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Sufficient and necessary conditions of stochastic permanence and extinction for stochastic logistic populations under regime switching ☆

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ABSTRACT

In this paper, we prove that a stochastic logistic population under regime switching controlled by a Markov chain is either stochastically permanent or extinctive, and we obtain the sufficient and necessary conditions for stochastic permanence and extinction under some assumptions. In the case of stochastic permanence we estimate the limit of the average in time of the sample path of the solution by two constants related to the stationary probability distribution of the Markov chain and the parameters of the subsystems of the population model. Finally, we illustrate our conclusions through two examples.

1. Introduction

A famous logistic population model is described by the ordinary differential equation (ODE)

$$\dot{N}(t) = N(t)\big(a - bN(t)\big), \tag{1.1}$$

where a is the rate of growth, a/b is the carrying capacity, and both a and b are positive constants. It is well known that the population survives indefinitely and there is a stable and globally attractive equilibrium point if there is no influence of environmental noise (see e.g. [1]). However if environmental noise is taken into account, the system will change significantly.

First of all, let us consider one type of environmental noise, namely white noise. Recently many authors have discussed population systems subject to white noise (see [2–12]). Recall that the parameter a in (1.1) represents the intrinsic growth rate of the population. In practice we usually estimate it by an average value plus an error which follows a normal distribution. If we still use a to denote the average growth rate, but incorporate white noise, then the intrinsic growth rate becomes

$$a \to a + \alpha \dot{B}(t),$$

where $\dot{B}(t)$ is white noise and α is a positive number representing the intensity of the noise. As a result, (1.1) becomes a stochastic differential equation (SDE)

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$$dN(t) = N(t)\big[\big(a - bN(t)\big)\,dt + \alpha\,dB(t)\big], \tag{1.2}$$

where $B(t)$ is the 1-dimensional standard Brownian motion with $B(0) = 0$. In [6], the authors considered a more complicated case corresponding to (1.2), namely that the coefficients of (1.2) are all periodic functions with period T. They obtained the stochastic permanence of (1.2) and global attractivity of one positive solution $N^p(t)$ satisfying $E[1/N^p(t)] = E[1/N^p(t+T)]$.

However, the assumption that all of the parameters of the stochastic differential equation are T-period periodic functions is not very reasonable since it implies regularity which is inconsistent with the random perturbation. As we know, there are various types of environmental noise. Let us now take a further step by considering another type of environmental noise, namely color noise, say telegraph noise (see e.g. [13,14]). In this context, telegraph noise can be described as a random switching between two or more environmental regimes, which differ in terms of factors such as nutrition or rainfall [15,16]. The switching is memoryless and the waiting time for the next switch has an exponential distribution. We can hence model the regime switching by a finite-state Markov chain. Assume that there are n regimes and the system obeys

$$dN(t) = N(t)\big[\big(a(1) - b(1)N(t)\big)\,dt + \alpha(1)\,dB(t)\big], \tag{1.3}$$

when it is in regime 1, while it obeys another stochastic logistic model

$$dN(t) = N(t)\big[\big(a(2) - b(2)N(t)\big)\,dt + \alpha(2)\,dB(t)\big], \tag{1.4}$$

in regime 2 and so on. Therefore, the system obeys

$$dN(t) = N(t)\big[\big(a(i) - b(i)N(t)\big)\,dt + \alpha(i)\,dB(t)\big], \tag{1.5}$$

in regime i $(1 \leqslant i \leqslant n)$. The switching between these n regimes is governed by a Markovian chain $r(t)$ on the state space $S = \{1, 2, \ldots, n\}$. The population system under regime switching can therefore be described by the following stochastic model

$$dN(t) = N(t)\big[\big(a\big(r(t)\big) - b\big(r(t)\big)N(t)\big)\,dt + \alpha\big(r(t)\big)\,dB(t)\big]. \tag{1.6}$$

This system is operated as follows: If $r(0) = i_0$, the system obeys Eq. (1.5) with $i = i_0$ until time τ_1 when the Markov chain jumps to i_1 from i_0; the system will then obey Eq. (1.5) with $i = i_1$ from τ_1 until τ_2 when the Markov chain jumps to i_2 from i_1. The system will continue to switch as long as the Markov chain jumps. In other words, Eq. (1.6) can be regarded as Eqs. (1.5) switching from one to another according to the law of the Markov chain. The different Eqs. (1.5) $(1 \leqslant i \leqslant n)$ are therefore referred to as the subsystems of Eq. (1.6).

Recently, Takeuchi et al. [13] investigated a 2-dimensional autonomous predator–prey Lotka–Volterra system with regime switching and revealed a very interesting and surprising result: If two equilibrium states of the subsystems are different, all positive trajectories of this system always exit from any compact set of R^2_+ with probability 1; on the other hand, if the two equilibrium states coincide, then the trajectory either leaves any compact set of R^2_+ or converges to the equilibrium state. In practice, two equilibrium states are usually different, in which case Takeuchi et al. [13] showed that the stochastic population system is neither permanent nor dissipative (see e.g. [17]). This is an important result as it reveals the significant effect of environmental noise on the population system: both its subsystems develop periodically but switching between them makes them become neither permanent nor dissipative. Therefore, these factors motivate us to consider the logistic population system subject to both white noise and color noise, described by (SDE)

$$dN(t) = N(t)\big[\big(a\big(r(t)\big) - b\big(r(t)\big)N(t)\big)\,dt + \alpha\big(r(t)\big)\,dB(t)\big], \tag{1.7}$$

where for each $i \in S$, $a(i)$, $b(i)$ and $\alpha(i)$ are all nonnegative constants. Our aim is to reveal how the environmental noise affects the population system.

In this paper, in order to understand better the dynamic properties of SDE (1.7), in Section 2 we will give the nature of its solution and show that the solution starting from anywhere in R^+ will remain in R^+ with probability 1. In the study of population systems, permanence and extinction are two important and interesting properties, respectively meaning that the population system will survive or die out in the future. One of our main aims is to investigate these two properties and their relationship. In Sections 3 and 4, we show that SDE (1.7) is either stochastically permanent or extinctive under some assumptions, and, moreover, that it is stochastically permanent if and only if a constant related to the stationary probability distribution of the Markov chain is positive. If SDE (1.7) is stochastically permanent, we estimate in Section 5 the limit of the average in time of the sample path of its solution by two constants related to the stationary distribution and the parameters of the population subsystems. Finally, in Section 6 we illustrate our main results through two examples.

2. The nature of global solutions

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B(t)$, $t \geqslant 0$, be a scalar standard Brownian motion defined on this probability space. We also denote by R^+ the open interval $(0, \infty)$, and denote by

$\bar{R}^+$ the interval $[0,\infty)$. Let $r(t)$ be a right-continuous Markov chain on the probability space, taking values in a finite state space $S=\{1,2,\dots,n\}$, with the generator $\Gamma=(\gamma_{uv})_{n\times n}$ given by

$$P\{r(t+\delta)=v \mid r(t)=u\}=\begin{cases}\gamma_{uv}\delta+o(\delta), & \text{if } u\neq v,\\ 1+\gamma_{uv}\delta+o(\delta), & \text{if } u=v,\end{cases}$$

where $\delta>0$. Here γ_{uv} is the transition rate from u to v and $\gamma_{uv}\geqslant 0$ if $u\neq v$, while

$$\gamma_{uu}=-\sum_{v\neq u}\gamma_{uv}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is well known that almost every sample path of $r(\cdot)$ is a right continuous step function with a finite number of jumps in any finite subinterval of $\bar{R}^+$. As a standing hypothesis we assume in this paper that the Markov chain $r(t)$ is *irreducible*. This is a very reasonable assumption, as it means that the system can switch from any regime to any other regime. This is equivalent to the condition that for any $u,v\in S$, one can find finite numbers $i_1,i_2,\dots,i_k\in S$ such that $\gamma_{u,i_1}\gamma_{i_1,i_2}\cdots\gamma_{i_k,v}>0$. Note that Γ always has an eigenvalue 0. The algebraic interpretation of irreducibility is that $\operatorname{rank}(\Gamma)=n-1$. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi=(\pi_1,\pi_2,\dots,\pi_n)\in R^{1\times n}$ which can be determined by solving the following linear equation

$$\pi\Gamma=0 \tag{2.1}$$

subject to

$$\sum_{i=1}^{n}\pi_i=1 \quad\text{and}\quad \pi_i>0,\quad \forall i\in S.$$

For convenience and simplicity in the following discussion, define

$$\hat{f}=\min_{i\in S}f(i),\qquad \check{f}=\max_{i\in S}f(i),$$

where $\{f(i)\}_{i\in S}$ is a constant vector. In this paper, we impose the following assumptions:

Assumption 1. For each $i\in S$, $b(i)>0$.

