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# Real submanifolds of codimension two of a complex space form

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ABSTRACT

We prove some classification theorems for real submanifolds of codimension two of a complex space form under the condition that $h(FX,Y)+h(X,FY)=0$, where $h$ is the second fundamental form of the submanifold and $F$ is the endomorphism induced from the almost complex structure $J$ on the tangent bundle of the submanifold.



## 1. Introduction

Let $M$ be a real submanifold of a complex manifold $\overline{M}$ and $J$ be the natural almost complex structure of $\overline{M}$. If the holomorphic tangent space $H_x(M)=JT_x(M)\cap T_x(M)$ has constant dimension with respect to $x\in M$, the submanifold $M$ is called a CR submanifold and the constant complex dimension is called the CR dimension of $M$ [3,6].

In this paper we study real submanifolds of codimension 2 of a complex manifold. It is clear that the codimension 2 case is fundamental in the study of even-dimensional real submanifolds of a complex manifold. In this direction, in [8], K. Yano and the second author of this paper studied submanifolds of codimension 2 of a complex Euclidean space. The known results show that the situation for submanifolds of codimension 2 is more complicated than in the case of real hypersurfaces. For example, a complex hypersurface, which is a CR submanifold of CR dimension $\frac{n-2}{2}$, is a real submanifold of codimension 2, but there also exist real submanifolds of codimension 2 which are not CR submanifolds (for example, an even-dimensional sphere of codimension 2 of an even-dimensional Euclidean space, see [8]). The aim of this paper is to extend the results obtained in [8] for complex Euclidean space and, moreover, to investigate real submanifolds of codimension 2, but not only of complex Euclidean space but also of other complex space forms.

In Section 2 we develop the theory of submanifolds of codimension 2 of a Kähler manifold and we derive some fundamental formulae for later use. We also prove that if a complex hypersurface satisfies the algebraic condition on the $(1,1)$-tensor, induced from the almost complex structure $J$, and the second fundamental form of the submanifold, then the submanifold is a totally geodesic complex hypersurface. In Section 3, we restrict our investigation to the case when the

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ambient manifold is a non-Euclidean complex space form. When the ambient manifold is a complex Euclidean space, in Section 4, we obtain similar, but more rigorous results than those presented in [8].

From the results derived in Sections 3 and 4, we conclude that the case $\lambda = 0$ is significant, where $\lambda$ is a function defined on the real submanifold of codimension 2. Therefore, in Section 5 we examine submanifolds $M$ of a complex Euclidean space, with $\lambda = 0$, and in Section 6 we study even more particular case, when there exists a totally umbilical hypersurface $M'$ of a complex Euclidean space such that $M \subset M'$. We recall here that K. Yano studied in [7] hypersurfaces of an odd-dimensional sphere satisfying a certain algebraic condition. However, the results obtained in [7] establish some properties of a vector field defined on the hypersurface and not of the hypersurface itself. Our purpose is to give a classification theorem for hypersurfaces $M \subset M' \subset \mathbf{C}^{\frac{n+2}{2}}$.

Throughout this paper we assume that all submanifolds are connected.

## 2. Submanifolds of codimension 2 of a complex manifold

Let $\overline{M}$ be a real $(n+2)$-dimensional complex manifold, $J$ its natural almost complex structure and $\bar{g}$ its Hermitian metric. Further, let $M$ be an $n$-dimensional submanifold of $\overline{M}$ with the immersion $\imath$ of $M$ into $\overline{M}$ where we also denote by $\imath$ the differential of the immersion, or we omit to mention $\imath$, for brevity of notation. Then the tangent bundle $T(M)$ is identified with a subbundle of $T(\overline{M})$ and a Riemannian metric $g$ of $M$ is induced from the Riemannian metric $\bar{g}$ of $\overline{M}$ in such a way that $g(X, Y) = \bar{g}(\imath X, \imath Y)$ where $X, Y \in T(M)$. Let $\xi_1$ and $\xi_2$ be mutually orthogonal unit normals to $M$. Then

$$J\imath X = \imath FX + \sum_{a=1}^{2} u^a(X)\xi_a = \imath FX + u^1(X)\xi_1 + u^2(X)\xi_2, \tag{2.1}$$

$$J\xi_a = -\imath U_a + \sum_{b=1}^{2} \lambda_{ab}\xi_b = -\imath U_a + \lambda_{a1}\xi_1 + \lambda_{a2}\xi_2, \tag{2.2}$$

that is,

$$J\xi_1 = -\imath U_1 + \lambda\xi_2, \qquad J\xi_2 = -\imath U_2 - \lambda\xi_1, \tag{2.3}$$

where $\lambda = \lambda_{12} = -\lambda_{21}$. Here, $F$ is a skew-symmetric endomorphism acting on $T(M)$, $U_a$, $a = 1, 2$ are local tangent vector fields and $u^a$, $a = 1, 2$ are local one forms on $M$. We note that $u^1$ and $u^2$ depend on the choice of normals $\xi_1$ and $\xi_2$, but the function $\lambda^2$, where $\lambda = \bar{g}(J\xi_1, \xi_2)$, does not depend on the choice of $\xi_1$ and $\xi_2$. More precisely, if we choose another pair of mutually orthogonal unit normals: $\xi_1'$ and $\xi_2'$, then $\xi_1' = \xi_1 \cos\theta - \xi_2 \sin\theta$, $\xi_2' = \xi_1 \sin\theta + \xi_2 \cos\theta$, or $\xi_1' = \xi_1 \cos\theta + \xi_2 \sin\theta$, $\xi_2' = \xi_1 \sin\theta - \xi_2 \cos\theta$, for some $\theta$. Consequently, if the orientation is preserved, then $\lambda' = \bar{g}(J\xi_1', \xi_2') = \lambda$. In the same manner we can see that $\lambda' = -\lambda$ if the orientation is not preserved.

Now, applying $J$ to (2.1) and (2.2), we have

$$-\imath X = \imath F^2X + \sum_{b=1}^{2} u^b(FX)\xi_b + \sum_{a=1}^{2} u^a(X)\left(-\imath U_a + \sum_{b=1}^{2} \lambda_{ab}\xi_b\right), \tag{2.4}$$

$$-\xi_a = -\imath\left(FU_a + \sum_{b=1}^{2} \lambda_{ab}U_b\right) - \sum_{c=1}^{2}\left\{u^c(U_a) - \sum_{b=1}^{2} \lambda_{ab}\lambda_{bc}\right\}\xi_c. \tag{2.5}$$

Comparing the tangential parts in (2.4) and (2.5), we obtain

$$F^2X = -X + \sum_{a=1}^{2} u^a(X)U_a = -X + u^1(X)U_1 + u^2(X)U_2, \tag{2.6}$$

$$FU_a = -\sum_{b=1}^{2} \lambda_{ab}U_b, \tag{2.7}$$

that is,

$$FU_1 = -\lambda U_2, \qquad FU_2 = \lambda U_1. \tag{2.8}$$

Also, using (2.5), we get $-\delta_a^b = -u^b(U_a) + \sum_{c=1}^{2} \lambda_{ac}\lambda_{cb}$ and therefore

$$u^1(U_1) = u^2(U_2) = 1 - \lambda^2, \qquad u^1(U_2) = u^2(U_1) = 0. \tag{2.9}$$

Since $J$ is a skew-symmetric operator, we calculate

$$g(U_a, X) = u^a(X), \quad a = 1, 2, \tag{2.10}$$

and consequently

$$g(U_1,U_1)=g(U_2,U_2)=1-\lambda^2, \qquad g(U_1,U_2)=0. \tag{2.11}$$

The subspace $H_x(M)=JT_x(M)\cap T_x(M)$ of the tangent space $T_x(M)$ is called the holomorphic tangent space. It is well-known that a holomorphic tangent space is the maximal $J$-invariant subspace of $T_x(M)$. If the dimension of the holomorphic tangent space is constant with respect to $x\in M$, the submanifold is called CR submanifold and its complex dimension is called the CR dimension of the submanifold [3,6]. Every $n$-dimensional real hypersurface of a complex manifold is a CR submanifold of CR dimension $\frac{n-1}{2}$.

