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Differential transform method for the system of two-dimensional nonlinear Volterra integro-differential equations

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ABSTRACT

In this paper, we develop the differential transform method (DTM) for solving a class of the system of two-dimensional linear and nonlinear Volterra integro-differential equations of the second kind. To this end, we give some preliminary results of the differential transform and describe the method of this paper. We also give some examples to demonstrate the accuracy of the presented method.

1. Introduction

The concept of differential transform was first introduced by Zhou [1] for solving linear and nonlinear initial value problems in electric analysis (see also [2]). Indeed the differential transform method is an iterative procedure to obtain Taylor series solutions of differential and integral equations (see [3]).

Up until now, the differential transform method has been developed for solving various types of differential and integral equations. For example in [4], this method has been used for solving a system of differential equations and in [5] for differential–algebraic equations. In [2,6], this method has been applied to partial differential equations and in [7,8] to one-dimensional Volterra integral and integro-differential equations. Also in [9] the DTM has been developed for solving two-dimensional Volterra integral equations.

On the other hand, there are many numerical methods for solving one-dimensional integral equations of the second kind, but in two-dimensional cases, a few works have been done (see, for example [10,11,9,12]).

The subject of the present paper is to apply the DTM for solving a system of two-dimensional linear and nonlinear Volterra integro-differential equations. For this propose we consider the system of two-dimensional Volterra integro-differential equations of the form

$$F_i(D_{11}^{(1)}u_1(x,t)+\cdots+D_{1m}^{(1)}u_m(x,t))-\lambda_i\int_{t_0}^{t}\int_{x_0}^{x}K_i(x,t,y,z)G_i(D_{11}^{(2)}u_1(y,z),\ldots,D_{1m}^{(2)}u_m(y,z))\mathrm{d}y\mathrm{d}z=f_i(x,t),\quad i=1,2,\ldots,m \tag{1.1}$$

with given supplementary conditions.

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Let us assume that K_i has the degenerate form

$$K_i(x,t,y,z)=\sum_{j=0}^{p} v_{ij}(x,t)w_{ij}(y,z),\quad i=1,2,\ldots,m \tag{1.2}$$

since otherwise it can be approximated by polynomials of this form.

2. Two-dimensional differential transform

We define the (m,n)th differential transform of the bivariate function $f(x,t)$ (see [4]) at (x_0,t_0) as

$$F(m,n)=\frac{1}{m!n!}\left[\frac{\partial^{m+n}f(x,t)}{\partial x^m\partial t^n}\right]_{x=x_0,t=t_0} \tag{2.1}$$

then its inverse transform is defined as

$$f(x,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}F(m,n)(x-x_0)^m(t-t_0)^n. \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$f(x,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!n!}\left[\frac{\partial^{m+n}f(x,t)}{\partial x^m\partial t^n}\right]_{x=x_0,t=t_0}(x-x_0)^m(t-t_0)^n \tag{2.3}$$

which is the Taylor series of the bivariate function $f(x,t)$ around (x_0,t_0).

In the following theorem, we summarize fundamental properties of two-dimensional differential transforms (see [4,8]).

Theorem 2.1. *Let $F(m,n)$, $U(m,n)$ and $V(m,n)$ be the differential transforms of the functions $f(x,t)$, $u(x,t)$ and $v(x,t)$ at $(0,0)$ respectively, then*

(a) *If $f(x,t)=u(x,t)\pm v(x,t)$ then*

$$F(m,n)=U(m,n)\pm V(m,n).$$

(b) *If $f(x,t)=au(x,t)$ then*

$$F(m,n)=aU(m,n).$$

(c) *If $f(x,t)=u(x,t)v(x,t)$ then*

$$F(m,n)=\sum_{l=0}^{n}\sum_{k=0}^{m}U(k,l)V(m-k,n-l).$$

(d) *If $f(x,t)=x^kt^l$ then*

$$F(m,n)=\delta_{m,k}\delta_{n,l}.$$

(e) *If $f(x,t)=x^k\sin(at+b)$ then*

$$F(m,n)=\frac{a^n}{n!}\delta_{m,k}\sin\left(\frac{n\pi}{2}+b\right).$$

(f) *If $f(x,t)=x^k\cos(at+b)$ then*

$$F(m,n)=\frac{a^n}{n!}\delta_{m,k}\cos\left(\frac{n\pi}{2}+b\right).$$

(g) *If $f(x,t)=x^ke^{at}$ then*

$$F(m,n)=\frac{a^n}{n!}\delta_{m,k}.$$

We state the following theorem to use the DTM to differential parts of Eq. (1.1) (see [2]).

