Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Multivariate Analysis 96 (2005) 311 – 331

Journal of Multivariate Analysis

www.elsevier.com/locate/jmva

Robust nonparametric estimators of monotone boundaries

Abdelaati Daouia[a,1], Léopold Simar[b,*,2]

[a]*GREMAQ, Université de Toulouse I and LSP, Université de Toulouse III, France*
[b]*Institut de Statistique, Université Catholique de Louvain, Voie du Roman Pays 20, B-1348 Louvain-la-Neuve, Belgium*

Received 7 November 2003
Available online 18 November 2004

Abstract

This paper revisits some asymptotic properties of the robust nonparametric estimators of order-m and order-α quantile frontiers and proposes isotonized version of these estimators. Previous convergence properties of the order-m frontier are extended (from weak uniform convergence to complete uniform convergence). Complete uniform convergence of the order-m (and of the quantile order-α) nonparametric estimators to the boundary is also established, for an appropriate choice of m (and of α, respectively) as a function of the sample size. The new isotonized estimators share the asymptotic properties of the original ones and a simulated example shows, as expected, that these new versions are even more robust than the original estimators. The procedure is also illustrated through a real data set.

AMS 1991 subject classification: 62G07; 62G35; 62G20

Keywords: Estimation of a monotone boundary; Robust nonparametric estimators; Isotonization procedure; Complete uniform convergence; Exponential probability inequalities

* Corresponding author. Fax: +32 10 47 30 32.
E-mail addresses: abdelaati.daouia@math.ups-tlse.fr (A. Daouia), simar@stat.ucl.ac.be (L. Simar).

1 This work was supported in part by "Actions Thématiques de l'Université Paul Sabatier", Toulouse III.

2 Research support from "Projet d'Actions de Recherche Concertées" (No. 98/03-217) and from the "Interuniversity Attraction Pole", Phase V (No. P5/24) from the Belgian Government are also acknowledged.

doi:10.1016/j.jmva.2004.10.005

1. Introduction and basic notation

Let Ψ be the support of the joint probability measure of a random vector $(X, Y) \in \mathbb{R}^p_+ \times \mathbb{R}_+$ and let $(\Omega, \mathcal{A}, P)$ be the probability space on which the vector X and the variable Y are defined. Consider the problem of estimating non parametrically the upper boundary of Ψ, where "upper" is in the direction of the univariate Y. This boundary is assumed to be a monotone nondecreasing[3] function of X and we have a sample $(X_1, Y_1), \dots, (X_n, Y_n)$ of independent random vectors with the same distribution as (X, Y).

Let us denote by $F(y|x) = F(x, y)/F_X(x)$ the conditional distribution function of Y given $X \leqslant x$, where F is the joint distribution function of (X, Y) and $F_X(x) = F(x, \infty)$. From now on we assume that $x \in \mathbb{R}^p_+$ is such that $F_X(x) > 0$. The monotone boundary of Ψ can then be characterized through the frontier function

$$\varphi(x) = \inf\{y \in \mathbb{R}_+ | F(y|x) = 1\},$$

which is the upper boundary of the support of the nonstandard conditional probability measure of Y given $X \leqslant x$.

This kind of problem appears naturally to be useful when analyzing production performance of firms, where X represents the vector of inputs (resources of production) and Y is the output (a quantity of produced goods). In this context, $\varphi(x)$ is the production frontier, i.e., the maximal achievable level of output for a firm working at the level of inputs x. The production efficiency of a firm operating at the level (x, y) can then be measured by the relative comparison of its output y with the reference frontier $\varphi(x)$.

Nonparametric envelopment estimators have been mostly used, like the Free Disposal Hull estimator (FDH, initiated by Deprins et al. [6] in the context of measuring the efficiency of enterprises),

$$\widehat{\varphi}_n(x) = \inf\{y \in \mathbb{R}_+ | \widehat{F}_n(y|x) = 1\} = \max_{i|X_i \leqslant x} Y_i,$$

where $\widehat{F}_n(y|x) = \widehat{F}_n(x, y)/\widehat{F}_{X,n}(x)$, with $\widehat{F}_n(x, y) = (1/n)\sum_{i=1}^n 1(X_i \leqslant x, Y_i \leqslant y)$ and $\widehat{F}_{X,n}(x) = \widehat{F}_n(x, \infty)$. The convex hull of the FDH frontier $\widehat{\varphi}_n$ provides the data envelopment analysis estimator (DEA, initiated by Farrell [7] and popularized as linear programming estimator by Charnes et al. [4]). The statistical inference based on these estimators is now mostly available either by using asymptotic results or by using the bootstrap (see [16] for a recent survey and [13] for a survey and more than 700 references on applications using these estimators). But, by construction, these estimators envelop all the data points and so, are very sensitive to extreme values.

Original robust nonparametric estimators have been suggested recently by Cazals et al. [3]. In place of looking for the full frontier, they estimate a partial frontier of order $m \geqslant 1$, which can be defined as follows. For a given level x, it is defined as the expected value of the maximum of m-independent random variables $Y^1, \dots, Y^m$, drawn from the conditional

[3] For two vectors x and x' in $\mathbb{R}^p$ the inequality $x \leqslant x'$ has to be understood componentwise. A real valued function r on $\mathbb{R}^p$ is then said to be monotone nondecreasing with respect to this partial order if $x \leqslant x'$ implies $r(x) \leqslant (x')$.

distribution of Y given $X \leqslant x$, i.e.,

$$\varphi_m(x) = E[\max(Y^1, \dots, Y^m)|X \leqslant x] = \int_0^\infty (1 - [F(y|x)]^m)\, dy.$$

For all finite integer $m \geqslant 1$, $\varphi_m(x) \leqslant \varphi(x)$ and $\lim_{m\to\infty} \varphi_m(x) = \varphi(x)$. This expected frontier function of order m can be estimated nonparametrically by plugging the empirical version $\widehat{F}_n(y|x)$ of the conditional distribution function $F(y|x)$ to obtain

$$\widehat{\varphi}_{m,n}(x) = \widehat{E}[\max(Y^1, \dots, Y^m)|X \leqslant x] = \int_0^\infty (1 - [\widehat{F}_n(y|x)]^m)\, dy.$$

An explicit formula is available in order to compute $\widehat{\varphi}_{m,n}(x)$, but in practice it is more easy to approximate the empirical expectation by a Monte-Carlo algorithm (see, e.g., [8]). To summarize the properties of these functions, we have

$$\widehat{\varphi}_{m,n}(x) \leqslant \widehat{\varphi}_n(x), \quad \lim_{m\to\infty} \widehat{\varphi}_{m,n}(x) = \widehat{\varphi}_n(x),$$

$$\sqrt{n}(\widehat{\varphi}_{m,n}(x) - \varphi_m(x)) \to N(0, \sigma^2(x, m)) \quad \text{as} \quad n \to \infty,$$

where an expression of $\sigma^2(x, m)$ is available. By choosing m appropriately as a function of the sample size n, $\widehat{\varphi}_{m(n),n}(x)$ estimates the true frontier function $\varphi(x)$ itself and is more robust to extreme values than the FDH since it does not envelop all the data points: it is computed as the expectation of a maximum and not as an observed maximum. An explicit formula of the order $m(n)$ is given in [3], to summarize, we must have $m(n) = O(n \log(n))$. In this case, this estimator keeps the asymptotic properties of the FDH estimator as derived in [12].

Similarly, Aragon et al. [1] introduce the concept of an order-α quantile frontier function, which increases w.r.t. the continuous order $\alpha \in [0, 1]$ and converges to the full frontier $\varphi(x)$ as $\alpha \nearrow 1$. It is defined, for a given level x, by the conditional α-quantile of the distribution of Y given $X \leqslant x$, i.e.,

$$q_\alpha(x) := F^{-1}(\alpha|x) = \inf\{y \in \mathbb{R}_+ | F(y|x) \geqslant \alpha\}.$$

A nonparametric estimator of $q_\alpha(x)$, which increases and converges to the FDH $\widehat{\varphi}_n(x)$ as $\alpha \nearrow 1$, is easily derived by inverting the empirical version of the conditional distribution function,

$$\widehat{q}_{\alpha,n}(x) := \widehat{F}_n^{-1}(\alpha|x) = \inf\{y \in \mathbb{R}_+ | \widehat{F}_n(y|x) \geqslant \alpha\}.$$

As pointed out in [1], this estimator is very fast to compute, very easy to interpret and satisfies very similar statistical properties to those of the nonparametric estimator $\widehat{\varphi}_{m,n}(x)$. In summary, it converges at the rate $\sqrt{n}$, is asymptotically unbiased and normally distributed. Moreover, when the order α is considered as a function of n such that $n^{(p+2)/(p+1)}(1 - \alpha(n)) \to 0$ as $n \to \infty$, $\widehat{q}_{\alpha(n),n}(x)$ estimates the true frontier function $\varphi(x)$ and shares the same asymptotic distribution of both the FDH estimator $\widehat{\varphi}_n(x)$ and the order-$m(n)$ frontier $\widehat{\varphi}_{m(n),n}(x)$.

The reliability of the two sequences of estimators $\{\widehat{q}_{\alpha,n}(x)\}$ and $\{\widehat{\varphi}_{m,n}(x)\}$ is analyzed from a robustness theory point of view in Daouia and Ruiz-Gazen [5]. Both of these nonparametric frontier estimators are qualitatively robust and bias-robust. But the order-α quantile frontiers can be more robust to extreme values than the order-m frontiers when estimating the true full frontier since the influence function is no longer bounded for order-m frontiers when m tends to infinity, while it remains bounded for the conditional quantile frontiers when the quantile order tends to one. The advantage of the order-m frontiers lies in the fact that they can be easily extended to the full multivariate case ($X \in \mathbb{R}_+^p$ and $Y \in \mathbb{R}_+^q$), where they can be computed by using a Monte-Carlo algorithm ([15]). This full multivariate extension has not been obtained for the order-α quantile frontiers.