**Proposition 2.1.** *Let $M$ be a real submanifold of codimension 2 of a complex manifold $\overline{M}$ and let $\lambda$ be the function defined by (2.3). Then*:

(1) *$M$ is a complex hypersurface if and only if $\lambda^2(x)=1$ for any $x\in M$.*
(2) *$M$ is a CR submanifold of CR dimension $\frac{n-2}{2}$ if $\lambda(x)=0$ for any $x\in M$.*

**Proof.** From (2.11) we conclude that $\lambda^2=1$ implies $U_1=U_2=0$. Using (2.1) and (2.6), we compute $J\imath X=\imath FX$ and $F^2X=-X$. Thus, $M$ is a $J$-invariant submanifold and $F$ is the induced almost complex structure from $J$. Since the ambient manifold is a complex manifold, the $J$-invariant submanifold $M$ is a complex manifold, that is, a complex hypersurface.

Let $\lambda=0$. Then, using (2.2) it follows $J\imath U_a=\xi_a$. For all $X$ orthogonal to $U_1$ and $U_2$, using (2.1) and (2.10), it follows $J\imath X=\imath FX$. Consequently, $JT_x(M)\cap T_x(M)=\{X\in T_x(M)\mid X\perp span\{U_1,U_2\}\}$ and therefore $\dim_{\mathbf{R}} H_x(M)=n-2$ for any $x\in M$. $\square$

**Remark 1.** In the following example we show that in (2) of Proposition 2.1 the converse is not true, that is, for a CR submanifold of CR dimension $\frac{n-2}{2}$ the function $\lambda$ does not always vanish.

**Example 2.1.** Let $M$ be an $n(=2m)$-dimensional submanifold of a complex Euclidean space $\mathbf{C}^{m+1}$ defined by

$$Re\, z^{m+1}=Im\, z^m, \qquad Im\, z^{m+1}=0,$$

that is, using the real coordinate system $(x^1,y^1,\dots,x^{m+1},y^{m+1})$, $M$ is defined by

$$\left(x^1,y^1,\dots,x^{m-1},y^{m-1},x^m,y^m,y^m,0\right).$$

Then $M$ is a CR submanifold of CR dimension $\frac{n-2}{2}$ and for the orthonormal vectors

$$\xi_1=\frac{\partial}{\partial y^{m+1}}, \qquad \xi_2=\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial y^m}-\frac{\partial}{\partial x^{m+1}}\right),$$

normal to $M$ we compute $\lambda=\langle J\xi_1,\xi_2\rangle=\frac{1}{\sqrt{2}}$.

Let $\overline{\nabla}$ be the covariant differentiation with respect to the Hermitian metric $\bar g$ of $\overline{M}$. Then the Gauss and Weingarten formulae are the following

$$\overline{\nabla}_X\imath Y=\imath\nabla_XY+h(X,Y)=\imath\nabla_XY+\sum_{a=1}^{2}h^a(X,Y)\xi_a, \tag{2.12}$$

$$\overline{\nabla}_X\xi_a=-\imath A_aX+\sum_{b=1}^{2}s_{ab}(X)\xi_b, \tag{2.13}$$

where $h(X,Y)$ is the second fundamental form, $A_a$ the shape operator with respect to the normal $\xi_a$ and $s_{ab}$ the third fundamental form. If we put $s=s_{12}$, then $s_{21}=-s$ and relation (2.13) reduces to

$$\overline{\nabla}_X\xi_1=-\imath A_1X+s(X)\xi_2, \qquad \overline{\nabla}_X\xi_2=-\imath A_2X-s(X)\xi_1. \tag{2.14}$$

Using $\bar g(\imath Y,\xi_a)=0$, (2.12) and (2.13), we compute $h^a(X,Y)=g(A_aX,Y)$ and therefore

$$h(X,Y)=\sum_{a=1}^{2}g(A_aX,Y)\xi_a. \tag{2.15}$$

In what follows we assume that the ambient manifold $\overline{M}$ is a Kähler manifold. Then, since $\overline{\nabla}J=0$, applying $\overline{\nabla}$ to $J\imath Y$, using (2.1), (2.2), (2.12), (2.13) and comparing the tangential and normal components of the obtained relations, we obtain

$$(\nabla_X F)Y = \sum_{a=1}^{2}\left\{u^a(Y)A_aX - g(A_aX, Y)U_a\right\},$$

$$\left(\nabla_X u^a\right)(Y) = -g(A_aX, FY) + \sum_{b=1}^{2}\left\{g(A_bX, Y)\lambda_{ba} - u^b(Y)s_{ba}(X)\right\}.$$

Now, applying $\overline{\nabla}$ to $J\xi_a$, using (2.2), (2.13), (2.1), (2.12) and comparing the tangential and normal components of the obtained relations, we get

$$\nabla_X U_a = FA_aX + \sum_{b=1}^{2}\left\{s_{ab}(X)U_b - \lambda_{ab}A_bX\right\}, \tag{2.16}$$

$$X\lambda_{ab} = g(A_bU_a - A_aU_b, X) - \sum_{c=1}^{2}\left\{\lambda_{ac}s_{cb}(X) - \lambda_{cb}s_{ac}(X)\right\}, \tag{2.17}$$

that is,

$$\nabla_X U_1 = FA_1X - \lambda A_2X + s(X)U_2, \qquad \nabla_X U_2 = FA_2X + \lambda A_1X - s(X)U_1, \tag{2.18}$$

$$X\lambda = g(A_2U_1 - A_1U_2, X), \tag{2.19}$$

where we used the fact that $\lambda_{ab}$ and $s_{ab}$ are both skew-symmetric with respect to $a$ and $b$.

Now we assume that $M$ satisfies the condition

$$h(FX, Y) + h(X, FY) = 0, \quad \text{for all } X, Y \in T(M). \tag{2.20}$$

Using (2.15) it follows that the condition (2.20) is equivalent to

$$A_aF = FA_a, \quad a = 1, 2, \tag{2.21}$$

that is, the linear map $F$ commutes with both shape operators, $A_1$ and $A_2$.

We begin our investigation with the case when the submanifold $M$ is a complex hypersurface, i.e. when the tangent space $T_x(M)$ and the normal space $T^{\perp}(M)$ are $J$-invariant. Consequently, we can choose the orthonormal vectors $\xi_1$, $\xi_2$ which are normal to $M$ in such a way that $\xi_2 = J\xi_1$. Using (2.14) we conclude $\overline{\nabla}_X\xi_2 = J\overline{\nabla}_X\xi_1 = -J\imath A_1X + s(X)J\xi_2 = -\imath FA_1X - s(X)\xi_1$ and therefore $A_2 = FA_1$.