Theorem 2.2. *Let $F(m,n)$, $U(m,n)$ and $V(m,n)$ be differential transforms of the functions $f(x,t)$, $u(x,t)$ and $v(x,t)$ around $(0,0)$ respectively, then*

(a) *If $f(x,t)=\frac{\partial u(x,t)}{\partial x}$ then*

$$F(m,n)=(m+1)U(m+1,n).$$

(b) *If* $f(x,t)=\frac{\partial u(x,t)}{\partial t}$ *then*

$$F(m,n)=(n+1)U(m,n+1).$$

(c) *If* $f(x,t)=\frac{\partial^{r+s}u(x,t)}{\partial^r x\partial^s t}$ *then*

$$F(m,n)=(m+1)(m+2)\cdots(m+r)(n+1)(n+2)\cdots(n+s)U(m+r,n+s).$$

(d) *If* $f(x,t)=\frac{\partial u(x,t)}{\partial x}\frac{\partial v(x,t)}{\partial x}$ *then*

$$F(m,n)=\sum_{r=0}^{m}\sum_{s=0}^{n}(r+1)(m-r+1)U(r+1,n-s)V(m-r+1,s).$$

(e) *If* $f(x,t)=u(x,t)v(x,t)w(x,t)$ *then*

$$F(m,n)=\sum_{r=0}^{m}\sum_{i=0}^{m-r}\sum_{s=0}^{n}\sum_{j=0}^{n-s}U(r,n-s-j)V(i,s)W(m-r-i,j).$$

Now we give the basic theorem of this paper.

Theorem 2.3. *Assume that $U(m,n)$, $V(m,n)$, $H(m,n)$ and $G(m,n)$ are the differential transforms of the functions $u(x,t)$, $v(x,t)$, $h(x,t)$ and $g(x,t)$ respectively, then we have*

(a) *If* $g(x,t)=\int_{t_0}^{t}\int_{x_0}^{x}u(y,z)v(y,z)\mathrm{d}y\mathrm{d}z$, *then*

$$G(m,0)=G(0,n)=0,\quad m,n=0,1,\ldots$$

$$G(m,n)=\frac{1}{mn}\sum_{l=0}^{n-1}\sum_{k=0}^{m-1}U(k,l)V(m-k-1,n-l-1),\quad m,n=1,2,\ldots. \tag{2.4}$$

(b) *If* $g(x,t)=h(x,t)\int_{t_0}^{t}\int_{x_0}^{x}u(y,z)\mathrm{d}y\mathrm{d}z$, *then*

$$G(m,0)=G(0,n)=0,\quad m,n=0,1,\ldots$$

$$G(m,n)=\sum_{l=0}^{n-1}\sum_{k=0}^{m-1}H(k,l)\frac{U(m-k-1,n-l-1)}{(m-k)(n-l)},\quad m,n=1,2,\ldots. \tag{2.5}$$

Proof. (a) From the definition of $g(x,t)$ it is obvious that $G(0,0)=0$.

Since

$$\frac{\partial^m g(x,t)}{\partial x^m}=\int_{t_0}^{t}\frac{\partial^{m-1}}{\partial x^{m-1}}(u(x,z)v(x,z))\mathrm{d}z,\quad\text{and}\quad\frac{\partial g^n(x,t)}{\partial t^n}=\int_{x_0}^{x}\frac{\partial^{n-1}}{\partial t^{n-1}}(u(y,t)v(y,t))\mathrm{d}y$$

hence

$$\left[\frac{\partial^m g(x,t)}{\partial x^m}\right]_{x=x_0,t=t_0}=0,\quad\text{and}\quad\left[\frac{\partial^n g(x,t)}{\partial t^n}\right]_{x=x_0,t=t_0}=0$$

therefore $G(m,0)=0$, $G(0,n)=0$, $m,n=0,1,\ldots$..

