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On expected factors in reduced decompositions in type B

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Abstract

The expected number of Yang–Baxter moves appearing in a reduced decomposition of the longest element of the Coxeter group of type B_n is computed to be $2 - 4/n$. For the same element, the expected number of 0101 or 1010 factors appearing in a reduced decomposition is $2/(n^2 - 2)$.

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1. Introduction

Let \mathfrak{S}_n^B denote the Coxeter group of type B on $\{\pm 1, \dots, \pm n\}$, also known as the hyperoctahedral group. This is the group of signed permutations w on $\{\pm 1, \dots, \pm n\}$, with the requirement that $w(-i) = -w(i)$. For ease of notation, a negative sign will be written beneath an integer, rather than before it. An element $w \in \mathfrak{S}_n^B$ will be written $w = w(1) \cdots w(n)$, where w maps $i \mapsto w(i)$ (and $\underline{i} \mapsto \underline{w(i)}$). Notice that the element w is completely determined by the values of $w(1), \dots, w(n)$. The group \mathfrak{S}_n^B is generated by the transpositions $\{s_i : 0 \leq i \leq n - 1\}$, which satisfy the Coxeter relations:

$$s_i^2 = 1 \quad \text{for all } i; \tag{1}$$

$$s_i s_j = s_j s_i \quad \text{if } |i - j| > 1; \tag{2}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{for } 1 \leq i \leq n - 2; \tag{3}$$

$$s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0. \tag{4}$$

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A map is written to the left of its input, meaning that for $i > 0$, the product $s_i w$ changes the positions of the values i and $i + 1$ (and \underline{i} and $\underline{i + 1}$) in w , whereas ws_i changes the values in positions i and $i + 1$ in w . The transposition s_0 changes the sign of the first entry, so $s_0 w$ interchanges $\underline{1}$ and 1 in w , and $ws_0 = \underline{w(1)}w(2) \cdots w(n)$. An extensive treatment of Coxeter groups can be found in [2].

Every element in \mathfrak{S}_n^B can be written as a product of the transpositions $\{s_i : 0 \leq i \leq n - 1\}$. The minimum number of transpositions required for a product to equal w is the *length* of w , denoted $\ell(w)$. The *longest element* in \mathfrak{S}_n^B is $w_0^B = \underline{1}2 \cdots n$, and $\ell(w_0^B) = n^2$.

Definition 1.1. A *reduced decomposition* for an element w of length $\ell = \ell(w)$ is a string $i_1 \cdots i_\ell$ such that $w = s_{i_1} \cdots s_{i_\ell}$. The set $R(w)$ consists of all reduced decompositions of w .

Definition 1.2. A consecutive substring of a reduced decomposition is a *factor*. For any $j > 0$, a factor of the form $j(j + 1)j$ or $(j + 1)j(j + 1)$ in a reduced decomposition will be called a *Yang–Baxter move*, and a factor 0101 or 1010 will be called a *01 move*.

The symmetric group \mathfrak{S}_n of unsigned permutations is the Coxeter group of type A . It is generated by the transpositions $\{s_i : 1 \leq i \leq n - 1\}$, which are subject to the relations in Eqs. (1)–(3). The longest element in \mathfrak{S}_n is $w_0 = n(n - 1) \cdots 1$, which has length $\binom{n}{2}$. In [4], Reiner computes the following somewhat surprising result.

Theorem 1.3 (Reiner). *The expected number of Yang–Baxter moves in a reduced decomposition of $w_0 \in \mathfrak{S}_n$ is 1 for all $n \geq 3$.*

This paper presents results for Coxeter groups of type B that are analogous to Theorem 1.3. In type A , the expectation of factors corresponding to the Coxeter relation in Eq. (3) was computed. For the hyperoctahedral group, factors corresponding to the Coxeter relations in each of Eqs. (3) and (4) will be treated. Theorem 3.1 calculates that the expected number of Yang–Baxter moves in a reduced decomposition of $w_0^B \in \mathfrak{S}_n^B$ is $2 - 4/n$, and Theorem 4.1 shows that the expected number of 01 moves is $2/(n^2 - 2)$. Unlike Reiner’s result, both of these expectations are dependent upon n . Moreover, in the context of Theorem 1.3, the value $2 - 4/n$ seems quite plausible since the length of $w_0^B \in \mathfrak{S}_n^B$ is approximately twice that of $w_0 \in \mathfrak{S}_n$.

A variety of tools are used to prove Theorems 3.1 and 4.1, several of which are discussed in Section 2. Section 3 computes the expected number of Yang–Baxter moves in elements of $R(w_0^B)$, and Section 4 does the same for 01 moves.