Moreover, if a complex hypersurface $M$ satisfies the condition (2.21), it follows $A_2^2 = FA_1FA_1 = F^2A_1^2 = -A_1^2$. Since $A_1$ and $A_2$ are both symmetric, the last equation shows that $A_1 = A_2 = 0$, namely, we have proved

**Theorem 2.1.** *If a complex hypersurface $M^n$ of a Kähler manifold $\overline{M}^{n+2}$ satisfies the condition (2.20), then $M^n$ is a totally geodesic submanifold.*

Now, we consider the following open submanifold of $M$ defined by

$$M_0 = \left\{x \in M \;\middle|\; \lambda(x)\left(\lambda^2(x) - 1\right) \neq 0\right\}. \tag{2.22}$$

**Lemma 2.1.** *Let $M_0$ be an opened submanifold of $M^n \subset \overline{M}^{n+2}$ defined by (2.22). If the condition (2.20) is satisfied, then $U_1$ and $U_2$ are eigenvectors of both $A_1$ and $A_2$ in $M_0$. More precisely,*

$$A_aU_b = \alpha_aU_b, \tag{2.23}$$

*that is,*

$$A_aU_1 = \alpha_aU_1, \qquad A_aU_2 = \alpha_aU_2, \quad a = 1, 2. \tag{2.24}$$

**Proof.** From (2.7) and (2.21), it follows $FA_aU_b = -\sum_{c=1}^{2}\lambda_{bc}A_aU_c$ and $F^2A_aU_b = \sum_{c,d=1}^{2}\lambda_{bc}\lambda_{cd}A_aU_d$. Therefore, using (2.6), we obtain

$$-A_aU_b + \sum_{c=1}^{2}u^c(A_aU_b)U_c = \sum_{c,d=1}^{2}\lambda_{bc}\lambda_{cd}A_aU_d. \tag{2.25}$$

Putting $b = 1$ in (2.25), we obtain

$$\left(1 - \lambda^2\right)A_aU_1 = g(A_aU_1, U_1)U_1 + g(A_aU_2, U_1)U_2. \tag{2.26}$$

In entirely the same way, putting $b=2$ in (2.25), we obtain

$$\left(1-\lambda^2\right)A_aU_2=g(A_aU_1,U_2)U_1+g(A_aU_2,U_2)U_2. \tag{2.27}$$

Hence, in $M_0$, we have

$$A_aU_1=\alpha_{11}^aU_1+\alpha_{12}^aU_2,\qquad A_aU_2=\alpha_{12}^aU_1+\alpha_{22}^aU_2,\quad a=1,2, \tag{2.28}$$

since $A_1$ and $A_2$ are symmetric operators. Applying $F$ to Eqs. (2.28) and using (2.8), we find

$$FA_aU_1=\lambda\left(-\alpha_{11}^aU_2+\alpha_{12}^aU_1\right).$$

On the other hand, from (2.21) and (2.8), it follows

$$FA_aU_1=A_aFU_1=-\lambda A_aU_2=-\lambda\left(\alpha_{12}^aU_1+\alpha_{22}^aU_2\right).$$

Comparing the above two equations, we obtain $\alpha_{11}^a=\alpha_{22}^a$ and $\alpha_{12}^a=0$, since $\lambda\neq 0$ in $M_0$. Hence, using (2.28), we obtain (2.23). □

## 3. Certain real submanifolds of codimension 2 of a complex space form

From now on, we assume that the ambient manifold $\overline{M}$ is a complex space form. Then the curvature tensor $\overline{R}$ of $\overline{M}$ is given by

$$\overline{R}(\overline{X},\overline{Y})\overline{Z}=k\left\{\bar g(\overline{Y},\overline{Z})\overline{X}-\bar g(\overline{X},\overline{Z})\overline{Y}+\bar g(J\overline{Y},\overline{Z})J\overline{X}-\bar g(J\overline{X},\overline{Z})J\overline{Y}-2\bar g(J\overline{X},\overline{Y})J\overline{Z}\right\},$$

for some constant $k$ and the Codazzi equation becomes

$$(\nabla_XA_a)Y-(\nabla_YA_a)X=k\left\{u^a(X)FY-u^a(Y)FX-2g(FX,Y)U_a\right\}+\sum_{b=1}^{2}\left\{s_{ab}(X)A_bY-s_{ab}(Y)A_bX\right\}. \tag{3.1}$$

Differentiating (2.23) covariantly and using (2.16) and (2.23), we obtain

$$(\nabla_XA_a)U_b+FA_aA_bX-\sum_{c=1}^{2}\lambda_{bc}A_aA_cX=(X\alpha_a)U_b+\alpha_a\left(FA_bX-\sum_{c=1}^{2}\lambda_{bc}A_cX\right). \tag{3.2}$$

Since $\nabla_XA_a$ is a symmetric operator, it follows

$$\begin{aligned}&g\left((\nabla_XA_a)Y-(\nabla_YA_a)X,U_b\right)+g(FA_aA_bX,Y)-g(FA_aA_bY,X)-\sum_{c=1}^{2}\left\{\lambda_{bc}g(A_aA_cX,Y)-\lambda_{bc}g(A_aA_cY,X)\right\}\\&\quad=(X\alpha_a)u^b(Y)-(Y\alpha_a)u^b(X)+\alpha_a\left\{g(FA_bX,Y)-g(FA_bY,X)\right\}.\end{aligned} \tag{3.3}$$

Further, using (2.21) and (2.7), the Codazzi equation (3.1) and relation (3.3) imply

$$\begin{aligned}&k\left\{u^a(X)\sum_{c=1}^{2}\lambda_{bc}u^c(Y)-u^a(Y)\sum_{c=1}^{2}\lambda_{bc}u^c(X)-2\left(1-\lambda^2\right)g(FX,Y)\delta_{ab}\right\}+\sum_{c=1}^{2}\alpha_c\left\{s_{ac}(X)u^b(Y)-s_{ac}(Y)u^b(X)\right\}\\&\qquad+g\left(F(A_aA_b+A_bA_a)X,Y\right)-\sum_{c=1}^{2}\lambda_{bc}g\left((A_aA_c-A_cA_a)X,Y\right)\\&\quad=(X\alpha_a)u^b(Y)-(Y\alpha_a)u^b(X)+2\alpha_ag(FA_bX,Y).\end{aligned} \tag{3.4}$$

**Lemma 3.1.** *Let $M_0$ be an open submanifold of $M^n\subset\overline{M}^{n+2}$ defined by (2.22). Then the eigenvalues $\alpha_1$ and $\alpha_2$, defined by (2.23), satisfy the following equations*:

$$X\alpha_1-\alpha_2s(X)=-3k\lambda u^2(X),\qquad X\alpha_2+\alpha_1s(X)=3k\lambda u^1(X). \tag{3.5}$$