Now for $m\geq 1$, $n\geq 1$ we have

$$\begin{aligned}\frac{\partial^{m+n}g(x,t)}{\partial x^m\partial t^n}&=\frac{\partial^{m+n-2}(u(x,t)v(x,t))}{\partial x^{m-1}\partial t^{n-1}}=\frac{\partial^{m-1}}{\partial x^{m-1}}\left[\frac{\partial^{n-1}u(x,t)v(x,t)}{\partial t^{n-1}}\right]\\&=\frac{\partial^{m-1}}{\partial x^{m-1}}\left[\sum_{l=0}^{n-1}\binom{n-1}{1}\frac{\partial^l u(x,t)}{\partial t^l}\frac{\partial^{n-1-l}v(x,t)}{\partial t^{n-1-l}}\right]\\&=\sum_{l=0}^{n-1}\binom{n-1}{l}\left[\frac{\partial^{m-1}}{\partial x^{m-1}}\left(\frac{\partial^l u(x,t)}{\partial t^l}\frac{\partial^{n-1-l}v(x,t)}{\partial t^{n-1-l}}\right)\right]\\&=\sum_{l=0}^{n-1}\binom{n-1}{l}\left[\sum_{k=0}^{m-1}\binom{m-1}{k}\frac{\partial^{k+l}u(x,t)}{\partial x^k\partial t^l}\frac{\partial^{m+n-l-k-2}v(x,t)}{\partial x^{m-k-1}\partial t^{n-l-1}}\right]\\&=\sum_{l=0}^{n-1}\sum_{k=0}^{m-1}\binom{n-1}{l}\binom{m-1}{k}\left(\frac{\partial^{k+l}u(x,t)}{\partial x^k\partial t^l}\right)\left(\frac{\partial^{m+n-l-k-2}v(x,t)}{\partial x^{m-k-1}\partial t^{n-l-1}}\right)\end{aligned}$$

therefore

$$\left[\frac{\partial^{m+n}g(x,t)}{\partial x^m\partial t^n}\right]_{x=x_0,t=t_0} = \sum_{l=0}^{n-1}\sum_{k=0}^{m-1}\binom{n-1}{l}\binom{m-1}{k}[k!l!U(k,l)] \times [(m-k-1)!(n-l-1)!V(m-k-1,n-l-1)] = (m-1)!(n-1)!\sum_{l=0}^{n-1}\sum_{k=0}^{m-1}U(k,l)V(m-k-1,n-l-1)$$

hence by (2.1)

$$G(m,n) = \frac{1}{mn}\sum_{l=0}^{n-1}\sum_{k=0}^{m-1}U(k,l)V(m-k-1,n-l-1).$$

(b) Analogously to part (a), we have $G(m,0) = G(0,n) = 0$ for $m,n = 0,1,\ldots$ Thus we assume that $m,n \geq 1$ and for convenience we set $v(x,t) = \int_{t_0}^{t}\int_{x_0}^{x} u(y,z)\mathrm{d}y\mathrm{d}z$ therefore $g(x,t) = h(x,t)v(x,t)$ and we have

$$\frac{\partial^{m+n}g(x,t)}{\partial x^m\partial t^n} = \frac{\partial^{m+n}}{\partial x^m\partial t^n}(h(x,t)v(x,t))$$

and similar to part (a)

$$G(m,n) = \sum_{l=0}^{n}\sum_{k=0}^{m}\binom{n}{l}\binom{m}{k}\left[\left(\frac{\partial^{k+l}h(x,t)}{\partial x^k\partial t^l}\right)\left(\frac{\partial^{m+n-l-k}v(x,t)}{\partial x^{m-k}\partial t^{n-l}}\right)\right]_{x=x_0,t=t_0}$$

but since for $m=k$ or $n=l$

$$\left[\frac{\partial^{m+n-l-k}v(x,t)}{\partial x^{m-k}\partial t^{n-l}}\right]_{x=x_0,t=t_0} = 0$$

