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A Class of Infinite-Rank Modules over Tame Hereditary Algebras

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Given a finite dimensional hereditary connected (basic-) algebra A of infinite representation type the finite dimensional indecomposable modules divide into three classes: preprojective, regular, and preinjective modules. The preprojective modules yield the torsion class of $\mathscr{P}^\infty$-torsion modules. Here a module $M \in$ Mod-A is called a $\mathscr{P}^\infty$-*torsion* module if M maps only trivially into preprojectives, equivalently, if M has no non-zero preprojective direct summand. This torsion theory was first introduced by Ringel [14, 2.4–2.7] (note that he called $\mathscr{P}^\infty$-torsionfree modules preprojective).

We will show that the full subcategory of $\mathscr{P}^\infty$-torsion modules has enough projective objects, so-called $\mathscr{P}^\infty$-*projective* modules. The $\mathscr{P}^\infty$-projectives divide into finitely many classes whose number depends on the number of regular partial tilting modules.

As an application of these results we show that every torsionfree module (for the defintion see, e.g., [14]) of countable rank is a direct sum of a $\mathscr{P}^\infty$-torsion and a $\mathscr{P}^\infty$-torsionfree module. Restricting this result to Baer modules we see that every Baer module of countable rank is the direct sum of a $\mathscr{P}^\infty$-projective and a $\mathscr{P}^\infty$-torsionfree module. Finally it should be noted that every purely simple module of infinite rank has to be a $\mathscr{P}^\infty$-projective module.

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Notation. The word algebra always denotes a finite dimensional, unitary basic algebra over some commutative field k. The letter A normally is reserved for a tame hereditary, connected algebra.

If A is an algebra, Mod-A (mod-A) denotes the category of (finitely generated) right A-modules. For a class $\mathscr{S}$ of modules we define Add($\mathscr{S}$)

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(add($\mathscr{S}$)) as the class of all direct summands of (finite) direct sums of modules from $\mathscr{S}$. We write Gen($\mathscr{S}$) for the class of all modules generated by modules of Add($\mathscr{S}$).

By a torsion theory in Mod-A (mod-A) we understand a pair $(\mathscr{F}, \mathscr{T})$, were $\mathscr{F}$ is the class of *torsionfree* modules and the class $\mathscr{T}$ consists of the *torsion modules*, together with the following properties:

(i) $\mathrm{Gen}(\mathscr{T}) = \mathscr{T}$

(ii) The class $\mathscr{T}$ is closed under extensions.

Every module M has a largest submodule $\mathscr{T}$ which is a torsion module. $\mathscr{T}$ can be regarded as a subfunctor of the identity functor and is called the torsion radical of the torsion theory. For every module M we have $\mathscr{T}\mathscr{M} \in \mathscr{T}$ and $M/\mathscr{T}\mathscr{M} \in \mathscr{F}$. To express that we are dealing with a fixed torsion theory $(\mathscr{F}, \mathscr{T})$ we will write "$\mathscr{T}$*-torsion* (resp. $\mathscr{T}$*-torsionfree*) modules." A $\mathscr{T}$-torsion module X is called $\mathscr{T}$*-projective* if $\mathrm{Ext}^i(X, \mathscr{T})$ is zero for all $i \geqslant 1$.

By $\Gamma(A)$ we denote the Auslander–Reiten quiver of A. The Auslander–Reiten translation will be denoted by τ and τ^-; to emphasize the algebra, we sometimes add a subscript, e.g., τ_A, τ_B^-.

We define the right- (left-) perpendicular category $\mathscr{S}^\perp$ (${}^\perp\mathscr{S}$), where $\mathscr{S}$ is some class of modules in Mod-A in the sense of Geigle and Lenzing [6] as the full subcategory of all modules M in Mod-A with $\mathrm{Hom}(\mathscr{S}, M) = 0$ and $\mathrm{Ext}^1(\mathscr{S}, M) = 0$ ($\mathrm{Hom}(M, \mathscr{S}) = \mathrm{Ext}^1(M, \mathscr{S}) = 0$).

An indecomposable partial tilting module is called *stone*. If $\mathscr{S}$ is a quasi-simple module over a tame hereditary algebra we denote by $\mathscr{S}(i)$ the indecomposable regular module of quasi-length i and quasi-socle $\mathscr{S}$.