Assumption 2. For some $u\in S$, $\gamma_{iu}>0$ $(\forall i\neq u)$.

Assumption 3. $\sum_{i=1}^{n}\pi_i[a(i)-\frac{1}{2}\alpha^2(i)]>0$.

Assumption 4. For each $i\in S$, $a(i)-\frac{1}{2}\alpha^2(i)>0$.

As the state $N(t)$ of SDE (1.7) is the size of the species in the system at time t, it should be nonnegative. We prove not only the global existence but also the precise nature of the solution to SDE (1.7), giving a positive $N(0)$.

Theorem 2.1. *There exists a unique continuous positive solution $N(t)$ to SDE* (1.7) *for any initial value $N(0)=N_0>0$, which is global and represented by*

$$N(t)=\frac{\exp\{\int_0^t[a(r(s))-\frac{1}{2}\alpha^2(r(s))]\,ds+\alpha(r(s))\,dB(s)\}}{\frac{1}{N_0}+\int_0^t b(r(s))\exp\{\int_0^s[a(r(u))-\frac{1}{2}\alpha^2(r(u))]\,du+\alpha(r(u))\,dB(u)\}\,ds}. \tag{2.2}$$

Proof. Since the coefficients of the equation are local Lipschitz continuous for any initial value $N_0>0$, there is a unique local solution $N(t)$ on $t\in[0,\tau_e)$, where τ_e is the explosion time (see [14]).

To show this solution is global, we will derive the nature of the solution. Let

$$\begin{aligned}U(t):=\exp&\left\{-\int_0^t\left[a\bigl(r(s)\bigr)-\frac{1}{2}\alpha^2\bigl(r(s)\bigr)\right]ds+\alpha\bigl(r(s)\bigr)\,dB(s)\right\}\\&\times\left[\frac{1}{N_0}+\int_0^t b\bigl(r(s)\bigr)\exp\left\{\int_0^s\left[a\bigl(r(u)\bigr)-\frac{1}{2}\alpha^2\bigl(r(u)\bigr)\right]du+\alpha\bigl(r(u)\bigr)\,dB(u)\right\}ds\right].\end{aligned} \tag{2.3}$$

Then by the generalized Itô formula (see [19]), $U(t)$ satisfies the equation

$$dU(t) = U(t)\big[\big(\alpha^2\big(r(t)\big) - a\big(r(t)\big)\big)\,dt - \alpha\big(r(t)\big)\,dB(t)\big] + b\big(r(t)\big)\,dt. \tag{2.4}$$

Let

$$N(t) := \frac{1}{U(t)},$$

so $N(t) > 0$ and $N(t)$ is continuous and global on $t \in [0, \infty)$. By the Itô formula

$$\begin{aligned} dN(t) &= -\frac{1}{x^2(t)}\,dx(t) + \frac{1}{x^3(t)}\big(dx(t)\big)^2 \\ &= -N(t)\big[\big(\alpha^2\big(r(t)\big) - a\big(r(t)\big)\big)\,dt - \alpha\big(r(t)\big)\,dB(t)\big] - b\big(r(t)\big)N^2(t)\,dt + \alpha^2\big(r(t)\big)N(t)\,dt \\ &= N(t)\big[\big(a\big(r(t)\big) - b\big(r(t)\big)N(t)\big)\,dt + \alpha\big(r(t)\big)\,dB(t)\big]. \end{aligned}$$

Thus $N(t)$ defined by (2.2) is a continuous positive solution of SDE (1.7) and is global on $t \in [0, \infty)$ (i.e. $\tau_e = \infty$). This completes the proof of Theorem 2.1. □

3. Stochastic permanence

Theorem 2.1 shows that the solution of SDE (1.7) with a positive initial value will remain positive. This nice property provides us with a great opportunity to discuss in more detail how the solution varies in R^+. In the study of population systems permanence is one of the most important and interesting characteristics, meaning that the population system will survive in the future. In this section, we firstly give the definition of the stochastic permanence and the stochastically ultimate boundedness of SDE (1.7), and then give some sufficient conditions which guarantee that SDE (1.7) is stochastically permanent.

Definition 3.1. SDE (1.7) is said to be stochastically permanent if for any $\varepsilon \in (0, 1)$, there exist positive constants $\delta = \delta(\epsilon)$, $\chi = \chi(\epsilon)$ such that

$$\liminf_{t\to+\infty} P\big\{N(t) \leqslant \chi\big\} \geqslant 1 - \epsilon, \qquad \liminf_{t\to+\infty} P\big\{N(t) \geqslant \delta\big\} \geqslant 1 - \epsilon,$$

where $N(t)$ is the solution of SDE (1.7) with any positive initial value.

Definition 3.2. The solutions of SDE (1.7) are called stochastically ultimately bounded, if for any $\epsilon \in (0, 1)$, there is a positive constant $\chi (= \chi(\epsilon))$, such that the solution of SDE (1.7) with any positive initial value has the property that

$$\limsup_{t\to+\infty} P\big\{N(t) > \chi\big\} < \epsilon.$$

It is obvious that if a stochastic equation is stochastically permanent, its solutions must be stochastically ultimately bounded. So we will begin with the following lemma and make use of it to obtain the stochastically ultimate boundedness of SDE (1.7).

Lemma 3.1. *Under Assumption* 1*, for an arbitrary given positive constant* p*, the solution* $N(t)$ *of SDE* (1.7) *with any given positive initial value has the property that*

$$\limsup_{t\to\infty} E\big(N(t)|^p\big) \leqslant K(p), \tag{3.1}$$

where

$$K(p) := \begin{cases} (\frac{\check{a}}{\hat{b}})^p, & \text{for } 0 < p < 1; \\ [\frac{\check{a} + \frac{1}{2}(p-1)\check{\alpha}^2}{\hat{b}}]^p, & \text{for } p \geqslant 1. \end{cases} \tag{3.2}$$

Proof. By the generalized Itô formula, we have

$$\begin{aligned} dN^p(t) &= pN^{p-1}(t)\,dN(t) + \frac{1}{2}p(p-1)N^{p-2}(t)\big(dN(t)\big)^2 \\ &= pN^p(t)\big[\big(a\big(r(t)\big) - b\big(r(t)\big)N(t)\big)\,dt + \alpha\big(r(t)\big)\,dB(t)\big] + \frac{1}{2}p(p-1)N^p(t)\alpha^2\big(r(t)\big)\,dt. \end{aligned}$$

Integrating it from 0 to t and taking expectations of both sides, we obtain that

$$E\big(N^p(t)\big) - E\big(N^p(0)\big) = \int_0^t pE\big[N^p(s)\big(a\big(r(s)\big) - b\big(r(s)\big)N(s)\big)\big]\,ds + \int_0^t \frac{1}{2}p(p-1)E\big[\alpha^2\big(r(s)\big)N^p(s)\big]\,ds.$$

Then we have

$$\frac{dE(N^p(t))}{dt} = pE\big[N^p(t)\big(a\big(r(t)\big) - b\big(r(t)\big)N(t)\big)\big] + \frac{1}{2}p(p-1)E\big[\alpha^2\big(r(t)\big)N^p(t)\big]. \tag{3.3}$$

If $0 < p < 1$, we obtain

$$\begin{aligned}\frac{dE(N^p(t))}{dt} &\leqslant p\breve{a}E\big(N^p(t)\big) - p\hat{b}E\big(N^{p+1}(t)\big)\\ &\leqslant p\breve{a}E\big(N^p(t)\big) - p\hat{b}\big[E\big(N^p(t)\big)\big]^{\frac{p+1}{p}}\\ &\leqslant pE\big(N^p(t)\big)\big\{\breve{a} - \hat{b}\big[E\big(N^p(t)\big)\big]^{\frac{1}{p}}\big\},\end{aligned} \tag{3.4}$$

while if $p \geqslant 1$, we obtain

$$\begin{aligned}\frac{dE(N^p(t))}{dt} &\leqslant p\breve{a}E\big(N^p(t)\big) - p\hat{b}E\big(N^{p+1}(t)\big) + \frac{1}{2}p(p-1)\breve{\alpha}^2E\big(N^p(t)\big)\\ &\leqslant p\breve{a}E\big(N^p(t)\big) - p\hat{b}\big[E\big(N^p(t)\big)\big]^{\frac{p+1}{p}} + \frac{1}{2}p(p-1)\breve{\alpha}^2E\big(N^p(t)\big)\\ &\leqslant pE\big(N^p(t)\big)\left\{\left[\breve{a} + \frac{1}{2}(p-1)\breve{\alpha}^2\right] - \hat{b}\big[E\big(N^p(t)\big)\big]^{\frac{1}{p}}\right\}.\end{aligned} \tag{3.5}$$

