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The classifying space of a bound quiver

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Abstract

We associate to a bound quiver (Q, I) a CW-complex which we denote by $\mathcal{B}(Q, I)$, and call the classifying space of (Q, I). We show that the fundamental group of $\mathcal{B}(Q, I)$ is isomorphic to the fundamental group of (Q, I). Moreover, we show that this construction behaves well with respect to coverings. On the other hand, we study the (co)homology groups of $\mathcal{B}(Q, I)$, and compare them with the simplicial and the Hochschild (co)homology groups of the algebra $A = kQ/I$. More precisely, we give sufficient conditions for these groups to be isomorphic. This generalizes a theorem due to Gerstenhaber and Schack [J. Pure Appl. Algebra 30 (1983) 143–156].

Keywords: Fundamental group; Bound quivers; Simply connected algebras; Schurian algebras; Incidence algebras; Hochschild cohomology; Simplicial cohomology of algebras

Introduction

Let A be an associative, finite dimensional algebra over an algebraically closed field k. It is well-known (see [8], for instance) that if A is basic and connected, then there exists a connected bound quiver (Q, I) such that $A \simeq kQ/I$, where kQ is the path algebra of Q and I is an admissible two sided ideal of kQ. The pair (Q, I) is then called a *presentation* of A. If Q contains no oriented cycles, then A is said to be a *triangular* k-algebra.

For each presentation (Q, I) of A, one can define its *fundamental group*, denoted by $\pi_1(Q, I)$ (see [21,25], for instance). A triangular k-algebra A is said to be *simply connected* if, for every presentation (Q, I) of A, the group $\pi_1(Q, I)$ is trivial. Simply connected

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algebras play an important role in the representation theory of algebras, since covering techniques often allow to reduce the study of indecomposable representations of an algebra A to the study of indecomposable representations of a suitably chosen simply connected algebra [8].

On the other hand, given an algebra $A = kQ/I$, its Hochschild cohomology groups $\mathrm{HH}^i(A)$ also give important information about the simple connectedness, as well as about the rigidity properties of A, see [13,19,33]. If moreover A admits a semi-normed basis, one can define the *simplicial homology* and *cohomology groups of* A with coefficients in some abelian group G, denoted by $\mathrm{SH}_i(A)$, and $\mathrm{SH}^i(A, G)$, respectively see [9] (also [24,26]).

Results in [2,9,10,15,20,23] exhibit several relations between the groups mentioned above. To a schurian triangular algebra A is associated a simplicial complex $|A|$, see [9] (and also [7,26]). It turns that in this case the groups $\pi_1(Q, I)$ and the simplicial (co)homology groups of A are respectively isomorphic to the fundamental group and the (co)homology groups of $|A|$ see [9,10,24].

The main aim of this work is to build a topological space which would be a geometrical model for studying the fundamental groups and coverings of bound quivers, as well as the simplicial and Hochschild (co)homology of not necessarily schurian algebras. The paper is organized as follows.

In Section 1, we fix notation and terminology, and recall the definition of the (natural) homotopy relation induced by an ideal I on the set of walks in a quiver Q. We also recall the definition of the fundamental group $\pi_1(Q, I)$.

We begin Section 2 with a motivating discussion about classifying spaces of small categories, in the sense of K-theory [28,32]. A particular case of that construction is the simplicial complex $|A|$ associated to an incidence algebra $A(\Sigma)$. The main part of this section is devoted to the construction of a CW-complex $\mathcal{B} = \mathcal{B}(Q, I)$, which we call the *classifying space* of the bound quiver (Q, I). Several examples are given.

In Section 3, we discuss some homotopy properties of $\mathcal{B}$, and we prove the first main result of this paper.

3.3. Theorem. *Let (Q, I) be a bound quiver, and $\mathcal{B} = \mathcal{B}(Q, I)$ be its classifying space. Then the groups $\pi_1(\mathcal{B})$ and $\pi_1(Q, I)$ are isomorphic.*

This generalizes previous similar results obtained independently for incidence algebras, and for schurian triangular algebras in [30] and [10], respectively. Moreover, this allows to obtain an adaptation of Van Kampen's theorem to the context of bound quivers (compare with [30]).

In Section 4, we deal with coverings of bound quivers, and of topological spaces. Theorem 4.3 says that a covering morphism of bound quivers $p : (\widehat{Q}, \hat{I}) \to (Q, I)$ induces a covering of topological spaces $\mathcal{B}p : \mathcal{B}(\widehat{Q}, \hat{I}) \to \mathcal{B}(Q, I)$. The main result in this section is the following:

4.5. Theorem. *Let $p : (\widehat{Q}, \hat{I}) \to (Q, I)$ be a Galois covering given by a group G. Then $(\widehat{\mathcal{B}}, \mathcal{B}p)$ is a regular covering of $\mathcal{B}$ with covering automorphism group isomorphic to G.*

In Section 5, we recall the definition of the simplicial (co)homology of an algebra $A = kQ/I$ having a semi-normed basis. We show that the homotopy relation is strongly related to the vectors of such a basis. Finally, we compare the simplicial homology and cohomology groups of A to the (cellular) homology and cohomology groups of $\mathcal{B}(Q, I)$ with coefficients in some abelian group G, $\mathrm{H}_i(\mathcal{B})$ and $\mathrm{H}^i(\mathcal{B}, G)$, respectively. More precisely, we show the following statement:

5.5. Corollary. *Let $A = kQ/I$ be an algebra having a semi-normed basis. Then, for each $i \geqslant 0$, there are isomorphisms of abelian groups*

$$\mathrm{SH}^i(A, G) \xrightarrow{\sim} \mathrm{H}^i(\mathcal{B}, G),$$
$$\mathrm{SH}_i(A) \xrightarrow{\sim} \mathrm{H}_i(\mathcal{B}).$$

An immediate consequence of this result is that one can think about the (co)homology theories of $\mathcal{B}(Q, I)$ as a generalization of the simplicial (co)homology theories of an algebra A, which are defined only for algebras having semi-normed bases.

Finally, in Section 6 we focus on the Hochschild cohomology groups of A. More precisely, following [20,24] we compare them with the simplicial cohomology groups of A. Strengthening Theorem 3 in [24], we then prove the following result, which generalizes a theorem due to Gerstenhaber and Schack [20] (see also [12]).

6.3. Theorem. *Let $A = kQ/I$ be a schurian triangular, semi-commutative algebra. Then, for each $i \geqslant 0$, there is an isomorphism of abelian groups*

$$\mathrm{H}^i(\varepsilon) : \mathrm{SH}^i\left(A, k^+\right) \xrightarrow{\sim} \mathrm{HH}^i(A).$$

It is worth to note that the isomorphisms of Corollary 5.5 and Theorem 6.3 are induced by isomorphisms of complexes which preserve canonical cup-products, thus the isomorphism above yields an isomorphism of graded rings. We use them to obtain new algebraic-topology flavored proofs of some known results about the Hochschild cohomology groups of monomial algebras [4,22].

1. Preliminaries

1.1. Notation and terminology

Let Q be a finite quiver. We denote by Q_0 and Q_1 the sets of vertices and arrows of Q, respectively. Given a commutative field k, the path algebra kQ is the k-vector space with basis all the paths of Q, including one stationary path e_x for each vertex x of Q. Two paths sharing source and target are said to be *parallel*. The multiplication of two basis elements of kQ is their composition whenever it is possible, and 0 otherwise. Let F be the two-sided ideal of kQ generated by the arrows of Q. A two-sided ideal I of kQ is called *admissible* if there exists an integer $m \geqslant 2$ such that $F^m \subseteq I \subseteq F^2$. The pair (Q, I) is a *bound quiver*. It is well-known that if A is a basic, connected, finite dimensional algebra over an algebraically closed field k, then there exists a unique finite connected quiver Q and a surjective morphism of k-algebras $\nu : kQ \to A$, which is not unique in general, with $I = \operatorname{Ker} \nu$ admissible [8].

Let (Q, I) be a bound quiver, and $A \simeq kQ/I$. It will sometimes be convenient to consider A as a locally bounded k-category, whose object class is Q_0, and, for x, y in Q_0, the morphism set $A(x, y)$ equals the quotient of the free k-module $kQ(x, y)$ with basis the set of paths from x to y, modulo the subspace $I(x, y) = I \cap kQ(x, y)$, see [8]. A path w from x to y is said to be a non-zero path if $w \notin I(x, y)$. It is easily seen that $A(x, y) = e_x A e_y$.

1.2. The fundamental group

Given a bound quiver (Q, I), its fundamental group is defined as follows [25]. For x, y in Q_0, a relation $\rho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y)$ (where $\lambda_i \in k_*$, and w_i are different paths from x to y) is said to be *minimal* if $m \geqslant 2$, and, for every proper subset J of $\{1, \dots, n\}$, we have $\sum_{i \in J} \lambda_i w_i \notin I(x, y)$.

We define the *homotopy relation* $\sim$ on the set of walks on (Q, I), as the smallest equivalence relation satisfying:

(1) For each arrow α from x to y, one has $\alpha\alpha^{-1} \sim e_x$ and $\alpha^{-1}\alpha \sim e_y$.
(2) For each minimal relation $\sum_{i=1}^{m} \lambda_i w_i$, one has $w_i \sim w_j$ for all i, j in $\{1, \dots, m\}$.
(3) If u, v, w and w' are walks, and $u \sim v$ then $wuw' \sim wvw'$, whenever these compositions are defined.

We denote by $\tilde{w}$ the homotopy class of a walk w. A closely related notion is that of *natural homotopy*. Two parallel paths p and q are said to be *naturally homotopic* if $p = q$ or there exists a sequence $p = p_0, p_1, \dots, p_s = q$ of parallel paths, and, for $i \in \{1, \dots, s\}$, paths u_i, v_i, v_i' and w_i such that $p_i = u_i v_i w_i$, $p_{i+1} = u_i v_i' w_i$ with v_i and v_i' appearing in the same minimal relation (compare with [3]). In that case we write $p \sim_\circ q$ and $\tilde{p}^\circ$ will denote the natural homotopy class of a path p. It is easily seen that natural homotopy is the smallest equivalence relation on the set of paths on (Q, I) satisfying conditions (2) and (3) (replacing, of course, *walks* by *paths* in condition (3)). Thus $p \sim_\circ q$ implies $p \sim q$ but the converse is not true (see 2.2, example (1)). Since the ideal I is admissible, for every arrow α in Q one has $\tilde{\alpha}^\circ = \{\alpha\}$. Moreover, note that if $A = kQ/I$ is schurian, that is for every $x, y \in Q_0$ one has $\dim_k e_x A e_y \leqslant 1$, then the relations $\sim$ and $\sim_\circ$ coincide.