**Proof.** Regarding relation (3.4), there are several cases to consider: $a=1$, $b=2$; $a=2$, $b=1$; $a=b=1$ and $a=b=2$. Therefore, we compute respectively:

$$\big\{X\alpha_1-\alpha_2 s(X)\big\}u^2(Y)-\big\{Y\alpha_1-\alpha_2 s(Y)\big\}u^2(X)=-2\alpha_1 g(FA_2X,Y)+g\big(F(A_1A_2+A_2A_1)X,Y\big), \tag{3.6}$$

$$\big\{X\alpha_2+\alpha_1 s(X)\big\}u^1(Y)-\big\{Y\alpha_2+\alpha_1 s(Y)\big\}u^1(X)=-2\alpha_2 g(FA_1X,Y)+g\big(F(A_2A_1+A_1A_2)X,Y\big), \tag{3.7}$$

$$\begin{aligned}&k\big\{\lambda u^1(X)u^2(Y)-\lambda u^1(Y)u^2(X)-2\big(1-\lambda^2\big)g(FX,Y)\big\}+2g\big(FA_1^2X,Y\big)-\lambda g\big((A_1A_2-A_2A_1)X,Y\big)\\&\quad=\big\{X\alpha_1-\alpha_2 s(X)\big\}u^1(Y)-\big\{Y\alpha_1-\alpha_2 s(Y)\big\}u^1(X)+2\alpha_1 g(FA_1X,Y),\end{aligned} \tag{3.8}$$

$$\begin{aligned}&k\big\{\lambda u^1(X)u^2(Y)-\lambda u^1(Y)u^2(X)-2\big(1-\lambda^2\big)g(FX,Y)\big\}+2g\big(FA_2^2X,Y\big)-\lambda g\big((A_1A_2-A_2A_1)X,Y\big)\\&\quad=\big\{X\alpha_2+\alpha_1 s(X)\big\}u^2(Y)-\big\{Y\alpha_2+\alpha_1 s(Y)\big\}u^2(X)+2\alpha_2 g(FA_2X,Y).\end{aligned} \tag{3.9}$$

Putting $X=U_1$ in (3.6), we obtain $\{U_1\alpha_1-\alpha_2 s(U_1)\}u^2(Y)=0$ and consequently

$$U_1\alpha_1-\alpha_2 s(U_1)=0. \tag{3.10}$$

In the same way, putting $X=U_2$ in (3.7), we get

$$U_2\alpha_2+\alpha_1 s(U_2)=0. \tag{3.11}$$

Then, putting $X=U_1$ in (3.8), $X=U_2$ in (3.9), and using (3.10) and (3.11), we obtain (3.5). □

Further, substituting (3.5) into (3.6) and (3.7) we get

$$F(A_1A_2+A_2A_1)X=2\alpha_1 FA_2X, \qquad F(A_1A_2+A_2A_1)X=2\alpha_2 FA_1X,$$

i.e. $\alpha_1 FA_2X=\alpha_2 FA_1X$. Consequently, relation (2.6) implies

$$\alpha_1 A_2X=\alpha_2 A_1X. \tag{3.12}$$

**Lemma 3.2.** *Under the above assumptions, if the complex space form $\overline{M}$ is not a complex Euclidean space, then $M_0=\emptyset$.*

**Proof.** Differentiating (3.12) covariantly and using (3.5) it follows

$$\big\{3k\lambda u^1(X)-\alpha_1 s(X)\big\}A_1Y+\alpha_2(\nabla_X A_1)Y=\big\{-3k\lambda u^2(X)+\alpha_2 s(X)\big\}A_2Y+\alpha_1(\nabla_X A_2)Y.$$

Interchanging $X$ and $Y$ and subtracting the obtained equations, we get

$$\begin{aligned}&\big\{3k\lambda u^1(X)-\alpha_1 s(X)\big\}A_1Y-\big\{3k\lambda u^1(Y)-\alpha_1 s(Y)\big\}A_1X+\alpha_2\big\{(\nabla_X A_1)Y-(\nabla_Y A_1)X\big\}\\&\quad=\big\{-3k\lambda u^2(X)+\alpha_2 s(X)\big\}A_2Y-\big\{-3k\lambda u^2(Y)+\alpha_2 s(Y)\big\}A_2X+\alpha_1\big\{(\nabla_X A_2)Y-(\nabla_Y A_2)X\big\}.\end{aligned}$$

Substituting (3.1) into the above equation, we compute

$$\begin{aligned}&3k\lambda\big\{u^1(X)A_1Y-u^1(Y)A_1X\big\}+\alpha_2 k\big\{u^1(X)FY-u^1(Y)FX-2g(FX,Y)U_1\big\}\\&\quad=-3k\lambda\big\{u^2(X)A_2Y-u^2(Y)A_2X\big\}+\alpha_1 k\big\{u^2(X)FY-u^2(Y)FX-2g(FX,Y)U_2\big\}.\end{aligned} \tag{3.13}$$

Putting $X=U_1$ in (3.13) and making use of (2.8), (2.9) and (2.24), we obtain

$$\big(1-\lambda^2\big)k\{3\lambda A_1Y+\alpha_2 FY\}-k\lambda\big\{3\alpha_1 u^1(Y)+\alpha_2 u^2(Y)\big\}U_1+k\lambda\big\{-3\alpha_1 u^2(Y)+\alpha_2 u^1(Y)\big\}U_2=0. \tag{3.14}$$

Since $\dim M\geqslant 4$, we can choose the eigenvector $Y$ of $A_1$ which is orthogonal to both $U_1$ and $U_2$. As $FY$ is orthogonal to $U_1$ and $U_2$, it follows that $A_1Y$, $FY$, $U_1$, $U_2$ are linearly independent and hence (3.14) implies

$$\alpha_2 k\big(1-\lambda^2\big)=0. \tag{3.15}$$

Next putting $X=U_2$ in (3.13), we compute

$$\big(1-\lambda^2\big)k\{3\lambda A_2Y-\alpha_1 FY\}-k\lambda\big\{3\alpha_2 u^1(Y)-\alpha_1 u^2(Y)\big\}U_1-k\lambda\big\{3\alpha_2 u^2(Y)+\alpha_1 u^1(Y)\big\}U_2=0.$$

Here we take the eigenvector $Y$ of $A_2$ and proceeding in entirely in the same way as to get (3.15), we obtain

$$\alpha_1 k\big(1-\lambda^2\big)=0. \tag{3.16}$$

If $M$ is a non-Euclidean complex space form, namely $k\neq 0$, relations (3.15) and (3.16) imply $\alpha_1=\alpha_2=0$ on $M_0$, contrary to (3.5). Hence $M_0=\emptyset$. □

**Theorem 3.1.** *Let $\overline{M}$ be a non-Euclidean complex space form. If a real submanifold $M$ of codimension two satisfies the condition (2.20), then one of the following holds.*

(1) *M is a totally geodesic complex hypersurface.*
(2) *M is a CR submanifold of CR dimension* $\frac{n-2}{2}$ *with* $\lambda = 0$.

**Proof.** By Lemma 3.2, it follows $M_0 = \emptyset$ which means that $1 - \lambda^2 = 0$ or $\lambda = 0$ in $M$. Combining this with Proposition 2.1 and Theorem 2.1, the theorem follows. □

## 4. Certain real submanifolds of codimension 2 of a complex Euclidean space

In this section, we consider a real submanifold $M^n$ of codimension 2 of a complex Euclidean space $\mathbf{C}^{\frac{n+2}{2}}$, which satisfies relation (2.20). Especially, we investigate its opened submanifold $M_0$, defined by relation (2.22).