The drawback of the concepts of these partial frontiers lies in the fact that they are not necessarily monotone with respect to x, whereas the full frontier is monotone. In this paper, we propose an isotonized version $\varphi_m^{\#}(x)$ of $\varphi_m(x)$ and $q_\alpha^{\#}(x)$ of $q_\alpha(x)$, respectively, which converges uniformly to the full frontier $\varphi(x)$ as $m \to \infty$ and as $\alpha \nearrow 1$, respectively. In the same way, we introduce monotone versions $\widehat{\varphi}_{m,n}^{\#}(x)$ and $\widehat{q}_{\alpha,n}^{\#}(x)$ of the initial estimators $\widehat{\varphi}_{m,n}(x)$ and $\widehat{q}_{\alpha,n}(x)$. We first extend, in Lemmas 3.2 and 3.3, the results obtained in [8] about weak uniform consistency of $\widehat{\varphi}_{m,n}$ and $\widehat{\varphi}_n$ to the complete [4] uniform convergence. We also establish the complete uniform convergence of both $\widehat{\varphi}_{m(n),n}$ and $\widehat{q}_{\alpha(n),n}$ to φ as $n \to \infty$. We then show that the isotone estimator $\widehat{\varphi}_{m,n}^{\#}$ converges completely and uniformly to the monotone order-m frontier $\varphi_m^{\#}$, and that the monotone versions $\widehat{\varphi}_{m(n),n}^{\#}$ and $\widehat{q}_{\alpha(n),n}^{\#}$ of the initial estimators $\widehat{\varphi}_{m(n),n}$ and $\widehat{q}_{\alpha(n),n}$ share the same strong uniform convergence property of the FDH estimator $\widehat{\varphi}_n$ to the full frontier φ. Finally, we show that $P(||\widehat{\varphi}_{m(n),n}^{\#} - \varphi|| > \varepsilon)$ and $P(||\widehat{q}_{\alpha(n),n}^{\#} - \varphi|| > \varepsilon)$, for $\varepsilon > 0$ converge to 0 at an exponential rate, where $||\cdot||$ stands for the sup-norm. We illustrate the method through some numerical examples with real and simulated data.

2. Monotone estimators of the upper boundary

The partial functions $\varphi_m(x)$ and $q_\alpha(x)$ converge to the nondecreasing full function $\varphi(x)$ as $m \to \infty$ and as $\alpha \nearrow 1$, respectively, but they are not nondecreasing themselves unless we assume that the conditional distribution function $F(y|x)$ is nonincreasing as a function of x (see [3, Theorem A.3, 1, Proposition 2.5], respectively). Our goal is to make these partial frontier functions monotone nondecreasing on some given subset D interior to the support of X in a more general setup, i.e. without relying on such an assumption.

This is achieved through the following isotonization method: we denote by $||\cdot||$ the sup-norm of a real-valued function over the domain D and we assume from now on that this domain is compact. For a real-valued function r defined on D, let us define the following three functions:

$$\begin{aligned} r^u(x) &= \sup_{x' \in D; x' \leqslant x} r(x'), \\ r^l(x) &= \inf_{x' \in D; x' \geqslant x} r(x'), \\ r^{\#}(x) &= (r^u(x) + r^l(x))/2. \end{aligned} \tag{1}$$

[4] Following Hsu and Robbins [10], we say that a sequence of random variables $\{X_n\}$ converges completely to a random variable X if $\sum_{n=1}^{\infty} \text{Prob}(|X_n - X| > \varepsilon) < \infty$ for every $\varepsilon > 0$.

It is clear that $r^u(x)$, $r^l(x)$ and $r^\#(x)$ are nondecreasing and that $r^l(x) \leqslant r(x) \leqslant r^u(x)$, for all x in their domain D.

A natural concept of a monotone order-m frontier can then be defined simply as the isotonized version $\varphi_m^\#(x)$ of $\varphi_m(x)$. This nondecreasing partial function can be estimated nonparametrically by the isotonized version $\widehat{\varphi}_{m,n}^\#(x)$ of $\widehat{\varphi}_{m,n}(x)$.

The basic idea of this monotonization procedure is not new. Mukerjee and Stern [11] use a similar principle to isotonize a Nadaraya–Watson kernel estimator of the regression function, and with a slight difference, which is in fact a computational artifact: in their approach, the sup and inf in (1) are taken over a discrete grid instead of the whole domain D. In the context of production efficiency measurement, Aragon et al. [2] use the same technique to isotonize a smoothed estimator of the nonstandard conditional distribution function $F(y|x)$ with respect to x, but in the nonincreasing sense. They prove that when the initial smoothed estimator is strongly uniformly consistent and the function $x \mapsto F(y|x)$ is nonincreasing for y fixed, then the isotonized estimator is also strongly uniformly consistent. Their argument is based on the fact that the # operator, which provides in their approach a nonincreasing version of r on D, is sup-norm contracting (see [2, Lemma 3.3]). In our setup, we only need to adapt this result to our # operator which rather provides a nondecreasing version of r on D. This ingenious property of the operator # allows to show that the monotone estimator $\widehat{\varphi}_{m(n),n}^\#$ converges uniformly and completely to the full frontier function φ and is globally closer to φ than the non-isotone estimator $\widehat{\varphi}_{m(n),n}$. This can be seen from the following theorem.

Theorem 2.1. *Assume that F_X, φ and φ_m are continuous on the compact D, for every $m \geqslant 1$, and that the upper boundary of the support of Y is finite. Then*

$$||\widehat{\varphi}_{m(n),n}^\# - \varphi|| \leqslant ||\widehat{\varphi}_{m(n),n} - \varphi|| \xrightarrow{\text{co.}} 0 \quad as \quad n \to \infty,$$

where the integer $m(n) \geqslant 1$ is such that

$$\lim_{n\to\infty} m(n) = \infty, \qquad \lim_{n\to\infty} m(n)(\log n/n)^{1/2} = 0.$$

Note that this result extends the weak pointwise consistency of $\widehat{\varphi}_{m(n),n}(x)$ for $\varphi(x)$ proved in [3] to the complete uniform convergence. The next result gives a more subtle convergence rate of $m(n)$ as n tends to infinity, but the stochastic convergence here is only in the almost sure sense.

Theorem 2.2. *Under the conditions of Theorem 2.1, we have*

$$||\widehat{\varphi}_{m(n),n}^\# - \varphi|| \leqslant ||\widehat{\varphi}_{m(n),n} - \varphi|| \xrightarrow{\text{a.s.}} 0 \quad as \quad n \to \infty,$$

where $\lim_{n\to\infty} m(n) = \infty$, *and* $\lim_{n\to\infty} m(n)(\log\log n/n)^{1/2} = 0$.

Likewise, by isotonizing the quantile frontiers, the global error estimation for estimating φ becomes weaker and converges uniformly and completely to 0 as n goes to ∞.

Theorem 2.3. *Under the conditions of Theorem 2.1, we have*

$$||\widehat{q}^{\#}_{\alpha(n),n} - \varphi|| \leqslant ||\widehat{q}_{\alpha(n),n} - \varphi|| \xrightarrow{\text{co.}} 0 \quad as \quad n \to \infty,$$

where the order $\alpha(n)$ *is such that* $n(1-\alpha(n)) \to 0$ *as* $n \to \infty$.

Here, the initial estimator $\widehat{q}_{\alpha(n),n}(x)$ and its isotone version $\widehat{q}^{\#}_{\alpha(n),n}(x)$ estimate the full frontier $\varphi(x)$ itself. As expected by Aragon et al. [1, Theorem 4.3], when the order $\alpha(n)$ converges to 1 at the rate $n^{(p+2)/(p+1)}$ as $n \to \infty$, the random error $n^{1/(p+1)}(\varphi(x) - \widehat{q}_{\alpha(n),n}(x))$ converges to a Weibull distribution whose parameters depend on the joint density of (X, Y) near the frontier point $(x, \varphi(x))$.

The next corollaries exhibit bounds on the related probabilities $P(||\widehat{\varphi}^{\#}_{m(n),n} - \varphi|| > \varepsilon)$ and $P(||\widehat{q}^{\#}_{\alpha(n),n} - \varphi|| > \varepsilon)$, for $\varepsilon > 0$, showing that they converge to 0 at an exponential rate. Naturally, this implies the uniform complete convergence results established above, but requires more powerful techniques of proof. To prove these results we shall use Lemma 3.4 (see next section).

Corollary 2.4. *Let* $m(n) \geqslant 1$ *be an integer such that* $\lim_{n\to\infty} m(n) = \infty$. *Given the conditions of Theorem 2.1 on* F_X, φ, $\{\varphi_m\}$ *and the upper boundary* ν *of the support of* Y, *there exists a finite positive constant* C *such that for all* $r > 0$, $\lambda > 1$ *and all* n *sufficiently large*

$$\begin{aligned} P(||\widehat{\varphi}^{\#}_{m(n),n} - \varphi|| > \lambda r) &\leqslant P(||\widehat{\varphi}_{m(n),n} - \varphi|| > \lambda r) \\ &\leqslant C \left\{ \exp\left(-nr^2 \left(\inf_{x\in D} F_X(x) \right)^2 \Big/ (4m(n)\nu)^2 \right) \right. \\ &\quad \left. + \exp\left(-n \left(1 - \frac{1}{\lambda}\right)^2 \left(\inf_{x\in D} F_X(x) \right)^2 \right) \right\}. \end{aligned}$$

Corollary 2.5. *Let* $\alpha(n) \in (0, 1)$ *be such that* $\lim_{n\to\infty} n(1-\alpha(n)) = 0$. *Under the conditions of Corollary 2.4, there exists a constant* $C \in (0, \infty)$ *such that for all* $r > 0$, $\lambda > 1$ *and all* n *large enough*

$$\begin{aligned} P(||\widehat{q}^{\#}_{\alpha(n),n} - \varphi|| > \lambda r) &\leqslant P(||\widehat{q}_{\alpha(n),n} - \varphi|| > \lambda r) \\ &\leqslant C \left\{ \exp\left(-nr^2 \left(\inf_{x\in D} F_X(x) \right)^2 \Big/ (8m(n)\nu)^2 \right) \right. \\ &\quad + \exp\left(-n \left(1 - \frac{1}{\lambda}\right)^2 \left(\inf_{x\in D} F_X(x) \right)^2 \right) \\ &\quad \left. + \exp\left(-n(\lambda-1)^2 \left(\sup_{x\in D} F_X(x) \right)^2 \right) \right\}. \end{aligned}$$

3. Lemmas and proofs

The following lemma asserts that the # operator is sup-norm contracting.