the upper indices in the above sums reduce to $m-1$ and $n-1$, hence

$$G(m,n) = \sum_{l=0}^{n-1}\sum_{k=0}^{m-1}H(k,l)\frac{U(m-k-1,n-l-1)}{(m-k)(n-l)}$$

and therefore the proof is completed. □

3. Application of the method

In this section, we describe application of the method for (1.1). To this end, we substitute from (1.2) into (1.1) and obtain

$$F_i(D_{11}^{(1)}u_1(x,t)+\cdots+D_{1m}^{(1)}u_m(x,t)) - \sum_{j=1}^{p}v_{ij}(x,t)\int_0^t\int_0^x w_{ij}(y,z)G_i(D_{11}^{(2)}u_1(y,z),\ldots,D_{1m}^{(2)}u_m(y,z))\mathrm{d}y\mathrm{d}z = f_i(x,t) \quad i=1,2,\ldots,m \tag{3.1}$$

which is solvable by using differential transform method.

By using Theorems 2.1–2.3 a recurrence relation is obtained for $U_i(m,n)$ (differential transform of $u_i(x,t)$). To find $U_i(m,n)$ by this relation we need some starting values of U_i that can be obtained from integro-differential equations and supplementary conditions.

We use (2.2) in truncated form

$$u_i(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N}U_i(m,n)(x-x_0)^m(t-t_0)^n \tag{3.2}$$

to obtain $u_i(x,t)$.

For the starting values of U_i we need the following lemma.

Lemma 3.1. *Let* $g(x,t) = \int_0^t\int_0^x w(y,z)u^q(y,z)\mathrm{d}y\mathrm{d}z$ *then for differential transforms of* $g(x,t)$ *we have*

(a) $G(m,0) = 0,\ m = 0,1,\ldots.$
(b) $G(0,n) = 0,\ n = 0,1,\ldots.$

Proof. (a) It is evident that $G(0, 0) = 0$, now we have

$$\frac{\partial^m g(x,t)}{\partial x^m} = \int_0^t \frac{\partial^{m-1}}{\partial x^{m-1}}[w(x,z)u^q(x,z)]\mathrm{d}z, \quad m = 1, 2, \ldots,$$

therefore

$$G(m, 0) = 0, \quad m = 1, 2, \ldots.$$

(b) Similarly to (a) we have

$$\frac{\partial^n g(x,t)}{\partial t^n} = \int_0^x \frac{\partial^{n-1}}{\partial x^{n-1}}[w(y,t)u^q(y,t)]\mathrm{d}y, \quad n = 1, 2, \ldots,$$

therefore

$$G(0, n) = 0, \quad n = 1, 2, \ldots. \quad \square$$

4. Examples

In this section, we give some examples to clarify the accuracy of the presented method. As we mentioned in the previous section, we first obtain a recurrence relation for differential transform of integro-differential equation; then solve it by programming in MAPLE environment. We also present numerical results for each example.

Example 1. Consider the system of integro-differential equations

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial x} - u_2(x,t) - \displaystyle\int_0^t \int_0^x y^2 u_1^2(y,z) u_2^2(y,z)\mathrm{d}y\mathrm{d}z = e^t - te^{-t} - \dfrac{1}{15}x^5 t^3 \\ u_1(x,t) + \dfrac{\partial u_2(x,t)}{\partial x} - \displaystyle\int_0^t \int_0^x z^2 u_1(y,z) u_2(y,z)\mathrm{d}y\mathrm{d}z = xe^t - \dfrac{1}{8}x^2 t^4 \end{cases} \tag{4.1}$$

for $x, t \in [0, 1]$ and with supplementary conditions

$$u_1(0,t) = 0, \qquad u_2(0,t) = te^{-t}, \quad t \in [0, 1]$$

which the exact solutions are $u_1(x,t) = xe^t$ and $u_2(x,t) = te^{-t}$.