**Lemma 4.1.** *Under the above assumptions, the sum* $\alpha_1^2 + \alpha_2^2$ *is constant, where* $\alpha_1$ *and* $\alpha_2$ *are defined by* (2.23).

**Proof.** Since the ambient manifold is a complex Euclidean space, the holomorphic sectional curvature vanishes identically, that is $k = 0$ and the equations in (3.5) become

$$X\alpha_1 = \alpha_2 s(X), \qquad X\alpha_2 = -\alpha_1 s(X). \tag{4.1}$$

Therefore, $X(\alpha_1^2 + \alpha_2^2) = 2(\alpha_1 X\alpha_1 + \alpha_2 X\alpha_2) = 0$, which completes the proof. □

We continue considering first the case $\alpha_1^2 + \alpha_2^2 \neq 0$. It is clear that

$$\xi_1' = \frac{1}{\sqrt{\alpha_1^2 + \alpha_2^2}}(\alpha_1\xi_1 + \alpha_2\xi_2), \qquad \xi_2' = -\frac{1}{\sqrt{\alpha_1^2 + \alpha_2^2}}(\alpha_2\xi_1 - \alpha_1\xi_2)$$

are orthonormal normals to $M_0$ for which $J\xi_1' = -\imath U_1' + \lambda\xi_2'$, $J\xi_2' = -\imath U_2' - \lambda\xi_1'$, where

$$U_1' = \frac{1}{\sqrt{\alpha_1^2 + \alpha_2^2}}(\alpha_1 U_1 + \alpha_2 U_2), \qquad U_2' = -\frac{1}{\sqrt{\alpha_1^2 + \alpha_2^2}}(\alpha_2 U_1 - \alpha_1 U_2). \tag{4.2}$$

Also, using (2.14) and (4.1), we compute

$$\overline{\nabla}_X \xi_1' = \frac{-1}{\sqrt{\alpha_1^2 + \alpha_2^2}}\imath(\alpha_1 A_1 + \alpha_2 A_2)X, \qquad \overline{\nabla}_X \xi_2' = \frac{1}{\sqrt{\alpha_1^2 + \alpha_2^2}}\imath(\alpha_2 A_1 - \alpha_1 A_2)X,$$

that is,

$$A_1'X = \frac{1}{\sqrt{\alpha_1^2 + \alpha_2^2}}(\alpha_1 A_1 + \alpha_2 A_2)X, \qquad A_2'X = -\frac{1}{\sqrt{\alpha_1^2 + \alpha_2^2}}(\alpha_2 A_1 - \alpha_1 A_2)X, \tag{4.3}$$

and

$$s'(X) = 0, \tag{4.4}$$

which means that we have chosen the orthonormal normals $\xi_1'$ and $\xi_2'$ in such a way that the normal connection is trivial.

Using relations (2.24) and (4.3), we compute $A_1'U_a = \sqrt{\alpha_1^2 + \alpha_2^2}U_a$, $A_2'U_a = 0$. Consequently, using (4.2), we obtain $A_1'U_a' = \sqrt{\alpha_1^2 + \alpha_2^2}U_a'$, $A_2'U_a' = 0$. This shows that the corresponding eigenvalues $\alpha_1'$, $\alpha_2'$ of $A_a'$ for $U_a'$ are

$$\alpha_1' = \sqrt{\alpha_1^2 + \alpha_2^2}, \qquad \alpha_2' = 0. \tag{4.5}$$

Since in all the considerations throughout the previous sections the orthonormal normals $\xi_1$ and $\xi_2$ were arbitrary, the corresponding relations are also satisfied for the orthonormal normals $\xi_1'$ and $\xi_2'$. Hence from (3.12) and (4.5), it follows

$$A_2'X = 0. \tag{4.6}$$

Therefore, as $\alpha_1^2 + \alpha_2^2 \neq 0$, we conclude that the first normal space $N_1(X)$ of $M_0$ in $\mathbf{C}^{\frac{n+2}{2}}$ is *span*$\{\xi_1\}$. Using (4.4), we conclude that $N_1(x)$ is invariant under parallel translation with respect to the normal connection. Therefore, we can apply the codimension reduction theorem by Erbacher [2] and obtain

**Lemma 4.2.** *Under the above assumptions, there exists an $(n+1)$-dimensional totally geodesic Euclidean subspace $\mathbf{E}^{n+1}$ of $\mathbf{C}^{\frac{n+2}{2}}$ such that $M_0$ is a hypersurface of $\mathbf{E}^{n+1}$.*

According to Lemma 4.2, we can regard the submanifold $M_0$ as a hypersurface of a Euclidean space $\mathbf{E}^{n+1}$. Let us denote by $\imath_1$ the immersion of $M_0$ into $\mathbf{E}^{n+1}$ and by $\imath_2$ the totally geodesic immersion of $\mathbf{E}^{n+1}$ into $\mathbf{C}^{\frac{n+2}{2}}$. Then from the Gauss formula (2.12), it follows $\nabla'_X \imath_1 Y = \imath_1 \nabla_X Y + g(AX, Y)\xi''$, where $\xi''$ is a unit normal vector field to $M_0$ in $\mathbf{E}^{n+1}$ and $A$ is the corresponding shape operator. Thus, using the Gauss formula and $\imath = \imath_2 \circ \imath_1$, we derive

$$\overline{\nabla}_X \imath_2 \circ \imath_1 Y = \imath_2 \nabla'_X \imath_1 Y = \imath_2\big(\imath_1 \nabla_X Y + g(AX, Y)\xi''\big), \tag{4.7}$$

since $\mathbf{E}^{n+1}$ is totally geodesic in $\mathbf{C}^{\frac{n+2}{2}}$. Comparing relation (4.7) with relation (2.12) and using (4.6), it follows $\xi'_1 = \imath_2 \xi''$ and $A = A'_1$.

Using relation (3.8) it follows $F{A'_1}^2 X = \alpha'_1 F A'_1 X$ and therefore

$${A'_1}^2 X = \alpha'_1 A'_1 X. \tag{4.8}$$

We conclude from (4.8), (4.5) and Lemma 4.1 that $A'_1$ has at most two constant distinct eigenvalues: $\alpha'_1$ and 0. Thus, from the hypersurface theory of Euclidean space (see for example Theorem 11.4 [1]), we conclude that $M_0$ is one of the following: open submanifold of an $n$-dimensional hypersphere $\mathbf{S}^n$, of $n$-dimensional hyperplane $\mathbf{E}^n$, of the product manifold of an $r$-dimensional sphere and an $(n-r)$-dimensional Euclidean space $\mathbf{S}^r \times \mathbf{E}^{n-r}$. On the other hand, since $A'_1 = A$, it follows $A'_1 F = F A'_1$, which implies that if $X$ is an eigenvector of $A'_1$, then $FX$ is also an eigenvector of $A'_1$ for the corresponding eigenvalue for $X$. Therefore, the multiplicities of the eigenvalues $\alpha'_1$ and 0 are both even numbers.