Lemma 3.1. *If r and s are two functions defined on D, then*

$$||r^{\#} - s^{\#}|| \leqslant ||r - s||.$$

Proof. Let $M = \sup_{x\in D} |r(x) - s(x)|$. The lemma will follow from the following sets of inequalities:

$$r^u - M \leqslant s^u \leqslant r^u + M,$$
$$r^l - M \leqslant s^l \leqslant r^l + M.$$

The two right inequalities follow from taking the $\sup_{x'\leqslant x}$ (resp., the $\inf_{x'\geqslant x}$) in the inequality $s(x') \leqslant r(x') + M$, and the left ones follow from taking the $\sup_{x'\leqslant x}$ (resp., the $\inf_{x'\geqslant x}$) in the inequality $r(x') - M \leqslant s(x')$. □

We know from Florens and Simar [8, see the appendix, Proof of Lemma A.1] that $\widehat{\varphi}_{m,n}$ converges uniformly in probability to φ_m as $n \to \infty$. By applying Lemma 3.1, we obtain

$$||\widehat{\varphi}^{\#}_{m,n} - \varphi^{\#}_m|| \leqslant ||\widehat{\varphi}_{m,n} - \varphi_m||,$$

which implies the weak uniform consistency of $\widehat{\varphi}^{\#}_{m,n}$ for $\varphi^{\#}_m$. This result can be improved to obtain the complete uniform convergence by using the following lemma.

Lemma 3.2. *Assume that F_X is continuous on the compact D and that the upper boundary of the support of Y is finite. Then,*

$$||\widehat{\varphi}_{m,n} - \varphi_m|| \xrightarrow{\text{co.}} 0 \quad as \quad n \to \infty.$$

Proof. Let $v < \infty$ be the upper boundary of the support of Y and let $x \in D$. Since $\widehat{\varphi}_n(x) \leqslant \varphi(x) \leqslant v$ with probability 1 (for a proof, see [1, Section 3]), we have with probability 1,

$$\widehat{\varphi}_{m,n}(x) = \int_0^{\widehat{\varphi}_n(x)} (1 - [\widehat{F}_n(y|x)]^m)\, dy = \int_0^v (1 - [\widehat{F}_n(y|x)]^m)\, dy.$$

We therefore obtain, with probability 1,

$$\begin{aligned}\widehat{\varphi}_{m,n}(x) - \varphi_m(x) &= \int_0^v ([F(y|x)]^m - [\widehat{F}_n(y|x)]^m)\, dy \\ &= \int_0^v (F(y|x) - \widehat{F}_n(y|x)) \sum_{j=0}^{m-1} [F(y|x)]^{m-1-j} [\widehat{F}_n(y|x)]^j\, dy.\end{aligned}$$

This implies, with probability 1,

$$\begin{aligned}
&|\widehat{\varphi}_{m,n}(x) - \varphi_m(x)| \\
&\quad\leqslant m\int_0^v |F(y|x) - \widehat{F}_n(y|x)|\,dy \\
&\quad= m\int_0^v \frac{|\widehat{F}_{X,n}(x)F(x,y) - F_X(x)\widehat{F}_n(x,y)|}{F_X(x)\widehat{F}_{X,n}(x)}\,dy \\
&\quad\leqslant m\int_0^v \frac{F(x,y)|\widehat{F}_{X,n}(x) - F_X(x)| + F_X(x)|\widehat{F}_n(x,y) - F(x,y)|}{F_X(x)\widehat{F}_{X,n}(x)}\,dy \\
&\quad\leqslant \frac{mv}{\widehat{F}_{X,n}(x)}(||\widehat{F}_{X,n} - F_X|| + ||\widehat{F}_n - F||).
\end{aligned}$$

Thus, we have with probability 1,

$$||\widehat{\varphi}_{m,n} - \varphi_m|| \leqslant \frac{mv}{\inf_{x\in D}\ \widehat{F}_{X,n}(x)}(||\widehat{F}_{X,n} - F_X|| + ||\widehat{F}_n - F||). \tag{2}$$

To complete the proof, it suffices to show that the term on the right-hand side of Inequality (2) converges completely to 0 as $n \to \infty$. We know from Glivenko–Cantelli theorem [14, see the proof of Theorem A, p. 61] that $||\widehat{F}_{X,n} - F_X||$ and $||\widehat{F}_n - F||$ converge completely to 0 as $n \to \infty$. Hence, it only remains to show that

$$\exists \delta > 0 \quad \text{such that} \quad \sum_{n=1}^{\infty} P\left(\inf_{x\in D}\ \widehat{F}_{X,n}(x) \leqslant \delta\right) < \infty. \tag{3}$$

Indeed, it can be easily seen that, if $\{V_n\}$ and $\{W_n\}$ are two sequences of random variables s.t. V_n converges completely to 0 and there exists $\delta > 0$ s.t. $\sum_{n=1}^{\infty} P(|W_n| \leqslant \delta) < \infty$, then V_n/W_n converges completely to 0.

Since $|\inf_{x\in D}\ \widehat{F}_{X,n}(x) - \inf_{x\in D}\ F_X(x)| \leqslant ||\widehat{F}_{X,n} - F_X||$ and $||\widehat{F}_{X,n} - F_X||$ converges completely to 0, we obtain $\sum_{n=1}^{\infty} P(|\inf_{x\in D}\ \widehat{F}_{X,n}(x) - \inf_{x\in D}\ F_X(x)| \geqslant \delta) < \infty$, for every $\delta > 0$. This yields $\sum_{n=1}^{\infty} P(\inf_{x\in D}\ \widehat{F}_{X,n}(x) \leqslant \inf_{x\in D}\ F_X(x) - \delta) < \infty$, $\forall \delta > 0$. Thus, we can end the proof by putting $\delta = \inf_{x\in D}\ F_X(x)/2 > 0$. $\square$

As an immediate consequence of Lemmas 3.1 and 3.2, we obtain the complete convergence of $||\widehat{\varphi}^{\#}_{m,n} - \varphi^{\#}_m||$ to 0 as $n \to \infty$. Furthermore, we know from Florens and Simar [8, see the proof of Lemma A.1] that φ_m converges uniformly to φ as $m \to \infty$, provided of course that φ and φ_m are continuous on D, for every $m \geqslant 1$. Therefore,

$$||\varphi^{\#}_m - \varphi|| = ||\varphi^{\#}_m - \varphi^{\#}|| \leqslant ||\varphi_m - \varphi|| \longrightarrow 0 \quad \text{as} \quad m \to \infty.$$

This indicates that the isotone partial order-m function $\varphi^{\#}_m$ is closer in sup-norm to the true frontier function φ than the initial version φ_m. This property remains valid with estimators $\widehat{\varphi}^{\#}_{m(n),n}$ and $\widehat{\varphi}_{m(n),n}$ of φ as it is shown by Theorem 2.1.

Proof of Theorem 2.1. First let us show that $||\widehat{\varphi}_{m(n),n} - \varphi_{m(n)}||$ converges completely to 0 as $n \to \infty$. Let $\varepsilon > 0$. We know from Kiefer's Inequality [14, Theorem B, p. 61] that there exist finite positive constants C_1 and C_2 (not depending on F and F_X) such that

$$P(||\widehat{F}_n - F|| > d) \leqslant C_1 e^{-nd^2}, \quad P(||\widehat{F}_{X,n} - F_X|| > d) \leqslant C_2 e^{-nd^2} \tag{4}$$

for every $d > 0$ and all $n \geqslant 1$. By taking $\varepsilon/m(n) > 0$ in place of d in the above inequalities, we obtain

$$P(m(n)||\widehat{F}_n - F|| > \varepsilon) \leqslant C_1 e^{-n\varepsilon^2/m^2(n)},$$
$$P(m(n)||\widehat{F}_{X,n} - F_X|| > \varepsilon) \leqslant C_2 e^{-n\varepsilon^2/m^2(n)}$$

for all $n \geqslant 1$. Since $\lim_{n\to\infty}(m^2(n)\log n)/n = 0$, we have $(m^2(n)\log n)/n \leqslant \varepsilon^2/2$, for n large enough. Hence $\exp(-n\varepsilon^2/m^2(n)) \leqslant n^{-2}$, for all n sufficiently large. This implies

$$\sum_{n=1}^{\infty} P(m(n)||\widehat{F}_n - F|| > \varepsilon) < \infty, \quad \sum_{n=1}^{\infty} P(m(n)||\widehat{F}_{X,n} - F_X|| > \varepsilon) < \infty$$

showing therefore that $m(n)||\widehat{F}_n - F||$ and $m(n)||\widehat{F}_{X,n} - F_X||$ converge completely to 0. Thus, $\widehat{\varphi}_{m(n),n}$ converges completely and uniformly to $\varphi_{m(n)}$ in view of (2) and (3). Since $\lim_{n\to\infty} m(n) = \infty$ and $\lim_{m\to\infty} ||\varphi_m - \varphi|| = 0$, we have $\lim_{n\to\infty} ||\varphi_{m(n)} - \varphi|| = 0$. Finally, we obtain the desired result by using the following inequalities:

$$||\widehat{\varphi}^{\#}_{m(n),n} - \varphi|| \leqslant ||\widehat{\varphi}_{m(n),n} - \varphi|| \leqslant ||\widehat{\varphi}_{m(n),n} - \varphi_{m(n)}|| + ||\varphi_{m(n)} - \varphi||. \tag{5}$$

This completes the proof. □

Proof of Theorem 2.2. We only need to show that $m(n)||\widehat{F}_n - F||$ and $m(n)||\widehat{F}_{X,n} - F_X||$ converge almost surely to 0, and then we follow the same setup used to prove the last result of Theorem 2.1. We have from the law of the iterated logarithm [14, Theorem B, p. 62],

$$||\widehat{F}_{X,n} - F_X|| \leqslant 2C(F_X)(\log\log n/n)^{1/2}, \quad ||\widehat{F}_n - F|| \leqslant 2C(F)(\log\log n/n)^{1/2}$$

for all n large enough, with probability 1, where $C(F_X)$ and $C(F)$ are two finite positive constants. Since $\lim_{n\to\infty} m(n)(\log\log n/n)^{1/2} = 0$, the conclusion follows directly from the above inequalities. □

Making use of Lemma 3.2, we also can improve the weak uniform consistency of the FDH estimator $\widehat{\varphi}_n$ by adapting the proof of Florens and Simar [8, Lemma A.1].