2. Vexillary elements, shapes, and hook-lengths in type B

In [9], Stanley shows that for a vexillary element $v \in \mathfrak{S}_n$,

$$\#R(v) = f^{\lambda(v)}, \tag{5}$$

where $f^{\lambda(v)}$ is the number of standard Young tableaux of a particular shape $\lambda(v)$. This result is central to the proof of Theorem 1.3.

There are numerous definitions of vexillarity in type A ; for example see [3,10]. One definition is that $w \in \mathfrak{S}_n$ is vexillary if it is 2143-avoiding. Billey and Lam define a notion of vexillarity for type B in [1]. Their definition is in terms of Stanley symmetric functions and Schur Q -functions, and they prove its equivalence with a statement about pattern avoidance, now of signed permutations. This latter statement will be given as the definition here, and it follows from the work of Billey and Lam that it generalizes Eq. (5) to type B in the appropriate way.

Definition 2.1. An element $w \in \mathfrak{S}_n^B$ is *vexillary for type B* if $w = w(1) \cdots w(n)$ avoids the following patterns:

$$\begin{array}{ccc} 21 & \underline{321} & \underline{2341} \\ \underline{2341} & \underline{3412} & \underline{3412} \\ \underline{3412} & \underline{4123} & \underline{4123} \end{array}$$

Note that patterns in signed permutations must maintain their signs. For example, $\underline{12}$ is not an instance of the pattern $\underline{21}$, even though $\underline{1} > \underline{2}$.

Example 2.2. $\underline{2143} \in \mathfrak{S}_4^B$ is vexillary for type B, but $\underline{2143} \in \mathfrak{S}_4^B$ is not.

To each element $w \in \mathfrak{S}_n^B$, Billey and Lam define a shifted shape $\lambda^B(w)$ as follows.

Definition 2.3. Let $w = w(1) \cdots w(n) \in \mathfrak{S}_n^B$.

- (1) Write $\{w(1), \dots, w(n)\}$ in increasing order and call this $u \in \mathfrak{S}_n^B$.
- (2) Let $v \in \mathfrak{S}_n$ be the (vexillary) permutation $u^{-1}w$.
- (3) Let μ be the partition with (distinct) parts $\{|u_i| : u_i < 0\}$.
- (4) Let U be any standard shifted Young tableau of shape μ , and let V be any standard Young tableau whose shape is the transpose of the partition with parts $\{c_1, \dots, c_n\}$, where

$$c_i = \#\{j : j > i \text{ and } v(j) < v(i)\}.$$

- (5) Embed U in the shifted shape $\delta = (n, n - 1, \dots, 1)$.
- (6) Fill in the rest of δ with $1', \dots, k'$ starting from the rightmost column and labeling each column from bottom to top. This gives the tableau R .
- (7) Obtain S by adding $|\mu|$ to each entry of V , and glue R to the left side of S to obtain T .
- (8) Delete the box containing $1'$ from T . If the remaining tableau is not shifted, apply jeu de taquin to fill in the box. Do likewise for the box containing $2'$, then $3'$, and so on, stopping after the procedure for the box containing k' .

The (shifted) shape of the resulting tableau is $\lambda^B(w)$.

Example 2.4. Suppose $w = \underline{2143} \in \mathfrak{S}_4$. Then $u = \underline{4123}$, $v = 3214$, $\mu = (4, 1)$, and the tableau V has shape $(2, 1)$. Five boxes of δ will be filled by primed numbers, and the final tableau has shifted shape $\lambda^B(w) = (6, 2)$.

Proposition 2.5 (Billey–Lam). If $w \in \mathfrak{S}_n^B$ is vexillary for type B, then

$$\#R(w) = f^{\lambda^B(w)} \tag{6}$$

where $f^{\lambda^B(w)}$ is the number of standard tableaux of shifted shape $\lambda^B(w)$.

Eq. (6) will play an analogous role in the proofs of this paper to that played by Eq. (5) in [4]. Hooks and hook-lengths for shifted shapes will also be important tools, as they facilitate the calculation of $f^{\lambda^B(w)}$. Recall the hook-length formula for straight shapes (see [8] for a more extensive treatment).

Proposition 2.6. For a shape $\lambda \vdash N$,

$$f^\lambda = \frac{N!}{\prod_{u \in \lambda} h(u)},$$

where $h(u)$ is the number of squares in λ that are in the same column as u but no higher, or in the same row as u but no farther to the left.

There is an analogous formula for shifted shapes (for more information, see [6]).