Therefore, letting $z(t) = E(N^p(t))$, we have

$$\frac{dz(t)}{dt} \leqslant \begin{cases} pz(t)[\breve{a} - \hat{b}z^{\frac{1}{p}}(t)], & \text{for } 0 < p < 1;\\ pz(t)[\breve{a} + \frac{1}{2}(p-1)\breve{\alpha}^2 - \hat{b}z^{\frac{1}{p}}(t)], & \text{for } p \geqslant 1.\end{cases} \tag{3.6}$$

Notice that if $0 < p < 1$ the solution of equation

$$\frac{d\bar{z}(t)}{dt} = p\bar{z}(t)\big[\breve{a} - \hat{b}\bar{z}^{\frac{1}{p}}(t)\big]$$

obeys

$$\bar{z}(t) \to \left(\frac{\breve{a}}{\hat{b}}\right)^p \quad \text{as } t \to \infty.$$

Also, if $p \geqslant 1$ the solution of equation

$$\frac{d\tilde{z}(t)}{dt} = p\tilde{z}(t)\left[\breve{a} + \frac{1}{2}(p-1)\breve{\alpha}^2 - \hat{b}\tilde{z}^{\frac{1}{p}}(t)\right],$$

as $t \to \infty$ is such that

$$\tilde{z}(t) \to \left[\frac{\breve{a} + \frac{1}{2}(p-1)\breve{\alpha}^2}{\hat{b}}\right]^p.$$

Thus by the comparison argument we get

$$\limsup_{t\to\infty} z(t) \leqslant \begin{cases} (\frac{\breve{a}}{\hat{b}})^p, & \text{for } 0 < p < 1;\\ [\frac{\breve{a}+\frac{1}{2}(p-1)\breve{\alpha}^2}{\hat{b}}]^p, & \text{for } p \geqslant 1.\end{cases}$$

By the definitions of $z(t)$, we obtain the assertion (3.1). □

Remark 3.1. From (3.1) of Lemma 3.1, there is a $T > 0$, such that

$$E\big(N^p(t)\big) \leqslant 2K(p) \quad \text{for all } t \geqslant T.$$

In addition, $E(N^p(t))$ is continuous, and there is a $\tilde{K}(p, N_0) > 0$ such that

$$E\big(N^p(t)\big) \leqslant \tilde{K}(p, N_0) \quad \text{for } t \in [0, T].$$

Let

$$L(p) = \max\big\{2K(p), \tilde{K}(p, N_0)\big\},$$

then we have

$$E\big(N^p(t)\big) \leqslant L(p, N_0) \quad \text{for all } t \in [0, \infty).$$

This means the pth moment of any positive solution of SDE (1.7) is bounded.

Theorem 3.1. *Solutions of Eq.* (1.7) *are stochastically ultimately bounded under Assumption* 1.

Proof. This can be easily verified by Chebyshev's inequality and Lemma 3.1. □

Based on the above result, we will prove the other equality in the definition of stochastic permanence. For convenience, define

$$\beta(i) = a(i) - \frac{1}{2}\alpha^2(i). \tag{3.7}$$

Under Assumption 3, we know

$$\sum_{i=1}^{n} \pi_i \beta(i) > 0.$$

Moreover, let G be a vector or matrix. By $G \gg 0$ we mean all elements of G are positive. We also adopt here the traditional notation by letting

$$Z^{n\times n} = \big\{A = (a_{ij})_{n\times n}\colon a_{ij} \leqslant 0,\ i \neq j\big\}.$$

We shall also need two classical results.

Lemma 3.2. *(See Mao [19, Lemma 5.3].) If $A = (a_{ij}) \in Z^{n\times n}$ has all of its row sums positive, that is*

$$\sum_{j=1}^{n} a_{ij} > 0 \quad \textit{for all } 1 \leqslant i \leqslant n,$$

then $\det A > 0$.

Lemma 3.3. *(See Mao [19, Theorem 2.10].) If $A \in Z^{n\times n}$, then the following statements are equivalent*:

(1) *A is a nonsingular M-matrix.*
(2) *All of the principal minors of A are positive*; *that is*

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0 \quad \textit{for every } k = 1, 2, \ldots, n.$$

(3) *A is semi-positive*; *that is, there exists $x \gg 0$ in R^n such that $Ax \gg 0$.*

The proof of stochastic permanence is rather technical, so we first present several useful lemmas.

Lemma 3.4. *Assumptions* 2 *and* 3 *imply that there exists a constant $\theta > 0$ such that the matrix*

$$A(\theta) = diag\big(\xi_1(\theta), \xi_2(\theta), \ldots, \xi_n(\theta)\big) - \Gamma \tag{3.8}$$

is a nonsingular M-matrix, where

$$\xi_i(\theta) = \theta\beta(i) - \theta^2 \frac{1}{2}\alpha^2(i), \quad \forall i \in S.$$

Proof. It is known that a determinant will not change its value if we switch the ith row with the jth row and then switch the ith column with the jth column. It is also known that given a nonsingular M-matrix, if we switch the ith row with the jth row and then switch the ith column with the jth column, then the new matrix is still a nonsingular M-matrix. We may therefore assume $u = n$ without loss of generality, that is

$$\gamma_{in} > 0, \quad \forall 1 \leqslant i \leqslant n-1,$$

instead of Assumption 2. It is easy to see that

$$\begin{aligned}\det A(\theta) &= \begin{vmatrix} \xi_1(\theta) & -\gamma_{12} & \cdots & -\gamma_{1n} \\ \xi_2(\theta) & \xi_2(\theta)-\gamma_{22} & \cdots & -\gamma_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \xi_{n-1}(\theta) & -\gamma_{n-1,2} & \cdots & -\gamma_{n-1,n} \\ \xi_n(\theta) & -\gamma_{n2} & \cdots & \xi_n(\theta)-\gamma_{nn} \end{vmatrix} \\ &= \sum_{i=1}^n \xi_i(\theta) M_i(\theta), \end{aligned} \tag{3.9}$$

where $M_i(\theta)$ is the corresponding minor of $\xi_i(\theta)$ in the first column. More precisely,

$$M_1(\theta) = (-1)^{1+1} \begin{vmatrix} \xi_2(\theta)-\gamma_{22} & \cdots & -\gamma_{2n} \\ \vdots & \cdots & \vdots \\ -\gamma_{n-1,2} & \cdots & -\gamma_{n-1,n} \\ -\gamma_{n2} & \cdots & \xi_n(\theta)-\gamma_{nn} \end{vmatrix},$$

$$\vdots$$

$$M_n(\theta) = (-1)^{n+1} \begin{vmatrix} -\gamma_{12} & \cdots & -\gamma_{1n} \\ \xi_2(\theta)-\gamma_{22} & \cdots & -\gamma_{2n} \\ \vdots & \cdots & \vdots \\ -\gamma_{n-1,2} & \cdots & -\gamma_{n-1,n} \end{vmatrix}.$$

Noting that

$$\xi_i(0) = 0 \quad \text{and} \quad \frac{d}{d\theta}\xi_i(0) = \beta(i),$$

we have

$$\frac{d}{d\theta}\det A(0) = \sum_{i=1}^n \beta(i) M_i(0),$$

which means that

$$\frac{d}{d\theta}\det A(0) = \begin{vmatrix} \beta_1 & -\gamma_{12} & \cdots & -\gamma_{1n} \\ \beta_2 & -\gamma_{22} & \cdots & -\gamma_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_n & -\gamma_{n2} & \cdots & -\gamma_{nn} \end{vmatrix}, \tag{3.10}$$

where we write $\beta(i) = \beta_i$. By Appendix A in reference [20], under Assumption 2, Assumption 3 is equivalent to

$$\begin{vmatrix} \beta_1 & -\gamma_{12} & \cdots & -\gamma_{1n} \\ \beta_2 & -\gamma_{22} & \cdots & -\gamma_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \beta_n & -\gamma_{n2} & \cdots & -\gamma_{nn} \end{vmatrix} > 0.$$

Together with (3.10), we obtain that

$$\frac{d}{d\theta}\det A(0) > 0. \tag{3.11}$$

It is easy to see that $\det A(0) = 0$. Hence, we can find a $\theta > 0$ sufficiently small for $\det A(\theta) > 0$ and

$$\xi_i(\theta) = \theta\beta(i) - \theta^2 \frac{1}{2}\alpha_i^2 > -\gamma_{in}, \quad 1 \leqslant i \leqslant n-1. \tag{3.12}$$

For each $k = 1, 2, \dots, n-1$, consider the leading principal sub-matrix

$$A_k(\theta) := \begin{vmatrix} \xi_1(\theta) - \gamma_{11} & -\gamma_{12} & \cdots & -\gamma_{1k} \\ -\gamma_{21} & \xi_2(\theta) - \gamma_{22} & \cdots & -\gamma_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \cdots & \xi_k(\theta) - \gamma_{kk} \end{vmatrix}$$

of $A(\theta)$. Clearly $A_k(\theta) \in Z^{k\times k}$. Moreover, by (3.12), each row of this sub-matrix has the sum

$$\xi_i(\theta) - \sum_{i=1}^{k} \gamma_{ij} \geqslant \xi_i(\theta) + \gamma_{in} > 0.$$