Lemma 3.3. *Under the same regularity conditions of Theorem 2.1, we have*

$$||\widehat{\varphi}_n - \varphi|| \xrightarrow{\text{co.}} 0 \quad as \quad n \to \infty.$$

Proof. Let $\varepsilon > 0$ and $n \geqslant 1$. Since φ_m converges uniformly to φ as $m \to \infty$, we have

$$\exists m_\varepsilon \text{ such that } ||\varphi_{m_\varepsilon} - \varphi|| < \varepsilon/2. \tag{6}$$

We also have in view of Lemma 3.2,

$$\sum_{n=1}^{\infty} P(||\widehat{\varphi}_{m_\varepsilon,n} - \varphi_{m_\varepsilon}|| > \varepsilon/2) < \infty. \tag{7}$$

We know that $\widehat{\varphi}_{m_\varepsilon,n}(x) \leqslant \widehat{\varphi}_n(x) \leqslant \varphi(x)$ with probability 1, for any $x \in D$. Here, we need to extend this result to show that

$$\forall x \in D,\ \widehat{\varphi}_{m_\varepsilon,n}(x) \leqslant \widehat{\varphi}_n(x) \leqslant \varphi(x) \tag{8}$$

with probability 1. We know that $y \leqslant \varphi(x)$ for any $(x, y) \in \Psi$ such that $F_X(x) > 0$. Since the random variable $F_X(X_i)$ is uniform on (0, 1), it is almost surely strictly positive, and since $(X_i, Y_i) \in \Psi$ almost surely, we have $Y_i \leqslant \varphi(X_i)$ with probability 1. Put $\Omega_i = \{Y_i \leqslant \varphi(X_i)\}$, $i = 1, \ldots, n$. We have $P(\Omega_i)$=1, for $i = 1, \ldots, n$. Let $\Omega_0 = \cap_{i=1}^n \Omega_i$. Then $P(\Omega_0) = 1$. To prove (8), it is sufficient to show that $\Omega_0 \subset \{\forall x \in D,\ \max_{i|X_i \leqslant x} Y_i \leqslant \varphi(x)\}$. If $\omega \in \Omega_0$, then $Y_i(\omega) \leqslant \varphi(X_i(\omega))$, for all $i = 1, \ldots, n$. In particular, we obtain by using the monotonicity of φ,

$$\forall x \in D \quad \forall i \text{ such that } X_i(\omega) \leqslant x \ :\ Y_i(\omega) \leqslant \varphi(X_i(\omega)) \leqslant \varphi(x).$$

Hence, $\max_{i|X_i(\omega) \leqslant x} Y_i(\omega) \leqslant \varphi(x)$ for any $x \in D$, and thus we obtain, $\omega \in \{\forall x \in D,\ \max_{i|X_i \leqslant x} Y_i \leqslant \varphi(x)\}$. This ends the proof of (8). Now we obtain by using (8),

$$||\widehat{\varphi}_n - \varphi|| \leqslant ||\widehat{\varphi}_{m_\varepsilon,n} - \varphi|| \leqslant ||\widehat{\varphi}_{m_\varepsilon,n} - \varphi_{m_\varepsilon}|| + ||\varphi_{m_\varepsilon} - \varphi|| \tag{9}$$

with probability 1. Combining with (6), we get

$$\begin{aligned} P(||\widehat{\varphi}_n - \varphi|| > \varepsilon) &\leqslant P(||\widehat{\varphi}_{m_\varepsilon,n} - \varphi_{m_\varepsilon}|| + \varepsilon/2 > ||\widehat{\varphi}_{m_\varepsilon,n} - \varphi_{m_\varepsilon}|| + ||\varphi_{m_\varepsilon} - \varphi|| > \varepsilon) \\ &\leqslant P(||\widehat{\varphi}_{m_\varepsilon,n} - \varphi_{m_\varepsilon}|| > \varepsilon/2). \end{aligned}$$

Thus $\sum_{n=1}^{\infty} P\left(||\widehat{\varphi}_n - \varphi|| > \varepsilon\right) < \infty$, in view of (7). □

Likewise, in place of looking to the α-quantile function $q_\alpha(x)$ and its estimator $\widehat{q}_{\alpha,n}(x)$, we rather concentrate on their isotonic versions $q^{\#}_\alpha(x)$ and $\widehat{q}^{\#}_{\alpha,n}(x)$. We know from Aragon et al. [1, Proposition 2.4] that, if $x \mapsto q_\alpha(x)$ is continuous on the compact D, for every $\alpha \in [0, 1]$, then q_α converges uniformly to φ as $\alpha \nearrow 1$ and so, we obtain by using Lemma 3.1,

$$||q^{\#}_\alpha - \varphi|| \leqslant ||q_\alpha - \varphi|| \longrightarrow 0 \quad \text{as} \quad \alpha \nearrow 1.$$

The estimators $\widehat{q}_{\alpha(n),n}$ and $\widehat{q}^{\#}_{\alpha(n),n}$ of φ fulfill the same property as indicated in Theorem 2.3.

Proof of Theorem 2.3. It follows from [1] (see the appendix: last inequality of the proof of Theorem 4.3) that, for any $\alpha > 0$ and all $x \in D$,

$$0 \leqslant \widehat{\varphi}_n(x) - \widehat{q}_{\alpha,n}(x) \leqslant n(1-\alpha)v\widehat{F}_{X,n}(x) \tag{10}$$

with probability 1, where $v < \infty$ denotes the upper boundary of the support of Y. This implies, for any $\alpha > 0$,

$$||\widehat{\varphi}_n - \widehat{q}_{\alpha,n}|| \leqslant n(1-\alpha)v(||\widehat{F}_{X,n} - F_X|| + ||F_X||)$$

with probability 1. Therefore, by choosing α as a function of n converging to 1, such that $n(1 - \alpha(n)) \to 0$ as $n \to \infty$, we obtain by using Glivenko–Cantelli Theorem and the continuity of F_X on D ($||F_X|| < \infty$),

$$||\widehat{\varphi}_n - \widehat{q}_{\alpha(n),n}|| \xrightarrow{\text{co.}} 0 \quad \text{as} \quad n \to \infty. \tag{11}$$

Thus, we get by applying Lemma 3.1,

$$||\widehat{q}^{\#}_{\alpha(n),n} - \varphi|| \leqslant ||\widehat{q}_{\alpha(n),n} - \varphi|| \leqslant ||\widehat{q}_{\alpha(n),n} - \widehat{\varphi}_n|| + ||\widehat{\varphi}_n - \varphi|| \tag{12}$$

which converges completely to 0 as $n \to \infty$, in view of (11) and Lemma 3.3. □

The following lemma provides a useful bound for probabilities of large deviations. It indicates that $P(||\widehat{\varphi}_{m,n} - \varphi_m|| > \varepsilon) \to 0$ exponentially fast.

Lemma 3.4. *Given the conditions of Lemma 3.2 on F_X and the upper boundary v of the support of Y, there exists a finite positive constant C (not depending on F) such that for all $r > 0$, $\lambda > 1$ and all $n \geqslant 1$,*

$$P(||\widehat{\varphi}_{m,n} - \varphi_m|| > \lambda r)$$
$$\leqslant C \left\{ \exp\left(-nr^2 \left(\inf_{x \in D} F_X(x) \right)^2 \Big/ (2mv)^2 \right) + \exp\left(-n \left(1 - \frac{1}{\lambda}\right)^2 \left(\inf_{x\in D} F_X(x)\right)^2 \right) \right\}.$$

Proof. We have from (2), with probability 1

$$||\widehat{\varphi}_{m,n} - \varphi_m|| \leqslant \frac{mv}{\inf_{x\in D} F_X(x)} \frac{\inf_{x\in D} F_X(x)}{\inf_{x\in D} \widehat{F}_{X,n}(x)} (||\widehat{F}_{X,n} - F_X|| + ||\widehat{F}_n - F||).$$

According to Kiefer's Inequality (4), there exist finite positive constants C_1 and C_2 such that for all $n \geqslant 1$ and $r > 0$,

$$P\left[\frac{mv}{\inf_{x\in D} F_X(x)} (||\widehat{F}_{X,n} - F_X|| + ||\widehat{F}_n - F||) > r \right]$$
$$\leqslant P\left[||\widehat{F}_n - F|| > \frac{(\inf_{x\in D} F_X(x))r}{2mv} \right] + P\left[||\widehat{F}_{X,n} - F_X|| > \frac{(\inf_{x\in D} F_X(x))r}{2mv} \right]$$
$$\leqslant (C_1 + C_2) \exp\left\{ -n \left(\frac{(\inf_{x\in D} F_X(x))r}{2mv} \right)^2 \right\}.$$

It also can be easily seen, for every $\lambda > 1$,

$$P\left[\frac{\inf_{x\in D} F_X(x)}{\inf_{x\in D} \widehat{F}_{X,n}(x)} > \lambda \right] = P\left[\inf_{x\in D} F_X(x) - \inf_{x\in D} \widehat{F}_{X,n}(x) > \frac{\lambda - 1}{\lambda} \inf_{x\in D} F_X(x) \right]$$
$$\leqslant P\left[||\widehat{F}_{X,n} - F_X|| > \frac{\lambda-1}{\lambda} \inf_{x\in D} F_X(x) \right]$$
$$\leqslant C_2 \exp\left\{ -n \left(\frac{\lambda - 1}{\lambda} \inf_{x\in D} F_X(x) \right)^2 \right\}.$$