For a fixed vertex $x_0 \in Q_0$, we denote by $\pi_1(Q, x_0)$ the fundamental group of the underlying graph of Q at the vertex x_0. Let $N(Q, I, x_0)$ be the normal subgroup of $\pi_1(Q, x_0)$ generated by all elements of the form $w^{-1}u^{-1}vw$, where w is a walk from x_0 to x, and u, v are two homotopic paths from x to y. The fundamental group $\pi_1(Q, I)$ is defined to be

$$\pi_1(Q, I) = \pi_1(Q, x_0)/N(Q, I, x_0).$$

Since the quiver Q is connected, this definition is independent of the choice of the base point x_0. An important remark is that the group defined above depends essentially on the minimal relations, which are given by the ideal. It is well-known that, for a k-algebra A, its presentation as a bound quiver algebra is not unique. Thus, the fundamental group is not an invariant of the algebra (see [1]). A triangular k-algebra A is said to be *simply connected* if, for every presentation $A \simeq kQ/I$ of A as a bound quiver algebra, we have $\pi_1(Q, I) = 1$.

However, it has been shown in [5] that if an algebra A is *constricted* (that is, if, for every arrow $\alpha : x \to y$ in Q_1, one has $\dim_k A(x, y) = 1$), then the fundamental group is independent of the presentation.

2. The classifying space $\mathcal{B}(Q, I)$

2.1. Background and motivation

To a schurian triangular algebra $A = kQ/I$, is associated a simplicial complex $|A|$ in the following way [9] (see also [7,26]): An n-simplex is a sequence $x_0, x_1, \dots, x_n$ of $n+1$ different vertices of Q such that for each j with $1 \leqslant j \leqslant n$, there is a morphism f_j in $A(x_{j-1}, x_j)$ with $f_n f_{n-1} \cdots f_1 \neq 0$. For instance, if $A = A(\Sigma)$ is the incidence algebra of a poset $(\Sigma, \leqslant)$, then $|A|$ is the simplicial complex of non empty chains of Σ (see [6,20]). Moreover, in this case, this construction is a particular case of that of the classifying space $\mathcal{B}C$ of a small category $\mathcal{C}$ (see [28,32]). More generally, let $\mathcal{C}$ be a small category. The space $\mathcal{BC}$ is a CW-complex with 0-cells corresponding to the objects of $\mathcal{C}$, and, for $n \geqslant 1$, one n-cell for each diagram

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n$$

in $\mathcal{C}$ where none of the f_i is an identity map. The corresponding n-cell is attached in the obvious way to any cell of smaller dimension obtained by deleting some X_i and, if $0 < i < n$, replacing f_i and f_{i+1} by the composition $f_{i+1} f_i$, whenever this composition is not an identity map. This construction leads to a functor from the category of small categories to that of CW-complexes.

For example, consider a poset $(\Sigma, \leqslant)$ as a category whose objects are the elements of Σ, and, for $x, y \in \Sigma$ there is a morphism $\rho_x^y : y \to x$ if and only if $x \leqslant y$ in Σ, with the obvious composition. With the above notation, the simplicial complex $|\Sigma|$ is equal to $\mathcal{B}\Sigma$. This leads us to the following.

2.2. Definition and examples

Recall that, given $n \geqslant 0$, the standard n-simplex is the set $\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1,\ t_i \geqslant 0\}$. Its jth face is $\partial_j \Delta^n = \{(t_0, t_1, \dots, t_n) \in \Delta^n \mid t_j = 0\}$, and, moreover $\partial \Delta^n = \bigcup_j \partial_j \Delta^n$. An open n-cell is an homeomorphic copy of $\Delta^n \backslash \partial \Delta^n$. Given two topological spaces X, and Y, a closed set $A \subset X$, and a continuous map $f : A \to Y$, the pushout of

$$X \overset{i}{\hookleftarrow} A \overset{f}{\to} Y$$

will be denoted by $X \amalg_f Y$.

We wish to build a CW-complex by successively attaching n-cells to a previously built $(n-1)$-dimensional complex. We begin by giving a description of the sets $\mathcal{C}_n$ of n-cells. Set $\mathcal{C}_0 = Q_0$, $\mathcal{C}_1 = \{\tilde{\sigma}^\circ \mid \sigma : x \rightsquigarrow y$ is a path in Q, $\sigma \notin I$, $\sigma \neq e_i\}$, and, for $n \geqslant 2$, $\mathcal{C}_n = \{(\tilde{\sigma}_1^\circ, \dots, \tilde{\sigma}_n^\circ) \mid \sigma_1\sigma_2\cdots\sigma_n$ is a path in Q, $\sigma_1\cdots\sigma_n \notin I$, $\sigma_i \neq e_i\}$. To avoid cumbersome notations, an element $(\tilde{\sigma}_1^\circ, \tilde{\sigma}_2^\circ, \dots, \tilde{\sigma}_n^\circ)$ of $\mathcal{C}_n$ will be denoted by $\boldsymbol{\sigma}^n$, or even $\boldsymbol{\sigma}$, if there is no risk of confusion.

Given $\boldsymbol{\sigma} = \tilde{\sigma}^\circ \in \mathcal{C}_1$, with, say, $\sigma : x \rightsquigarrow y$, define $\partial_0^1(\boldsymbol{\sigma}) = y$ and $\partial_1^1(\boldsymbol{\sigma}) = x$. More generally, for $n \geqslant 2$ and $i \in \{0, \dots, n\}$ define maps $\partial_i^n : \mathcal{C}_n \to \mathcal{C}_{n-1}$ in the following way: given $\boldsymbol{\sigma} = (\tilde{\sigma}_1^\circ, \tilde{\sigma}_2^\circ, \dots, \tilde{\sigma}_n^\circ)$ in $\mathcal{C}_n$, set: $\partial_0^n(\boldsymbol{\sigma}) = (\tilde{\sigma}_2^\circ, \dots, \tilde{\sigma}_n^\circ)$, $\dots$, $\partial_i^n(\boldsymbol{\sigma}) = (\tilde{\sigma}_1^\circ, \dots, \widetilde{\sigma_i\sigma_{i+1}}^\circ, \dots, \tilde{\sigma}_n^\circ), \dots,$ and $\partial_n^n(\boldsymbol{\sigma}) = (\tilde{\sigma}_1^\circ, \dots, \tilde{\sigma}_{n-1}^\circ)$. Again, we shall write ∂_i instead of ∂_i^n. With these notations we build a CW-complex as follows:

- 0-cells: Set $\mathcal{B}_0 = \bigcup_{x \in Q_0} \Delta_x^0$.
- 1-cells: We attach one 1-cell $\Delta_{\boldsymbol{\sigma}}^1$ for each $\boldsymbol{\sigma} \in \mathcal{C}_1$. More precisely, given $\boldsymbol{\sigma} \in \mathcal{C}_1$ define $f_{\boldsymbol{\sigma}} : \partial\Delta_{\boldsymbol{\sigma}}^1 \to \mathcal{B}_0$ by $f_{\boldsymbol{\sigma}}(\partial_i \Delta_{\boldsymbol{\sigma}}^1) = \Delta^0_{\partial_i^1 \boldsymbol{\sigma}}$. Consider the co-product $f_1 = \coprod_{\boldsymbol{\sigma} \in \mathcal{C}_1} f_{\boldsymbol{\sigma}}$, define
$$\mathcal{B}_1 = \left(\coprod_{\boldsymbol{\sigma} \in \mathcal{C}_1} \Delta_{\boldsymbol{\sigma}}^1\right) \amalg_{f_1} \mathcal{B}_0$$
and let p_1 be the canonical projection $p_1 : (\coprod_{\mathcal{C}_1} \Delta_{\boldsymbol{\sigma}}^1) \amalg \mathcal{B}_0 \to \mathcal{B}_1$.
- n-cells: Assume $\mathcal{B}_{n-1}$ has already been built. Given $\boldsymbol{\sigma} \in \mathcal{C}_n$, we have $\partial_j(\boldsymbol{\sigma}) \in \mathcal{C}_{n-1}$ for $j \in \{0, \dots, n\}$. The corresponding $(n-1)$-cells are $\Delta^{n-1}_{\partial_j \boldsymbol{\sigma}}$. Denote by $q_{\partial_i \boldsymbol{\sigma}} : \Delta^{n-1}_{\partial_i \boldsymbol{\sigma}} \to \coprod_{\mathcal{C}_{n-1}} \Delta_{\boldsymbol{\tau}}^{n-1}$ the inclusions, and define $f_{\boldsymbol{\sigma}} : \partial\Delta_{\boldsymbol{\sigma}}^n \to \mathcal{B}_{n-1}$ by
$$f_{\boldsymbol{\sigma}}\left(\partial_j \Delta_{\boldsymbol{\sigma}}^n\right) = p_{n-1} q_{\partial_j \boldsymbol{\sigma}}\left(\Delta^{n-1}_{\partial_j \boldsymbol{\sigma}}\right),$$
which is a continuous function on each $\partial_j \Delta_{\boldsymbol{\sigma}}^n$, thus continuous in $\partial\Delta_{\boldsymbol{\sigma}}^n$. Consider the co-product $f_n = \coprod_{\boldsymbol{\sigma} \in \mathcal{C}_n} f_{\boldsymbol{\sigma}}$, define
$$\mathcal{B}_n = \left(\coprod_{\boldsymbol{\sigma} \in \mathcal{C}_n} \Delta_{\boldsymbol{\sigma}}^n\right) \amalg_{f_n} \mathcal{B}_{n-1}$$
and let p_n be the canonical projection
$$p_n : \left(\coprod_{\mathcal{C}_n} \Delta_{\boldsymbol{\sigma}}^n\right) \amalg \mathcal{B}_{n-1} \to \mathcal{B}_n.$$

Note that since Q is a finite quiver and I is an admissible ideal, there are only finitely many $k \in \mathbb{N}$ such that $\mathcal{C}_k \neq \emptyset$.

Definition. Let (Q, I) be a bound quiver. The *CW*-complex obtained by the preceding construction is the *classifying space of* (Q, I), and is denoted by $\mathcal{B}(Q, I)$.