Applying differential transform on both sides of (4.1) equations and using Theorems 2.1–2.3, for first equation of system we obtain

$$\begin{aligned} &(m+1)U_1(m+1,n) - U_2(m,n) - \frac{1}{mn}\sum_{l=0}^{n-1}\sum_{k=0}^{m-1}\sum_{s=0}^{n-l-1}\sum_{r=0}^{m-k-1}\sum_{j=0}^{s}\sum_{i=0}^{r}\sum_{q=0}^{n-l-s-1}\sum_{p=0}^{m-k-r-1} \\ &\quad \times [\delta_{k,2}\delta_{l,0}U_1(i,j)U_1(r-i,s-j)U_2(p,q)U_2(m-k-r-p-1,n-l-s-q-1)] \\ &\quad = \frac{1}{n!}\delta_{m,0} - \frac{(-1)^{n-1}}{(n-1)!} - \frac{1}{15}\delta_{m,5}\delta_{n,3}, \quad m = 1, 2, \ldots, N-1,\ n = 1, 2, \ldots, N \end{aligned} \tag{4.2}$$

and for second equation

$$\begin{aligned} &U_1(m,n) + (m+1)U_2(m+1,n) - \frac{1}{mn}\sum_{l=0}^{n-1}\sum_{k=0}^{m-1}\sum_{s=0}^{n-l-1}\sum_{r=0}^{m-k-1}[\delta_{k,0}\delta_{l,2}U_1(r,s)U_2(m-k-r-1,n-l-s-1)] \\ &\quad = \frac{1}{n!}\delta_{m,1} - \frac{1}{8}\delta_{m,2}\delta_{n,4}, \quad m = 1, 2, \ldots, N-1,\ n = 1, 2, \ldots, N. \end{aligned} \tag{4.3}$$

Also from the first supplementary condition we have

$$U_1(0,n) = 0, \quad n = 0, 1, 2, \ldots, N$$

and from the second condition

$$U_2(0,0) = 0, \qquad U_2(0,n) = \frac{(-1)^{n-1}}{(n-1)!}, \quad n = 1, 2, \ldots, N.$$

Now by substituting $x = 0$ in the first equation of (4.1) and using the second condition we obtain

$$\frac{\partial u_1}{\partial x}(0,t) = e^t.$$

Therefore

$$\frac{\partial^{1+n}u_1}{\partial x\partial t^n}(0,t)=e^t,\quad n=0,1,\ldots,N$$

hence

$$U_1[1,n]=\frac{1}{n!},\quad n=0,1,\ldots,N.$$

Similarly by substituting $x=0$ in the second equation of (4.1) and using the first condition we obtain

$$\frac{\partial u_2}{\partial x}(0,t)=0\Rightarrow\frac{\partial^{1+n}u_2}{\partial x\partial t^n}(0,t)=0.$$

Therefore

$$U_2[1,n]=0,\quad n=0,1,\ldots,N.$$

By solving (4.2) and (4.3) for the cases $N=10$, $N=12$ and $N=14$ we obtain the approximate solutions as

$$\begin{cases}u_{1N}(x,t)=x\left(1+t+\frac{1}{2!}t^2+\cdots+\frac{1}{N!}t^N\right)\\ u_{2N}(x,t)=t-t^2+\frac{1}{2!}t^3-\cdots+\frac{(-1)^{N-1}}{(N-1)!}t^N\end{cases}$$

that u_{1N} and u_{2N} denote approximates of u_1 and u_2 respectively and are parts of Taylor's series of the exact solutions. Table 1 shows the absolute errors at the some points.

Example 2. We consider in this example the system

$$\begin{cases}\frac{\partial u_1(x,t)}{\partial t}+u_2(x,t)-\int_0^t\int_0^x y\sin z(u_1^2(y,z)-u_2^2(y,z))\mathrm{d}y\mathrm{d}z=\frac{1}{12}(1+2\cos^3 t-3\cos t)x^4\\ \frac{\partial u_1(x,t)}{\partial t}+\frac{\partial u_2(x,t)}{\partial t}+u_1(x,t)-\int_0^t\int_0^x y\cos z\left(u_1(y,z)-\frac{\partial u_2(y,z)}{\partial z}\right)\mathrm{d}y\mathrm{d}z=x(2\cos t-\sin t)\end{cases}\tag{4.4}$$

with $x,t\in[0,1]$ and supplementary conditions

$$u_1(x,0)=x,\qquad u_2(x,0)=0,\quad x\in[0,1]$$

which has the exact solutions $u_1(x,t)=x\cos t$ and $u_2(x,t)=x\sin t$.

