A Pointwise Abelian Ergodic Theorem for $L_p$ Semigroups, 
$1 \leq p < \infty^*$

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Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space and $L_p(\mu) = L_p(X, \Sigma, \mu)$, $1 \leq p < \infty$, the usual Banach spaces of complex-valued functions. Let $\{T_t : t > 0\}$ be a strongly continuous semigroup of positive $L_p(\mu)$ operators for some $1 \leq p < \infty$. Denote by $R_h$ the resolvent of $\{T_t\}$. We show that $f \in L_p(\mu)$ implies $\lambda R_h f(\lambda) \to f(\lambda)$ a.e. as $\lambda \to \infty$.

INTRODUCTION

Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space and $L_p(\mu) = L_p(X, \Sigma, \mu)$, $1 \leq p \leq \infty$, the usual Banach spaces of complex-valued functions. Let $\{T_t : t \geq 0\}$ be a strongly continuous semigroup of positive $L_p(\mu)$ operators for some $1 \leq p < \infty$. This means that (i) $\|T_t f\|_p < \infty$, $t \geq 0$; (ii) $T_{s+t} = T_s T_t$, $s, t \geq 0$; (iii) $\|T_s f - T_t f\|_p \to 0$ as $s \to t$ for any $f \in L_p(\mu)$; (iv) $0 \leq f \in L_p(\mu) \Rightarrow T_t f \geq 0$, $t \geq 0$. To simplify the notation we assume $T_0 = I$; all results hold with appropriate modification if $T_0 \neq I$.

There exists $M > 0$, $\alpha \geq 0$ such that $\|T_t\|_p \leq M e^{\alpha t}$, $t \geq 0$ [6, p. 232]. For $\lambda > \alpha$ and $f \in L_p(\mu)$, we define

$$R_\lambda = \int_0^\infty e^{-\lambda t} T_t f dt,$$

where the integral is defined in the sense of Bochner [3, p. 79]. It is well known (see [2, p. 196; 5, p. 103]) that given $f \in L_p(\mu)$ the vector $T_t f$ has a representation $T_t f(\lambda)$, defined on $R^+ \times X$, such that $T_t f(\cdot)$ is in the equivalence class of $T_t f$ for all $t \geq 0$. This representation is unique modulo sets of product measure zero. For all $\lambda > \alpha$, $e^{-\lambda t} T_t f(\lambda)$

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is integrable with respect to the product measure on $R^+ \times X$. Additionally there is a $\mu$-null set $E(f)$, independent of $\lambda$, outside which $\int_0^\infty e^{-\lambda \tau} f(x) \, dt$ exists and, as a function of $x$, is in the equivalence class of $\int_0^\infty e^{-\lambda \tau} f(x) \, dt$ for any $\lambda > 0$. We define $R_\lambda f(x) = \int_0^\infty e^{-\lambda \tau} f(x) \, dt$ for $x \notin E(f)$ and define it arbitrarily for $x \in E(f)$.

Our goal in this note is to prove the following:

$$\lim_{\lambda \to \infty} \lambda R_\lambda f(x) = f(x) \quad \text{a.e. (})$$

for any $f \in L_p(\mu)$ and $\{T_t\}$ a strongly continuous semigroup of positive $L_p(\mu)$ operators. This result generalizes a theorem proved by the author in [4]. There he showed that if $p > 1$ (*) holds for a strongly continuous semigroup of positive contractions. In case $p = 1$ he proved (*) for an arbitrary strongly continuous semigroup of $L_1(\mu)$ contractions. Earlier results involving pointwise convergence appeared in [1, p. 178]. The question of the existence of $\lim_{\lambda \to \infty} \lambda R_\lambda f$ with respect to some topology is considered in [3] and elsewhere.

**Main Results**

Before establishing our main result we prove one preliminary lemma. We set $S_t = e^{-\beta t} T_t$ for some $\beta > 0$.