By Lemma 3.2, $\det A_k(\theta) > 0$. In other words, we have shown that all the leading principal minors of $A(\theta)$ are positive. By Lemma 3.3, we obtain the required assertion. □

Lemma 3.5. *Assumption 4 implies that there exists a constant $\theta > 0$ such that the matrix $A(\theta)$ is a nonsingular M-matrix.*

Proof. Note that for every $i \in S$,

$$\xi_i(0) = 0 \quad \text{and} \quad \frac{d}{d\theta}\xi_i(0) = \beta(i) > 0.$$

We can then choose $\theta > 0$ so small that $\xi_i(\theta) > 0$ for all $1 \leqslant i \leqslant n$. Consequently, every row of $A(\theta)$ has a positive sum. By Lemma 3.2, we see easily that all the leading principal minors of $A(\theta)$ are positive, so $A(\theta)$ is a nonsingular M-matrix. □

Lemma 3.6. *If there exists a constant $\theta > 0$ such that $A(\theta)$ is a nonsingular M-matrix, then the solution $N(t)$ of SDE (1.7) with any positive initial value has the property that*

$$\limsup_{t\to\infty} E\left(\frac{1}{N^\theta(t)}\right) \leqslant H, \tag{3.13}$$

where H is a fixed positive constant (defined by (3.21) in the proof).

Proof. By Theorem 2.1, the solution $N(t)$ with positive initial value will remain in R^+. Define

$$U(t) = \frac{1}{N(t)} \quad \text{on } t \geqslant 0. \tag{3.14}$$

We derive from (2.4) that

$$dU(t) = U(t)\big[-a\big(r(t)\big) + \alpha^2\big(r(t)\big) + b\big(r(t)\big)N(t)\big]\,dt - \alpha\big(r(t)\big)U(t)\,dB(t). \tag{3.15}$$

By Lemma 3.3, for given θ, there is a vector $\vec{q} = (q_1, q_2, \dots, q_n)^T \gg 0$ such that

$$\vec{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)^T := A(\theta)\vec{q} \gg 0,$$

namely,

$$q_i\left(\theta\beta(i) - \theta^2\frac{\alpha^2(i)}{2}\right) - \sum_{j=1}^{n} \gamma_{ij} q_j > 0 \quad \text{for all } 1 \leqslant i \leqslant n. \tag{3.16}$$

Define the function $V : R_+ \times S$ by

$$V(U, i) = q_i(1+U)^\theta. \tag{3.17}$$

Applying the generalized Itô formula, we have

$$EV\big(U(t), r(t)\big) = V\big(U(0), r(0)\big) + E\int_0^t LV\big(U(s), r(s)\big)\,ds,$$

where

$$\begin{aligned}
LV(U,i) &= q_i\theta(1+U)^{\theta-1}U\big[-a(i)+\alpha^2(i)+b(i)N\big]+q_i\frac{\theta(\theta-1)}{2}(1+U)^{\theta-2}\alpha^2(i)U^2+\sum_{j=1}^n\gamma_{ij}q_j(1+U)^\theta \\
&= (1+U)^{\theta-2}\left\{q_i\theta(1+U)U\big[-a(i)+\alpha^2(i)+b(i)N\big]+q_i\frac{\theta(\theta-1)}{2}\alpha_i^2U^2+\sum_{j=1}^n\gamma_{ij}q_j(1+U)^2\right\} \\
&\leqslant (1+U)^{\theta-2}\left\{-U^2\left[q_i\left(\theta\beta(i)-\theta^2\frac{\alpha^2(i)}{2}\right)-\sum_{j=1}^n\gamma_{ij}q_j\right]\right. \\
&\quad\left.+U\left[q_i\theta\big(b(i)+\alpha^2(i)\big)+2\sum_{j=1}^n\gamma_{ij}q_j\right]+\left[q_i\theta b(i)+\sum_{j=1}^n\gamma_{ij}q_j\right]\right\}. \qquad (3.18)
\end{aligned}$$

Now, choose a constant $\kappa > 0$ sufficiently small such that it satisfies

$$\vec{\lambda}-\kappa\vec{q}\gg 0,$$

i.e.

$$q_i\left(\theta\beta(i)-\theta^2\frac{\alpha^2(i)}{2}\right)-\sum_{j=1}^n\gamma_{ij}q_j-\kappa q_i>0 \quad \text{for all } 1\leqslant i\leqslant n. \qquad (3.19)$$

Then, by the generalized Itô formula again,

$$E\big[e^{\kappa t}V\big(U(t),r(t)\big)\big]=V\big(U(0),r(0)\big)+E\int_0^t L\big[e^{\kappa s}V\big(U(s),r(s)\big)\big]\,ds,$$

where

$$\begin{aligned}
L\big[e^{\kappa t}V(U,i)\big] &= \kappa e^{\kappa t}V(U,i)+e^{\kappa t}LV(U,i) \\
&\leqslant e^{\kappa t}(1+U)^{\theta-2}\left\{\kappa q_i(1+U)^2-U^2\left[q_i\left(\theta\beta(i)-\theta^2\frac{\alpha^2(i)}{2}\right)-\sum_{j=1}^n\gamma_{ij}q_j\right]\right. \\
&\quad\left.+U\left[q_i\theta\big(b(i)+\alpha^2(i)\big)+2\sum_{j=1}^n\gamma_{ij}q_j\right]+\left[q_i\theta b(i)+\sum_{j=1}^n\gamma_{ij}q_j\right]\right\} \\
&\leqslant e^{\kappa t}(1+U)^{\theta-2}\left\{-U^2\left[q_i\left(\theta\beta(i)-\theta^2\frac{\alpha^2(i)}{2}\right)-\sum_{j=1}^n\gamma_{ij}q_j-\kappa q_i\right]\right. \\
&\quad\left.+U\left[q_i\theta\big(b(i)+\alpha^2(i)\big)+2\sum_{j=1}^n\gamma_{ij}q_j+2\kappa q_i\right]+\left[q_i\theta b(i)+\sum_{j=1}^n\gamma_{ij}q_j+\kappa q_i\right]\right\} \\
&\leqslant \hat{q}\kappa He^{\kappa t}, \qquad (3.20)
\end{aligned}$$

$$\begin{aligned}
H=\frac{1}{\hat{q}\kappa}\max_{1\leqslant i\leqslant n}&\left\{\sup_{x\in R^+}\left\{(1+x)^{\theta-2}\left\{-x^2\left[q_i\left(\theta\beta(i)-\theta^2\frac{\alpha^2(i)}{2}\right)-\sum_{j=1}^n\gamma_{ij}q_j-\kappa q_i\right]\right.\right.\right. \\
&\left.\left.\left.+x\left[q_i\theta\big(b(i)+\alpha^2(i)\big)+2\sum_{j=1}^n\gamma_{ij}q_j+2\kappa q_i\right]+\left[q_i\theta b(i)+\sum_{j=1}^n\gamma_{ij}q_j+\kappa q_i\right]\right\}\right\},1\right\}, \qquad (3.21)
\end{aligned}$$

in which we put 1 in order to make H positive. This implies

$$\hat{q}E\big[e^{\kappa t}\big(1+U(t)\big)^\theta\big]\leqslant\check{q}\left(1+\frac{1}{N_0}\right)^\theta+\hat{q}He^{\kappa t}.$$

Then

$$\limsup_{t\to\infty}E\big[U^\theta(t)\big]\leqslant\limsup_{t\to\infty}E\big[\big(1+U(t)\big)^\theta\big]\leqslant H. \qquad (3.22)$$

Recalling the definition of $U(t)$, we obtain the required assertion (3.13). $\square$

Theorem 3.2. *Under Assumptions* 1, 2 *and* 3, *SDE* (1.7) *is stochastically permanent.*

Proof. Let $N(t)$ be the solution of SDE (1.7) with any given positive initial value. By Lemmas 3.4 and 3.6, we know

$$\limsup_{t\to\infty} E\left(\frac{1}{N^\theta(t)}\right) \leqslant H.$$

Now, for any $\epsilon > 0$, let $\delta = (\frac{\epsilon}{H})^{\frac{1}{\theta}}$. Then

$$P\{|N(t)| < \delta\} = P\left\{\frac{1}{|N(t)|} > \frac{1}{\delta}\right\} \leqslant \frac{E(\frac{1}{|N(t)|^\theta})}{\frac{1}{\delta^\theta}} = \delta^\theta E\left(\frac{1}{|N(t)|^\theta}\right) = \delta^\theta E\left(\frac{1}{N^\theta(t)}\right).$$

Hence,

$$\limsup_{t\to+\infty} P\{|N(t)| < \delta\} \leqslant \delta^\theta H = \epsilon.$$

This implies

$$\liminf_{t\to+\infty} P\{|N(t)| \geqslant \delta\} \geqslant 1 - \epsilon.$$

The other part of Definition 3.1 required for Theorem 3.2 follows from Theorem 3.1. $\square$

Theorem 3.3. *Under Assumptions* 1 *and* 4, *SDE* (1.7) *is stochastically permanent.*

Corollary 3.1. *Assume for some $i \in S$, $b(i) > 0$, $a(i) > \frac{1}{2}\alpha^2(i)$. Then the subsystem*

$$dN(t) = N(t)\big[\big(a(i) - b(i)N(t)\big)\,dt + \alpha(i)\,dB(t)\big] \tag{3.23}$$

is stochastically permanent.