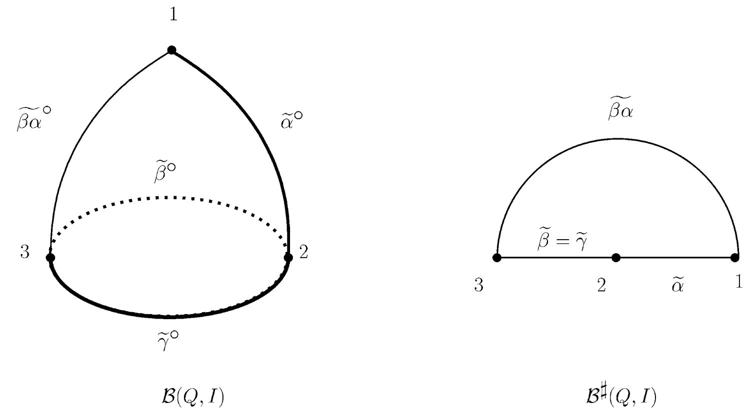


Fig. 1.

A slightly different approach consists in considering homotopy classes of paths, instead of *natural* homotopy classes to attach the cells. The complex obtained in this way will be denoted $\mathcal{B}^\sharp(Q, I)$, and called the *total classifying space of* (Q, I).

Remarks.

(1) If the algebra $A = kQ/I$ is *almost triangular*, that is $e_x(\operatorname{rad} A)e_y \neq 0$ implies $e_y(\operatorname{rad} A)e_x = 0$ for all vertices $x, y \in Q_0$ (compare with the definition in [10]), then the spaces $\mathcal{B}(Q, I)$ and $\mathcal{B}^\sharp(Q, I)$ are regular CW-complexes.

(2) Since for every arrow α of Q we have that $\tilde{\alpha}^\circ = \{\alpha\}$, the underlying graph of Q can be considered in a natural way as a subspace of the classifying space $\mathcal{B}(Q, I)$. This is not the case with the total classifying space (see example (1) below).

Examples. (1) Consider the quiver

$$3 \underset{\gamma}{\overset{\beta}{\rightrightarrows}} 2 \xrightarrow{\alpha} 1$$

bound by $I = \langle \beta\alpha - \gamma\alpha \rangle$. The arrows β and γ are homotopic, but not naturally homotopic. The spaces $\mathcal{B}(Q, I)$ and $\mathcal{B}^\sharp(Q, I)$ look as in Fig. 1.

(2) Let $A = kQ/I$ be a schurian algebra. As noted before, in this case homotopy and natural homotopy coincide, thus $\mathcal{B}(Q, I) = \mathcal{B}^\sharp(Q, I)$. For $x, y \in Q_0$, there is a 1-cell joining them if and only if there is a non-zero path w from, say, x to y. Moreover, if there is another such path w', then, since A is schurian, one has $w \sim_\circ w'$. Thus different paths give the same 1-cell, and one can identify it with the pair (x, y). In a similar way, given an n-cell corresponding to $(\tilde{\sigma}_1^\circ, \dots, \tilde{\sigma}_n^\circ)$, with $\sigma_i \in A(x_{i-1}, x_i)$, one can identify it with the sequence of $n+1$ points $x_0, x_1, \dots, x_n$ in Q. This shows that for schurian algebras $A = kQ/I$, the cellular complex $\mathcal{B}(Q, I)$ is precisely the simplicial complex $|A|$ of 2.1, above (see also [24]).

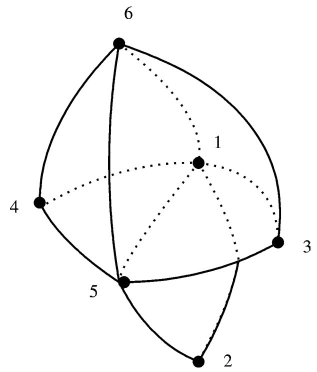
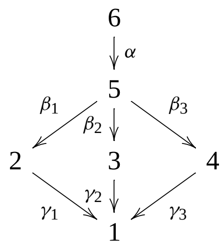


Fig. 2.

(3) Consider the following quiver



bound by the ideal generated by $\alpha\beta_1$ and $\sum_{i=1}^{3}\beta_i\gamma_i$. The cells of $\mathcal{B}(Q, I)$ are the following:

The 1-cells are given by $\tilde{\alpha}^\circ$, $\tilde{\beta}_i^\circ$, $\tilde{\gamma}_i^\circ$, $\widetilde{\beta_1\gamma_1}^\circ = \widetilde{\beta_2\gamma_2}^\circ = \widetilde{\beta_3\gamma_3}^\circ$, $\widetilde{\alpha\beta_2}^\circ$, $\widetilde{\alpha\beta_3}^\circ$ and $\widetilde{\alpha\beta_2\gamma_2}^\circ = \widetilde{\alpha\beta_3\gamma_3}^\circ$

The 2-cells are given by $(\tilde{\alpha}^\circ, \tilde{\beta}_2^\circ)$, $(\tilde{\alpha}^\circ, \tilde{\beta}_3^\circ)$, $(\tilde{\alpha}^\circ, \widetilde{\beta_2\gamma_2}^\circ)$, $(\tilde{\alpha}^\circ, \widetilde{\beta_3\gamma_3}^\circ)$, $(\widetilde{\alpha\beta_2}^\circ, \tilde{\gamma}_2^\circ)$, $(\widetilde{\alpha\beta_3}^\circ, \tilde{\gamma}_3^\circ)$, and $(\tilde{\beta}_i^\circ, \tilde{\gamma}_i^\circ)$, for $i \in \{1, 2, 3\}$.

The 3-cells are given by $(\tilde{\alpha}^\circ, \tilde{\beta}_2^\circ, \tilde{\gamma}_2^\circ)$ and $(\tilde{\alpha}^\circ, \tilde{\beta}_3^\circ, \tilde{\gamma}_3^\circ)$.

Note that the boundaries of the 2-cells $(\tilde{\beta}_i^\circ, \tilde{\gamma}_i^\circ)$ are the union of the 1-cells $\tilde{\gamma}_i^\circ$, $\tilde{\beta}_i^\circ$, and $\widetilde{\beta_i\gamma_i}^\circ$. Since $\beta_1\gamma_1 \sim_\circ \beta_2\gamma_2 \sim_\circ \beta_3\gamma_3$, the three 2-cells have a whole 1-face in common. The same argument shows that the two cells of dimension 3 share a whole face of dimension 2. The geometric realisation of $\mathcal{B}(Q, I)$ looks as the space shown in Fig. 2.

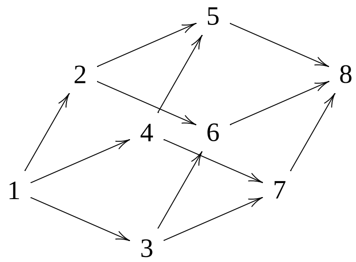
Note that since $\beta_2\gamma_2 \sim_\circ \beta_1\gamma_1$, one has $(\tilde{\alpha}^\circ, \widetilde{\beta_2\gamma_2}^\circ) = (\tilde{\alpha}^\circ, \widetilde{\beta_1\gamma_1}^\circ)$, even though the path $\alpha\beta_1\gamma_1$ belongs to I. However, $\alpha\beta_1\gamma_1 \sim_\circ \alpha\beta_2\gamma_2$ and the latter is not zero. Moreover, in this case we have $\mathcal{B}(Q, I) = \mathcal{B}^\sharp(Q, I)$.

(4) Let Q be a quiver, and I be the ideal of kQ generated by paths of length 2. In this case there are no minimal relations, so homotopy, and natural homotopy are trivial relations thus $\mathcal{B}(Q, I) = \mathcal{B}^\sharp(Q, I)$. Each arrow $\alpha : x \to y$ of Q gives rise to a 1-cell. Moreover, since the only non-zero paths are the arrows, these are the only cells. For the same reason,

there are no higher dimensional cells, so the space $\mathcal{B}(Q, I)$ is homeomorphic to $\overline{Q}$, the underlying graph of Q.

Remarks. (1) Let $\mathcal{P}(Q)$ denote the path category of Q. That is, the object class of $\mathcal{P}(Q)$ is Q_0, and for $x, y \in Q_0$, the morphism set $\mathcal{P}(Q)(x, y)$ is the set of paths from x to y in Q. The composition is the obvious one. Moreover, let $\mathcal{P}(Q, I) = \mathcal{P}(Q)/\sim_\circ$ be the quotient category modulo the natural homotopy relation induced by I. The complex $\mathcal{B}(Q, I)$ is a subcomplex of the classifying space of $\mathcal{P}(Q, I)$. The n-cells of $\mathcal{B}(\mathcal{P}(Q, I))$ are in bijection with n-tuples $(\tilde{\sigma}_1^\circ, \dots, \tilde{\sigma}_n^\circ)$ of composable morphisms, regardless whether their composition is zero or not. However, if there are no monomial relations in I, the complex $\mathcal{B}(Q, I)$ is exactly the classifying space $\mathcal{B}(\mathcal{P}(Q, I))$. The same applies for $\mathcal{B}^\sharp(Q, I)$ with respect to the classifying space of the category $\mathcal{P}(Q)/\sim$.

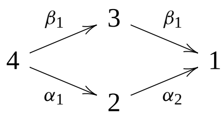
(2) Consider the quiver



bound by all the commutativity relations and all the paths of length 3. The space $\mathcal{B}(Q, I)$ is homeomorphic to the 3-dimensional sphere $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$. The commutativity relations tell how to "glue" the 2-dimensional cells. The existence of monomial relations implies that there are no 3-cells to "fill the hole". On the other hand, the space $\mathcal{B}(\mathcal{P}(Q, I))$ is homeomorphic to the ball B^3.

Let (Q, I') be the same quiver bound by the commutativity relations (and only these). In particular, the natural homotopy relations induced by I and I' are the same. However, the space $\mathcal{B}(Q, I')$ is homeomorphic to $B^3 = \{x \in \mathbb{R}^3 \mid \|x\| \leqslant 1\}$, hence does not have the same homotopy type as $\mathcal{B}(Q, I)$. This shows that monomial relations play an important role in the construction of $\mathcal{B}(Q, I)$, even though they are taken into account to define the neither the natural homotopy nor the homotopy relations.