In general we follow the notations used in [14] with one difference: The word "preprojective" is reseved for finitely generated (not necessary indecomposable) modules.

1. Basic Properties and Definitions

In the first section we draw attention to the functors $\mathscr{P}_\alpha$, $\mathscr{P}$, and $\mathscr{P}^\infty$ defined by Ringel in [14]. The functor $\mathscr{P}^\infty$ gives rise to a torsion theory; with respect to this torsion theory we will call the torsionfree modules $\mathscr{P}^\infty$-torsionfree modules. Furthermore, we introduce the rank function, needed in the last section of this paper. In the following sections we concentrate our interest on the projective objects in the torsion class of the functor $\mathscr{P}^\infty$; we will call them $\mathscr{P}^\infty$-projectives. As usefull tools we will therefore recall (1.3) and (1.4) from [9].

First we define the subfunctors $\mathscr{P}_\alpha$ and $\mathscr{P}$ of the identity functor. We call a set α of indecomposable preprojective modules *predecessor closed*, if for every indecompocable module P with $\operatorname{Hom}(P, P')$ non-zero for some $P' \in \alpha$ the set α contains a module isomorphic to P. For a predecessor closed set of preprojective modules we define for every $M \in \text{Mod-}A$ the submodule

$$\mathscr{P}_\alpha M := \bigcap_{f: M \to P} \ker f$$

with the intersection taken over all P in α. The following was proved in [14, 2.1]: Let α be a finite predecessor closed set of preprojective modules. Then every module $M \in \text{Mod-}A$ has decomposition $M = \mathscr{P}_\alpha M \oplus M'$ with $M' \in \operatorname{Add}(\alpha)$.

If the set α contains all indecomposable preprojective modules we will write $\mathscr{P}$ instead of $\mathscr{P}_\alpha$. Note that the functor $\mathscr{P}$ does not split, in general; moreover, it does not induce a torsion theory. The problem is that there exist modules M such that $\mathscr{P}\mathscr{P}M$ is a proper submodule of $\mathscr{P}M$, see [14, 2.6]. The largest submodule U of M with $\mathscr{P}U = U$ is denoted by $\mathscr{P}^\infty M$. $\mathscr{P}^\infty$ is a torsion radical and modules M of the torsion class are characterized by the property $\operatorname{Hom}(M, P) = 0$ for all preprojective modules P or, equivalently, $\operatorname{Ext}^1(P, M) = 0$ for all preprojective modules P, see, for example, [9, 1.5].

Next we define the rank function using an idea of Dean [4]. For every tame hereditary algebra there exists a generic module—by [15] uniquely determined—that is, an infinite dimensional indecomposable module Q which is finitely generated over its endomorphism ring $\operatorname{End}(Q)$ which is a division ring (for the relevance of generic modules see [3]). Defining for a module $M \in \text{Mod-}A$ the rank $\operatorname{rk} M$ of M as the cardinality of an $\operatorname{End}(Q)$-basis of $\operatorname{Hom}(M, Q)$ we obtain the rank function. This definition coincides with the usual definition as given, e.g., in [14] if and only if M has finite rank.

We write $\mathscr{T}_0$ for the torsion class of all modules which are generated by finite dimensional regular modules. If there is no danger of confusion we say *torsionfree* (resp. *torsion*) if we mean a $\mathscr{T}_0$-torsionfree ($\mathscr{T}_0$-torsion) module. The following lemma is easy to prove (see [5]).

LEMMA 1.1. *Let X, Y be torsionfree modules of finite rank and*

$$0 \to X \to Y \to Z \to 0$$

be a short exact sequence in Mod-A:

(a) *If Z is torsionfree too, we have* $\operatorname{rk} X + \operatorname{rk} Z = \operatorname{rk} Y$.

(b) *If Z is a torsion module we have* $\operatorname{rk} Y \leqslant \operatorname{rk} X$ *and the equality only holds if Z is a regular torsion module.*

As an application we prove the following lemma needed in the sequel.