Now we consider the case $\alpha_1^2 + \alpha_2^2 = 0$, that is, $\alpha_1 = \alpha_2 = 0$. Taking $k = 0$ and $\alpha_1 = \alpha_2 = 0$ in (3.4), we obtain

$$F(A_a A_b + A_b A_a)X - \sum_{c=1}^{2} \lambda_{bc}(A_a A_c - A_c A_a)X = 0. \tag{4.9}$$

Putting $a = b = 1$, $a = b = 2$ and $a = 1$, $b = 2$ in (4.9) we get, respectively,

$$2FA_1^2 X - \lambda(A_1 A_2 - A_2 A_1)X = 0, \tag{4.10}$$

$$2FA_2^2 X - \lambda(A_1 A_2 - A_2 A_1)X = 0, \tag{4.11}$$

$$(A_1 A_2 + A_2 A_1)FX = 0. \tag{4.12}$$

Using (4.10), (4.11) and (2.21), it follows $A_1^2 FX = A_2^2 FX$ and since $\alpha_1 = \alpha_2 = 0$, we conclude

$$A_1^2 X = A_2^2 X, \qquad (A_1 A_2 + A_2 A_1)X = 0. \tag{4.13}$$

Substituting the second equation of (4.13) into the first equation of (4.10) and using (2.21), we compute

$$A_1^2 FX = -\lambda A_2 A_1 X. \tag{4.14}$$

Now, let us suppose that there exists a non-zero eigenvalue $\beta$ of $A_1$ and let $X$ be the corresponding eigenvector, that is, $A_1 X = \beta X$. Then, (2.21) yields that $FX$ is also an eigenvector of $A_1$, corresponding to $\beta$. Therefore, using (4.14), we compute $\beta^2 FX = -\lambda\beta A_2 X$ and $\beta^2 A_2 FX = -\lambda\beta A_2^2 X = -\lambda\beta A_1^2 X$, that is,

$$A_2 FX = -\lambda\beta X. \tag{4.15}$$

On the other hand, from the second equation of (4.13), it follows $A_1 A_2 FX = -A_2 A_1 FX = -\beta A_2 FX$. Substituting (4.15) into the last equation, we have $2\lambda\beta X = 0$ and hence $\lambda = 0$. Then from (4.10), we conclude $A_1^2 = A_2^2 = 0$, since $\beta \neq 0$. Consequently, $A_1 = A_2 = 0$, submanifold $M_0$ is totally geodesic and all eigenvalues of $A_1$ and $A_2$ are 0, which contradicts our assumption that there exists a non-zero eigenvalue $\beta$ of $A_1$.

A slight change in the proof shows that there does not exist a non-zero eigenvalue of $A_2$. Therefore, it follows that $M_0$ is totally geodesic and $M_0$ is an open submanifold of an $n$-dimensional Euclidean space $\mathbf{E}^n$.

**Theorem 4.1.** *Let $M$ be a connected real submanifold of codimension 2 of a complex Euclidean space $\overline{M} = \mathbf{C}^{\frac{n+2}{2}}$. If $M$ satisfies the condition (2.20), then $M$ is one of the following:*

(1) *$n$-dimensional sphere $\mathbf{S}^n$,*
(2) *$n$-dimensional Euclidean space $\mathbf{E}^n$,*
(3) *product manifold of an $r$-dimensional sphere and an $(n-r)$-dimensional Euclidean space $\mathbf{S}^r \times \mathbf{E}^{n-r}$, where $r$ is an even number,*
(4) *CR submanifold of CR dimension $\frac{n-2}{2}$ with $\lambda = 0$.*

**Proof.** Let $M_1 = \{x \in M \mid \lambda(x)(1-\lambda^2(x)) = 0\}$. Then, $M = M_0 \cup M_1$, $M_0 \cap M_1 = \emptyset$. If $M_1$ is an open set, then $M = M_1$ or $M = M_0$, since $M$ is connected. When $M = M_1$, then on $M$ we have $\lambda = 0$ or $\lambda^2 = 1$. Using the first case in Proposition 2.1, we obtain (4) and using the second case, it follows that $M$ is a complex hypersurface $\mathbf{E}^n$, which is a special case of (2). When $M = M_0$, we have (1), (2), (3). If $M_1$ is not an open set, then by definition, $M_1$ is a closed set in $M$ and $\dim M_1 < n$ and $M_1$ is a subset of measure 0 in $M$. Hence, $M$ is one of (1), (2), (3), which completes the proof. □

## 5. Real submanifolds of codimension 2 of a complex space form, with $\lambda = 0$

Having in mind the facts and theorems proved in Sections 3 and 4, we proceed with the study of real submanifolds of codimension 2 of a complex space form, with $\lambda = 0$.

The following example provides a large class of real submanifolds of codimension 2 of a complex space form satisfying $\lambda = 0$, since there are many real hypersurfaces of a complex Euclidean space.

**Example 5.1.** Let $M_1'$ and $M_2'$ be complex manifolds and $J_1$ and $J_2$ the natural almost complex structure of $M_1'$ and $M_2'$ respectively. Then $\overline{M} = M_1' \times M_2'$ is a complex manifold with the almost complex structure $J = J_1 \otimes J_2$. For real hypersurfaces $M_a$ of $M_a'$, $a = 1, 2$ with unit normals $\xi_a'$ to $M_a$, the product $M = M_1 \times M_2$ is a submanifold of codimension 2 of $\overline{M}$ and $\xi_1 = (\xi_1', 0)$ and $\xi_2 = (0, \xi_2')$ are orthonormal unit normals to $M$. Then $M$ is a submanifold of codimension 2 with $\lambda = 0$. Especially, for a complex Euclidean space $\mathbf{C}^{\frac{n+2}{2}}$, the product manifold $M$ of respective real hypersurfaces $M_1$ and $M_2$ of mutually orthogonal complex subspaces $\mathbf{C}^p$ and $\mathbf{C}^q$ is a submanifold of codimension 2 with $\lambda = 0$.

If we take $\lambda = 0$ in (2.26) and (2.27), we obtain

$$A_a U_b = \sum_{c=1}^{2} \alpha^a_{bc} U_c, \quad a, b = 1, 2. \tag{5.1}$$

Since $A_a$ is symmetric, it follows $\alpha^a_{bc} = g(A_a U_b, U_c) = \alpha^a_{cb}$. Differentiating relation (5.1) covariantly, we compute

$$(\nabla_X A_a)U_b + A_a \nabla_X U_b = \sum_{c=1}^{2} \{X\alpha^a_{bc} U_c + \alpha^a_{bc} \nabla_X U_c\}. \tag{5.2}$$

Substituting (2.18) into (5.2) and using (2.26), (2.27), (2.21) and $\lambda = 0$, we obtain

$$(\nabla_X A_a)U_b + F A_a A_b X + \sum_{d,e=1}^{2} s_{bd}(X)\alpha^a_{de} U_e = \sum_{d=1}^{2} \left\{ \left( X\alpha^a_{bd} + \sum_{c=1}^{2} \alpha^a_{bc} s_{cd}(X) \right) U_d \right\} + \sum_{c=1}^{2} \alpha^a_{bc} F A_c X. \tag{5.3}$$