(3) The fact that $\mathcal{B}(Q, I)$ is not really the classifying space of a category implies that this construction is not functorial, as the following example shows: Consider the quiver Q

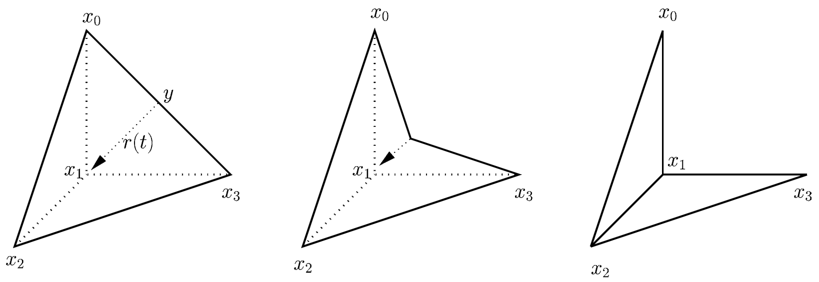


and let $I = \langle \alpha_1\alpha_2 + \beta_1\beta_2 \rangle$, $I' = \langle \alpha_1\alpha_2, \beta_1\beta_2 \rangle$. The identity map on Q yields a bound quiver morphism $f : (Q, I) \to (Q, I')$. However the induced map $g : (Q, I') \to (Q, I)$ is not a bound quiver morphism since $\alpha_1\alpha_2 \in I'$, but $\alpha_1\alpha_2 \notin I$. In this case, the space $\mathcal{B}(Q, I)$ is homeomorphic to $B^2 = \{x \in \mathbb{R}^2 \mid \|x\| \leqslant 1\}$. On the other hand, $\mathcal{B}(Q, I')$ is homeomorphic to S^1. Thus, there is no non homotopically trivial map from $\mathcal{B}(Q, I)$ to $\mathcal{B}(Q, I')$.

3. Homotopy

3.1. Proposition. *Let (Q, I) be a bound quiver with I a monomial ideal such that $= kQ/I$ is almost triangular. Then the underlying graph $\overline{Q}$ is a deformation retract of $\mathcal{B}(Q, I)$.*

Proof. Since I is monomial, the natural homotopy and the homotopy relations are trivial. Thus, the n-cells are given by n-tuples $(\sigma_1, \dots, \sigma_n)$ of paths such that $\sigma_1\sigma_2 \cdots \sigma_n \notin I$. If $I = F^2$, there is nothing to prove (see example (4), above). If this is not the case, then there are non-zero paths of length greater or equal than 2. Let $\alpha_1 \cdots \alpha_n$ be a maximal non-zero path. It gives a n-cell $\boldsymbol{\alpha}$ which is maximal. For $i \in \{0, \dots, n-1\}$, let x_i be the source of α_{i+1}, and x_n be the target of α_n. Since A is almost triangular, $x_i \neq x_j$ whenever $i \neq j$, thus we may consider the n-cell $\boldsymbol{\alpha}$ as the standard n-simplex $[x_0, \dots, x_n]$. Let $y = (x_0 + x_n)/2$, thus $[x_0, \dots, x_n] = [x_0, \dots, x_{n-1}, y] \cup [y, x_1, \dots, x_n]$.



$t = 0$ $t = \frac{1}{2}$ $t = 1$

For $t \in [0, 1]$ define $r(t) = (1 - t)y + tx_1$, that is the path joining y to x_1. Using r one can "crush" the simplex $[x_0, \dots, x_n]$ onto its faces $[x_0, \dots, x_{n-1}]$ and $[x_1, \dots, x_n]$, which, by the maximality assumption, are the only faces in the complex that are not free (use the barycentric coordinates, and the division of the simplex given below). The space obtained is $\mathcal{B}(Q, I)$, in which we have crushed the cell $\boldsymbol{\alpha}$ onto the $(n-1)$-cells $\partial_0(\boldsymbol{\alpha})$ and $\partial_n(\boldsymbol{\alpha})$. It is easily seen that this space is precisely $\mathcal{B}(Q, I_1)$ where $I_1 = I + \langle \alpha_1 \cdots \alpha_n \rangle$, which is again a monomial ideal. If $I_1 = F^2$, we have finished, otherwise we repeat the process above with another maximal dimensional cell of $\mathcal{B}(Q, I_1)$. This must end in a finite number of steps, since I is admissible. □

A first immediate consequence is that if I is a monomial ideal, then the fundamental group $\pi_1(\mathcal{B})$ of the topological space $\mathcal{B}$, is isomorphic to the fundamental group $\pi_1(Q, I)$ of the bound quiver (Q, I). In fact, this is a particular case of a much more general situation.

Given a cell complex X, its fundamental group $\pi_1(X)$ can be described in the following convenient way (see [27], for instance). Fix a 0-cell, x_0, and a maximal tree M, that is, a subcomplex of dimension smaller or equal than 1, which is acyclic and maximal with respect to this property. For every 2-cell e^2_λ of X, let α_e be a path class which starts at some fixed 0-cell x in its boundary, ∂e^2_λ, and goes around ∂e^2_λ exactly once. Moreover, let β_x be the unique path class in X which goes from x_0 to x along the tree M. Finally set $\gamma_e = \beta_x \alpha_e \beta_x^{-1}$ (compare with the definition of a *parade data* in [17]).

Let G be the free group on the set of 1-cells of X, and N be the normal subgroup of G generated by the following elements:

(1) The cells of M.
(2) The elements γ_e, as constructed above.

With the above notation, reformulating Theorem 2.1, p. 213 of [27], one has $\pi_1(X) \simeq G/N$.

Moreover, since we are interested in complexes of the form $\mathcal{B} = \mathcal{B}(Q, I)$, and in this case all the 2-cells are of the form $(\tilde{\sigma}_1^\circ, \tilde{\sigma}_2^\circ)$, the boundary of such a cell being $\tilde{\sigma}_2^\circ$, $\widetilde{\sigma_1\sigma_2}^\circ$, $\tilde{\sigma}_1^\circ$, we can improve the preceding presentation for $\pi_1(\mathcal{B})$.

3.2. Lemma. *Let (Q, I) be a bound quiver and T be a maximal tree in Q. Then $\pi_1(\mathcal{B}) \simeq F/K$, where F is the free group with basis the set of arrows of Q, and K is the normal subgroup of F generated by the elements of the following two types*:

(1) *α, for every arrow α in T,*
(2) *$(\alpha_1\alpha_2\cdots\alpha_r)(\beta_1\beta_2\cdots\beta_s)^{-1}$ whenever $\alpha_1\alpha_2\cdots\alpha_r$ and $\beta_1\beta_2\cdots\beta_s$ are two paths appearing in the same minimal relation.*

Proof. Let T be a maximal tree in the quiver Q. In particular T is a set of arrows of Q. It follows from the construction of $\mathcal{B}(Q, I)$ that an arrow $\alpha : x \to y$ in T gives rise to a 1-cell $\tilde{\alpha}^\circ$ in $\mathcal{B}(Q, I)$. The set of 1-cells obtained from the arrows of T forms a maximal tree M in $\mathcal{B}(Q, I)$. We work with respect to these maximal trees. Moreover, let G and N be as before.

Consider the map $\phi : F \to G/N$ defined by $\alpha \mapsto \tilde{\alpha}N$. It is straightforward to see that $K \subseteq \operatorname{Ker}\phi$, so we obtain a group homomorphism $\Phi : F/K \to G/N$ defined by $\alpha K \mapsto \tilde{\alpha}N$.

We show that Φ is an isomorphism by constructing its inverse. If $\tilde{w}^\circ$ is a 1-cell in $\mathcal{B}(Q, I)$, then there are arrows $\alpha_1, \alpha_2, \ldots, \alpha_r$ in Q such that $\alpha_1\alpha_2\cdots\alpha_r \sim_\circ w$, and $\alpha_1\alpha_2\cdots\alpha_r \notin I$. Define the map $\psi : G \to F/K$ by $\tilde{w} \mapsto \alpha_1\alpha_2\cdots\alpha_r K$. It is not hard to see that this is a well-defined map, and, moreover, that $N \subseteq \operatorname{Ker}\psi$. Thus, we obtain a group homomorphism $\Psi : G/N \to F/K$ defined by $\tilde{w}^\circ N \mapsto \alpha_1\cdots\alpha_r K$. Finally, it is straightforward to check that Φ and Ψ are mutually inverse. □

3.3. Theorem. *Let (Q, I) be a bound quiver, with Q triangular and $\mathcal{B} = \mathcal{B}(Q, I)$. Then the groups $\pi_1(\mathcal{B})$ and $\pi_1(Q, I)$ are isomorphic.*

Proof. This is an easy consequence of the preceding lemma, and the description of $\pi_1(Q, I)$ given in Section 1.2. □

Remark. A similar argument applies to the fundamental group of $\mathcal{B}^\sharp(Q, I)$. Thus we obtain $\pi_1(\mathcal{B}^\sharp) \simeq \pi_1(Q, I)$.

3.4. Corollary. *Let $A = kQ/I$ be a triangular algebra. Then A is simply connected if and only if for every presentation (Q, I), $\mathcal{B}(Q, I)$ or, equivalently $\mathcal{B}^\sharp(Q, I)$, is a simply connected topological space.*

Given a bound quiver (Q, I), and a full convex subquiver Q^i, let I^i denote the ideal $I \cap kQ^i$ of kQ^i, that is, the restriction of I to Q^i. With these notations we have the following result (see also [30]).

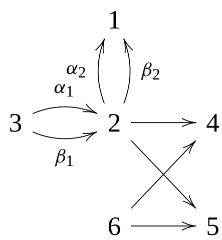
3.5. Corollary. *Let (Q, I) be a bound quiver, Q^1 and Q^2 be two full convex subquivers of Q such that every non-zero path of Q lies entirely in either Q^1 or Q^2, and $Q^0 = Q^1 \cap Q^2$ is connected. Then, $\pi_1(Q, I)$ is the pushout of the diagram*

$$\pi_1\bigl(Q^1, I^1\bigr) \leftarrow \pi_1\bigl(Q^0, I^0\bigr) \rightarrow \pi_1\bigl(Q^2, I^2\bigr)$$

where the arrows are the maps induced by the inclusions.

Proof. It follows from the hypotheses made on Q^1 and Q^2, that $\mathcal{B}(Q^1, I^1) \cup \mathcal{B}(Q^2, I^2) = \mathcal{B}(Q, I)$, and that $\mathcal{B}(Q^1, I^1) \cap \mathcal{B}(Q^2, I^2)$ is connected. The result then follows from Theorem 3.3, and Van Kampen theorem for topological spaces (see [31], for instance). □

Example. Consider the quiver Q



bound by $I = \langle \alpha_1\alpha_2 - \beta_1\beta_2, \alpha_1\beta_2 - \beta_1\alpha_2 \rangle$, and let Q^1 and Q^2 be the full subquivers of Q generated be the vertices 2, 3, 4, 5, 6, and 1, 2, 3, respectively. An easy computation gives $\pi_1(Q^1, I^1) = \mathbb{Z} * \mathbb{Z}$, $\pi_1(Q^2, I^2) = \mathbb{Z}_2$, and $\pi_1(Q^0, I^0) = \mathbb{Z}$, and this yields $\pi_1(Q, I) = \mathbb{Z} * \mathbb{Z}_2$.