4. Extinction

In the previous sections we have shown that under certain conditions, the original non-autonomous equation (1.1) and the associated SDE (1.7) behave similarly in the sense that both have positive solutions which will not explode to infinity in a finite time and, in fact, will be ultimately bounded and permanent. In other words, we show that under certain conditions the noise will not spoil these nice properties. However, we will show in this section that if the noise is sufficiently large, the solution to the associated stochastic SDE (1.7) will become extinct with probability one, although the solution to the original equation (1.1) may be persistent. It is well known that if $a > 0$, $b > 0$, then the solution $N(t)$ of (1.1) is persistent, because

$$\lim_{t\to\infty} N(t) = \frac{a}{b}.$$

However, consider its associated stochastic equation

$$dN(t) = N(t)\big[\big(a - bN(t)\big)\,dt + \sigma\,dB(t)\big], \quad t \geqslant 0, \tag{4.1}$$

where $\sigma > 0$. Theorem 4.1 shows that if $\sigma^2 > 2b$, then the solution to this stochastic equation will become extinctive with probability one, namely

$$\lim_{t\to\infty} N(t) = 0 \quad \text{a.s.}$$

In other words, the following theorem reveals the important fact that environmental noise may make the population extinct.

Theorem 4.1. *The solution $N(t)$ of SDE* (1.7) *with any positive initial value has the property that*

$$\limsup_{t\to\infty} \frac{\log N(t)}{t} \leqslant \sum_{i=1}^n \pi_i\left[a(i) - \frac{1}{2}\alpha^2(i)\right] \quad a.s. \tag{4.2}$$

Particularly, if $\sum_{i=1}^n \pi_i[a(i) - \frac{1}{2}\alpha^2(i)] < 0$ holds, then

$$\lim_{t\to\infty} N(t) = 0 \quad a.s.$$

Proof. By Theorem 2.1, the solution $N(t)$ with initial value $N_0 \in R^+$ will remain in R^+ with probability one. By the generalized Itô formula, we drive from (1.7) that

$$d(\log N(t)) = \left[a(r(t)) - \frac{1}{2}\alpha^2(r(t)) - b(r(t))N(t)\right]dt + \alpha(r(t))\,dB(t). \tag{4.3}$$

Hence,

$$\begin{aligned}\log N(t) &= \log N_0 + \int_0^t \left[a(r(s)) - \frac{1}{2}\alpha^2(r(s))\right]ds - \int_0^t b(r(s))N(s)\,ds + \int_0^t \alpha(r(s))\,dB(s) \\ &\leqslant \log N_0 + \int_0^t \left[a(r(s)) - \frac{1}{2}\alpha^2(r(s))\right]ds + M(t),\end{aligned} \tag{4.4}$$

where $M(t)$ is a martingale defined by

$$M(t) = \int_0^t \alpha(r(s))\,dB(s).$$

The quadratic variation of this martingale is

$$\langle M, M\rangle_t = \int_0^t \alpha^2(r(s))\,ds \leqslant \breve{\alpha}^2 t.$$

By the strong law of large numbers for martingales (see [18,19]), we therefore have

$$\lim_{t\to\infty} \frac{M(t)}{t} = 0 \quad \text{a.s.}$$

It finally follows from (4.4), by dividing by t on both sides and then letting $t \to \infty$, that

$$\limsup_{t\to\infty} \frac{\log N(t)}{t} \leqslant \limsup_{t\to\infty} \frac{1}{t}\int_0^t \left[a(r(s)) - \frac{1}{2}\alpha^2(r(s))\right]ds = \sum_{i=1}^n \pi_i\left[a(i) - \frac{1}{2}\alpha^2(i)\right] \quad \text{a.s.,}$$

which is the required assertion (4.2). □

Corollary 4.1. *Assume for some $i \in S$, $a(i) < \frac{1}{2}\alpha^2(i)$. Then solutions of subsystem*

$$dN(t) = N(t)\left[\left(a(i) - b(i)N(t)\right)dt + \alpha(i)\,dB(t)\right] \tag{4.5}$$

tend to zero a.s.

5. Asymptotic properties

Lemma 5.1. *Under Assumption* 1, *the solution $N(t)$ of SDE* (1.7) *with any positive initial value has the property*

$$\limsup_{t\to\infty} \frac{\log(N(t))}{\log t} \leqslant 1 \quad \text{a.s.} \tag{5.1}$$

Proof. By Theorem 2.1, the solution $N(t)$ with positive initial value will remain in R^+. We know that

$$\begin{aligned}dN(t) &= N(t)\left[\left(a(r(t)) - b(r(t))N(t)\right)dt + \alpha(r(t))\,dB(t)\right] \\ &\leqslant \breve{a}N(t)\,dt + \alpha(r(t))N(t)\,dB(t).\end{aligned}$$

We can also derive from this that

$$E\Big(\sup_{t\leqslant u\leqslant t+1} N(u)\Big) \leqslant E(N(t)) + \breve{a}\int_t^{t+1} E(N(s))\,ds + E\left(\sup_{t\leqslant u\leqslant t+1} \int_t^u \alpha(r(s))N(s)\,dB(s)\right).$$

From (3.1) of Lemma 3.1, we know that

$$\limsup_{t\to\infty} E\big(N(t)\big) \leqslant K(1). \tag{5.2}$$

But, by the well-known Burkholder–Davis–Gundy inequality (see [18,19]) and the Hölder inequality, we derive that

$$\begin{aligned}
E\left(\sup_{t\leqslant u\leqslant t+1}\int_t^u \alpha\big(r(s)\big)N(s)\,dB(s)\right) &\leqslant 3E\left[\int_t^{t+1}\big(\alpha\big(r(s)\big)N(s)\big)^2\,ds\right]^{\frac{1}{2}}\\
&\leqslant E\left[9\breve{\alpha}^2\int_t^{t+1}N^2(s)\,ds\right]^{\frac{1}{2}}\\
&\leqslant E\left[\sup_{t\leqslant u\leqslant t+1}N(u)\cdot 9\breve{\alpha}^2\int_t^{t+1}N(s)\,ds\right]^{\frac{1}{2}}\\
&\leqslant E\left[\left(\frac{1}{2}\sup_{t\leqslant u\leqslant t+1}N(u)\right)^2+\left(9\breve{\alpha}^2\int_t^{t+1}N(s)\,ds\right)^2\right]^{\frac{1}{2}}\\
&\leqslant E\left[\frac{1}{2}\sup_{t\leqslant u\leqslant t+1}N(u)+9\breve{\alpha}^2\int_t^{t+1}N(s)\,ds\right]\\
&\leqslant \frac{1}{2}E\Big(\sup_{t\leqslant u\leqslant t+1}N(u)\Big)+9\breve{\alpha}^2\int_t^{t+1}E\big(N(s)\big)\,ds.
\end{aligned} \tag{5.3}$$

Therefore

$$E\Big(\sup_{t\leqslant u\leqslant t+1}N(u)\Big)\leqslant 2E\big(N(t)\big)+2\breve{a}\int_t^{t+1}E\big(N(s)\big)\,ds+18\breve{\alpha}^2\int_t^{t+1}E\big(N(s)\big)\,ds.$$

This, together with (5.2), yields

$$\limsup_{t\to\infty}E\Big(\sup_{t\leqslant u\leqslant t+1}N(u)\Big)\leqslant 2\big(1+\breve{a}+9\breve{\alpha}^2\big)K(1). \tag{5.4}$$

To prove assertion (5.1), we observe from (5.4) that there is a positive constant $\bar{K}$ such that

$$E\Big(\sup_{k\leqslant t\leqslant k+1}N(t)\Big)\leqslant \bar{K},\quad k=1,2,\ldots.$$

Let $\epsilon>0$ be arbitrary. Then, by the well-known Chebyshev inequality, we have

$$P\Big\{\sup_{k\leqslant t\leqslant k+1}N(t)>k^{1+\epsilon}\Big\}\leqslant\frac{\bar{K}}{k^{1+\epsilon}},\quad k=1,2,\ldots.$$