Therefore, we have for all $r > 0$, $\lambda > 1$ and all $n \geqslant 1$

$$\begin{aligned}
&P[||\widehat{\varphi}_{m,n} - \varphi_m|| > \lambda r] \\
&\quad\leqslant P\left[\frac{mv(||\widehat{F}_{X,n} - F_X|| + ||\widehat{F}_n - F||)}{\inf_{x\in D} F_X(x)} > r\right] + P\left[\frac{\inf_{x\in D} F_X(x)}{\inf_{x\in D} \widehat{F}_{X,n}(x)} > \lambda\right] \\
&\quad\leqslant (C_1 + C_2)\exp\left\{-nr^2\left(\inf_{x\in D} F_X(x)/2mv\right)^2\right\} \\
&\qquad + C_2 \exp\left\{-n\left(1-\frac{1}{\lambda}\right)^2\left(\inf_{x\in D} F_X(x)\right)^2\right\}.
\end{aligned}$$

This ends the proof by putting $C = C_1 + C_2$. □

By applying the fact that the # operator is sup-norm contracting, we get the same exponential bound for $P(||\widehat{\varphi}^{\#}_{m,n} - \varphi^{\#}_m|| > \lambda r)$. Even more strongly, we have for every $N \geqslant 1$

$$\begin{aligned}
P\left(\sup_{n\geqslant N} ||\widehat{\varphi}^{\#}_{m,n} - \varphi^{\#}_m|| > \lambda r\right) &\leqslant \sum_{n\geqslant N} P(||\widehat{\varphi}^{\#}_{m,n} - \varphi^{\#}_m|| > \lambda r) \\
&\leqslant \frac{Ce^{-Nr^2(\inf_{x\in D} F_X(x))^2/(2mv)^2}}{1 - e^{-r^2(\inf_{x\in D} F_X(x))^2/(2mv)^2}} \\
&\quad + \frac{Ce^{-N(1-\frac{1}{\lambda})^2(\inf_{x\in D} F_X(x))^2}}{1 - e^{-(1-\frac{1}{\lambda})^2(\inf_{x\in D} F_X(x))^2}}.
\end{aligned}$$

As a consequence of Lemma 3.4, we also can prove Corollaries 2.4 and 2.5.

Proof of Corollary 2.4. We have from Inequality (5), for any $\lambda > 1$, $r > 0$ and $n \geqslant 1$

$$\begin{aligned}
&P[||\widehat{\varphi}_{m(n),n} - \varphi|| > \lambda r] \\
&\quad\leqslant P[||\widehat{\varphi}_{m(n),n} - \varphi_{m(n)}|| > \lambda r/2] + P[||\varphi_{m(n)} - \varphi|| > \lambda r/2].
\end{aligned}$$

Since $||\varphi_{m(n)} - \varphi|| \to 0$ as $n \to \infty$, the second probability of the term on the right-hand side is zero for n large enough. Therefore $P[||\widehat{\varphi}_{m(n),n} - \varphi|| > \lambda r] \leqslant P[||\widehat{\varphi}_{m(n),n} - \varphi_{m(n)}|| > \lambda r/2]$ for all $\lambda > 1$, $r > 0$ and all n sufficiently large. The desired result follows thus by applying Lemma 3.4. □

Proof of Corollary 2.5. We know from the proof of Lemma 3.3 that $||\widehat{\varphi}_n - \varphi|| \leqslant ||\widehat{\varphi}_{m(n),n} - \varphi||$ with probability 1 (it suffices to replace m_ε by $m(n)$ in (9)). Therefore, by making use of Corollary 2.4, we obtain for all $\lambda > 1$, $r > 0$ and all n sufficiently large

$$\begin{aligned}
&P[||\widehat{\varphi}_n - \varphi|| > \lambda r/2] \\
&\quad\leqslant C\left\{e^{-nr^2(\inf_{x\in D} F_X(x)/8m(n)v)^2} + e^{-n(1-\frac{1}{\lambda})^2(\inf_{x\in D} F_X(x))^2}\right\}, \qquad (13)
\end{aligned}$$

where $C > 0$ is a finite constant. We also have from the proof of Theorem 2.3 (see (10))

$$||\widehat{\varphi}_n - \widehat{q}_{\alpha(n),n}|| \leqslant n(1-\alpha(n))v\left(\sup_{x\in D} \widehat{F}_{X,n}(x) \Big/ \sup_{x\in D} F_X(x)\right)$$

with probability 1. Hence

$$P[||\widehat{\varphi}_n - \widehat{q}_{\alpha(n),n}|| > \lambda r/2] \leqslant P\left[\frac{\sup_{x\in D} \widehat{F}_{X,n}(x)}{\sup_{x\in D} F_X(x)} > \lambda] + P[n(1-\alpha(n))v > r/2\right]. \tag{14}$$

Since $n(1-\alpha(n)) \to 0$ as $n \to \infty$, we obtain $n(1-\alpha(n))v < r/2$ for n sufficiently large. On the other hand, we have via (4)

$$\begin{aligned} P\left[\frac{\sup_{x\in D} \widehat{F}_{X,n}(x)}{\sup_{x\in D} F_X(x)} > \lambda\right] &= P\left[\sup_{x\in D} \widehat{F}_{X,n}(x) - \sup_{x\in D} F_X(x) > (\lambda-1)\sup_{x\in D} F_X(x)\right] \\ &\leqslant P\left[||\widehat{F}_{X,n} - F_X|| > (\lambda-1)\sup_{x\in D} F_X(x)\right] \\ &\leqslant C_2 \exp\left\{-n(\lambda-1)^2\left(\sup_{x\in D} F_X(x)\right)^2\right\}. \end{aligned}$$

We finally conclude by using (12) in conjunction with (13) and (14). □

4. Algorithms for practical computation

In practice, to compute the monotone frontier $\widehat{\varphi}^{\#}_{m,n}$ (in the same way $\widehat{q}^{\#}_{\alpha,n}$), we use a discrete grid instead of the whole domain D. For instance, we could consider the minimal rectangular set with edges parallel to the coordinate axes that covers all the observations X_i, and then choose a discrete grid $D_n = \{x_{n,1}, \ldots, x_{n,k}\}$ in this rectangular set containing the unique minimal and maximal (with respect to the partial order "$\leqslant$") points of this set (we could choose D_n to be simply the set of the observation points $\{X_i\}$ besides the minimal and maximal points of the minimal envelopment rectangular set). Such a choice makes it easier to compute both $\widehat{\varphi}_{m,n}(x)$ and $\widehat{\varphi}^{\#}_{m,n}(x)$ over the rectangular set. For example, if $p = 1$ and $x_{n,1} \leqslant \cdots \leqslant x_{n,k}$, then $\widehat{\varphi}^{l}_{m,n}$ and $\widehat{\varphi}^{u}_{m,n}$ are constant between successive points such that

$$\begin{aligned} \widehat{\varphi}^{l}_{m,n}(x_{n,i}) &= \widehat{\varphi}^{l}_{m,n}(x_{n,i+1}) \wedge \widehat{\varphi}_{m,n}(x_{n,i}), \\ \widehat{\varphi}^{u}_{m,n}(x_{n,i+1}) &= \widehat{\varphi}^{u}_{m,n}(x_{n,i}) \vee \widehat{\varphi}_{m,n}(x_{n,i+1}) \end{aligned}$$

for all $i = 1, \ldots, k-1$. Note that in this case, the choice of $D_n = \{X_i\}$ happens to be more natural for the quantile framework since the initial frontier $\widehat{q}_{\alpha,n}$ is by construction constant between successive observations X_i. For the general case ($p \geqslant 1$), first compute $\widehat{\varphi}^{u}_{m,n}$ successively along D_n starting from its minimal point, using the fact that

$$\widehat{\varphi}^{u}_{m,n}(x_{n,i}) = \widehat{\varphi}_{m,n}(x_{n,i}) \vee \max\{\widehat{\varphi}^{u}_{m,n}(x_{n,j}) : \ x_{n,j} \text{ is an immediate predecessor of } x_{n,i}\}$$

for all $x_{n,i} \in D_n$. Compute also $\widehat{\varphi}^{l}_{m,n}$ successively along D_n starting this time from its maximal point, using the fact that

$$\widehat{\varphi}^{l}_{m,n}(x_{n,i}) = \widehat{\varphi}_{m,n}(x_{n,i}) \wedge \min\{\widehat{\varphi}^{l}_{m,n}(x_{n,j}) : \ x_{n,j} \text{ is an immediate successor of } x_{n,i}\}.$$

The isotonic order-m frontier $\widehat{\varphi}^{\#}_{m,n}(x)$ can be therfore easily computed, for any x in the rectangular set, as the mean of

$$\widehat{\varphi}^{u}_{m,n}(x) = \max_{x_{n,i} \in D_n | x_{n,i} \leqslant x} \widehat{\varphi}^{u}_{m,n}(x_{n,i}) \quad \text{and} \quad \widehat{\varphi}^{l}_{m,n}(x) = \min_{x_{n,i} \in D_n | x_{n,i} \geqslant x} \widehat{\varphi}^{l}_{m,n}(x_{n,i}).$$

It is clear that a large value of k is necessary to get a good result in practice. We will see a numerical illustration in Section 5.

Mukerjee and Stern [11] perform a very closely similar isotonization algorithm by using an appropriate choice of D_n that leads to the strong uniform consistency of their isotonic estimator. We can easily adapt their setup to our problem by taking φ, $\widehat{\varphi}_{m(n),n}$, $\widehat{\varphi}^{\#}_{m(n),n}$, $\widehat{\varphi}^{u}_{m(n),n}$, $\widehat{\varphi}^{l}_{m(n),n}$ and D in place of the quantities τ, $\widehat{\tau}_n$, G_n, G_{1n}, G_{2n} and H in [11] (see Section 2), respectively (the same construction can be done for the quantile framework):

For $\delta > 0$, let $D_\delta \supset D$ be the closed δ-neighborhood of D which we assume to be interior to the support of X. Let the initial estimator $\widehat{\varphi}_{m(n),n}(x)$ of the monotone upper boundary $\varphi(x)$ be defined on D_δ with $\widehat{\varphi}_{m(n),n}(x) = 0$ if $\widehat{F}_{X,n}(x) = 0$. Consider a positive sequence $\{b_n\}$ tending to 0, and let D_n be the set of vectors in D_δ with components that are integral multiples of b_n. For $\widehat{\varphi}^{\#}_{m(n),n}(x)$ to be well defined for $x \in D$ (see [11, Eq. (2)]), we assume that n is large enough.