4. Coverings

As we saw in the example before Proposition 3.1, the construction of $\mathcal{B}(Q, I)$ does not lead to a functor from the category of bound quivers with bound quivers morphisms to the category of CW-complexes with cellular maps. However, as we now see, if we consider the category of bound quivers with covering morphisms, then $\mathcal{B}$ gives rise to a covering morphism of topological spaces.

We refer the reader to [8,14,18], for references about coverings and their uses in representation theory of algebras.

Given a bound quiver Q and a vertex $x \in Q_0$, let, as usual, x^+ (or x^-) be the set of all arrows leaving (or entering, respectively) the vertex x. A bound quiver morphism $p:(\widehat{Q}, \hat{I}) \to (Q, I)$ induces in an obvious way a k-linear map $p:k\widehat{Q} \to kQ$. Recall from [14], for instance, that a bound quiver morphism $p:(\widehat{Q}, \hat{I}) \to (Q, I)$ is called a *covering morphism* if

(1) $p^{-1}(x) \neq \emptyset$ for every $x \in Q_0$.
(2) For every $x \in Q_0$, and $\hat{x} \in p^{-1}(x)$, the map p induces bijections $\hat{x}^+ \to x^+$ and $\hat{x}^- \to x^-$.
(3) For every $x, y \in Q_0$, every relation $\rho \in I(x, y)$ and every $\hat{x} \in p^{-1}(x)$ there exists $\hat{y} \in p^{-1}(y)$ and $\hat{\rho} \in \hat{I}(\hat{x}, \hat{y})$ such that $p(\hat{\rho}) = \rho$.

Conditions (2) and (3) ensure that covering morphisms behave well with respect to homotopy relations. We have the following straightforward lemmata whose easy proofs will be omitted.

4.1. Lemma. *Let $p:(\widehat{Q}, \hat{I}) \to (Q, I)$ be a covering morphism, x be a vertex of Q, and w_1, w_2 be two paths with source x. Moreover, let $\hat{x} \in p^{-1}(x)$, and $\hat{w}_1$, $\hat{w}_2$ be two paths with source $\hat{x}$ such that $p(\hat{w}_i) = w_i$, for $i = 1, 2$. Then $w_1 \sim_\circ w_2$ if and only if $\hat{w}_1 \sim_\circ \hat{w}_2$.*

4.2. Lemma. *Let $p:(\widehat{Q}, \hat{I}) \to (Q, I)$ be a covering morphism. For $x_0 \in Q_0$, and every $\hat{x}_0 \in p^{-1}(x_0)$, there is a bijective correspondence between the set of n-cells of $\mathcal{B} = \mathcal{B}(Q, I)$ having x_0 as boundary point, and the set of n-cells of $\widehat{\mathcal{B}} = \mathcal{B}(\widehat{Q}, \hat{I})$ having $\hat{x}_0$ as boundary point.*

In light of the preceding result, we can define $\mathcal{B}p:\widehat{\mathcal{B}} \to \mathcal{B}$ as the map which maps homeomorphically an n-cell $(\tilde{s}_1^\circ, \dots, \tilde{s}_n^\circ)$ onto the cell $(\widetilde{p(s_1)}^\circ, \dots, \widetilde{p(s_n)}^\circ)$.

Recall from [31], for instance, that if X is a topological space, then a covering space of X is a pair $(\widehat{X}, p)$ where

(1) $\widehat{X}$ is an arcwise connected topological space.
(2) $p:\widehat{X} \to X$ is a continuous map.
(3) Each $x \in X$ has an open neighborhood U_x such that $p^{-1}(U_x) = \bigcup_{i \in I} \widehat{U}_i$, with $\widehat{U}_i$ disjoint open sets, and $p|_{\widehat{U}}:\widehat{U}_i \to U_x$ an homeomorphism, for every $i \in I$.

This yields the following result.

4.3. Theorem. *Let $p:(\widehat{Q}, \hat{I}) \to (Q, I)$ be a covering morphism of bound quivers, with Q, $\widehat{Q}$ connected. Then $(\widehat{\mathcal{B}}, \mathcal{B}p)$ is a covering space of $\mathcal{B}(Q, I)$.*

Proof. Since $\widehat{Q}$ is a connected quiver, $\widehat{\mathcal{B}}$ is an arcwise connected space. Moreover, it follows from its definition that $\mathcal{B}p$ is a continuous map. Thus, there only remains to prove that the third condition of the definition is satisfied. But this follows from the fact that the open cells of $\widehat{\mathcal{B}}$ are disjoint open sets, and from the definition of $\mathcal{B}p$, whose restriction the each such cell is an homeomorphism. $\square$

Among all the covers of a bound quiver (Q, I), there is one of particular interest. The *universal cover* of (Q, I) is a cover map $p:(\widehat{Q}, \hat{I}) \to (Q, I)$ such that for any other cover $p':(\overline{Q}, \overline{I}) \to (Q, I)$ there exists a covering map $\pi:(\widehat{Q}, \hat{I}) \to (\overline{Q}, \overline{I})$ satisfying $p = p'\pi$ (see [25]).

4.4. Corollary. *If $p:(\widehat{Q}, \hat{I}) \to (Q, I)$ is the universal cover of (Q, I), then $(\widehat{\mathcal{B}}, \mathcal{B}p)$ is the universal cover of $\mathcal{B}$.*

Recall from [14], for instance, that a cover of bound quivers $p:(\widehat{Q}, \hat{I}) \to (Q, I)$ is said to be a *Galois cover* defined by the action of a group G of automorphisms of $(\widehat{Q}, \hat{I})$ if

(4) $pg = p$ for all $g \in G$.
(5) $p^{-1}(x) = Gp^{-1}(x)$ and $p^{-1}(\alpha) = Gp^{-1}(\alpha)$, for all vertices x in Q_0 and arrows $\alpha \in Q_1$.
(6) G acts without fixed points on $\widehat{Q}$.

Moreover, in this situation there exists a normal subgroup H of $\pi_1(Q, I)$ such that $\pi_1(\widehat{Q}, \hat{I}) \simeq H$ and $\pi_1(Q, I)/H \simeq G$ (see [14]).

As before, an automorphism g of $(\widehat{Q}, \hat{I})$ induces a map from the set of paths of $(\widehat{Q}, \hat{I})$ to the set of paths of (Q, I). This allows to define a cellular map $\mathcal{B}g:\widehat{\mathcal{B}} \to \mathcal{B}$ as the continuous function that maps homeomorphically the cell $(\tilde{\sigma}_1^\circ, \ldots, \tilde{\sigma}_n^\circ)$ onto the cell $(\widetilde{g\sigma}_1^\circ, \ldots, \widetilde{g\sigma}_n^\circ)$. The fact that $\mathcal{B}g$ is well-defined follows from the fact that g is invertible. Moreover, it is straightforward to check that $\mathcal{B}g\mathcal{B}p = \mathcal{B}p$. An important remark is that the restriction of $\mathcal{B}g$ to the 0-cells of $\widehat{\mathcal{B}}$ is precisely g. An immediate consequence of this is that if g_1 and g_2 are automorphisms of $(\widehat{Q}, \hat{I})$ with $g_1 \neq g_2$, then $\mathcal{B}g_1 \neq \mathcal{B}g_2$.

On the other hand, given a covering space $(\widehat{X}, p)$ of a topological space X, we denote by p_* the group homomorphism $\pi_1(p):\pi_1(\widehat{X}) \to \pi_1(X)$. In particular, p_* is always a monomorphism (see [31]). If $\operatorname{Im} p_*$ is a normal subgroup of $\pi_1(X)$ then the covering $(\widehat{X}, p)$ is said to be *regular*. The set of all homeomorphisms $\phi:\widehat{X} \to \widehat{X}$ such $p\phi = p$ is a group, which is called the group of covering automorphisms of $(\widehat{X}, p)$, and is denoted by $\operatorname{Cov}(\widehat{X}/X)$. It is well known that a covering $(\widehat{X}, p)$ is regular if and only if $\operatorname{Cov}(\widehat{X}/X)$ acts transitively on $p^{-1}(x_0)$, the fiber over the base point (see [31], for instance).

Note that the set $\{\mathcal{B}g \mid g \in G\}$ is a subgroup of $\operatorname{Cov}(\widehat{\mathcal{B}}/\mathcal{B})$, which is isomorphic to G. In fact, we have the following stronger result.

4.5. Theorem. *Let $p:(\widehat{Q}, \hat{I}) \to (Q, I)$ be a Galois covering given by a group G. Then $(\widehat{\mathcal{B}}, \mathcal{B}p)$ is a regular covering of $\mathcal{B}$ and* $\operatorname{Cov}(\widehat{\mathcal{B}}/\mathcal{B}) \simeq G$.

Proof. First of all, fix a vertex $x_0 \in Q_0$, which is also a 0-cell of $\mathcal{B}$. These will be the base points with respect the fundamental groups that will be considered.

In order to show that $(\widehat{\mathcal{B}}, \mathcal{B}p)$ is a regular covering of $\mathcal{B}$, it is enough to show that $\mathrm{Cov}(\widehat{\mathcal{B}}/\mathcal{B})$ acts transitively on the fiber $p^{-1}(x_0)$. This follows immediately from condition (5) in the definition of a Galois covering, and the fact that G is isomorphic to a subgroup of $\mathrm{Cov}(\widehat{\mathcal{B}}/\mathcal{B})$.

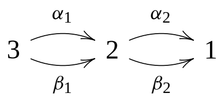
On the other hand

$$\mathrm{Cov}\big(\widehat{\mathcal{B}}/\mathcal{B}\big) \simeq \frac{\pi_1(\mathcal{B})}{(\mathcal{B}p)_*\pi_1(\widehat{\mathcal{B}})} \simeq \frac{\pi_1(Q, I)}{\pi_1(\widehat{Q}, \hat{I})} \simeq G$$

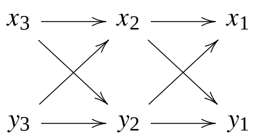
where the first isomorphism is given by Corollary 10.28, p. 294 in [31]. □

Remark. Theorem 10.19, p. 290 in [31] states that a nontrivial covering automorphism $h \in \mathrm{Cov}(\widehat{X}/X)$ acts without fixed points on $\widehat{X}$. Thus, condition (6) in the definition of Galois coverings is redundant.