Proposition 2.7. For a shifted shape $\lambda^B \vdash N$,

$$f^{\lambda^B} = \frac{N!}{\prod_{u \in \lambda^B} h^B(u)},$$

where $h^B(u)$ is the total number of the squares in λ^B that are

- (1) in the same column as u but no higher;
- (2) in the same row as u but no farther to the left; or
- (3) in the $(k + 1)$ st row of λ^B if u is in the k th column of λ^B .

Fig. 1 depicts the hook-lengths for the shifted shape $(5,4,1)$, and uses Proposition 2.7 to determine the number of standard tableaux of this shape. The final preliminary to proving the main results of this paper is the following lemma. The proof is straightforward and is omitted here.

Lemma 2.8. For $w_0^B \in \mathfrak{S}_n^B$ and $i \in [0, n - 1]$,

$$s_i w_0^B s_i = w_0^B.$$

This indicates a $\mathbb{Z}/n^2\mathbb{Z}$ -action on the set $R(w_0^B)$ defined by

$$s_{i_1} s_{i_2} \cdots s_{i_{n_2}} \mapsto s_{i_2} \cdots s_{i_{n_2}} s_{i_1}.$$

As with the other machinery discussed in this section, Lemma 2.8 has an analogous (though not identical) statement in type A which is used in [4].

3. Expectation of Yang–Baxter moves

Consider the set $R(w_0^B)$ with uniform probability distribution. Let X_n^B be the random variable on reduced decompositions of $w_0^B \in \mathfrak{S}_n^B$ which counts the number of Yang–Baxter moves.

Theorem 3.1. For all $n \geq 3$, $E(X_n^B) = 2 - 4/n$.

Proof. Fix $n \geq 3$. For $k > 0$, let $X_n^B[j, k]$ be the indicator random variable which determines whether the factor $i_j i_{j+1} i_{j+2}$ in a reduced decomposition $i_1 \cdots i_{n_2}$ in $R(w_0^B)$ is of either form $k(k + 1)k$ or $(k + 1)k(k + 1)$. Therefore

$$E(X_n^B) = \sum_{j=1}^{n^2-2} \sum_{k=1}^{n-2} E(X_n^B[j, k]).$$

The variables $X_n^B[j, k]$ and $X_n^B[j', k]$ have the same distribution by Lemma 2.8, so in fact

$$E(X_n^B) = (n^2 - 2) \sum_{k=1}^{n-2} E(X_n^B[1, k]).$$

If $X_n^B[1, k](\mathbf{i}) = 1$ for $\mathbf{i} = i_1 \cdots i_{n_2} \in R(w_0^B)$, then

$$\mathbf{i} \in \{k(k + 1)ki_4 \cdots i_{n_2}, (k + 1)k(k + 1)i_4 \cdots i_{n_2}\}.$$

9	6	5	3	2
	5	4	2	1
		1		

Fig. 1. Hook-lengths for $\lambda^B = (5, 4, 1)$. $f^{\lambda^B} = 56$.

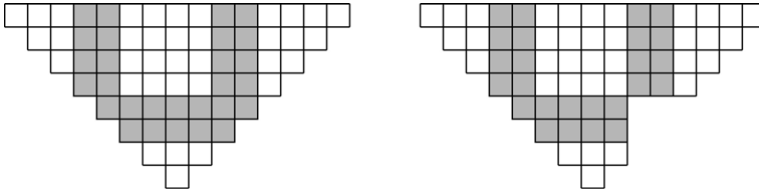


Fig. 2. The shifted shapes $\lambda^B(w_0^B)$ and $\lambda^B(w_k)$ for $n = 8$ and $k = 2$. The shaded boxes are where the hook-lengths are unequal.

In both cases, $i_4 \cdots i_{n^2}$ is a reduced decomposition of

$$w_k := s_k s_{k+1} s_k w_0^B = \underline{1} \cdots \underline{(k-1)(k+2)(k+1)(k)(k+3)} \cdots \underline{n}.$$

Notice that w_k is vexillary for type B for all k . Therefore, by Proposition 2.5,

$$E(X_n^B) = 2(n^2 - 2) \sum_{k=1}^{n-2} \frac{\#R(w_k)}{\#R(w_0^B)} = 2(n^2 - 2) \sum_{k=1}^{n-2} \frac{f^{\lambda^B(w_k)}}{f^{\lambda^B(w_0^B)}}. \tag{7}$$

The shifted shapes $\lambda^B(w_0^B)$ and $\lambda^B(w_k)$ are easy to determine, as the signed permutation u in Definition 2.3 is $\underline{n} \cdots \underline{1}$ in both cases, so no boxes contain primed entries in the shifted tableau T . Thus, the shifted shapes are

$$\begin{aligned} \lambda^B(w_0^B) &= (2n - 1, 2n - 3, \dots, 3, 1) \quad \text{and} \\ \lambda^B(w_k) &= (2n - 1, 2n - 3, \dots, 2k + 5, 2k + 1, 2k, 2k - 1, \dots, 3, 1). \end{aligned} \tag{8}$$

Recall the hook-length formula of Proposition 2.7, particularly the definition of the hooks h^B in shifted shapes. The only hook-lengths that do not cancel in the ratio $f^{\lambda^B(w_k)} / f^{\lambda^B(w_0^B)}$ are as indicated in Fig. 2.