LEMMA 1.2. *Every $\mathscr{P}^{\infty}$-torsionfree module M of finite rank is a direct sum of finitely many preprojective modules.*

Proof. Let $\alpha_1 \subset \alpha_2 \subset \alpha_3 \subset \cdots$ be an ascending chain of finite predecessor closed subsets of preprojective modules such that every indecomposable preprojective module is isomorphic to a module in $\bigcup_{n \in \mathbb{N}} \alpha_n$. This ascending chain gives a descending chain

$$\mathscr{P}_{\alpha_1} M \supset \mathscr{P}_{\alpha_2} M \supset \mathscr{P}_{\alpha_3} M \supset \cdots$$

of direct summands of M. For every i we can find a module $C_{i+1} \in \mathrm{Add}(\alpha_{i+1})$ such that $\mathscr{P}_{\alpha_i} M = \mathscr{P}_{\alpha_{i+1}} M \oplus C_{i+1}$, especially we have $\mathrm{rk}\, \mathscr{P}_{\alpha_i} M > \mathscr{P}_{\alpha_{i+1}} M$ by (1.1(a)) if $\mathscr{P}_{\alpha_{i+1}} M$ is a proper submodule of $\mathscr{P}_{\alpha_i} M$. Since the rank of M is finite, the descending chain $\mathscr{P}_{\alpha_1} M \supset \mathscr{P}_{\alpha_2} M \supset \cdots$ becomes stationary for some natural number n. Therefore the functors $\mathscr{P}$ and $\mathscr{P}_{\alpha_n}$ coincide for the module M; that is, $\mathscr{P}M$ is a direct summand of M. But then $\mathscr{P}M = \mathscr{P}^{\infty} M = 0$ and we obtain that M is a direct sum of finitely many preprojective modules by [14, 2.1]. ∎

In the following, the next lemma will be helpful.

LEMMA 1.3. *Let $M \in \mathrm{Mod}\text{-}A$ be a module with $\mathrm{Ext}(M, M^{(I)}) = 0$ for all sets I. Then $\mathrm{Gen}(M)$ is a torsion class.*

Proof. In the case of M being finite dimensional $\mathrm{Gen}(M)$ is a torsion class (also in Mod-A) if $\mathrm{Ext}(M, M) = 0$ by [2, 7]. A straightforward inspection of this proof, given in [9, 1.1] shows that it remains valid, in general, if we use the stronger condition $\mathrm{Ext}(M, M^{(I)}) = 0$. ∎

Given a torsion theory $(\mathscr{F}, \mathscr{T})$ we ask now, if there exist $\mathscr{T}$-projective modules. Assume that there is given a class $\mathscr{S}$ of modules with $\mathrm{Ext}^1(\mathscr{S}, \mathscr{T}) = 0$ for all $S \in \mathscr{S}$. We define the class $\mathscr{E}(\mathscr{S})$ as the class of all modules M which are the well-ordered union of submodules M_λ such that $M_0 = 0$, $M_{\lambda+1}/M_\lambda \in \mathscr{S}$ and $M_\lambda = \bigcup_{\mu < \lambda} M_\mu$ if λ is a limit ordinal. For this class we proved in [9, 3.1]:

THEOREM 1.4. *Let $(\mathscr{T}, \mathscr{F})$ be a torsion theory in* Mod-R, *where R is some ring and $\mathscr{S}$ is a class of modules with* $\mathrm{Ext}^1(\mathscr{S}, \mathscr{T}) = 0$:

(a) *For every torsion module M with $M \in \mathscr{E}(\mathscr{S})$ we have* $\mathrm{Ext}^1(M, \mathscr{T}) = 0$; *that is, M is $t_{\mathscr{T}}$-projective.*

(b) *If there exists a short exact sequence*

$$0 \to R \to T_1 \to T_2 \to 0$$

with T_1, T_2 $t_{\mathscr{T}}$-projective, then $\mathrm{Ext}^1(N, \mathscr{T}) = 0$ *for every $N \in \mathscr{E}(\mathscr{S})$.*

2. Construction of $\mathcal{P}^\infty$-projective Modules

The aim of this section is to prove the following theorem:

THEOREM 2.1. *Let A be a tame hereditary algebra*:

(a) *Let Y be a regular partial tilting module. Then there exists a nonzero $\mathcal{P}^\infty$-projective module M with* $\mathrm{Hom}(M, Y)=0$.