As stated by Mukerjee and Stern, if D is rectangular with edges parallel to the coordinate axes, as is often the case, then we could consider only the minimal subset of D_n that covers D by convex combinations. The minimal and maximal points of this subset being unique, we then can isotonize $\widehat{\varphi}_{m(n),n}(x)$ over D, for a given order $m(n)$, by applying the computation method described above.

From a theoretical point of view, since D_n is not contained in D, we cannot apply Lemma 3.1 to obtain the complete uniform convergence of $\widehat{\varphi}^{\#}_{m(n),n}$ to φ on D (see Theorem 2.1). However, we can easily adapt the proof of Mukerjee and Stern to keep this asymptotic property. But such technique of proof requires more stringent conditions compared with those of Theorem 2.1. Indeed, if φ is uniformly continuous on D_δ and φ_m is continuous on this compact for every $m \geqslant 1$, then the same arguments used by Mukerjee and Stern (see [11, the paragraph after Eq. (4), p. 78]) show that

$$||\widehat{\varphi}^{\#}_{m(n),n} - \varphi|| \leqslant \sup_{x \in D_\delta} |\widehat{\varphi}_{m(n),n}(x) - \varphi(x)| + R_n,$$

where the remainder $R_n = o(1)$ in view of the appropriate characterization of D_n and the uniform continuity of φ on D_δ (for more details see [11, Theorem 2, the proof of Eq. (6)]). Finally, using the fact that $\sup_{x \in D_\delta} |\widehat{\varphi}_{m(n),n}(x) - \varphi(x)| \overset{\text{co.}}{\to} 0$ (replace D by D_δ in the proof of Theorem 2.1 to obtain this result), we obtain the complete uniform convergence of $\widehat{\varphi}^{\#}_{m(n),n}$ to φ on D. Under the same regularity conditions, we also get the complete uniform convergence of $\widehat{q}^{\#}_{\alpha(n),n}$ to φ on D by using similar arguments.

5. Numerical illustration

In this section, we illustrate our concept of monotone partial frontiers through three examples, one with simulated samples and two with real data sets.

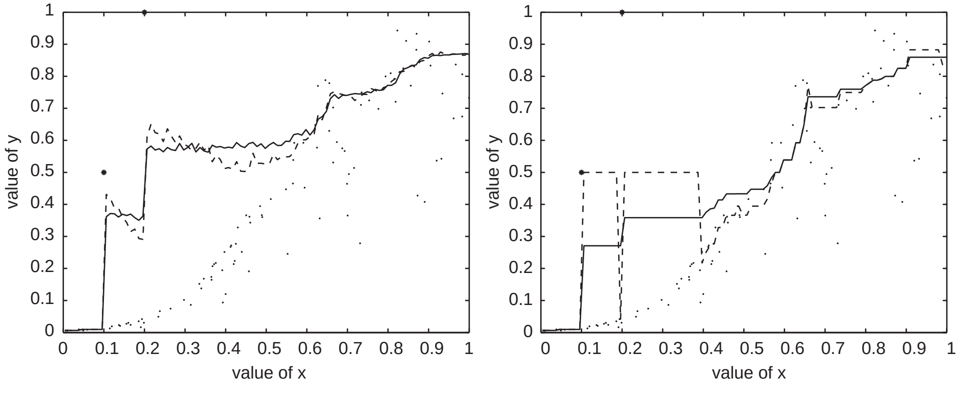

Fig. 1. $n = 102$ (2 outliers included): On the left, the initial frontier $\widehat{\varphi}_{13,n}$ in dashed line and its isotonized version $\widehat{\varphi}^{\#}_{13,n}$ in solid line. On the right, the quantile frontiers $\widehat{q}_{.94,n}$ in dashed line and $\widehat{q}^{\#}_{.94,n}$ in solid line.

5.1. Simulated data

First, we simulate a sample of 100 observations (x_i, y_i) according the data generating process $Y = \exp(-5 + 10X)/(1 + \exp(-5 + 10X)) \exp(-U)$, where X is uniform on $(0, 1)$ and U is exponential with mean $\frac{1}{3}$. In order to test the robustness of the isotonic estimators with respect to the initial ones, we add in the data set two outliers indicated by "*" in Fig. 1. We plot in dashed lines the initial frontiers $\widehat{\varphi}_{13,n}$ on the left panel (computed with $B = 500$ Monte-Carlo draws) and $\widehat{q}_{.94,n}$ on the right panel. The isotonized versions of these frontiers are displayed in solid lines. For the computations, we simply define D as a discrete grid of 100 points equispaced between the min and the max of the observations x_i. Note that a larger grid and more bootstrap loops are necessary to get a better quality of the monotone frontier $\widehat{\varphi}^{\#}_{13,n}$, which is not the case for the quantile frontier. This is due to the Monte-Carlo approximations.

We remark that both initial frontiers are more attracted by the two outliers than the isotone ones. This is natural since, by construction (see Eq. (1)), the monotone function $r^{\#}$ is everywhere below the monotone upper boundary r^u of the initial function r.

In Fig. 2, we simulate a sample of 100 observations according the Cobb–Douglas log-linear frontier model given by $Y = X^{0.5} \exp(-U)$, where X is uniform on $(0, 1)$ and U, independent of X, is exponential with mean $\frac{1}{3}$. On the left panel, we add an outlying point and we plot the quantile frontiers $\widehat{q}_{.94,n}$ and $\widehat{q}^{\#}_{.94,n}$ in dashed and solid lines, respectively. On the right panel, we add three outliers and we plot the frontiers $\widehat{\varphi}_{25,n}$ and $\widehat{\varphi}^{\#}_{25,n}$ in dashed and solid lines, respectively. When U is independent of X, an explicit formula is available in [8] (resp., [5]) in order to compute the true function $\varphi_m(x)$ (resp., $q_\alpha(x)$). In Fig. 2, the true frontiers φ_{25} and $q_{.94}$ are plotted in dash-dotted lines.

Here also, we remark that the isotone estimators are more resistant to the outlying points than the unconstrained ones. It is also clear, in this particular example, that the monotone quantile frontier (solid line) is everywhere closer to the true frontier (dash-dotted).

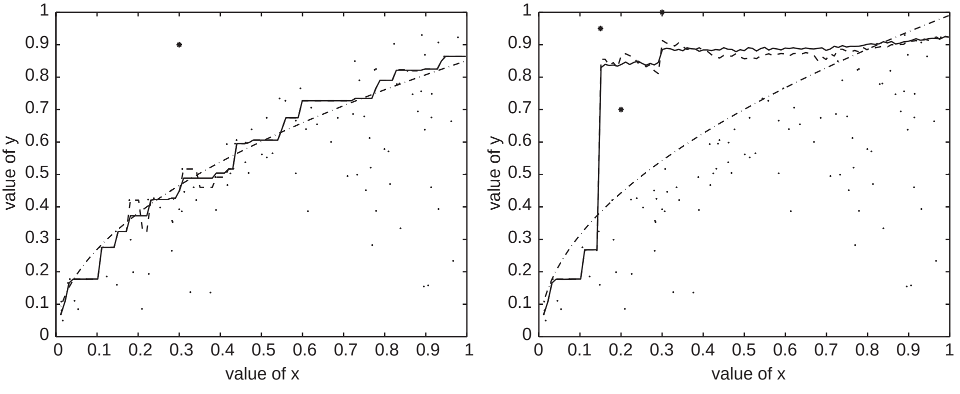

Fig. 2. On the left, $n = 101$ with 1 outlier included: the frontiers $q_{.94}$ in dash-dotted line, $\widehat{q}_{.94,n}$ in dashed line and $\widehat{q}^{\#}_{.94,n}$ in solid line. On the right, $n = 103$ with 3 outliers included: φ_{25} in dash-dotted line, $\widehat{\varphi}_{25,n}$ in dashed line and $\widehat{\varphi}^{\#}_{25,n}$ in solid line.

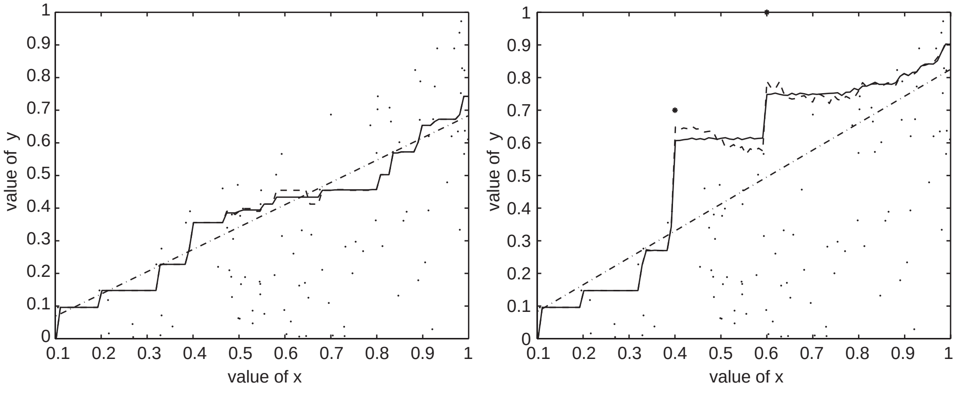

Fig. 3. On the left, $n = 100$ (no outlier): the frontiers $q_{.9}$ in dash-dotted line, $\widehat{q}_{.9,n}$ in dashed line and $\widehat{q}^{\#}_{.9,n}$ in solid line. On the right, $n = 102$ with 2 outliers included: φ_{25} in dash-dotted line, $\widehat{\varphi}_{25,n}$ in dashed line and $\widehat{\varphi}^{\#}_{25,n}$ in solid line.