**Lemma.** Let $\{T_t; t \geq 0\}$ be a strongly continuous semigroup of positive $L_p(\mu)$ operators for some $p$, $1 \leq p < \infty$. There exists a measure $m$, equivalent to $\mu$, and a positive function $h$ such that, if we define $P_t f = S_t(fh)/h$ for $f \in L_p(m)$, then $P_t$ can be extended by continuity to an $L_1(m)$ operator and $\{P_t\}$ becomes a strongly continuous semigroup of positive $L_1(m)$ contractions. The measure $m$ is such that $f \in L_p(\mu)$ implies $f'h \in L_p(m)$.

**Proof.** Suppose first that $p > 1$. We note that the adjoint semigroup $\{S_t^*\}$ of positive $L_q(\mu)$ operators, where $q = p/(p - 1)$, is weakly continuous. Since $L_q(\mu)$ is reflexive $\{S_t^*\}$ is strongly continuous. Set $g = \int_0^\infty S_t^* g' \, dt$ for some $0 < g' \in L_q(\mu)$. Then $g \in L_q(\mu)$ and for any $t \geq 0$,

$$S_t^* g = \int_t^\infty S_r^* g' \, dr \leq \int_0^\infty S_r^* g' \, dr = g.$$

Also for any $f \in L_p^+(\mu), f \neq 0$, we have $(f, g) = (\int_0^\infty S_t f \, dt, g) > 0$ since $\int_0^\infty S_t f \, dt \geq 0$ and $\neq 0$. Hence $g > 0$ a.e. on $X$. Setting $h = g^{1/(p-1)}$ we have $0 < h$ and $h \in L_p(\mu)$ since $h^p = g$. We define
m(A) = \int_A h^p \, d\mu, \ A \in \Sigma, \text{ and } P_t f = S_t(fh)/h \text{ for } f \in L_p(m). \text{ Clearly } m \sim \mu. \text{ We note that the map } V: f \rightarrow fh \text{ is an isometry of } L_p(m) \text{ onto } L_p(\mu) \text{ and that } P_t \text{ is simply } V^{-1} S_t V. \text{ It is thus obvious that } \{P_t\} \text{ is a strongly continuous } L_p(m) \text{ semigroup with } \|P_t\|_p = \|S_t\|_p. \text{ Since the adjoint } V^* : L_q(\mu) \rightarrow L_q(m) \text{ is given by } V^* f = f/h^{p-1} \text{ it follows that } P_t^*, \text{ which equals } V^* S_t^* (V^{-1})^*, \text{ is given by}

\[ P_t^* f = [S_t^* (fh^{p-1})]/h^{p-1} = S_t^* (fg)/g \]

for \( f \in L_q(m). \text{ Thus } P_t^*(1) \leq 1, \ t \geq 0, \text{ which implies } \|P_t^*\|_{p} \leq 1 \text{ since } P_t^* \text{ is positive; it follows that } \|P_t f\|_1 \leq \|f\|_1 \text{ for } f \in L_p(m) \text{ and consequently that } P_t \text{ can be extended to a positive contraction of } L_1(m). \text{ It is easy to show directly, by an approximation argument, that } \{P_t\}, \text{ regarded as an } L_1(m) \text{ semigroup, is strongly continuous. Finally, it is clear that } f \in L_p(\mu) \text{ implies } fh \in L_p(m). \)

Now consider the case \( p = 1. \text{ Set } S_t' = e^{-\beta't} T_t \text{ for some } \alpha < \beta' < \beta \text{ and let } u = \int_0^\infty S_t' f \, dt \text{ for some } 0 < f \in L_1(\mu). \text{ We note that } u > 0 \text{ a.e. on } X, \ u \in L_1(\mu), \text{ and } S_t' u \leq u \text{ for all } t \geq 0. \text{ Defining } \nu(A) = \int_A u \, d\mu \text{ for } A \in \Sigma \text{ and}

\[ Q_t f = S_t(fu)/u = e^{-(\beta-\beta')t} S_t'(fu)/u \]