(b) *There exists a short exact sequence,*

$$0 \to A \to A_{\mathcal{P}^\infty} \to A_{\mathcal{P}^\infty}/A \to 0,$$

with $\mathcal{P}^\infty$-projective modules $A_{\mathcal{P}^\infty}$ and $A_{\mathcal{P}^\infty}/A$. The module $A_{\mathcal{P}^\infty}$ is a union over a chain of finite dimensional modules $A_1 \subset A_2 \subset A_3 \subset \cdots$ with A_{n+1}/A_n torsionfree for all $n \in \mathbb{N}$.

For the proof the lemma will be helpfull

LEMMA 2.2. *Let P_1, P_2 be nonzero preprojective modules*:

(a) *There exists a short exact sequence*

$$0 \to P_1 \to Y \to Z \to 0$$

with Z preprojective and $\mathrm{Hom}(Y, P_2)=0$.

(b) *There exists $N \in \mathbb{N}$ such that $\tau^{-n}P_2$ cogenerates P_1 for all $n \geqslant N$.*

Proof. Assertion (a) is proved in [14, 2.5]; (b) is shown in [9, 6.2]. ∎

Before we prove the theorem let us formulate a consequence of the above lemma which will be needed in the next section:

COROLLARY 2.3. *Every nonzero $\mathcal{P}^\infty$-torsion module N without nonzero regular and nonzero preinjective submodules cogenerates every preprojective module.*

Proof. By (2.2(b)) it is enough to show that for every $n \in \mathbb{N}$, N has preprojective submodules P with $\mathrm{Hom}(\tau^{-n}A, P) \neq 0$. Since $\tau^{-n}A$ is a tilting module there exists a short exact sequence

$$0 \to A \to T_1 \to T_2 \to 0$$

with $T_1, T_2 \in \mathrm{add}(\tau^{-n}A)$. Applying $\mathrm{Hom}(-, N)$ to this sequence shows that $\mathrm{Hom}(T_1, N) \neq 0$ as $\mathrm{Ext}(T_2, N)=0$ (for $\mathrm{Ext}^1(T_2, N)=0$ see, for example, [9, 1.5]). The image of a nonzero map $\tau^{-n}A \to N$ is preprojective because of N having no nonzero regular and no nonzero preinjective submodules. ▮

Proof of Theorem 2.1. By [6] the full subcategory of finitely generated modules in the left perpendicular category ${}^{\perp}Y$ is equivalent to mod-B for some (not necessarily connected) algebra B. In B-mod there is exactly one infinite preprojective component which is the preprojective component of a tame hereditary connected algebra C. It consits of A-preprojective modules. First we construct a nonzero $\mathscr{P}^{\infty}$-projective module M in Mod-C. We define $M_0 = 0$ and $M_1 = C$. When M_n is defined we define M_{n+1} applying (2.2(a)) as the middle term of a short exact sequence

$$0 \to M_n \to M_{n+1} \to Z \to 0$$

with Z preprojective and $\operatorname{Hom}(M_{n+1}, \tau_C^{1-n}C) = 0$. We consider the modules M_n as modules in mod-A and define the A-module

$$M = \bigcup_{n \in \mathbb{N}} M_n.$$

Given any homomorphism $f: M \to P$ with P preprojective in mod-A by (2.1(b)) we can find $N \in \mathbb{N}$ such that P is cogenerated by $\tau_C^{-n}C$ for all $n > N$. By construction we have $\operatorname{Hom}(M_{n+1}, \tau_C^{1-n}C) = 0$ for all $n > N$ and therefore we obtain $f(M_n) = 0$ for all $n \in \mathbb{N}$. We see that M is a $\mathscr{P}^{\infty}$-torsion module in Mod-A and now (1.4(a)) implies that M is $\mathscr{P}^{\infty}$-projective in Mod-A. The property $\operatorname{Hom}(M, Y) = 0$ is trivial because any M_n satisfies it.