Example. Consider the quiver Q



bound by the ideal $I = \langle \alpha_1\alpha_2 - \beta_1\beta_2, \alpha_1\beta_2 - \beta_1\alpha_2 \rangle$. The space $\mathcal{B}(Q, I)$ is homeomorphic to the real projective plane $\mathbb{R}P^2$. On the other hand, the universal cover of (Q, I) is given by the quiver Q'



bound by the ideal I' generated by all the possible commutativity relations. The covering map $p : Q' \to Q$ is given by $p(x_i) = p(y_i) = i$, for $i \in \{1, 2, 3\}$. The space $\mathcal{B}(\widehat{Q}, \hat{I})$ is homeomorphic to the sphere S^2, and the map $\mathcal{B}p$ identifies antipodal points. It is well-known that the pair $(S^2, \mathcal{B}p)$ is a covering space of $\mathbb{R}P^2$.

Remarks. (1) Again, with the obvious changes in Lemmata 4.1, 4.2 and Theorem 4.3 we obtain a similar result for the total classifying space $\mathcal{B}^\sharp(Q, I)$, and its covering spaces.

(2) In light of the results in the last two sections, it would be natural to define the higher homotopy groups of a bound quiver as the corresponding homotopy groups of the space $\mathcal{B}(Q, I)$.

5. (Co)Homologies

5.1. (Co)Homology of $\mathcal{B}(Q, I)$

Let $(\mathrm{C}_\bullet(\mathcal{B}), \delta)$ be the complex defined by letting $\mathrm{C}_i(\mathcal{B})$ be the free abelian group with basis the set of i-cells, $\mathcal{C}_i$, and by defining $\delta_n : \mathrm{C}_n(\mathcal{B}) \to \mathrm{C}_{n-1}(\mathcal{B})$ on the basis elements by the rule $\delta_n(\boldsymbol{\sigma}) = \sum_{i=0}^{n+1}(-1)^i \partial_i^n(\boldsymbol{\sigma})$.

With these notations, the homology groups $\mathrm{H}_i(\mathcal{B})$ of $\mathcal{B}$ are the homology groups of $(\mathrm{C}_\bullet(\mathcal{B}), \delta)$. Moreover, if G is an abelian group, then the cohomology groups $\mathrm{H}^i(\mathcal{B}, G)$ of $\mathcal{B}$ with coefficients in G, are the cohomology groups of $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{C}_\bullet(\mathcal{B}), G)$. In an analogous way, we define the corresponding complex $(\mathrm{C}_\bullet(\mathcal{B}^\sharp), \delta)$, and obtain the (co)homology groups of $\mathcal{B}^\sharp$.

5.2. Simplicial (co)homology of algebras

On the other hand, recall that the simplicial homology of a schurian algebra was defined in [9] (see also [24,26] for generalization to the non-schurian case) in the following way. For every pair of vertices i, j of Q, let B_{ij} be a k-basis of $e_i A e_j$. The set $B = \bigcup_{i,j} B_{ij}$ is said to be a *semi-normed basis* if:

(1) $e_i \in B_{ii}$, for every vertex i.
(2) $\bar{\alpha} \in B_{ij}$, for every arrow $\alpha : i \to j$.
(3) For σ, σ' in B, either $\sigma\sigma' = 0$ or there exist $\lambda_{\sigma,\sigma'} \in k$, and $b(\sigma, \sigma') \in B$ such that $\sigma\sigma' = \lambda_{\sigma,\sigma'} b(\sigma, \sigma')$.

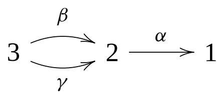
Define the chain complex $(\mathrm{SC}_\bullet(A), d)$ in the following way: $\mathrm{SC}_0(A)$, and $\mathrm{SC}_1(A)$ are the free abelian groups with basis Q_0, and B, respectively. For $n \geqslant 2$, let $\mathrm{SC}_n(A)$ be the free abelian group with basis the set of n-tuples $(\sigma_1, \dots, \sigma_n)$ of B^n such that $\sigma_1\sigma_2 \cdots \sigma_n \neq 0$, and $\sigma_i \neq e_j$, for all i, for all $j \in Q_0$. The differential is induced by the rules $d_1(\sigma) = y - x$ when $\sigma \in B_{xy}$, and, for $n \geqslant 1$,

$$d_n(\sigma_1\sigma_2 \cdots \sigma_n) = (\sigma_2, \dots, \sigma_n) + \sum_{j=1}^{n-1}(-1)^j \big(\sigma_1, \dots, b(\sigma_j, \sigma_{j+1}), \dots, \sigma_n\big) + (-1)^n(\sigma_1, \dots, \sigma_{n-1})$$

where $b(\sigma_j, \sigma_{j+1})$ is as in condition (3), above. The nth homology group $\mathrm{SH}_n(A)$ of $(\mathrm{SC}_\bullet(A, d))$ is called the nth *simplicial homology group of* A with respect to the basis B. For an abelian group M, the nth cohomology group $\mathrm{SH}^n(A, M)$, of the cochain complex $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{SC}_\bullet(A), M)$, nth *simplicial cohomology group of* A, with coefficients in M, with respect to the basis B. Moreover, by d^n, we denote the induced differential $\mathrm{Hom}_{\mathbb{Z}}(d_n, M)$.

Remarks. (1) Not every algebra of the form $A = kQ/I$ admits a semi-normed basis. However, it is shown in [24] that if the universal cover of (Q, I) is schurian, then A admits a semi-normed basis. In particular all schurian algebras admit semi-normed basis.

(2) On the other hand, assume A has a semi-normed basis B, the groups SC_i depend essentially on the way B is related to I. Hence, as for fundamental groups, different presentations of the algebra may lead to different simplicial homology groups, as the following well-known example shows: Consider the quiver Q



Let I_1 be the ideal generated by $\beta\alpha$, and $A_{\nu_1} = kQ/I_1$. Associated to this presentation, there is a semi-normed basis $\{e_1, e_2, e_3, \overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\gamma\alpha}\}$, and with respect to this basis, $\mathrm{SC}_2(A_{\nu_1})$ has as basis $\{(\overline{\gamma}, \overline{\alpha})\}$, and this leads to $\mathrm{SH}_1(A_{\nu_1}) = \mathbb{Z}$.

On the other hand, let I_2 be the ideal generated by $(\beta - \gamma)\alpha$, and $A_{\nu_2} = kQ/I_2$. It is easy to see that $A_{\nu_1} \simeq A_{\nu_2}$. With this presentation, one has the same semi-normed basis, however $\mathrm{SC}_2(A_{\nu_2})$ has a basis $\{(\overline{\gamma}, \overline{\alpha}), (\overline{\beta}, \overline{\alpha})\}$, and this leads to $\mathrm{SH}_1(A_{\nu_2}) = 0$.

(3) As noted in [24], in case A is schurian one can identify a basis element $(\sigma_1, \dots, \sigma_n)$ of $\mathrm{SC}_n(A)$ with the $(n+1)$-tuple $(x_0, x_1, \dots, x_n)$ of vertices of Q, where $\sigma_i \in e_{x_{i-1}} A e_{x_i}$, for $1 \leqslant i \leqslant n$ (compare with example (2) in Section 2.2). Thus, for schurian algebras the simplicial homology groups are independent of the semi-normed basis with respect to which they are computed. Moreover, it is straightforward that in this case one has, for every $i \geqslant 0$, $\mathrm{H}_i(\mathcal{B}) \simeq \mathrm{SH}_i(A)$, and $\mathrm{H}^i(\mathcal{B}, G) \simeq \mathrm{SH}^i(A, G)$ for every abelian group G.

The last remark is in fact a particular case of a more general result. Before stating and proving it we need the following technical lemma.

5.3. Lemma. *Let $A = kQ/I$ be an algebra having a semi-normed basis B, then*:

(a) *For every non-zero path w there exist a unique basis element $b(w)$ and a scalar λ such that $\overline{w} = \lambda b(w)$.*
(b) *For every basis element $v \in B$, there exist a non-zero path $p(v)$ and a scalar μ such that $v = \mu\overline{p(v)}$. Moreover, if $p(v)$ and $p'(v)$ are two such paths, then they are naturally homotopic.*
(c) *If w_1, w_2 are two non-zero paths with $w_1 \sim_\circ w_2$, then there is a scalar $\lambda \in k$ such that $\overline{w}_2 = \lambda\overline{w}_1$.*

Proof. (a) This follows from an easy induction on the length $l(w)$ of the path w.

(b) Let $\sigma \in B_{xy}$, and $w_1, w_2, \dots, w_r$ be paths from x to y such that $\{\overline{w_i} \mid 1 \leqslant i \leqslant r\}$ is a basis of $e_x A e_y$. It follows from (a) that there exists $\lambda_i \in k\backslash\{0\}$ and $b(w_i) \in B_{xy}$ such that $\overline{w_i} = \lambda_i b(w_i)$. Hence, $B_{xy} = \{b(w_1), \dots, b(w_r)\}$ so $\sigma = b(w_{i_0}) = \lambda_{i_0} w_{i_0}$ for some i_0 such that $1 \leqslant i_0 \leqslant r$. Then set $p(\sigma) = w_{i_0}$. Moreover, if $p(\sigma)$ and $p'(\sigma)$ are two different such paths and μ, μ' scalars such that $\sigma = \mu\overline{p(\sigma)} = \mu'\overline{p'(\sigma)}$ we get $\mu\overline{p(\sigma)} - \mu'\overline{p'(\sigma)} = \sigma - \sigma = 0$ which is a minimal relation, and this shows that $p(\sigma)$ and $p'(\sigma)$ are naturally homotopic.