To confirm still more these benefits of isotonized frontiers, we now consider a case where the monotone boundary of the support of (X, Y) is linear. We choose (X, Y) uniformly distributed over the region $D = \{(x, y)|0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x\}$ and simulate 100 observations (x_i, y_i). For this example also, an exact formula of $q_\alpha(x)$ (resp: $\varphi_m(x)$) is available in [5] (resp., [8]). On the left panel of Fig. 3, we plot the quantile frontiers $q_{.9}$, $\widehat{q}_{.9,n}$ and $\widehat{q}^{\#}_{.9,n}$ in absence of any outlier. It is clear that the curves of $\widehat{q}_{.9,n}$ and $\widehat{q}^{\#}_{.9,n}$ are very similar. Nevertheless, $\widehat{q}^{\#}_{.9,n}$ is better than $\widehat{q}_{.9,n}$ on the interval (0.6, 0.7) since it is monotone and closer to the true frontier $q_{.9}$.

On the right panel of Fig. 3, we add two outliers in the data set and plot the frontiers φ_{25}, $\widehat{\varphi}_{25,n}$ and $\widehat{\varphi}^{\#}_{25,n}$. Here again the isotone order-*m* frontier is less sensitive to the outlying points

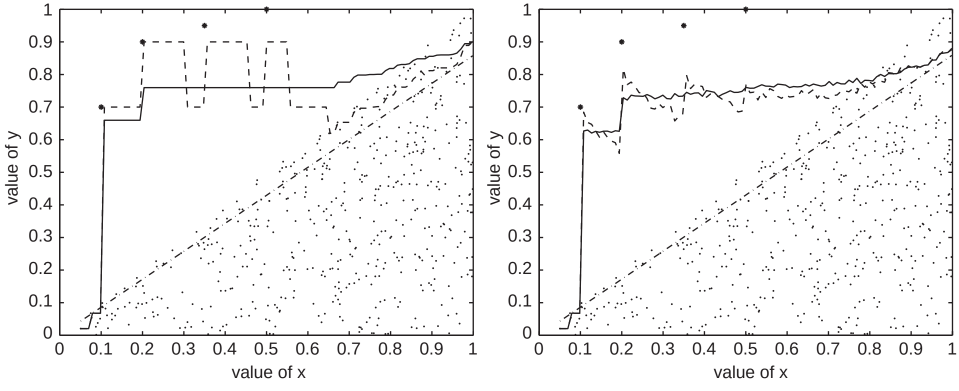

Fig. 4. $n = 504$ with 4 outliers included. On the left-hand side, the frontiers $q_{.98}$ in dash-dotted line, $\widehat{q}_{.98,n}$ in dashed line and $\widehat{q}^{\#}_{.98,n}$ in solid line. On the right-hand side, the frontiers φ_{35} in dash-dotted line, $\widehat{\varphi}_{35,n}$ in dashed line and $\widehat{\varphi}^{\#}_{35,n}$ in solid line.

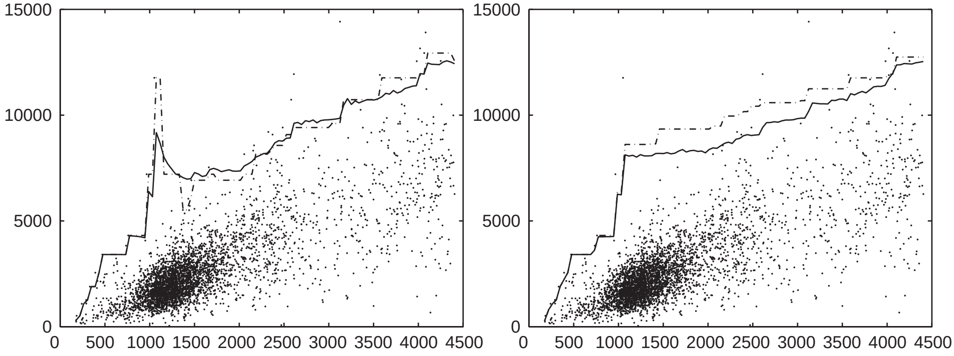

Fig. 5. $n = 4000$: On the left, the frontiers $\widehat{q}_{.999,n}$ in dash-dotted and $\widehat{\varphi}_{600,n}$ in solid line. On the right, the monotone frontiers $\widehat{q}^{\#}_{.999,n}$ in dash-dotted and $\widehat{\varphi}^{\#}_{600,n}$ in solid line.

than the unconstrained one. We repeated the same exercise, this time with 500 observations and 4 outliers as illustrated in Fig. 4, leading to the same kind of results.

5.2. *French post offices data*

We examine here real data in a bivariate case: the data are also used by Cazals et al. [3] and Aragon et al. [1] on frontier analysis of 9521 French post offices observed in 1994, with X as the quantity of labor and Y as the volume of delivered mail.

In this illustration, we only consider the $n = 4000$ observed post offices with the smallest levels x_i plotted in Fig. 5 on the left panel, along with the quantile frontier $\widehat{q}_{\alpha,n}$ of order $\alpha = .999$ in dash-dotted line, and the frontier $\widehat{\varphi}_{m,n}$ of order $m = 600$ in solid line ($B = 1000$). The isotonized versions of these extreme frontiers are displayed on the right panel.

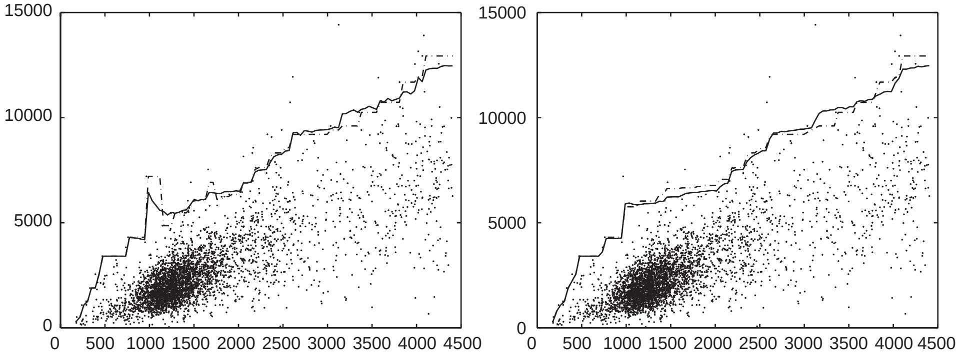

Fig. 6. $n = 3999$: On the left, the frontiers $\widehat{q}_{.999,n}$ in dash-dotted and $\widehat{\varphi}_{600,n}$ in solid line. On the right, the monotone frontiers $\widehat{q}^{\#}_{.999,n}$ in dash-dotted and $\widehat{\varphi}^{\#}_{600,n}$ in solid line.

Here, we use a discrete grid of 200 points equispaced between the min and the max of the first 4000 observations.

It is clear that the monotone estimators $\widehat{q}^{\#}_{.999,n}$ and $\widehat{\varphi}^{\#}_{600,n}$ are more resistant to the super-efficient post office than their initial versions $\widehat{q}_{.999}$ and $\widehat{\varphi}_{600,n}$.

More generally, for any orders α and m, the isotonized partial frontiers $\widehat{q}^{\#}_{\alpha,n}$ and $\widehat{\varphi}^{\#}_{m,n}$ are more robust to extreme values than the initial versions $\widehat{q}_{\alpha,n}$ and $\widehat{\varphi}_{m,n}$ introduced by Aragon et al. [1] and Cazals et al. [3], respectively, due to the average in the definition of the # operator.

We redo the calculation without the super-efficient post-office, the results are displayed in Fig. 6. The difference between the unisotonized and isotonized estimators is less important but still, the later are monotone and, as expected, more resistant to other extreme observations.

5.3. *Activity of Spanish electricity distributors*

The data set used in [2,9] contains information concerning the production of electricity by 61 firms in Spain. The output (y) is the amount of low, medium and high-voltage electricity distributed (GWh) and the 3 inputs are the population density (x^1), the substation transformer capacity from medium voltage-to-low voltage (x^2) and the length in km of voltage lines (x^3). The results are shown in Table 1.

As pointed out in [2] many of the firms are FDH-efficient (the FDH efficiency measure $\widehat{\lambda_n}^{-1}$ often equal to one). This is due to the high dimensionality of the space $(3 + 1)$ and to the small sample size ($n = 61$). All of these production units belong to the efficient surface (FDH frontier) of the smallest free disposal set containing all the data.

We remark that the quantile surfaces $\widehat{q}_{\alpha,n}$ and $\widehat{q}^{\#}_{\alpha,n}$ coincide everywhere except for the x_i's of units 11, 20, 41, 44 and 59. The unconstrained surface $\widehat{q}_{\alpha,n}$ is more influenced by the efficient extreme FDH points 11, 44 and 59 than the isotonized surface $\widehat{q}^{\#}_{\alpha,n}$ since $\widehat{q}_{\alpha,n}(x) > \widehat{q}^{\#}_{\alpha,n}(x)$ for each $x \in \{x_{11}, x_{44}, x_{59}\}$. The fact that $\widehat{q}_{\alpha,n}(x) < \widehat{q}^{\#}_{\alpha,n}(x)$ for $x \in \{x_{20}, x_{41}\}$ is quite natural and confirm that units 11, 44 and 59 are highly influent in the direction of Y. It

Table 1
$n = 61$ Spanish electricity distributors $\{(x_i, y_i)\}$, $\alpha = .93$, $m = 20$, $\widehat{\lambda}_n^{-1}(x_i, y_i) = y_i/\widehat{\varphi}_n(x_i)$, $\Delta\alpha = \widehat{q}_{\alpha,n} - \widehat{q}^{\#}_{\alpha,n}$ and $\Delta m = \widehat{\varphi}_{m,n} - \widehat{\varphi}^{\#}_{m,n}$