Using similar methods as in Example 1, we obtain

$$\begin{aligned}&(n+1)U_1(m,n+1)+U_2(m,n)-\frac{1}{mn}\sum_{l=0}^{n-1}\sum_{k=0}^{m-1}\sum_{s=0}^{l}\sum_{r=0}^{k}\sum_{q=0}^{n-l-1}\sum_{p=0}^{m-k-1}\frac{\delta_{r,1}\delta_{s,0}\delta_{k-r,0}}{(l-s)!}\sin\left(\frac{(l-s)\pi}{2}\right)\\&\quad\times[U_1(p,q)U_1(m-k-p-1,n-l-q-1)-U_2(p,q)U_2(m-k-p-1,n-l-q-1)]\\&\quad=\frac{1}{6}\sum_{l=0}^{n}\sum_{k=0}^{m}\sum_{s=0}^{n-l}\sum_{r=0}^{m-k}\sum_{q=0}^{n-l-s}\sum_{p=0}^{m-k-r}\frac{\delta_{k,4}\delta_{l,0}\delta_{r,0}\delta_{p,0}\delta_{m-k-r-p,0}}{s!q!(n-l-s-q)!}\cos\left(\frac{s\pi}{2}\right)\cos\left(\frac{q\pi}{2}\right)\cos\left(\frac{(n-l-s-q)\pi}{2}\right)\\&\quad+\frac{1}{12}\delta_{m,4}\delta_{n,0}-\frac{1}{4}\sum_{l=0}^{n}\sum_{k=0}^{m}\frac{\delta_{k,4}\delta_{l,0}\delta_{m-k,0}}{(n-l)!}\cos\left(\frac{(n-l)\pi}{2}\right),\quad m=1,2,\ldots,N,\ n=1,2,\ldots,N-1\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}&(n+1)U_1(m,n+1)+(n+1)U_2(m,n+1)+U_1(m,n)-\frac{1}{mn}\sum_{l=0}^{n-1}\sum_{k=0}^{m-1}\sum_{s=0}^{l}\sum_{r=0}^{k}\frac{\delta_{r,1}\delta_{s,0}\delta_{k-r,0}}{(l-s)!}\\&\quad\times\cos\left(\frac{(l-s)\pi}{2}\right)[U_1(m-k-1,n-l-1)-(n-l)U_2(m-k-1,n-l)]\\&\quad=\sum_{l=0}^{n}\sum_{k=0}^{m}\frac{\delta_{k,1}\delta_{l,0}\delta_{m-k,0}}{(n-l)!}\left[2\cos\left(\frac{(n-l)\pi}{2}\right)-\sin\left(\frac{(n-l)\pi}{2}\right)\right],\quad m=1,2,\ldots,N,\ n=1,2,\ldots,N-1.\end{aligned}\tag{4.6}$$

Also from the supplementary condition $u_1(x,0)=x$ we have

$$U_1(0,0)=0,\qquad U_1(1,0)=1,\qquad U_1(m,0)=0,\quad m=2,3,\ldots,N$$

Table 1
Numerical results of Example 1.

(x, t)	$(N = 10)$	$(N = 12)$	$(N = 14)$
$(u_1(x, t))$			
(0.2, 0.2)	0.104340e−15	0.200000e−19	0.100000e−20
(0.4, 0.6)	0.382607e−10	0.876419e−13	0.149430e−15
(0.4, 0.8)	0.921914e−9	0.374454e−11	0.113270e−13
(0.6, 0.4)	0.652138e−12	0.665590e−15	0.500000e−18
(0.8, 0.6)	0.765215e−10	0.175284e−12	0.298800e−15
(1, 1)	0.273127e−7	0.172877e−9	0.815487e−12
$(u_2(x, t))$			
(0.2, 0.2)	0.554281e−14	0.168000e−17	0.100000e−19
(0.4, 0.6)	0.947844e−9	0.260599e−11	0.518545e−14
(0.4, 0.8)	0.220581e−7	0.108091e−9	0.385094e−12
(0.6, 0.4)	0.111517e−10	0.135911e−13	0.119900e−16
(0.8, 0.6)	0.947844e−9	0.260599e−11	0.518545e−14
(1, 1)	0.252459e−6	0.193784e−8	0.107512e−10

Table 2
Numerical results of Example 2.