for \( f \in L_1(\nu), \text{ we see that } \|Q_t\|_1 = \|S_t\|_1 \leq Me^{-(\beta-\alpha)t} \text{ and } \|Q_t\|_\infty \leq e^{-(\beta-\beta')t}. \text{ By the Riesz convexity theorem, } \|Q_t\|_p \leq Me^{-(\beta-\beta')t} \text{ (we may assume } M > 1). \text{ We now fix } p, 1 < p < \infty. \text{ By the argument in } [2, \text{ p. 689}] \text{ we have that } \{Q_t\}, \text{ regarded as an } L_p(\nu) \text{ semigroup, is strongly continuous. Set } g = \int_0^\infty Q_t g' \, dt \text{ for some } 0 < g' \in L_q(\nu) \cap L_\infty(\nu). \text{ We note that } Q_t^* g' \leq g \text{ for all } t \geq 0 \text{ and that for any } A \in \Sigma, \text{ and}

\[ (\chi_A, g) = \int_0^\infty (Q_t \chi_A, g') \, dt \]

\[ \leq M\nu(A) \|g'\|_\infty \int_0^\infty e^{-(\beta-\alpha)t} \]

\[ \leq M\nu(A) \|g'\|_\infty / (\beta - \alpha). \]

Hence \( \|g\|_\infty < \infty. \text{ Now define } v = g^{1/p-1}, \ m(A) = \int_A v^p \, dv, \text{ and } P_t f = Q_t(fv)/v \text{ for } f \in L_p(m). \text{ Then, as in the first paragraph, } \{P_t\} \text{ is a strongly continuous semigroup of positive } L_1(m) \text{ contractions. Setting } h = uv, \text{ we have}

\[ P_t f = Q_t(fv)/v - S_t(fuv)/uv = S_t(fh)/h \]
for \( f \in L_p^p(m) \). Clearly \( m \sim \mu \) since \( m \sim \nu \) and \( \nu \sim \mu \). Also \( \frac{dm}{d\nu}(d\nu/d\mu) \frac{d\mu}{d\nu} = \nu^p u \) which implies that if \( f \in L_\lambda^\nu(\mu) \) then \( f/h \in L_1(m) \) since

\[
\int |f/h| \, dm = \int |f| \, \nu^{p-1} \, d\mu \leq \|\nu^{p-1}\|_\nu \int |f| \, d\mu.
\]

Q.E.D.

**Theorem.** Let \( \{T_t; t \geq 0\} \) be a strongly continuous semigroup of positive \( L_p^p(\mu) \) operators for some \( 1 \leq p < \infty \). Then \( \lim_{h \to 0} \lambda R_{\lambda} f(x) = f(x) \) a.e. for any \( f \in L_\lambda^\nu(\mu) \).

**Proof.** By our lemma and the theorem in [4] we have \( \lim_{h \to 0} \lambda \int f(x) = f(x) \) a.e. for any \( f \in L_1(m) \). We have used \( J_\lambda f \) to denote \( \int_0^\infty e^{-\lambda t} P_t f \, dt \). We have that \( f \in L_p^p(\mu) \) implies \( f/h \in L_1(m) \). Also

\[
J_\lambda(f/h) = \left[ \int_0^\infty e^{-\lambda t} S_t f \, dt \right]/h = [R_{\lambda+\beta} f]/h
\]

for all \( \lambda > 0 \). Hence

\[
\lim_{\lambda \to 0} \lambda R_{\lambda} f(x) = \lim_{\lambda \to 0} (\lambda + \beta) R_{\lambda+\beta} f(x)
\]

\[
= h(x) \lim_{\lambda \to 0} (\lambda + \beta) J_\lambda(f/h)(x)
\]

\[
= h(x) \lim_{\lambda \to 0} \lambda J_\lambda(f/h)(x)
\]

\[
= h(x)(f/h)
\]

\[
= f(x) \quad \text{a.e.}
\]

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**References**