Consequently, Eq. (7) can be written as

$$E(X_n^B) = \frac{1}{3} \binom{n^2}{2}^{-1} \sum_{k=1}^{n-2} C_k, \tag{9}$$

where

$$C_k = \frac{3 \cdot 5 \cdots (2k + 3)}{2 \cdot 4 \cdots (2k)} \cdot \frac{3 \cdot 5 \cdots (2n - 2k - 1)}{2 \cdot 4 \cdots (2n - 2k - 4)} \cdot \frac{(2k + 4)(2k + 6) \cdots (4k + 4)}{(2k + 1)(2k + 3) \cdots (4k + 1)} \cdot \frac{(4k + 8)(4k + 10) \cdots (2n + 2k + 2)}{(4k + 5)(4k + 7) \cdots (2n + 2k - 1)},$$

and empty products are defined to be 1.

Notice that

$$\frac{C_{k+1}}{C_k} = \frac{(2k + 3)(4k + 7)(2k + 1)(n - k - 2)(n + k + 2)}{(4k + 3)(k + 2)(2n + 2k + 1)(2n - 2k - 1)(k + 1)}$$

is a rational function in k . Therefore, $\sum_{k=1}^{n-2} C_k$ is a hypergeometric series. Following the notation in [5], Eq. (9) can be rewritten as

$$E(X_n^B) = \frac{1}{3} \binom{n^2}{2}^{-1} C_0 \left({}_5F_4 \left(\begin{matrix} 3/2, 7/4, 1/2, 2 - n, 2 + n \\ 3/4, 2, 1/2 + n, 1/2 - n \end{matrix} \right) - 1 \right).$$

The hypergeometric series in question can be computed via Dougall’s theorem, as discussed in [7]. The theorem states that

$$\begin{aligned} & {}_5F_4 \left(\begin{matrix} a, 1 + a/2, b, c, d \\ a/2, 1 + a - b, 1 + a - c, 1 + a - d \end{matrix} \right) \\ &= \frac{\Gamma(1 + a - b)\Gamma(1 + a - c)\Gamma(1 + a - d)\Gamma(1 + a - b - c - d)}{\Gamma(1 + a)\Gamma(1 + a - b - c)\Gamma(1 + a - b - d)\Gamma(1 + a - c - d)}. \end{aligned}$$

It is not immediately obvious that Dougall’s theorem applies to this particular series because of a potential pole. However, the theorem does show that

$$\begin{aligned} {}_5F_4 \left(\begin{matrix} 3/2, 7/4, 1/2, 2 - n, 2 + x \\ 3/4, 2, 1/2 + n, 1/2 - x \end{matrix} \right) &= \frac{\Gamma(2)\Gamma(1/2 + n)\Gamma(1/2 - x)\Gamma(n - x - 2)}{\Gamma(5/2)\Gamma(n)\Gamma(-x)\Gamma(n - x - 3/2)} \\ &= \frac{(-x)_{n-2}(5/2)_{n-2}}{(n - 1)!(1/2 - x)_{n-2}}. \end{aligned}$$

Therefore there is no pole in this situation. Letting x approach n shows that the desired hypergeometric series has sum $n/2$.

Finally,

$$C_0 = 3 \cdot \frac{3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdots (2n - 4)} \cdot \frac{4}{1} \cdot \frac{8 \cdot 10 \cdots (2n + 2)}{5 \cdot 7 \cdots (2n - 1)} = 6n(n^2 - 1),$$

which completes the proof:

$$E(X_n^B) = \frac{1}{3} \binom{n^2}{2}^{-1} 6n(n^2 - 1)(n/2 - 1) = 2 - 4/n. \quad \square$$

As suggested earlier, it is appropriate that Yang–Baxter moves are approximately twice as common in elements of $R(w_0^B)$ as in elements of $R(w_0)$, as

$$\ell(w_0^B) = n^2 \approx 2 \binom{n}{2} = 2\ell(w_0).$$

4. Expectation of 01 moves

As in the previous section, consider the set $R(w_0^B)$ with uniform probability distribution. Let Y_n^B be the random variable on reduced decompositions of $w_0^B \in \mathfrak{S}_n^B$ which counts the number of 01 moves.