(x, t)	$(N = 10)$	$(N = 12)$	$(N = 14)$
$(u_1(x, t))$			
(0.2, 0.4)	0.699893e−14	0.615000e−17	0.100000e−19
(0.4, 0.4)	0.139978e−13	0.123000e−16	0.200000e−19
(0.6, 0.8)	0.857765e−10	0.301888e−12	0.805500e−15
(0.8, 0.2)	0.685000e−17	0.100000e−19	0.100000e−19
(0.8, 0.8)	0.114369e−9	0.402517e−12	0.107400e−14
(1, 1)	0.207625e−8	0.114231e−10	0.476390e−13
$(u_2(x, t))$			
(0.2, 0.4)	0.209937e−12	0.215377e−15	0.164000e−18
(0.4, 0.4)	0.419874e−12	0.430760e−15	0.320000e−18
(0.6, 0.8)	0.128590e−8	0.528102e−11	0.161057e−13
(0.8, 0.2)	0.410340e−15	0.110000e−18	0.100000e−20
(0.8, 0.8)	0.171453e−8	0.704137e−11	0.214743e−13
(1, 1)	0.248923e−7	0.159828e−9	0.761913e−12

and from the condition $u_2(x, 0) = 0$

$$U_2(m, 0) = 0, \quad m = 0, 1, 2, \ldots, N.$$

Now by substituting $t = 0$ in the first equation of (4.4) and using the second condition we obtain

$$\frac{\partial u_1}{\partial t}(x, 0) = 0 \Rightarrow \frac{\partial^{m+1} u_1}{\partial x^m \partial t}(x, 0) = 0.$$

Therefore

$$U_1(m, 1) = 0, \quad m = 0, 1, \ldots, N.$$

Similarly by substituting $t = 0$ in the second equation of (4.4) and using the first condition and the above relation we obtain

$$\frac{\partial u_2}{\partial t}(x, 0) = x \Rightarrow \frac{\partial^2 u_2}{\partial x \partial t}(x, 0) = 1 \quad \text{and} \quad \frac{\partial^{m+1} u_2}{\partial x^m \partial t}(x, 0) = 0.$$

Therefore

$$U_2[0, 1] = 0, \qquad U_2[1, 1] = 1, \qquad U_2[m, 1] = 0, \quad m = 2, 3, \ldots, N.$$

Similar to the previous example by solving (4.5) and (4.6) for the cases $N = 10$, $N = 12$ and $N = 14$ we obtain the approximate solutions which are parts of Taylor's series of the exact solutions. Table 2 shows the absolute errors at some points.

Example 3. In this example we consider a nonlinear equation of the form

$$\frac{\partial^2 u(x, t)}{\partial t^2} + u(x, t) - \int_0^t \int_0^x (y + \cos z) u^2(y, z) \mathrm{d}y \mathrm{d}z = \frac{1}{8} x^4 \sin t \cos t - \frac{1}{8} x^4 t - \frac{1}{9} x^3 \sin^3 t, \quad x, t \epsilon [0, 1] \tag{4.7}$$

Table 3
Numerical results of Example 3.