(c) Let w_1, w_2 be two parallel homotopic paths, and suppose that $\overline{w}_1$ and $\overline{w}_2$ are linearly independent. Without loss of generality, we can assume that w_1 and w_2

appear in the same minimal relation. One then has $\sum_{i=1}^{n} \lambda_i \overline{w}_i = 0$, where $\lambda_i \in k^*$, and the paths w_i are parallel. Assume, without loss of generality, that $\{\overline{w}_1, \overline{w}_2, \dots, \overline{w}_r\}$ is a maximal linearly independent set of $\{\overline{w}_1, \dots, \overline{w}_n\}$, and that, after re-ordering if necessary (recall that A has a semi normed basis), there are scalars a_j such that $\overline{w}_{r+1} = a_{r+1}\overline{w}_1, \dots, \overline{w}_{r+i_1} = a_{r+i_1}\overline{w}_1, \overline{w}_{r+i_1+1} = a_{r+i_1+1}\overline{w}_2, \dots, \overline{w}_n = a_n\overline{w}_r$. Thus, replacing in $\sum_{i=1}^{n} \lambda_i \overline{w}_i = 0$ implies that

$$\lambda_1 + \lambda_{r+1}a_{r+1} + \cdots + \lambda_{r+i_1}a_{r+i_1} = 0$$

and then $\lambda_1\overline{w}_1 + \lambda_{r+1}\overline{w}_{r+1} + \cdots + \lambda_{r+i_1}\overline{w}_{r+i_1} = 0$, which is a contradiction to the minimality of the original relation. □

5.4. Theorem. *Let $A = kQ/I$ be an algebra having a semi-normed basis, then*:

(a) *There exists an epimorphism of complexes $\phi_\bullet^\sharp : \mathrm{SC}_\bullet(A) \to \mathrm{C}_\bullet(\mathcal{B}^\sharp)$*,
(b) *There exists an isomorphism of complexes $\phi_\bullet : \mathrm{SC}_\bullet(A) \to \mathrm{C}_\bullet(\mathcal{B})$.*

Proof. It is clear that $\mathrm{C}_0(\mathcal{B}) \simeq \mathrm{C}_0(\mathcal{B}^\sharp) \simeq \mathrm{SC}_0(A)$. For $n \geqslant 1$, consider the morphisms $\phi_n^\sharp$ and ϕ_n defined by

$$\phi_n^\sharp : \mathrm{SC}_n(A) \to \mathrm{C}_n\big(\mathcal{B}^\sharp\big), \quad (\sigma_1, \dots, \sigma_n) \mapsto \Big(\widetilde{p(\sigma_1)}, \dots, \widetilde{p(\sigma_n)}\Big),$$
$$\phi_n : \mathrm{SC}_n(A) \to \mathrm{C}_n(\mathcal{B}), \quad (\sigma_1, \dots, \sigma_n) \mapsto \Big(\widetilde{p(\sigma_1)}^\circ, \dots, \widetilde{p(\sigma_n)}^\circ\Big).$$

It follows from the second statement in Lemma 5.3 that $\phi_n^\sharp$ and ϕ_n are epimorphisms of abelian groups. Moreover, it is clear that they commute with the boundary operators involved. This proves the first statement. In order to prove the second, consider the map

$$\psi_n : \mathrm{C}_n(\mathcal{B}) \to \mathrm{SC}_n(A), \quad \big(\tilde{\sigma}_1^\circ, \dots, \tilde{\sigma}_n^\circ\big) \mapsto \big(b(\sigma_1), \dots, b(\sigma_n)\big).$$

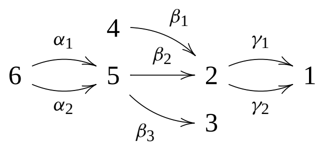
The fact that the definition of ψ_n is not ambiguous follows from the third statement in Lemma 5.3. It is clear that ψ_n commutes with the boundary operators, so it defines a morphism of complexes. Finally, it is straightforward to check that $\phi_\bullet$ and $\psi_\bullet$ are mutually inverse. □

5.5. Corollary. *With the above hypotheses, for each $i \geqslant 0$, there are isomorphisms of abelian groups*:

(a) $\mathrm{H}_i(\mathcal{B}) \simeq \mathrm{SH}_i(A)$, *and*
(b) $\mathrm{H}^i(\mathcal{B}, G) \simeq \mathrm{SH}^i(A, G)$, *for every abelian group G.*

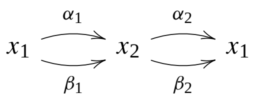
Remark. In case G is a commutative ring R, then the complexes involved are endowed with a canonical cup product. This provides to $\mathrm{H}^\bullet(A, R) = \bigoplus_{i \geqslant 0} \mathrm{H}^i(A, R)$, and $\mathrm{H}^\bullet(\mathcal{B}, R) = \bigoplus_{i \geqslant 0} \mathrm{H}^i(\mathcal{B}, R)$, a graded commutative ring structure. A direct computation shows that the morphisms induced in cohomology $\phi_\bullet$, $\phi^\sharp_\bullet$, and $\psi_\bullet$ preserve these products. Thus, the morphisms in the corollary above are ring homomorphisms.

Examples. (1) Consider the quiver Q



bound by $I = \langle(\alpha_1 - \alpha_2)\beta_3, \beta_1(\gamma_1 - \gamma_2)\rangle$. Even if in this case the complexes $\mathrm{SC}_\bullet(A)$ and $\mathrm{C}_\bullet(\mathcal{B}^\sharp)$ are not isomorphic, they still have the same homology groups. Let $\mathrm{K}_\bullet = \mathrm{Ker}\,\phi^\sharp_\bullet$. We have that K_n has a basis $\{(\sigma_1, \ldots, \sigma_n) - (\sigma'_1, \ldots, \sigma'_n) \mid \sigma_i \sim \sigma'_i$ for all i such that $1 \leqslant i \leqslant n\}$. More precisely, $\mathrm{rk}\,\mathrm{K}_0 = 0$, $\mathrm{rk}\,\mathrm{K}_1 = 7$, $\mathrm{rk}\,\mathrm{K}_2 = 10$, and $\mathrm{rk}\,\mathrm{K}_3 = 3$. We leave to the reader the definition of a contracting homotopy $s_\bullet : \mathrm{K}_\bullet \to \mathrm{K}_\bullet[1]$.

(2) Consider the quiver Q



bound by the ideal $I = \langle\alpha_1\beta_2 + \beta_1\beta_2 - \beta_1\alpha_2, \alpha_1\alpha_2 + \beta_1\alpha_2 - \beta_1\beta_2\rangle$. The sets of cells of $\mathcal{B}$ are the following: $\mathcal{C}_0 = \{x_1, x_2, x_3\}$, $\mathcal{C}_1 = \{\tilde{\alpha}_1^\circ, \tilde{\alpha}_2^\circ, \tilde{\beta}_1^\circ, \tilde{\beta}_2^\circ, \widetilde{\alpha_1\alpha_2}^\circ\}$, and $\mathcal{C}_2 = \{(\tilde{\alpha}_1^\circ, \tilde{\alpha}_2^\circ), (\tilde{\beta}_1^\circ, \tilde{\alpha}_2^\circ), (\tilde{\alpha}_1^\circ, \tilde{\beta}_2^\circ), (\tilde{\beta}_1^\circ, \tilde{\beta}_2^\circ)\}$. On the other hand, the sets of cells of $\mathcal{B}^\sharp$ are $\mathcal{C}_0^\sharp = \{x_1, x_2, x_3\}$, $\mathcal{C}_1^\sharp = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \widetilde{\alpha_1\alpha_2}\}$, and $\mathcal{C}_2^\sharp = \{(\tilde{\alpha}_1, \tilde{\alpha}_2)\}$. An straightforward computation yields to

$$\mathrm{H}_i(\mathcal{B}) = \begin{cases} \mathbb{Z} & \text{if } i \in \{0, 2\}, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathrm{H}_i\big(\mathcal{B}^\sharp\big) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{B}$ and $\mathcal{B}^\sharp$ do not have the same homotopy type. However, note that $A = kQ/I$, does not have a semi-normed basis.

Recall that, given an arcwise topological space X, the Hurewicz–Poincaré theorem (see [31], for instance) states that its first homology group, $\mathrm{H}_1(X)$ is the abelianisation of its fundamental group $\pi_1(X)$. This allows to prove the following:

5.6. Corollary. *Let $A = kQ/I$ be a constricted algebra having a semi-normed basis. Then its first simplicial* (*co*)*homology groups are independent of the presentation of A.*

Proof. For such an algebra one has

$$\mathrm{SH}_1(A) \simeq \mathrm{H}_1(\mathcal{B}) \simeq \frac{\pi_1(\mathcal{B})}{[\pi_1(\mathcal{B}), \pi_1(\mathcal{B})]} \simeq \frac{\pi_1(Q, I)}{[\pi_1(Q, I), \pi_1(Q, I)]}.$$

On the other hand, the universal coefficients theorem gives

$$\begin{aligned}\mathrm{SH}^1(A,G) &\simeq \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{SH}_1(A),G\big)\oplus \mathrm{Ext}^1_{\mathbb{Z}}\big(\mathrm{SH}_0(A),G\big)\\ &\simeq \mathrm{Hom}_{\mathbb{Z}}\big(\mathrm{SH}_1(A),G\big). \qquad \square\end{aligned}$$

We now turn to the Hochschild cohomology of algebras.

6. Hochschild cohomology

Recall that for an arbitrary k-algebra A, its enveloping algebra is the tensor product $A^{\mathrm{e}} = A \otimes_k A^{\mathrm{op}}$. Thus, an $(A{-}A)$-bimodule can be seen equivalently as an A^{e}-module. The Hochschild cohomology groups $\mathrm{HH}^i(A,M)$ of an algebra A with coefficients in some $(A{-}A)$-bimodule M, are, by definition the groups $\mathrm{Ext}^i_{A^{\mathrm{e}}}(A,M)$. In case M is the $(A{-}A)$-bimodule ${}_AA_A$, we simply denote them by $\mathrm{HH}^i(A)$. We refer the reader to [11,22,29], for instance, for general results about Hochschild (co)homology of algebras.

6.1. A convenient resolution

In [12], Cibils gave a convenient projective resolution of A over A^{e}. Let E be the subalgebra of A generated by the vertices of Q. Note that E is semi-simple, and that $A = \operatorname{rad} A \oplus E$ as $(E{-}E)$-bimodule. Let $\operatorname{rad} A^{\otimes n}$ denote the nth tensor power of $\operatorname{rad} A$ with itself over E. With these notations, one has a projective resolution of A as $(A{-}A)$-module:

$$\begin{aligned}\cdots \to A\otimes_E \operatorname{rad} A^{\otimes n}\otimes_E A &\xrightarrow{b_n} A\otimes_E \operatorname{rad} A^{\otimes n-1}\otimes_E A \xrightarrow{b_{n-1}} \cdots\\ \to A\otimes_E \operatorname{rad} A\otimes_E A &\xrightarrow{b_1} A\otimes_E A \xrightarrow{b_0} A \to 0\end{aligned}$$

where d_0 is the multiplication and

$$\begin{aligned}b_n(a\otimes\sigma_1\otimes\cdots\otimes\sigma_n\otimes b) = a\sigma_1\otimes\cdots\otimes\sigma_n\otimes b\\ &+\sum_{j=1}^{n-1}(-1)^j a\otimes\sigma_1\otimes\cdots\otimes\sigma_j\sigma_{j+1}\otimes\cdots\otimes\sigma_n b\\ &+(-1)^n a\otimes\sigma_1\otimes\cdots\otimes\sigma_n b.\end{aligned}$$

Moreover, there is an obvious natural isomorphism $\mathrm{Hom}_{A^{\mathrm{e}}}(A\otimes_E \operatorname{rad} A^{\otimes n}\otimes_E A, A) \simeq \mathrm{Hom}_{E^{\mathrm{e}}}(\operatorname{rad} A^{\otimes n}, A)$. This will be useful later. We denote by b_n the corresponding boundary operator, and, moreover, we let $b^n = \mathrm{Hom}_{E^{\mathrm{e}}}(b_n, A)$.