(b) If we choose $Y = 0$ in the construction of part (a), we obtain $A_{\mathscr{P}^{\infty}} = M$. The assertion follows now again from (1.4(a)). ▮

Remark. The first example for a $\mathscr{P}^{\infty}$-projective module I found in Okoh's paper [10, 1.3]. Given the Kronecker algebra over the complex numbers there is the source map $P_1 \to P_2 \oplus P_2$ of the simple projective module P_1. If $P_2 \to \tau^- P_1^2$ is the source map starting in P_2 he considers the sequence of monomorphisms

$$P_1 \to P_2^2 \to (\tau^- P_1)^4 \to (\tau^- P_2)^8 \to \cdots.$$

The union of these modules is $\mathscr{P}^{\infty}$-projective.

3. Classification of $\mathscr{P}^{\infty}$-projective Modules

In the second section we have constructed nonzero $\mathscr{P}^{\infty}$-projective modules, but there seems to be no hope to find nonzero $\mathscr{P}^{\infty}$-projective modules with a local endomorphisms ring. Given two $\mathscr{P}^{\infty}$-projective modules M, N we therefore consider the classes $\mathrm{Add}(M)$ and $\mathrm{Add}(N)$ and we want to find out conditions for one of these classes being a subclass of the other. As a main result of this section we obtain the following criterium:

Theorem 3.1. *Let M, N be two nonzero $\mathscr{P}^{\infty}$-projective modules. Then the following are equivalent:*

(i) $\mathrm{Add}(N) \subset \mathrm{Add}(M)$

(ii) *If S is a regular partial tilting module with* $\mathrm{Hom}(M, S) = 0$ *then* $\mathrm{Hom}(N, S) = 0$.

Given a tame hereditary algebra A, the module category mod-A contains only finitely many regular stones. This implies that there exist only finitely many different classes of $\mathscr{P}^{\infty}$-projective modules. In case A is a tame hereditary algebra with only two simple modules, we know that $\mathrm{Add}(M)$ contains every $\mathscr{P}^{\infty}$-projective module, if M is nonzero $\mathscr{P}^{\infty}$-projective. Note that this holds for every wild hereditary algebra, too [9, 6.1]. The proof of the theorem will use the following lemmas:

LEMMA 3.2. *Let M, N be two nonzero $\mathscr{P}^{\infty}$-projective modules. Then the following are equivalent:*

(i) $N \in \mathrm{Gen}(M)$

(ii) $N \in \mathrm{Add}(M)$.

Especially, every $\mathscr{P}^{\infty}$-projective module is contained in $\mathrm{Add}(A_{\mathscr{P}^{\infty}})$.

Proof. Assume M generates N. This implies that the homomorphism $(f_i): M^{(I)} \to N$ is an epimorphism if $(f_i)_{i \in I}$ is a k-basis of $\mathrm{Hom}(M, N)$. We apply the functor $\mathrm{Hom}(M, -)$ to the short exact sequence

$$0 \to K \to M^{(I)} \to N \to 0$$

and obtain

$$\cdots \to (M, M^{(I)}) \xrightarrow{(M, (f_i))} (M, N) \to \mathrm{Ext}(M, K) \to \mathrm{Ext}(M, M^{(I)}) \to \cdots.$$

Since (f_i) is a k-basis of $\mathrm{Hom}(M, N)$ the map $(M, (f_i))$ is surjective. This gives $\mathrm{Ext}(M, K) = 0$ as $\mathrm{Ext}(M, M^{(I)}) = 0$. By (2.3) we know that M cogenerates every preprojective module. As an application we obtain that K is a $\mathscr{P}^{\infty}$-torsion module and the above sequence splits by the $\mathscr{P}^{\infty}$-projectivity of N.

It remains to show that $A_{\mathscr{P}^\infty}$ generates every $\mathscr{P}^\infty$-torsion module N. Applying the functor $\mathrm{Hom}(-, N)$ to the short exact sequence of (2.1)

$$0 \to A \to A_{\mathscr{P}^\infty} \to A_{\mathscr{P}^\infty}/A \to 0,$$

we obtain the exact sequence

$$\cdots \to \mathrm{Hom}(A_{\mathscr{P}^\infty}, N) \to \mathrm{Hom}(A, N) \to \mathrm{Ext}^1(A_{\mathscr{P}^\infty}/A, N) \to \cdots.$$

As N is $\mathscr{P}^\infty$-torsion we obtain $\mathrm{Ext}(A_{\mathscr{P}^\infty}, N) = 0$ and therefore every map $A \to N$ can be extended to a map $A_{\mathscr{P}^\infty} \to N$, especially we have $N \in \mathrm{Gen}(A_{\mathscr{P}^\infty})$. ∎