Theorem 4.1. For all $n \geq 2$, $E(Y_n^B) = 2/(n^2 - 2)$.

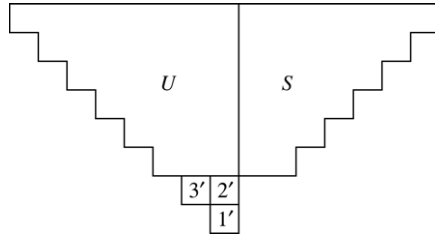


Fig. 3. Step (7) of Definition 2.3 applied to w' .

Proof. As in the proof of Theorem 3.1, let $Y_n^B[j]$ be the indicator random variable which determines whether the factor $i_j i_{j+1} i_{j+2} i_{j+3}$ in a reduced decomposition $i_1 \cdots i_{n^2}$ in $R(w_0^B)$ is of either form 0101 or 1010. Lemma 2.8 similarly applies, so

$$E(Y_n^B) = \sum_{j=1}^{n^2-3} E(Y_n^B[j]) = (n^2 - 3)E(Y_n^B[1]).$$

If $Y_n^B[1](i) = 1$ for $i = i_1 \cdots i_{n^2} \in R(w_0^B)$, then i is either $0101i_5 \cdots i_{n^2}$ or $1010i_5 \cdots i_{n^2}$. The string $i_5 \cdots i_{n^2}$ is a reduced decomposition of

$$w' = 1234 \cdots \underline{n}$$

in both situations. As with w_k , the signed permutation w' is vexillary for type B . Therefore

$$E(Y_n^B) = 2(n^2 - 3) \frac{\#R(w')}{\#R(w_0^B)} = 2(n^2 - 3) \frac{f^{\lambda^B(w')}}{f^{\lambda^B(w_0^B)}}.$$

The shifted shape $\lambda^B(w_0^B)$ is as in Eq. (8). Applying Definition 2.3 to w' proceeds as follows:

- (1) The signed permutation u is $\underline{n} \cdots \underline{3}12$.
- (2) The vexillary permutation v is $(n - 1)n(n - 2)(n - 3) \cdots 21$.
- (3) The partition μ is $(n, n - 1, \dots, 4, 3)$.
- (4) The shifted tableau U has shape $(n, n - 1, \dots, 4, 3)$ and the straight tableau V has shape $(n - 1, n - 2, \dots, 3, 2)$.

Unlike in the cases of w_0^B or w^k in the proof of Theorem 3.1, there will be boxes of T containing primed numbers, specifically $1'$, $2'$, and $3'$, as in Fig. 3. However, removing $1'$ leaves a shifted tableau so jeu de taquin is not applied. Similarly, $2'$ and then $3'$ can each be removed without performing jeu de taquin. Thus

$$\lambda^B(w') = (2n - 1, 2n - 3, \dots, 7, 5).$$

Having determined $\lambda^B(w')$, it remains to compute the ratio $f^{\lambda^B(w')}/f^{\lambda^B(w_0^B)}$ via Proposition 2.7. As in the proof of Theorem 3.1, many of the hook-lengths cancel. Fig. 4 depicts the only boxes in the two shapes where the hook-lengths differ.

From here it is not hard to compute that

$$\begin{aligned} E(Y_n^B) &= 2(n^2 - 3) \frac{(n^2 - 4)!}{(n^2)!} \cdot 3 \cdot (2n) \cdot \frac{(2n - 2)(2n)(2n + 2)}{2 \cdot 4 \cdot 6} \\ &= \binom{n^2}{3}^{-1} \cdot (2n) \cdot \frac{(n - 1)(n)(n + 1)}{1 \cdot 2 \cdot 3} \\ &= \frac{2}{n^2 - 2}. \quad \square \end{aligned}$$

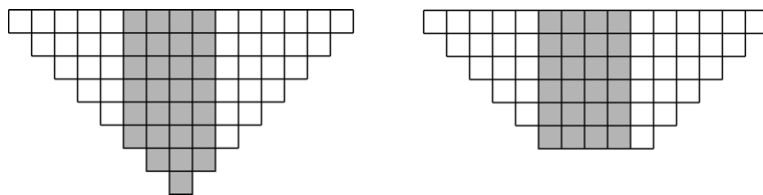


Fig. 4. The shifted shapes $\lambda^B(w_0^B)$ and $\lambda^B(w')$ for $n = 8$. The shaded boxes indicate unequal hook-lengths.

Acknowledgments

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