Since $\nabla_X A_a$ is symmetric, we have $g((\nabla_X A_a)Y, U_b) = g((\nabla_X A_a)U_b, Y)$. Therefore, using (5.3), we compute

$$\begin{aligned} & g\big((\nabla_X A_a)Y - (\nabla_Y A_a)X, U_b\big) + g(F A_a A_b X, Y) - g(F A_a A_b Y, X) + \sum_{d,e=1}^{2} \{s_{bd}(X)\alpha^a_{de} u^e(Y) - s_{bd}(Y)\alpha^a_{de} u^e(X)\} \\ &= \sum_{d=1}^{2} \left\{ \left( X\alpha^a_{bd} + \sum_{c=1}^{2} \alpha^a_{bc} s_{cd}(X) \right) u^d(Y) - \left( Y\alpha^a_{bd} + \sum_{c=1}^{2} \alpha^a_{bc} s_{cd}(Y) \right) u^d(X) \right\} \\ &\quad + \sum_{d=1}^{2} \{\alpha^a_{bd} g(F A_d X, Y) - \alpha^a_{bd} g(F A_d Y, X)\}. \end{aligned}$$

Using the Codazzi equation (3.1) we have

$$\begin{aligned} & -2k g(FX, Y)\delta_{ab} + g\big(F(A_a A_b + A_b A_a)X, Y\big) - 2\sum_{c=1}^{2} \alpha^a_{bc} g(F A_c X, Y) \\ &= \sum_{d=1}^{2} \left[ X\alpha^a_{bd} + \sum_{c=1}^{2} \{\alpha^a_{bc} s_{cd}(X) - \alpha^a_{cd} s_{bc}(X) - \alpha^c_{bd} s_{ac}(X)\} \right] u^d(Y) \\ &\quad - \sum_{d=1}^{2} \left[ Y\alpha^a_{bd} + \sum_{c=1}^{2} \{\alpha^a_{bc} s_{cd}(Y) - \alpha^a_{cd} s_{bc}(Y) - \alpha^c_{bd} s_{ac}(Y)\} \right] u^d(X). \end{aligned} \tag{5.4}$$

If in (5.4) we put $Y = U_e$, then, since $\lambda = 0$, relation (2.8) implies

$$X\alpha^a_{be} + \sum_{c=1}^{2}\{\alpha^a_{bc}s_{ce}(X) - \alpha^a_{ce}s_{bc}(X) - \alpha^c_{be}s_{ac}(X)\}$$
$$= \sum_{d=1}^{2}\left[U_e\alpha^a_{bd} + \sum_{c=1}^{2}\{\alpha^a_{bc}s_{cd}(U_e) - \alpha^a_{cd}s_{bc}(U_e) - \alpha^c_{bd}s_{ac}(U_e)\}\right]u^d(X). \tag{5.5}$$

Substituting (5.5) into (5.4), we obtain

$$-2kg(FX,Y)\delta_{ab} + g\big((A_aA_b + A_bA_a)FX,Y\big) - 2\sum_{c=1}^{2}\alpha^a_{bc}g(A_cFX,Y) = \sum_{e,d=1}^{2}\gamma^a_{ebd}u^d(X)u^e(Y), \tag{5.6}$$

where

$$\gamma^a_{ebd} = \beta^a_{ebd} - \beta^a_{dbe},$$
$$\beta^a_{ebd} = U_e\alpha^a_{bd} + \sum_{c=1}^{2}\{\alpha^a_{bc}s_{cd}(U_e) - \alpha^a_{cd}s_{bc}(U_e) - \alpha^c_{bd}s_{ac}(U_e)\}.$$

Replacing $Y$ by $U_f$ in (5.6) and using (2.21), we obtain

$$\sum_{d,e=1}^{2}\gamma^a_{ebd}u^d(X)\delta^e_f = \sum_{d=1}^{2}\gamma^a_{fbd}u^d(X) = 0. \tag{5.7}$$

Substituting (5.7) into (5.6), we get

$$-2kg(FX,Y)\delta_{ab} + g\big((A_aA_b + A_bA_a)FX,Y\big) - 2\sum_{c=1}^{2}\alpha^a_{bc}g(FA_cX,Y) = 0. \tag{5.8}$$

Taking $a = b$ and $a \neq b$ in (5.8), we compute

$$-kFX + A_a^2FX - \sum_{c=1}^{2}\alpha^a_{ac}A_cFX = 0, \tag{5.9}$$

$$(A_aA_b + A_bA_a)FX - 2\sum_{c=1}^{2}\alpha^a_{bc}A_cFX = 0, \quad a \neq b. \tag{5.10}$$

**Lemma 5.1.** *Let $\overline{M}$ be a complex space form. If a real submanifold $M$ of $\overline{M}$ of codimension 2, with $\lambda = 0$, satisfies the condition (2.20), then relations (5.9) and (5.10) hold.*

## 6. The case when $M$ is a hypersurface of a totally umbilical hypersurface $M' \subset \mathbf{C}^{\frac{n+2}{2}}$

In this section, we consider real submanifolds $M^n$ of $\overline{M} = \mathbf{C}^{\frac{n+2}{2}}$ with $\lambda = 0$, such that there exists a totally umbilical hypersurface $M'$ of $\mathbf{C}^{\frac{n+2}{2}}$ such that $M \subset M'$.

Let us denote by $\xi'_1$ the unit normal vector field of the immersion $\imath_1 : M \to M'$ and by $\xi'_2$ the unit normal vector field of the immersion $\imath_2 : M' \to \mathbf{C}^{\frac{n+2}{2}}$. Consequently, the immersion $\imath : M \to \mathbf{C}^{\frac{n+2}{2}}$ is $\imath = \imath_2 \circ \imath_1$. Since $M'$ is totally umbilical, the shape operator $A'$ of $M'$ satisfies $A' = cI$, where $I$ is the identity map and $c$ is constant, since the ambient manifold is a Euclidean space. Then, using the Weingarten formula (2.13), we have for $X \in T(M)$,

$$\overline{\nabla}_X\xi'_2 = -\imath_2A'\imath_1X = -\imath_2c\imath_1X = -\imath cX. \tag{6.1}$$

Choosing the orthonormals to $M$ in $\mathbf{C}^{\frac{n+2}{2}}$ in such a way that $\xi_1 = \imath_2\xi'_1$ and $\xi_2 = \xi'_2$, we obtain

$$\overline{\nabla}_X\xi_1 = \overline{\nabla}_X\imath_2\xi'_1 = \imath_2\nabla'_X\xi'_1 + h'\big(\imath_1X,\xi'_1\big) = -\imath_2\circ\imath_1AX + cg'(\imath_1X,\xi_1)\xi'_2 = -\imath AX, \tag{6.2}$$

where $A$ is the shape operator of $M$ in $M'$ and $h'$ and $g'$ are respectively the second fundamental form and the induced Riemannian metric of $M' \subset \mathbf{C}^{\frac{n+2}{2}}$. Comparing (6.1) and (6.2) with (2.14), we obtain that $A = A_1$ and $s = 0$. Since we discuss the case $\lambda = 0$, using (2.19), we compute $A_2U_1 = A_1U_2$. Therefore, having in mind the notation from (2.28), it follows $\alpha^1_{21} = \alpha^1_{12} = c$, $\alpha^1_{22} = 0$ and

$$A_1U_1 = \alpha^1_{11}U_1 + cU_2, \qquad A_1U_2 = cU_1. \tag{6.3}$$

Since $A_2 = cI$, relation (5.9) reduces to

$$A_1^2 FX - \alpha^1_{11}A_1FX - c^2FX = 0,$$

for $a = 1$. In the sequel we use the notation $\alpha^1_{11} = \alpha$. Further, using (6.3), we compute

$$A_1^2 U_a - \alpha A_1U_a - c^2U_a = 0, \quad a = 1, 2.$$

Thus we proved that

$$A^2X - \alpha AX - c^2X = 0 \tag{6.4}$$

holds for any $X \in T(M)$.