LEMMA 3.3 *Let M be a nonzero $\mathscr{P}^\infty$-projective module*:

(a) *If N is a nonzero $\mathscr{P}^\infty$-torsion module without nonzero regular and nonzero preinjective submodules then* $\mathrm{Hom}(M, N)$ *is nonzero.*

(b) *Every regular tube Π if $\Gamma(A)$ contains at least one quasi-simple module S with $S \in \mathrm{Gen}(M)$.*

(c) *Every $\mathscr{P}^\infty$-torsion module Y with* $\mathrm{Hom}(M, Y) = 0$ *is a direct sum of regular stones.*

Proof. For (b) it is enough to show that every regular tube of $\Gamma(A)$ contains a module N_j with $\mathrm{Hom}(M, N_j) \neq 0$, since the image of every nonzero map $M \to N_j$ has to be a regular module. We use the following common property of the modules in a regular tube and the module N in part (a) to prove (a) and (b) simultaneously:

(α) N cogenerates every preprojective module

(β) For every preprojective module P and every regular tube Π there exists a monomorphism $P \to X$ into a module X of $\mathrm{add}(\Pi)$.

By (3.2) M is in $\mathrm{Add}(A_{\mathscr{P}^\infty})$, so we only have to construct a monomorphism $A_{\mathscr{P}^\infty} \to Z$ for some $Z \in \mathrm{Add}(N)$ ($Z \in \mathrm{Add}(\Pi)$, respectively). We use the filtration of $A_{\mathscr{P}^\infty}$ described in (2.1(b)) and choose monomorphisms

$$f_1 : A \to Z_1,\ f_2 : A_2/A \to Z_2, \ldots, f_{n+1} : A_{n+1}/A_n \to Z_{n+1}, \ldots$$

for $Z_n \in \mathrm{Add}(N)$ ($Z_n \in \mathrm{Add}(\Pi)$, respectively) for all $n \geqslant 1$. Applying $\mathrm{Hom}(-, Z_{n+1})$ to the short exact sequence

$$0 \to A_{n+1}/A_n \to A_{\mathscr{P}^\infty}/A_n \to A_{\mathscr{P}^\infty}/A_{n+1} \to 0$$

and using that $A_{\mathscr{P}^\infty}/A_{n+1}$ is $\mathscr{P}^\infty$-projective we see that f_n can be lifted to a map

$$\tilde{f}_n : A_{\mathscr{P}^\infty}/A_n \to Z_{n+1}.$$

Defining

$$g_n: A_{\mathscr{P}\infty} \to A_{\mathscr{P}\infty}/A_n \to Z_{n+1},$$

where $A_{\mathscr{P}\infty} \to A_{\mathscr{P}\infty}/A_n$ is the canonical projection, we obtain the monomorphism

$$A_{\mathscr{P}\infty} \xrightarrow{(g_n)} \bigoplus_{n \in \mathbb{N}} Z_n.$$

(c) First we prove that Y is a torsion module. If we denote by $\mathscr{T}_0\mathscr{M}$ the torsion submodule of M and apply $\operatorname{Hom}(M, -)$ to the sequence

$$0 \to \mathscr{T}_0 Y \to Y \to Y/\mathscr{T}_0 Y \to 0,$$

we obtain

$$\cdots \to \operatorname{Hom}(M, Y) \to \operatorname{Hom}(M, Y/\mathscr{T}_0 Y) \to \operatorname{Ext}(M, \mathscr{T}_0 Y) \to \cdots.$$

Since $\operatorname{Hom}(M, Y) = \operatorname{Ext}(M, \mathscr{T}_0 Y) = 0$ we conclude that $Y/\mathscr{T}_0 Y = 0$ with the help of part (a). If S is a quasi-simple module in a homogenous tube, we conclude from $S \in \operatorname{Gen}(M)$ that there exists a nonzero map from M to every nonzero preinjective module. Therefore Y does not have a nonzero preinjective submodule.

