p_n z^n\right) \left(1 + \sum_{n=1}^{\infty} [(n + 1) a_{n+1} - n b_n] z^n\right) = 1 + \sum_{n=1}^{\infty} [(n + 1) a_{n+1} + n b_n] z^n. \tag{2.3}
\]
Equating the coefficients of \( z^n \) on either side of (2.3), we get
\[
2 nb_n = p_n + \sum_{k=1}^{n-1} (p_{n-k} [(k + 1) a_{k+1} - k b_k]). \tag{2.4}
\]
On the other hand, if \( p(z) \in \mathcal{P} \) we know that \( |p_k| \leq 2 \) satisfies for all \( k \in \mathbb{N} \). By using this inequality in (2.4), we obtain
\[
n|b_n| \leq \sum_{k=1}^{n} (k|a_k| + (k - 1)|b_{k-1}|) \quad (|a_1| = 1, |b_0| = 0). \tag{2.5}
\]
To prove (2.1), we use the induction principle. Consider inequalities (2.5) and
\[
n|b_n| \leq \sum_{k=1}^{n} k 2^{n-k} |a_k| \quad (|a_1| = 1). \tag{2.6}
\]
The right-hand sides of these inequalities are the same, as:
It is obvious for \( n = 1 \). Also,
\[
2|b_2| \leq 1 + 2|a_2| + |b_1| \leq 2 + 2|a_2|, \quad 2|b_2| \leq 2 + 2|a_2|
\]
for \( n = 2 \).
Suppose that this result is true for \( n = p \); then we have
\[
p|b_p| \leq \sum_{k=1}^{p} (k|a_k| + (k - 1)|b_{k-1}|) \quad (|a_1| = 1, |b_0| = 0), \tag{2.7}
\]
\[
p|b_p| \leq \sum_{k=1}^{p} k 2^{p-k} |a_k|. \tag{2.8}
\]
From (2.7) and the induction hypothesis, we have
\[
\sum_{k=1}^{p} (k|a_k| + (k - 1)|b_{k-1}|) = \sum_{k=1}^{p} k 2^{p-k} |a_k| \quad (|a_1| = 1, |b_0| = 0). \tag{2.8}
\]
By using equality (2.8) we obtain
\[
\sum_{k=1}^{p+1} (k|a_k| + (k - 1)|b_{k-1}|) = \sum_{k=1}^{p} (k|a_k| + (k - 1)|b_{k-1}|) + (p + 1)|a_{p+1}| + p|b_p|
\]
\[
= \sum_{k=1}^{p} k 2^{p-k} |a_k| + (p + 1)|a_{p+1}| + p|b_p|
\]
\[
= \frac{1}{2} \left( \sum_{k=1}^{p} k 2^{p+1-k} |a_k| + (p + 1)|a_{p+1}| \right) + \frac{1}{2} ((p + 1)|a_{p+1}| + 2p|b_p|)
\]
\[
= \frac{1}{2} \sum_{k=1}^{p+1} k 2^{p+1-k} |a_k| + \frac{1}{2} (p + 1)|a_{p+1}| + \sum_{k=1}^{p} k 2^{p+1-k} |a_k|
\]

\[\begin{align*}
&= \frac{1}{2} \sum_{k=1}^{p+1} k2^{p+1-k} |a_k| + \frac{1}{2} \sum_{k=1}^{p+1} k2^{p+1-k} |a_k| \\
&= \sum_{k=1}^{p+1} k2^{p+1-k} |a_k|.
\end{align*}\]

Therefore this assumption is true for \( n = p + 1 \). So we have
\[
|b_n| \leq \frac{1}{n} \sum_{k=1}^{n} k2^{n-k} |a_k|. \tag{2.9}
\]

If \( f \in \mathfrak{S}^2_{\mathfrak{S}(p)} \), then \( |a_k| \leq k \) \((k \in \mathbb{N}) \) [2]. By using this inequality in (2.9) we obtain
\[
|b_n| \leq \frac{1}{n} \sum_{k=1}^{n} k^2 2^{n-k} = \frac{2^n}{n} \sum_{k=1}^{n} k^2.
\]

It is easy to show that sum of this series is
\[
\sum_{k=1}^{n} \frac{k^2}{2^k} = 6 - \frac{1}{2^n} (n^2 + 4n + 6).
\]

Therefore we have
\[
|b_n| \leq \frac{2^n}{n} \left( 6 - \frac{1}{2^n} (n^2 + 4n + 6) \right).
\]

This completes the proof of the theorem. \( \square \)

References