Unit	y	x^1	x^2	x^3	$\widehat{\lambda}_n^{-1}$	$\widehat{q}_{\alpha,n}$	$\widehat{q}^{\#}_{\alpha,n}$	$\Delta\alpha$	$\widehat{\varphi}_{m,n}$	$\widehat{\varphi}^{\#}_{m,n}$	Δm
1	1241	28.83076923	439	7007	1	1227	1227	0	1195	1193	2
2	3334	50.27580645	1165	5577	1	1793	1793	0	2617	2372	245
3	1871	23.12402875	865	5960	1	1227	1227	0	1584	1533	51
4	1489	30.97287894	843	6840	0.8533	1489	1489	0	1597	1573	23
5	1450	23.33291345	728	5586	1	1227	1227	0	1316	1315	1
6	2724	91.25431152	1118	604	1	1793	1793	0	2161	2188	−27
7	4500	50.00441721	1935	17050	1	2127	2127	0	3478	3400	78
8	684	7.47295423	314	4272	1	684	684	0	0662	0643	19
9	504	7.085373364	237	3774	1	504	504	0	0494	0485	8
10	2177	276.1740644	1012	3859	1	1793	1793	0	1929	1887	42
11	968	9.927531182	407	5459	1	968	949.5	18.5	0951	0947	4
12	316	51.87694145	142	4383	0.81443	388	388	0	0379	0379	−1
13	1227	11.94137353	404	5239	1	931	931	0	1112	1113	0
14	1097	12.46605886	869	7692	0.89405	1097	1097	0	1165	1156	8
15	297	4.96336056	147	4370	0.76546	388	388	0	0383	0379	4
16	388	3.90584575	110	2169	1	388	388	0	0388	0371	17
17	358	9.212554927	393	2961	0.92268	388	388	0	0385	0382	4
18	1036	12.44840598	306	4869	1	1036	1036	0	1000	0969	31
19	971	17.39387475	460	4102	1	971	971	0	0872	0889	−17
20	1267	1407.346153	654	9182	0.59567	1267	1530	−263	1824	1786	37
21	415	16.79987577	277	5871	0.44576	931	931	0	0820	0842	−22
22	1393	15.43905681	365	9829	1	1036	1036	0	1271	1255	15
23	23	4.116877045	12	721	1	23	23	0	0023	0023	0
24	931	9.428855657	250	4690	1	931	931	0	0893	0826	66
25	705	16.69379752	675	8463	0.57457	1036	1036	0	1130	1144	−14
26	95	10.77475363	54	1406	1	95	95	0	95	95	0
27	809	17.49376518	269	5685	0.86896	931	931	0	893	878	16
28	501	15.6301784	306	3746	1	501	501	0	484	481	3
29	212	4.603279324	63	1774	1	212	212	0	0212	0209	3
30	87	4.888839285	23	1781	1	87	87	0	0087	0087	0
31	1745	25.9270113	700	5192	1	1036	1036	0	1510	1442	68
32	410	11.6827005	162	4711	1	410	410	0	0406	0404	1
33	22	6.847996695	10	1272	1	22	22	0	0022	0022	0
34	3476	1348.198148	1729	8594	1	2724	2724	0	3080	3050	30
35	2844	184.3938193	840	7038	1	1793	1793	0	2303	2204	100
36	1872	30.81301394	1080	8089	1	1745	1745	0	1797	1766	32
37	1868	169.2686671	1344	6058	0.56029	1871	1871	0	2567	2662	−95
38	93	46.11206896	76	973	1	93	93	0	93	93	0
39	435	74.6214605	251	3745	1	435	435	0	422	424	−2
40	150	10.22512234	118	438	1	150	150	0	150	149	1
41	913	10.28438	628	8071	0.94318	931	949.5	−18.5	959	952	7
42	3317	49.61331626	1309	9165	1	1872	1872	0	2519	2449	70
43	4397	154.5850094	1259	20925	1	2724	2724	0	3467	3419	47
44	2127	46.01906334	581	6784	1	1793	1530	263	1841	1762	78
45	7049	110.339922	1932	17353	1	3334	3334	0	5048	4842	206
46	270	3.437563171	62	2410	1	270	270	0	270	266	4
47	855	15.88256346	446	5427	0.69682	1036	1036	0	1148	1136	12
48	750	9.157086772	322	4681	1	750	750	0	738	729	9
49	4858	71.23629629	1494	14214	1	3317	3317	0	3808	3568	239

Table 1 (*Continued*)

Unit	y	x^1	x^2	x^3	$\widehat{\lambda}_n^{-1}$	$\widehat{q}_{\alpha,n}$	$\widehat{q}^{\#}_{\alpha,n}$	$\Delta\alpha$	$\widehat{\varphi}_{m,n}$	$\widehat{\varphi}^{\#}_{m,n}$	Δm
50	212	148.0787878	97	1394	1	212	212	0	211	210	1
51	339	15.30875576	143	6186	0.87371	388	388	0	379	383	−4
52	732	28.7513053	432	9602	0.59658	1036	1036	0	1124	1144	−20
53	2080	84.46452476	988	10075	0.9779	1871	1871	0	1954	1956	−2
54	957	32.76927651	327	3196	1	957	957	0	910	847	63
55	10470	184.5294044	3266	22811	1	4858	4858	0	6763	6632	131
56	6065	417.2896551	4610	16179	1	4858	4858	0	5014	5026	−12
57	3347	49.05046844	829	13977	1	1793	1793	0	2483	2486	−3
58	5	2.517326732	4	35	1	5	5	0	5	5	0
59	1793	45.42970036	531	3208	1	1793	1530	263	1667	1509	158
60	4992	164.1621212	1759	7426	1	2724	2724	0	3662	3418	244
61	5362	243.5128552	3612	7621	1	2844	2844	0	4551	4180	371

is due to the isotonization procedure of the nonmonotone surface $\widehat{q}_{\alpha,n}$. Indeed, we remark that this surface is attracted by the efficient FDH unit 11 (since, for instance, $x_{58} < x_{11}$ and $\widehat{q}_{\alpha,n}(x_{58}) < \widehat{q}_{\alpha,n}(x)_{11}$) and then comes back down to pass through the non-FDH unit 41 (since $x_{11} < x_{41}$ whereas $\widehat{q}_{\alpha,n}(x_{11}) > \widehat{q}_{\alpha,n}(x_{41})$) before to be attracted again (for instance by the extreme FDH point 42 since $x_{41} < x_{42}$ and $\widehat{q}_{\alpha,n}(x_{41}) < \widehat{q}_{\alpha,n}(x)_{42}$). This explains why for example $\widehat{q}_{\alpha,n}(x_{41}) < \widehat{q}^{\#}_{\alpha,n}(x_{41})$. The same analysis could be done for the order-*m* surfaces. Here, the superiority of $\widehat{\varphi}^{\#}_{m,n}$ with respect to $\widehat{\varphi}_{m,n}$ is clear since $\Delta m < 0$ only for 12 observations x_i, whereas it is strictly positive for 42 observations. Moreover, $\max_i(\widehat{\varphi}^{\#}_{m,n}(x_i) - \widehat{\varphi}_{m,n}(x_i))$ does not exceed the level 95 (only $\Delta m(x_{37}) = -95$), whereas $(\widehat{\varphi}_{m,n}(x_i) - \widehat{\varphi}^{\#}_{m,n}(x_i))$ even exceeds the level 200 for 5 observations.

6. Conclusions

Order-*m* frontier and order-α quantile frontier functions are very useful to provide nonparametric estimators of boundaries which are more robust to outliers and/or extreme values than the usual envelopment estimators (FDH/DEA).

Their monotonized versions proposed in this paper are very easy to compute and provide estimators sharing the same properties as the original ones.

These new estimators appear to be even more robust to outliers than their original versions.

References

[1] Y. Aragon, A. Daouia, C. Thomas-Agnan, Nonparametric frontier estimation: a conditional quantile-based approach, Discussion paper, GREMAQ et LSP, Université de Toulouse, Econom. Theory, Forthcoming, http://www.univ-tlse1.fr/GREMAQ/Statistique/adt1202.pdf, 2002.

[2] Y. Aragon, A. Daouia, C. Thomas-Agnan, Efficiency measurement: a nonparametric approach, Discussion paper, GREMAQ et LSP, Université de Toulouse, 2003, http://www.univ-tlse1.fr/GREMAQ/Statistique/mesure.pdf.

[3] C. Cazals, J.P. Florens, L. Simar, Nonparametric frontier estimation: a robust approach, J. Econom. 106 (2002) 1–25.
[4] A. Charnes, W.W. Cooper, E. Rhodes, Measuring the inefficiency of decision making units, Europ. J. Oper. Res. 2 (1978) 429–444.
[5] A. Daouia, A. Ruiz-Gazen, Robust nonparametric frontier estimators: influence function and qualitative robustness, Discussion paper, 2003, http://www.univ-tlse1.fr/GREMAQ/Statistique/ar0903.pdf.
[6] D. Deprins, L. Simar, H. Tulkens, Measuring labor inefficiency in post offices, in: M. Marchand, P. Pestieau, H. Tulkens (Eds.), The Performance of Public Enterprises: Concepts and Measurements, North-Holland, Amsterdam, 1984, pp. 243–267.
[7] M.J. Farrell, The measurement of productive efficiency, J. Roy. Statist. Soc. Ser. A 120 (1957) 253–281.
[8] J-P. Florens, L. Simar, Parametric approximations of nonparametric frontiers, Discussion paper #0222, Institut de Statistique, UCL, Belgium, J. Econom., Forthcoming, http://www.stat.ucl.ac.be.
[9] E. Griffel-Tatjé, C.A.K. Lovell, The Managers versus the Consultants, Sixth European Workshop on Efficiency and Productivity Analysis, 29–31 October, 1999, University of Copenhagen, Denmark, 1999.
[10] P.L. Hsu, H. Robbins, Complete convergence and the law of large numbers, Proc. Natl. Acad. Sci. USA 33 (1947) 25–31.
[11] H. Mukerjee, S. Stern, Feasible nonparametric estimation of multiargument monotone functions, J. Amer. Statist. Assoc. 89 (425) (1994) 77–80.
[12] B. Park, L. Simar, Ch. Weiner, The FDH estimator for productivity efficiency scores: asymptotic properties, Econom. Theory 16 (2000) 855–877.
[13] L.M. Seiford, Data envelopment analysis: the evolution of the state-of-the-art (1978–1995), J. Prod. Anal. 7 (2/3) (1996) 99–138.
[14] R.J. Serfling, Approximation Theorems of Mathematical Statistics, Wiley, New York, 1980.
[15] L. Simar, Detecting outliers in frontiers models: a simple approach, J. Prod. Anal. 20 (2003) 391–424.
[16] L. Simar, P.W. Wilson, Statistical inference in nonparametric frontier models: the state of the art, J. Prod. Anal. 13 (2000) 49–78.