Applying the well-known Borel–Cantelli lemma (see e.g. [18,19]), we obtain that for almost all $\omega\in\Omega$

$$\sup_{k\leqslant t\leqslant k+1}N(t)\leqslant k^{1+\epsilon} \tag{5.5}$$

holds for all but finitely many k. Hence, there exists a $k_0(\omega)$, for almost all $\omega\in\Omega$, for which (5.5) holds whenever $k\geqslant k_0$. Consequently, for almost all $\omega\in\Omega$, if $k\geqslant k_0$ and $k\leqslant t\leqslant k+1$,

$$\frac{\log(N(t))}{\log t}\leqslant\frac{(1+\epsilon)\log k}{\log k}=1+\epsilon.$$

Therefore

$$\limsup_{t\to\infty}\frac{\log(N(t))}{\log t}\leqslant 1+\epsilon\quad\text{a.s.}$$

Letting $\epsilon\to 0$ we obtain the desired assertion (5.1). The proof is therefore complete. □

Lemma 5.2. *If there exists a constant $\theta > 0$ such that $A(\theta)$ is a nonsingular M-matrix, then the solution $N(t)$ of SDE* (1.7) *with any positive initial value has the property that*

$$\liminf_{t\to\infty} \frac{\log(N(t))}{\log t} \geqslant -\frac{1}{\theta} \quad a.s. \tag{5.6}$$

Proof. Applying the generalized Itô formula, for the fixed constant $\theta > 0$, we derive from (3.15) that

$$\begin{aligned} d\big[\big(1+U(t)\big)^{\theta}\big] &\leqslant \theta\big(1+U(t)\big)^{\theta-2}\left\{-U^2(t)\left[\beta\big(r(t)\big)-\frac{1}{2}\theta\alpha^2\big(r(t)\big)\right]+U(t)\big[b\big(r(t)\big)+\alpha^2\big(r(t)\big)\big]+b\big(r(t)\big)\right\}dt \\ &\quad -\theta\big(1+U(t)\big)^{\theta-1}U(t)\alpha\big(r(t)\big)\,dB(t) \\ &\leqslant \theta\big(1+U(t)\big)^{\theta-2}\left\{-U^2(t)\left[\hat\beta-\frac{1}{2}\theta\breve\alpha^2\right]+U(t)\big[\breve b+\breve\alpha^2\big]+\breve b\right\}dt \\ &\quad -\theta\big(1+U(t)\big)^{\theta-1}U(t)\alpha\big(r(t)\big)\,dB(t), \end{aligned} \tag{5.7}$$

where $U(t)$ is defined by (3.14). By Lemma 3.6, there exists a positive constant M such that

$$E\big[\big(1+U(t)\big)^{\theta}\big] \leqslant M \quad \text{on } t \geqslant 0. \tag{5.8}$$

Let $\delta > 0$ be sufficiently small for

$$\theta\left\{\left[\hat\beta+2\breve b+\frac{1}{2}(\theta+2)\breve\alpha^2\right]\delta+3\breve\alpha\delta^{\frac{1}{2}}\right\} < \frac{1}{2}. \tag{5.9}$$

Let $k = 1, 2, \ldots$. Then (5.7) implies that

$$\begin{aligned} &E\Big[\sup_{(k-1)\delta\leqslant t\leqslant k\delta}\big(1+U(t)\big)^{\theta}\Big] \\ &\quad \leqslant E\big[\big(1+U\big((k-1)\delta\big)\big)^{\theta}\big] \\ &\qquad + E\left(\sup_{(k-1)\delta\leqslant t\leqslant k\delta}\left|\int_{(k-1)\delta}^{t}\theta\big(1+U(s)\big)^{\theta-2}\left\{-U^2(s)\left[\hat\beta-\frac{1}{2}\theta\breve\alpha^2\right]+U(s)\big[\breve b+\breve\alpha^2\big]+\breve b\right\}ds\right|\right) \\ &\qquad + E\left(\sup_{(k-1)\delta\leqslant t\leqslant k\delta}\left|\int_{(k-1)\delta}^{t}\theta\big(1+U(s)\big)^{\theta-1}U(s)\alpha\big(r(s)\big)\,dB(s)\right|\right). \end{aligned} \tag{5.10}$$

We compute

$$\begin{aligned} &E\left(\sup_{(k-1)\delta\leqslant t\leqslant k\delta}\left|\int_{(k-1)\delta}^{t}\theta\big(1+U(s)\big)^{\theta-2}\left\{-U^2(s)\left[\hat\beta-\frac{1}{2}\theta\breve\alpha^2\right]+U(s)\big[\breve b+\breve\alpha^2\big]+\breve b\right\}ds\right|\right) \\ &\quad \leqslant E\left(\int_{(k-1)\delta}^{k\delta}\left|\theta\big(1+U(s)\big)^{\theta-2}\left\{-U^2(s)\left[\hat\beta-\frac{1}{2}\theta\breve\alpha^2\right]+U(s)\big[\breve b+\breve\alpha^2\big]+\breve b\right\}\right|ds\right) \\ &\quad \leqslant \theta E\left(\int_{(k-1)\delta}^{k\delta}\left\{\big(1+U(s)\big)^{\theta}\left[\hat\beta+\frac{1}{2}\theta\breve\alpha^2\right]+\big(1+U(s)\big)^{\theta-1}\big[\breve b+\breve\alpha^2\big]+\big(1+U(s)\big)^{\theta-2}\breve b\right\}ds\right) \\ &\quad \leqslant \theta E\left(\int_{(k-1)\delta}^{k\delta}\left[\hat\beta+\frac{1}{2}\theta\breve\alpha^2+\breve b+\breve\alpha^2+\breve b\right]\big(1+U(s)\big)^{\theta}ds\right) \\ &\quad \leqslant \theta\left[\hat\beta+2\breve b+\frac{1}{2}(\theta+2)\breve\alpha^2\right]E\left(\int_{(k-1)\delta}^{k\delta}\sup_{(k-1)\delta\leqslant s\leqslant k\delta}\big(1+U(s)\big)^{\theta}ds\right) \\ &\quad \leqslant \theta\left[\hat\beta+2\breve b+\frac{1}{2}(\theta+2)\breve\alpha^2\right]\delta E\Big(\sup_{(k-1)\delta\leqslant t\leqslant k\delta}\big(1+U(t)\big)^{\theta}\Big). \end{aligned} \tag{5.11}$$

By the Burkholder–Davis–Gundy inequality, we derive that

$$E\left(\sup_{(k-1)\delta\leqslant t\leqslant k\delta}\left|\int_{(k-1)\delta}^{t}\theta\big(1+U(s)\big)^{\theta-1}U(s)\alpha\big(r(s)\big)\,dB(s)\right|\right)\leqslant 3E\left(\int_{(k-1)\delta}^{k\delta}\theta^2\big(1+U(s)\big)^{2(\theta-1)}U^2(s)\alpha^2\big(r(s)\big)\,ds\right)^{\frac{1}{2}}$$
$$\leqslant 3\theta\breve{\alpha}E\left(\int_{(k-1)\delta}^{k\delta}\big(1+U(s)\big)^{2\theta}\,ds\right)^{\frac{1}{2}}$$
$$\leqslant 3\theta\breve{\alpha}\delta^{\frac{1}{2}}E\Big(\sup_{(k-1)\delta\leqslant t\leqslant k\delta}\big(1+U(t)\big)^{2\theta}\Big)^{\frac{1}{2}}$$
$$\leqslant 3\theta\breve{\alpha}\delta^{\frac{1}{2}}E\Big(\sup_{(k-1)\delta\leqslant t\leqslant k\delta}\big(1+U(t)\big)^{\theta}\Big).$$

Substituting this and (5.11) into (5.10) gives

$$E\Big[\sup_{(k-1)\delta\leqslant t\leqslant k\delta}\big(1+U(t)\big)^{\theta}\Big]\leqslant E\big[\big(1+U\big((k-1)\delta\big)\big)^{\theta}\big]$$
$$+\theta\left\{\left[\hat{\beta}+2\breve{b}+\frac{1}{2}(\theta+2)\breve{\alpha}^2\right]\delta+3\breve{\alpha}\delta^{\frac{1}{2}}\right\}E\Big(\sup_{(k-1)\delta\leqslant t\leqslant k\delta}\big(1+U(t)\big)^{\theta}\Big). \tag{5.12}$$

Making use of (5.8) and (5.9) we obtain that

$$E\Big[\sup_{(k-1)\delta\leqslant t\leqslant k\delta}\big(1+U(t)\big)^{\theta}\Big]\leqslant 2M. \tag{5.13}$$