For the next step we assume that Y has a finitely generated submodule $S(n)$ with S quasi-simple, $n \in \mathbb{N}$ such that $\operatorname{Ext}(S(n), S(n)) \neq 0$. An easy calculation shows that the modules $S(i)$ are in the abelian subcategory $M^{\perp}$ for $1 \leqslant i \leqslant n$. But then the whole mouth of the tube, $S(i+1)/S(i)$, $1 \leqslant i \leqslant n-1$, is in $M^{\perp}$ in contradiction to (b).

By [14, G] there exists a pure submodule U of Y which is a direct sum of finitely generated regular modules such that Y/U has no nonzero finitely generated direct summand. We apply the functor $\operatorname{Hom}(M, -)$ to the short exact sequence

$$0 \to U \to Y \to Y/U \to 0$$

and obtain

$$\cdots \to \operatorname{Hom}(M, U) \to \operatorname{Hom}(M, Y/U) \to \operatorname{Ext}(M, U) \to \cdots.$$

Since U is a regular torsion module we have $\operatorname{Ext}(M, U) = 0$ and therefore $\operatorname{Hom}(M, Y/U) = 0$. The module Y/U is a direct sum of Prüfer modules [14, 4.8] and therefore we have $Y/U = 0$, as M maps nontrivially into every Prüfer module. So we have shown that $Y = U$ is a direct sum of finitely generated modules, which have to be regular stones. ∎

Proof of Theorem 3.1. We assume property (ii). By (3.2) it is enough to show that N is in Gen(M). Let $t_M N$ be the largest submodule of N which is in Gen(M) (see (1.3)). The module $Y = N/t_M N$ is a $\mathscr{P}^\infty$-torsion module with $\mathrm{Hom}(M, Y) = 0$ and therefore a direct sum of regular stones (3.3(c)). But this implies $Y = 0$ by assumption. ∎

4. Baer Modules and the Splitting Property of $\mathscr{P}^\infty$

The aim of this section is to show connections between $\mathscr{P}^\infty$-projective modules, Baer modules, and purely simple modules of infinite rank. Following Okoh [12] we call a module M *Baer module*, if $\mathrm{Ext}(M, T) = 0$ for every torsion module T. Since every P^∞-projective module M satisfies the stronger condition $\mathrm{Ext}(M, N) = 0$ for every $\mathscr{P}^\infty$-torsion module N, trivially every $\mathscr{P}^\infty$-projective module is a Baer module which is $\mathscr{P}^\infty$-torsion in addition.

Lemma 4.1. *For a torsionfree module M of countable rank there are equivalents*:

(i) $\mathrm{Ext}(M, N) = 0$ *for all $\mathscr{P}^\infty$-torsion modules*

(ii) *If U is a finitely generated submodule of M, then also $\mathscr{T}_0(M/U)$ is finitely generated* (*equivalent*: *Every submodule of finite rank is finitely generated*).

Proof. (i) ⇒ (ii) We only give a sketch of the proof since it is similar to the proof of [9, 5.1]. Let

$$0 \to M \to M' \to A_{\mathscr{P}^\infty}^{(I)} \to 0$$

be universal exact sequence, that is, a sequence such that the connection homomorphism $\mathrm{Hom}(A_{\mathscr{P}^\infty}, A_{\mathscr{P}^\infty}^{(I)}) \to \mathrm{Ext}^1(A_{\mathscr{P}^\infty}, M)$ is surjective. Applying the functor $\mathrm{Hom}(A_{\mathscr{P}^\infty}, -)$ to this sequence gives

$$\mathrm{Hom}(A_{\mathscr{P}^\infty}, A_{\mathscr{P}^\infty}^{(I)}) \xrightarrow{\delta} \mathrm{Ext}(A_{\mathscr{P}^\infty}, M) \longrightarrow \mathrm{Ext}(A_{\mathscr{P}^\infty}, M') \longrightarrow \mathrm{Ext}(A_{\mathscr{P}^\infty}, A_{\mathscr{P}^\infty}^{(I)}).$$

Since δ is surjective we obtain $\mathrm{Ext}(A_{\mathscr{P}^\infty}, M') = 0$ as $\mathrm{Ext}(A_{\mathscr{P}^\infty}, A_{\mathscr{P}^\infty}^{(I)}) = 0$. The same argument as in the proof of (3.2) shows that M' is a $\mathscr{P}^\infty$-torsion module. M' is $\mathscr{P}^\infty$-projective and by (3.2) in $\mathrm{Add}(A_{\mathscr{P}^\infty})$. Now it is easy to see that $A_{\mathscr{P}^\infty}$ has property (ii) and the class of modules satisfying (ii) is closed with respect to direct sums and submodules. This shows that M satisfies (ii).