**Lemma 6.1.** *Let $M^n$ be a real submanifold of $\overline{M} = \mathbf{C}^{\frac{n+2}{2}}$ which satisfies the condition (2.20), with $\lambda = 0$, such that there exists a totally umbilical hypersurface $M'$ of $\mathbf{C}^{\frac{n+2}{2}}$, i.e. $A' = cI$, with $M \subset M'$. If $c \neq 0$, then the function $\alpha$ is constant.*

**Proof.** Since $s = 0$ and $\lambda = 0$, relation (5.5) becomes

$$X\alpha = \beta u^1(X), \tag{6.5}$$

where $\beta = U_1\alpha$. Then, from the first equation of (2.18) and (2.21), we obtain

$$[X, Y]\alpha = XY\alpha - YX\alpha = (X\beta)u^1(Y) - (Y\beta)u^1(X) - 2\beta g(AFX, Y) + \beta u^1\big([X, Y]\big). \tag{6.6}$$

Using again (6.5), it follows from (6.6)

$$(X\beta)u^1(Y) - (Y\beta)u^1(X) = 2\beta g(AFX, Y). \tag{6.7}$$

Since $\lambda = 0$, using (2.9) and (2.8), if we put $Y = U_1$ in (6.7), we compute $X\beta = (U_1\beta)u^1(X)$. Substituting this into (6.7), we conclude $\beta = 0$ or $AFX = 0$. However, if $AFX = 0$, using (6.4), we get $c = 0$, which is a contradiction. □

**Theorem 6.1.** *Let $M^n$ be a real submanifold of codimension two of a complex Euclidean space $\mathbf{C}^{\frac{n+2}{2}}$ with $\lambda = 0$ which satisfies the condition (2.20). If there exists a totally umbilical hypersurface $M'$ of $\mathbf{C}^{\frac{n+2}{2}}$, i.e. $A' = cI$, $c \neq 0$, such that $M \subset M'$, then $M$ is a product of two odd-dimensional spheres.*

**Proof.** Since the shape operator $A$ satisfies relation (6.4) for a constant $\alpha$, we can apply Lemma 1.1 in [4] (cited as Theorem 13.2 in [1]) and obtain $\nabla A = 0$. Hence, by theorem of Ryan [5], we obtain that $M$ is a product of two spheres.

On the other hand, Lemma 6.1 implies that $M$ has exactly two constant principal curvatures $k_1$ and $k_2$. It is not possible that $A = A_1$ has only one principal curvature $k$, because, using (6.3), we compute $cU_1 = kU_2$, which is impossible since $U_1$ and $U_2$ are mutually orthogonal. Moreover, these principal curvatures satisfy

$$k_1 + k_2 = \alpha, \qquad k_1k_2 = -c^2. \tag{6.8}$$

For $V_1 = k_1U_1 + cU_2$, $V_2 = cU_1 - k_1U_2$, using (6.8), it is easily verified that $AV_1 = k_1V_1$, $AV_2 = k_2V_2$. For such an $X \in T(M)$ that $AX = k_aX$, $(a = 1, 2)$, using (6.4), it follows $AFX = k_aFX$ $(a = 1, 2)$ respectively. This shows that the distributions defined by the eigenspaces corresponding to $k_1$ and $k_2$ are both odd-dimensional. Since the spheres $S_1$ and $S_2$ are the integral submanifolds of these distributions (p. 85 in [1]), they are both odd-dimensional, which completes the proof. □

Now we consider the case $c = 0$. This means that $M'$ is a totally geodesic hypersurface of $\mathbf{C}^{\frac{n+2}{2}}$, that is, there exists a hyperplane $\mathbf{E}^{n+1}$ such that $M \subset \mathbf{E}^{n+1} \subset \mathbf{C}^{\frac{n+2}{2}}$ and in this case the shape operator $A$ satisfies

$$A^2X - \alpha AX = 0. \tag{6.9}$$

Here, if $\alpha = 0$, $M$ is a totally geodesic hypersurface of $\mathbf{E}^{n+1}$ and $M$ is a Euclidean space $\mathbf{E}^n$.

If $\alpha \neq 0$, relation (6.9) implies that $M$ has exactly 2 distinct principal curvatures: $\alpha$ and 0. Let

$$M_\alpha = \{x \in M \mid \alpha(x) \neq 0\},$$
$$T_\alpha(x) = \{X_x \in T_x(M_\alpha) \mid A_x X_x = \alpha X_x\},$$
$$T_0(x) = \{X_x \in T_x(M_0) \mid A_x X_x = 0\},$$

namely, $M_\alpha$ is an open submanifold of $M$, $T_\alpha(x)$ and $T_0(x)$ make distributions $T_\alpha$ and $T_0$ of $M_\alpha$, respectively.

Further, for $X, Y \in T_\alpha$, using the Codazzi equation for a hypersurface of a Euclidean space, we have

$$\begin{aligned} A[X,Y] &= A\nabla_X Y - A\nabla_Y X = \nabla_X(AY) - (\nabla_X A)Y - \nabla_Y(AX) + (\nabla_Y A)X \\ &= (X\alpha)Y + \alpha\nabla_X Y - (Y\alpha)YX - \alpha\nabla_Y X = (X\alpha)Y - (Y\alpha)X + \alpha[X,Y], \end{aligned}$$

that is,

$$(A - \alpha I)[X,Y] = (X\alpha)Y - (Y\alpha)X. \tag{6.10}$$

Since $(A - \alpha I)[X,Y] = (A - \alpha I)([X,Y]_\alpha + [X,Y]_0) = -\alpha[X,Y]_0$, the left-hand side of (6.10) belongs to $T_0$ and the right-hand side belongs to $T_\alpha$. This shows that $\alpha$ is constant on $M_\alpha$ and $A[X,Y] = \alpha[X,Y]$.

Since $\alpha$ is differentiable, $\alpha$ is constant on $M$. From (6.3), it follows $U_2 \in T_0(x)$ which shows that $M$ cannot be a totally umbilical hypersurface of $\mathbf{E}^{n+1}$. Thus, if $\alpha \neq 0$, then $A$ has exactly two distinct constant eigenvalues and, by standard argument, we know that $M$ is a product of $m$-dimensional sphere and an $(n-m)$-dimensional Euclidean space. Discussion similar to that in the proof of Theorem 6.1 shows that the multiplicity of $\alpha$ is the odd number. If $\alpha = 0$, then $M$ is a totally geodesic hypersurface. Thus we have proved

**Theorem 6.2.** *Let $M$ be a real submanifold of codimension two of a complex Euclidean space $\mathbf{C}^{\frac{n+2}{2}}$ with $\lambda = 0$ which satisfies the condition (2.20). If there exists a totally geodesic hypersurface $M'$ of $\mathbf{C}^{\frac{n+2}{2}}$ such that $M \subset M'$, then $M$ is one of the following*:

(1) *$n$-dimensional hyperplane $\mathbf{E}^n$*,
(2) *product manifold of an odd-dimensional sphere and a Euclidean space*: $\mathbf{S}^{2p+1} \times \mathbf{E}^{n-2p-1}$.

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