Let $\epsilon>0$ be arbitrary. Then, by the Chebyshev inequality, we have

$$P\Big\{\omega\colon \sup_{(k-1)\delta\leqslant t\leqslant k\delta}\big(1+U(t)\big)^{\theta}>(k\delta)^{1+\epsilon}\Big\}\leqslant\frac{2M}{(k\delta)^{1+\epsilon}},\quad k=1,2,\ldots.$$

Applying the Borel–Cantelli lemma, we obtain that for almost all $\omega\in\Omega$

$$\sup_{(k-1)\delta\leqslant t\leqslant k\delta}\big(1+U(t)\big)^{\theta}\leqslant(k\delta)^{1+\epsilon} \tag{5.14}$$

holds for all but finitely many k. Hence, there exists an integer $k_0(\omega)>1/\delta+2$, for almost all $\omega\in\Omega$, for which (5.14) holds whenever $k\geqslant k_0$. Consequently, for almost all $\omega\in\Omega$, if $k\geqslant k_0$ and $(k-1)\delta\leqslant t\leqslant k\delta$,

$$\frac{\log(1+U(t))^{\theta}}{\log t}\leqslant\frac{(1+\epsilon)\log(k\delta)}{\log((k-1)\delta)}=1+\epsilon.$$

Therefore

$$\limsup_{t\to\infty}\frac{\log(1+U(t))^{\theta}}{\log t}\leqslant 1+\epsilon\quad\text{a.s.}$$

Letting $\epsilon\to 0$, we obtain the desired assertion

$$\limsup_{t\to\infty}\frac{\log(1+U(t))^{\theta}}{\log t}\leqslant 1\quad\text{a.s.}$$

Recalling the definition of $U(t)$, this yields

$$\limsup_{t\to\infty}\frac{\log(\frac{1}{N^{\theta}(t)})}{\log t}\leqslant 1\quad\text{a.s.},$$

which further implies

$$\liminf_{t\to\infty}\frac{\log(N(t))}{\log t}\geqslant-\frac{1}{\theta}\quad\text{a.s.}$$

This is our required assertion (5.6). $\square$

Theorem 5.1. *Under Assumptions* 1, 2 *and* 3, *the solution* $N(t)$ *of SDE* (1.7) *with any positive initial value obeys*

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t N(s)\,ds\leqslant\frac{1}{\hat{b}}\sum_{i=1}^n\pi_i\left[a(i)-\frac{1}{2}\alpha^2(i)\right]\quad a.s. \tag{5.15}$$

and

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t N(s)\,ds\geqslant\frac{1}{\check{b}}\sum_{i=1}^n\pi_i\left[a(i)-\frac{1}{2}\alpha^2(i)\right]\quad a.s. \tag{5.16}$$

Proof. By Theorem 2.1, the solution $N(t)$ with any positive initial value will remain in R^+. From Lemmas 5.1, 3.4 and 5.2, we know that

$$\lim_{t\to+\infty}\frac{\log N(t)}{t}=0\quad \text{a.s.} \tag{5.17}$$

We derive from (4.3) that

$$\log N(t)=\log N_0+\int_0^t\left[a\big(r(s)\big)-\frac{1}{2}\alpha^2\big(r(s)\big)\right]ds-\int_0^t b\big(r(s)\big)N(s)\,ds+\int_0^t\alpha\big(r(s)\big)\,dB(s). \tag{5.18}$$

Dividing by t on both sides, then we have

$$\frac{\log N(t)}{t}=\frac{\log N_0}{t}+\frac{1}{t}\int_0^t\left[a\big(r(s)\big)-\frac{1}{2}\alpha^2\big(r(s)\big)\right]ds-\frac{1}{t}\int_0^t b\big(r(s)\big)N(s)\,ds+\frac{1}{t}\int_0^t\alpha\big(r(s)\big)\,dB(s).$$

Letting $t\to\infty$, by the strong law of large numbers for martingales and (5.17), we therefore have

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t N(s)\,ds\leqslant\frac{1}{\hat{b}}\sum_{i=1}^n\pi_i\left[a(i)-\frac{1}{2}\alpha^2(i)\right]\quad \text{a.s.}$$

and

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t N(s)\,ds\geqslant\frac{1}{\check{b}}\sum_{i=1}^n\pi_i\left[a(i)-\frac{1}{2}\alpha^2(i)\right]\quad \text{a.s.},$$

which are the required assertions (5.15) and (5.16). □

Similarly, using Lemmas 5.1, 3.5 and 5.2, we can show:

Theorem 5.2. *Under Assumptions* 1 *and* 4, *the solution* $N(t)$ *of SDE* (1.7) *with any positive initial value obeys*

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t N(s)\,ds\leqslant\frac{1}{\hat{b}}\sum_{i=1}^n\pi_i\left[a(i)-\frac{1}{2}\alpha^2(i)\right]\quad a.s. \tag{5.19}$$

and

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t N(s)\,ds\geqslant\frac{1}{\check{b}}\sum_{i=1}^n\pi_i\left[a(i)-\frac{1}{2}\alpha^2(i)\right]\quad a.s. \tag{5.20}$$

Corollary 5.1. *Assume for some* $i\in S$, $b(i)>0$, $a(i)>\frac{1}{2}\alpha^2(i)$. *Then the solution with positive initial value to subsystem*

$$dN(t)=N(t)\big[\big(a(i)-b(i)N(t)\big)\,dt+\alpha(i)\,dB(t)\big] \tag{5.21}$$

has the property that

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t N(s)\,ds=\frac{a(i)-\frac{1}{2}\alpha^2(i)}{b(i)}\quad a.s.$$

We observe that if the growth rates $b(i)$ are the same in different regimes, then the results in Theorems 5.1 and 5.2 become limits. More precisely, consider the logistic population system subject to both white noise and color noise described by

$$dN(t) = N(t)\big[\big(a\big(r(t)\big) - bN(t)\big)\,dt + \alpha\big(r(t)\big)\,dB(t)\big], \tag{5.22}$$

where for each $i \in S$, $a(i)$, $\alpha(i)$ are all nonnegative constants and $b > 0$.

Corollary 5.2. *Under Assumptions* 2 *and* 3, *the solution* $N(t)$ *of SDE* (5.22) *with any positive initial value has the property that*

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t N(s)\,ds = \frac{1}{b}\sum_{i=1}^{n} \pi_i\left[a(i) - \frac{1}{2}\alpha^2(i)\right] \quad \textit{a.s.}$$

6. Conclusions and examples

It is interesting to find that if $b(i) > 0$ and $a(i) > \frac{1}{2}\alpha^2(i)$ for some $i \in S$, then the equation

$$dN(t) = N(t)\big[\big(a(i) - b(i)N(t)\big)\,dt + \alpha(i)\,dB(t)\big] \tag{6.1}$$

is stochastically permanent. Hence Theorem 3.3 tells us that if every individual equation

$$dN(t) = N(t)\big[\big(a(i) - b(i)N(t)\big)\,dt + \alpha(i)\,dB(t)\big], \quad 1 \leqslant i \leqslant n, \tag{6.2}$$

is stochastically permanent, then as the result of Markovian switching, the overall behavior, i.e. SDE (1.7), remains stochastically permanent. On the other hand, if $a(i) < \frac{1}{2}\alpha^2(i)$ for some $i \in S$, then Eq. (6.1) is extinctive. Hence Theorem 4.1 tells us that if every individual Eq. (6.2) is extinctive, then as the result of Markovian switching, the overall behavior of SDE (1.7) remains extinctive. However, Theorems 3.2 and 4.1 provide a more interesting result that if some individual equations in (6.2) are stochastically permanent while some are extinctive, again as the result of Markovian switching, the overall behavior of SDE (1.7) may be stochastically permanent or extinctive, depending on the sign of the value $\sum_{i=1}^n \pi_i[a(i) - \frac{1}{2}\alpha^2(i)]$. In order to see this point clearly, we state the following necessary and sufficient conditions for stochastic permanence or extinction of SDE (1.7) which follow from Theorems 3.2 and 4.1.

Theorem 6.1. *Let Assumptions* 1 *and* 2 *hold and assume* $\sum_{i=1}^n \pi_i[a(i) - \frac{1}{2}\alpha^2(i)] \neq 0$. *Then the SDE* (1.7) *is either stochastically permanent or extinctive. That is, it is stochastically permanent if and only if* $\sum_{i=1}^n \pi_i[a(i) - \frac{1}{2}\alpha^2(i)] > 0$, *while it is extinctive if and only if* $\sum_{i=1}^n \pi_i[a(i) - \frac{1}{2}\alpha^2(i)] < 0$.

Making use of Theorems 5.1 and 5.2, we can also estimate the limit of the average in time of the sample path of the solution by two constants related to the stationary distribution and the parameters $a(i)$, $b(i)$, $\alpha(i)$, $i \in S$. We shall illustrate these conclusions through the following examples.