(x, t)	$Error(N = 10)$	$Error(N = 12)$	$Error(N = 14)$
(0.1, 0.1)	0.250000e−19	0.100000e−21	0.100000e−21
(0.2, 0.2)	0.102586e−15	0.270000e−19	0.100000e−20
(0.3, 0.3)	0.133060e−13	0.7679e−17	0.200000e−20
(0.4, 0.4)	0.419874e−12	0.430760e−15	0.320000e−18
(0.5, 0.5)	0.610645e−11	0.979001e−14	0.116500e−16
(0.6, 0.6)	0.544074e−10	0.125600e−12	0.215450e−15
(0.7, 0.7)	0.345667e−9	0.108662e−11	0.253681e−14
(0.8, 0.8)	0.171453e−8	0.704137e−11	0.214743e−13
(0.9, 0.9)	0.703886e−8	0.365966e−10	0.141282e−12
(1, 1)	0.273127e−7	0.159828e−9	0.761913e−12

with supplementary conditions

$$u(x,0) = 0$$
$$\frac{\partial u}{\partial t}(x,0) = x$$

with the exact solution $u(x,t) = x\sin t$.

By the same way of previous examples we have

$$\begin{aligned}(n+1)(n+2)U(m,n+2) = {} & -U(m,n) - \frac{1}{8}\delta_{m,4}\delta_{n,1} + \frac{1}{mn}\sum_{l=0}^{n-1}\sum_{k=0}^{m-1}\sum_{s=0}^{n-l-1}\sum_{r=0}^{m-k-1} \\ & \times \left[\left(\delta_{k,1}\delta_{l,0} + \frac{l}{l!}\delta_{k,0}\cos\frac{l\pi}{2}\right)U(r,s)U(m-k-r-1,n-l-s-1)\right] \\ & - \frac{1}{9}\sum_{l=0}^{n}\sum_{k=0}^{m}\sum_{s=0}^{n-l}\sum_{r=0}^{m-k}\left[\frac{1}{l!s!(n-l-s)!}\delta_{k,3}\delta_{r,0}\delta_{m-k-r,0}\sin\frac{l\pi}{2}\sin\frac{s\pi}{2}\right. \\ & \left.\times\ \sin\frac{(n-l-s)\pi}{2}\right] + \frac{1}{8}\sum_{l=0}^{n}\sum_{k=0}^{m}\left[\frac{1}{l!(n-l)!}\delta_{k,4}\delta_{m-k,0}\sin\frac{l\pi}{2}\cos\frac{(n-l)\pi}{2}\right], \\ & m = 1,2,\ldots,N,\ n = 1,2,\ldots,N-2. \end{aligned} \tag{4.8}$$

And from condition $u(x,0) = 0$ we obtain

$$U(m,0) = 0, \quad m = 0,1,2,\ldots.$$

Also from condition $\frac{\partial u}{\partial t}(x,0) = x$ we have

$$U(0,1) = 0, \qquad U(1,1) = 1, \qquad U(m,1) = 0, \quad m = 2,3,\ldots.$$

Now by differentiating Eq. (4.7) of order $m = 0,1,\ldots$ with respect to x and using Lemma 3.1 we obtain

$$U(m,2) = \frac{1}{2}[F(m,0) - U(m,0)], \quad m = 0,1,2,\ldots$$

similarly by differentiating Eq. (4.7) of order $n = 0,1,\ldots$ with respect to t and using Lemma 3.1 we obtain

$$U(0,n+2) = \frac{1}{(n+1)(n+2)}[F(0,n) - U(0,n)], \quad n = 0,1,2,\ldots.$$

That implies the solution

$$u(x,t) = xt - \frac{1}{3!}xt^3 + \frac{1}{5!}xt^5 - \frac{1}{7!}xt^7 + \frac{1}{9!}xt^9\cdots = x\sin t$$

which is the exact solution of equation.

Table 3 shows the absolute errors in points

$$(x,t) = ((0.1)i, (0.1)i), \quad i = 1,2,\ldots,10.$$

5. Conclusion

We applied two-dimensional differential transform for solving two-dimensional linear and nonlinear Volterra integro-differential equations. The advantages of this method are given as follows.

1. The results of examples showed that this method have high accuracy.
2. This method also can be applied to many linear and nonlinear two-dimensional Volterra integro-differential equations without linearization, discretization and perturbation.
3. In this method, we obtain solution from a recursive relation, therefore it is a very fast method and we can obtain arbitrary numbers in terms of Taylor expansion of solution.
4. Finally, since this is a simple method, it can be used in applied science and engineering.

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