(ii) ⇒ (i) Proposition 1.3 of [12] states the existence of an ascending chain

$$M_1 \subset M_2 \subset M_3 \subset M_4 \subset \cdots$$

of finite dimensional submodules of M, such that M_{i+1}/M_i is preprojective for all natural numbers i. Now the assertion follows from (1.4). ▮

Remark. Note that an infinite dimensional module of finite rank does not satisfy property (ii) of (4.1), therefore nonzero $\mathcal{P}^\infty$-projectives have infinite rank.

LEMMA 4.2. *Every $\mathcal{P}^\infty$-torsionfree module M satisfies property* (ii) *of* (4.1).

Proof. Assume that M has a finitely generated submodule U such that $\mathcal{T}(M/U)$ is not finitely generated. By [14, 3.7] $\mathcal{T}_0(M/U)$ is a direct sum of preinjective modules and a regular torsion module C. Lemma 3.7 of [14] implies that C is an infinite dimensional module. By the pullback-construction along the inclusion of C in M/U

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & M & \longrightarrow & M/U & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & U & \longrightarrow & M' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

we obtain an infinite dimensional submodule M' of M. By (1.1(b)) this module has finite rank, and since M' is $\mathcal{P}^\infty$-torsionfree we obtain by (1.2) that M is a finite direct sum of preprojective modules. This is a contradiction to the fact that M' is not finitely generated. ▮

Combining the last two lemmas we obtain the following theorem.

THEOREM 4.3. *Let M be a module of countable rank over a tame hereditary algebra A*:

(a) *If M is $\mathcal{P}^\infty$-torsionfree we have* $\mathrm{Ext}(M, N) = 0$ *for every $\mathcal{P}^\infty$-torsion module N.*

(b) *The submodule $\mathcal{P}^\infty M$ is a direct summand of M.*

(c) *If M is a Baer module, then M is a direct sum of a $\mathcal{P}^\infty$-projective and a $\mathcal{P}^\infty$-torsionfree module.*

Proof. (a) is a direct consequence of (4.1) and (4.2).

(b) The short exact sequence

$$0 \to \mathcal{P}^\infty M \to M \to M/\mathcal{P}^\infty M \to 0$$

splits by (a), because of $M/\mathcal{P}^\infty M$ being $\mathcal{P}^\infty$-torsionfree.

(c) By (b) we only have to show that $\mathscr{P}^{\infty}M$ is $\mathscr{P}^{\infty}$-projective. From [13] we know that every Baer module satisfies property (ii) of (4.1). Since the submodule $\mathscr{P}^{\infty}M$ satisfies this property, too, the assertion follows from (4.2). ∎

Remark. For the Kronecker algebra over the field of complex numbers Okoh has shown in [12]: Let α be a set of indecomposable preprojective modules such that infinitely many isomorphism-classes of indecomposable preprojectives are represented by modules of α. Then the module $\prod_{P \in \alpha} P$ is not a Baer module. This shows that (4.3(b)) fails in general if the modules have uncountable k-dimension. We also see that a product of $\mathscr{P}^{\infty}$-projectives is not $\mathscr{P}^{\infty}$-projective, in general.

Dealing with infinite dimensional modules the notation of indecomposability is not as relevant as in the finite dimensional case. The reason is that the correct generalisation of indecomposability is the concept of purely simple modules. Here a module is called *purely simple* if it has only the trivial modules as pure submodules. In [11] Okoh proved for Kronecker algebras (for the general case see [1]) that every purely simple module M of infinite rank has a chain of submodules

$$M_1 \subset M_2 \subset M_3 \subset \cdots$$

such that M_{i+1}/M_i is preprojective and $\bigcup_i M_i = M$. But then M is a $\mathscr{P}^{\infty}$-projective module. From the properties of $\mathscr{P}^{\infty}$-projectives we deduce that there exists at most finitely many isomorphism classes of purely simple modules of infinite rank and local endomorphism ring.

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