Example 6.1. To obtain more precise conditions to guarantee that SDE (1.7) is stochastically permanent or extinctive, let us assume that the Markov chain $r(t)$ is on the state space $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{pmatrix} -\gamma_{12} & \gamma_{12} \\ \gamma_{21} & -\gamma_{21} \end{pmatrix},$$

where $\gamma_{12} > 0$ and $\gamma_{21} > 0$. It is easy to see that the Markov chain has its stationary probability distribution $\pi = (\pi_1, \pi_2)$ given by

$$\pi_1 = \frac{\gamma_{21}}{\gamma_{12} + \gamma_{21}} \quad \text{and} \quad \pi_2 = \frac{\gamma_{12}}{\gamma_{12} + \gamma_{21}},$$

noting that $\sum_{i=1}^n \pi_i[a(i) - \frac{1}{2}\alpha^2(i)]$ has the form

$$\sum_{i=1}^{n} \pi_i\left[a(i) - \frac{1}{2}\alpha^2(i)\right] = \frac{\gamma_{21}}{\gamma_{12} + \gamma_{21}}\left[a(1) - \frac{1}{2}\alpha^2(1)\right] + \frac{\gamma_{12}}{\gamma_{12} + \gamma_{21}}\left[a(2) - \frac{1}{2}\alpha^2(2)\right].$$

As pointed out in Section 1, we may regard SDE (1.7) as the result of the following two equations:

$$dN(t) = N(t)\big[\big(a(1) - b(1)N(t)\big)\,dt + \alpha(1)\,dB(t)\big], \tag{6.3}$$

where $b(1) > 0$ and $a(1) - \frac{1}{2}\alpha^2(1) > 0$, and

$$dN(t) = N(t)\big[\big(a(2) - b(2)N(t)\big)\,dt + \alpha(2)\,dB(t)\big], \tag{6.4}$$

where $b(2) > 0$ and $a(2) - \frac{1}{2}\alpha^2(2) < 0$, switching from one to the other according to the movement of the Markov chain $r(t)$. We observe that Eq. (6.3) is stochastically permanent while Eq. (6.4) is extinctive. However, as the result of Markovian switching, the overall behavior of SDE (1.7) will be stochastically permanent as long as the transition rate γ_{21} from extinctive Eq. (6.4) to permanent Eq. (6.3) is greater than $\frac{\alpha^2(2)-2a(2)}{2a(1)-\alpha^2(1)}$ times the transition rate γ_{12} from permanent Eq. (6.3) to extinctive Eq. (6.4). On the other hand, as the result of Markovian switching, the overall behavior of SDE (1.7) will be extinctive as long as the transition rate γ_{21} from extinctive Eq. (6.4) to permanent Eq. (6.3) is less than $\frac{\alpha^2(2)-2a(2)}{2a(1)-\alpha^2(1)}$ times the transition rate γ_{12} from permanent Eq. (6.3) to extinctive Eq. (6.4).

Example 6.2. Consider a 3-dimensional stochastic differential equation with Markovian switching of the form

$$dN(t) = N(t)\big[\big(a\big(r(t)\big) - b\big(r(t)\big)N(t)\big)\,dt + \alpha\big(r(t)\big)\,dB(t)\big] \quad \text{on } t \geqslant 0, \tag{6.5}$$

where $r(t)$ is a right-continuous Markov chain taking values in $S = \{1, 2, 3\}$, and $r(t)$ and $B(t)$ are independent. Here

$$\begin{aligned} &a(1) = 2, \qquad b(1) = 3, \qquad \alpha(1) = 1;\\ &a(2) = 1, \qquad b(2) = 2, \qquad \alpha(2) = 2;\\ &a(3) = 4, \qquad b(3) = 1, \qquad \alpha(3) = 3. \end{aligned}$$

We compute

$$a(1) - \frac{1}{2}\alpha^2(1) = \frac{2}{3}; \qquad a(2) - \frac{1}{2}\alpha^2(2) = -1; \qquad a(3) - \frac{1}{2}\alpha^2(3) = -\frac{1}{2}.$$

Case 1. Let the generator of the Markov chain $r(t)$ be

$$\Gamma = \begin{pmatrix} -2 & 1 & 1 \\ 3 & -4 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

By solving the linear equation (2.1) we obtain the unique stationary (probability) distribution

$$\pi = (\pi_1, \pi_2, \pi_3) = \left(\frac{7}{15}, \frac{1}{5}, \frac{1}{3}\right).$$

Then

$$\sum_{i=1}^{3} \pi_i \left[a(i) - \frac{1}{2}\alpha^2(i)\right] = \frac{1}{3} > 0.$$

Therefore, by Theorems 5.1 and 6.1, Eq. (6.5) is stochastically permanent and its solution $N(t)$ with any positive initial value has the following properties:

$$\frac{1}{9} \leqslant \liminf_{t\to+\infty} \frac{1}{t}\int_0^t N(s)\,ds \leqslant \limsup_{t\to+\infty} \frac{1}{t}\int_0^t N(s)\,ds \leqslant \frac{1}{3} \quad \text{a.s.}$$

Case 2. Let the generator of the Markov chain $r(t)$ be

$$\Gamma = \begin{pmatrix} -5 & 2 & 3 \\ 1 & -1 & 0 \\ 3 & 0 & -3 \end{pmatrix}.$$

By solving the linear equation (2.1) we obtain the unique stationary distribution

$$\pi = (\pi_1, \pi_2, \pi_3) = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right).$$

Then

$$\sum_{i=1}^{3} \pi_i \left[a(i) - \frac{1}{2}\alpha^2(i)\right] = -\frac{1}{4} < 0.$$

Therefore, by Theorem 6.1, Eq. (6.5) is extinctive.

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References

[1] S. Ahmad, A.C. Lazer, Average conditions for global asymptotic stability in a nonautonomous Lotka–Volterra system, Nonlinear Anal. 40 (2000) 37–49.
[2] X. Mao, G. Marion, E. Renshaw, Environmental Brownian noise suppresses explosions in population dynamics, Stochastic Process. Appl. 97 (2002) 95–110.
[3] X. Mao, G. Marion, E. Renshaw, Asymptotic behavior of the stochastic Lotka–Volterra model, J. Math. Anal. Appl. 287 (2003) 141–156.
[4] X. Mao, Delay population dynamics and environmental noise, Stoch. Dyn. 5 (2) (2005) 149–162.
[5] D. Jiang, N. Shi, A note on nonautonomous logistic equation with random perturbation, J. Math. Anal. Appl. 303 (2005) 164–172.
[6] D. Jiang, N. Shi, X. Li, Global stability and stochastic permanence of a non-autonomous logistic equation with random perturbation, J. Math. Anal. Appl. 340 (2008) 588–597.
[7] T.C. Gard, Persistence in stochastic food web models, Bull. Math. Biol. 46 (1984) 357–370.
[8] T.C. Gard, Stability for multispecies population models in random environments, Nonlinear Anal. 10 (1986) 1411–1419.
[9] T.C. Gard, Introduction to Stochastic Differential Equations, Marcel Dekker, 1998.
[10] A. Bahar, X. Mao, Stochastic delay Lotka–Volterra model, J. Math. Anal. Appl. 292 (2004) 364–380.
[11] A. Bahar, X. Mao, Stochastic delay population dynamics, Int. J. Pure Appl. Math. 11 (2004) 377–400.
[12] S. Pang, F. Deng, X. Mao, Asymptotic property of stochastic population dynamics, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 15 (2008) 602–620.
[13] Y. Takeuchi, N.H. Du, N.T. Hieu, K. Sato, Evolution of predator–prey systems described by a Lotka–Volterra equation under random environment, J. Math. Anal. Appl. 323 (2006) 938–957.
[14] Q. Luo, X. Mao, Stochastic population dynamics under regime switching, J. Math. Anal. Appl. 334 (2007) 69–84.
[15] N.H. Du, R. Kon, K. Sato, Y. Takeuchi, Dynamical behavior of Lotka–Volterra competition systems: Non-autonomous bistable case and the effect of telegraph noise, J. Comput. Appl. Math. 170 (2004) 399–422.
[16] M. Slatkin, The dynamics of a population in a Markovian environment, Ecology 59 (1978) 249–256.
[17] H.I. Freedman, S. Ruan, Uniform persistence in functional differential equations, J. Differential Equations 115 (1995) 173–192.
[18] X. Mao, Stochastic Differential Equations and Applications, Horwood Publishing, Chichester, 1997.
[19] X. Mao, C. Yuan, Stochastic Differential Equations with Markovian Switching, Imperial College Press, 2006.
[20] X. Mao, G. Yin, C. Yuan, Stabilization and destabilization of hybrid systems of stochastic differential equations, Automatica 43 (2007) 264–273.