Remark. Note that the tensor products are taken over E. Thus, if $\sigma_1, \sigma_2 \in \operatorname{rad} A$, with, say $\sigma_1 \in e_iAe_j$, and $\sigma_2 \in e_lAe_m$ then, in $\operatorname{rad} A^{\otimes 2}$, one has

$$\sigma_1\otimes\sigma_2 = \sigma_1 e_j\otimes e_l\sigma_2 = \sigma_1\otimes e_je_l\sigma_2$$

and this vanishes if $j \neq l$. The same argument shows that rad $A^{\otimes n}$ is generated by elements of the form $\sigma_1 \otimes \cdots \otimes \sigma_n$ where $\sigma_i \in e_{i-1}Ae_i$ for $1 < i \leqslant n$. Moreover, if A admits a semi-normed basis B, then using k-linearity, one can assume that each σ_i is an element of B.

Following [24], for $n \geqslant 1$, define $\varepsilon_n : \mathrm{Hom}_{\mathbb{Z}}(\mathrm{SC}_n(A), k^+) \to \mathrm{Hom}_{E^e}(\mathrm{rad}\, A^{\otimes n}, A)$ in the following way: for $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{SC}_n(A), k^+)$, and a basis element $(\sigma_1, \ldots, \sigma_n)$, put $\varepsilon_n(f)(\sigma_1 \otimes \cdots \otimes \sigma_n) = f(\sigma_1, \ldots, \sigma_n)\sigma_1 \cdots \sigma_n$ whenever $(\sigma_1, \ldots, \sigma_n) \in \mathrm{SC}_n(A)$, and 0 otherwise.

Also, for $n \geqslant 1$, define $\mu_n : \mathrm{Hom}_{E^e}(\mathrm{rad}\, A^{\otimes n}, A) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{SC}_n(A), k^+)$ as follows: for a basis element $(\sigma_1, \ldots, \sigma_n)$ in $\mathrm{SC}_n(A)$, we have $\sigma_1\sigma_2 \cdots \sigma_n \neq 0$, and lies in, say, e_0Ae_n, which can be written as the direct sum of k-vector spaces $\langle \sigma_1\sigma_2 \cdots \sigma_n \rangle \oplus A'_{0n}$. Moreover, $\sigma_1 \otimes \cdots \otimes \sigma_n \in \mathrm{rad}\, A^{\otimes n}$, thus, for $g \in \mathrm{Hom}_{E^e}(\mathrm{rad}\, A^{\otimes n}, A)$ we have:

$$g(\sigma_1 \otimes \cdots \otimes \sigma_n) = g(e_0\sigma_1 \otimes \cdots \otimes \sigma_n e_n) = e_0 g(\sigma_1 \otimes \cdots \otimes \sigma_n) e_n$$

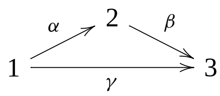
so that $g(\sigma_1 \otimes \cdots \otimes \sigma_n) \in e_0Ae_n$, and there is a scalar λ and $a_0 \in A'_{0n}$ such that $g(\sigma_1 \otimes \cdots \otimes \sigma_n) = \lambda\sigma_1\sigma_2 \cdots \sigma_n + a_0$. Define $\mu_n(g)(\sigma_1, \ldots, \sigma_n) = \lambda$.

6.2. Lemma [24]. *With the above notations, one has*:

(a) $\mu_n\varepsilon_n = \mathrm{id}$, *for* $n \geqslant 1$.
(b) $\varepsilon_\bullet$ *is a morphism of complexes.*
(c) *If* A *is schurian, then* $\mu_\bullet$ *is a morphism of complexes.*

Remark. In [20], it was shown that for incidence algebras, the morphisms $\mathrm{H}^i(\varepsilon)$ are isomorphisms. Moreover, as a consequence of a result of [15] (see also [10]), $\mathrm{H}^1(\varepsilon)$ is also an isomorphism for schurian triangular algebras. Thus, in light of [24] (or Lemma 6.2 above), one may naturally ask if in case A is schurian, the monomorphisms $\mathrm{H}^i(\varepsilon)$ are isomorphisms. As the following example shows, this is not always the case.

Example. Consider the quiver



bound by the ideal $I = \langle \alpha\beta \rangle$. The algebra $A = kQ/I$ is schurian, and one can easily compute

$$\mathrm{SH}^i(A, k^+) = \begin{cases} k & \text{if } i = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, using for instance, Happel's long exact sequence [22], one gets

$$\mathrm{HH}^i(A) = \begin{cases} k & \text{if } i = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

Recall that an algebra $A = kQ/I$ is said to be *semi-commutative* [22] whenever for w, w' paths sharing origin and terminus in Q, then $w \in I$ if and only if $w' \in I$. For instance, incidence algebras are semi-commutative. Schurian triangular, semi-commutative algebras are also called *weakly transitive* [16]. The algebra in the preceding example is schurian, but not semi-commutative. This leads to the main result of this section, which is a generalization of a result of Gerstenhaber and Schack [20], and makes more precise a result of Martins and de la Peña (Theorem 3 in [24]).

6.3. Theorem. *Let $A = kQ/I$ be a schurian triangular, semi-commutative algebra. Then, for each $i \geqslant 0$, there is an isomorphism of abelian groups*

$$\mathrm{H}^i(\varepsilon) : \mathrm{SH}^i\big(A, k^+\big) \xrightarrow{\sim} \mathrm{HH}^i(A).$$

Proof. In light of Lemma 6.2, there only remains to show that if A is semi-commutative then $\varepsilon_n\mu_n = \mathrm{id}$ for $n \geqslant 1$. Let $f \in \mathrm{Hom}_{E^e}(\mathrm{rad}\, A^{\otimes n}, A)$, and $\sigma_1 \otimes \cdots \otimes \sigma_n$ be a basis element in $\mathrm{rad}\, A^{\otimes n}$, with, say $\sigma_1\sigma_2 \cdots \sigma_n \in e_0 A e_n$. Assume $\sigma_1\sigma_2 \cdots \sigma_n \neq 0$. Since A is schurian, there exists some scalar λ such that $f(\sigma_1 \otimes \cdots \otimes \sigma_n) = \lambda\sigma_1 \cdots \sigma_n$, thus $(\mu_n f)(\sigma_1, \ldots, \sigma_n) = \lambda$, and

$$\begin{aligned}(\varepsilon_n\mu_n f)(\sigma_1 \otimes \cdots \otimes \sigma_n) &= \big((\mu_n f)(\sigma_1, \ldots, \sigma_n)\big)\sigma_1 \cdots \sigma_n = \lambda\sigma_1 \cdots \sigma_n \\ &= f(\sigma_1 \otimes \cdots \otimes \sigma_n).\end{aligned}$$

On the other hand, if $\sigma_1 \cdots \sigma_n = 0$, then,

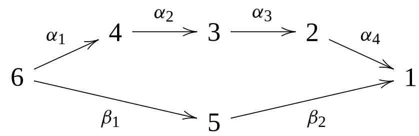
$$(\varepsilon_n\mu_n f)(\sigma_1 \otimes \cdots \otimes \sigma_n) = \big((\mu_n f)(\sigma_1, \ldots, \sigma_n)\big)\sigma_1 \cdots \sigma_n = 0.$$

Moreover, since A is semi-commutative we have $e_0 A e_n = 0$, and therefore $f(\sigma_1 \otimes \cdots \otimes \sigma_n) = 0$. □

Remark. Again, it is straightforward to check that $\varepsilon_\bullet$ are $\mu_\bullet$ preserve cup-products, thus the isomorphism above induce a ring isomorphism.

6.4. Corollary [20]. *Let $(\Sigma, \leqslant)$ be a finite poset and $A = A(\Sigma)$ be its incidence algebra, then, for each $i \geqslant 0$, there is an isomorphism of abelian groups* $\mathrm{H}^i(\mathcal{B}\Sigma, k^+) \simeq \mathrm{HH}^i(A)$.

Remark. There exist algebras which are not semi-commutative, but there are still an isomorphisms $\mathrm{H}^i(\varepsilon) : \mathrm{SH}^i(A, k^+) \xrightarrow{\sim} \mathrm{HH}^i(A)$ for all $i \geqslant 0$. Consider the following quiver Q



and let $A = kQ/I$ where $I = \langle\alpha_1\alpha_2, \alpha_3\alpha_4\rangle$. This algebra is schurian, but not semi-commutative. One can easily compute

$$\mathrm{SH}^i\big(A, k^+\big) = \mathrm{HH}^i(A) = \begin{cases} k & \text{if } i = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Keeping in mind the last theorem, and Proposition 3.1, we can get new short algebraic-topology flavored proofs of some well-known results in [5,22] about the Hochschild cohomology groups of monomial algebras. Let $\chi(Q)$ be the Euler characteristic of Q, that is, let $\chi(Q) = 1 - |Q_0| + |Q_1|$.

6.5. Corollary [22]. *Let $A = kQ/I$ be a monomial semi-commutative schurian algebra, then*

(a) $\mathrm{HH}^0(A) = k$.
(b) $\dim_k \mathrm{HH}^1(A) = \chi(Q)$.
(c) $\mathrm{HH}^i(A) = 0$ *for* $i \geqslant 2$.

Proof. With the above hypotheses, the graph $\overline{Q}$ is a strong deformation retract of $\mathcal{B}(Q, I)$, and Theorem 6.3 holds. The results follow directly. □

6.6. Corollary [4]. *Let $A = kQ/I$ be a monomial algebra, then the following are equivalent*:

(a) $\mathrm{HH}^i(A) = 0$ *for* $i > 0$.
(b) $\mathrm{HH}^1(A) = 0$.
(c) *Q is a tree.*

Proof. It is trivial that (a) implies (b). In order to show that (b) implies (c) assume that Q is not a tree. Again, since I is monomial, $\overline{Q}$ is a strong deformation retract of $\mathcal{B}(Q, I)$, and, since it is not a tree, we have

$$\dim_k \mathrm{SH}^1\big(A, k^+\big) = \chi(Q) > 0.$$

The result then follows from Lemma 6.2. Finally, we show that (c) implies (a). If Q is a tree, then A is schurian semi-commutative, thus Theorem 6.3 applies. But $\overline{Q}$, which is a strong deformation retract of $\mathcal{B}(Q, I)$, is an acyclic 1-dimensional complex. The result follows directly. □

Acknowledgments

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