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Commuting traces and commutativity preserving maps on triangular algebras

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Abstract

Let $\mathfrak{A}$ be a triangular algebra. The problem of describing the form of a bilinear map $B:\mathfrak{A}\times\mathfrak{A}\to\mathfrak{A}$ satisfying $B(x,x)x = xB(x,x)$ for all $x\in\mathfrak{A}$ is considered. As an application, commutativity preserving maps and Lie isomorphisms of certain triangular algebras (e.g., upper triangular matrix algebras and nest algebras) are determined.

Keywords: Triangular algebra; Nest algebra; Commuting trace of a bilinear map; Lie isomorphism; Commutativity preserving map

1. Introduction

Let $\mathcal{A}$ be an algebra over R, a commutative ring with unity. By $Z(\mathcal{A})$ we denote the center of $\mathcal{A}$. An R-linear map $f:\mathcal{A}\to\mathcal{A}$ is said to be *commuting* if it satisfies $[f(x),x]=0$ for all $x\in\mathcal{A}$ (we denote $xy-yx$ by $[x,y]$, the commutator of x and y). Each commuting R-linear map of the form $f(x)=\lambda x+\mu(x)$, where λ is a central element in $\mathcal{A}$ and $\mu:\mathcal{A}\to Z(\mathcal{A})$ is an R-linear map, will be called *proper*. A trace of a bilinear map is a map of the form $x\mapsto B(x,x)$, where $B:\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ is some bilinear map. We say that a commuting

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trace q is *proper* if it can be written as $q(x) = \lambda x^2 + \mu(x)x + \nu(x)$ for some central element λ in $\mathcal{A}$, an R-linear map $\mu : \mathcal{A} \to Z(\mathcal{A})$, and a trace $\nu : \mathcal{A} \to Z(\mathcal{A})$ of some bilinear map. Commuting maps which are not proper will be called improper. For an account on commuting maps we refer the reader to the forthcoming survey paper [20].

At the beginning of the 90s Brešar described the form of commuting additive maps [17], and also the form of commuting traces of biadditive maps [18] (see also [23]) on prime rings. These results have initiated the theory of functional identities, which deals with maps of rings satisfying some identical relations. We refer the reader to [19] for the survey of the theory of functional identities. More recently Cheung [27] considered commuting linear maps on triangular algebras (e.g., on upper triangular matrix algebras and nest algebras). He determined the class of triangular algebras for which every commuting linear map is proper. Motivated by the results of Brešar and Cheung we consider commuting traces of bilinear maps on triangular algebras. The main purpose of this paper is to find a certain class of triangular algebras for which every commuting trace is proper (Theorem 3.1). Consequently, we will be able to consider commuting traces of bilinear maps of upper triangular matrix algebras and nest algebras. It should be mentioned that the form of commuting traces of multilinear maps of upper triangular matrix algebras has already been described by Beidar, Brešar, and Chebotar [3].

Another important motivation for the present paper is the study of Lie isomorphisms. Let us mention that the first functional identity on prime rings which has turned out to be important because of its applications was the one concerning commuting traces of biadditive maps. Namely, in [18] the long-standing Herstein's conjecture on Lie isomorphisms of prime rings was settled using this identity. This initiated a series of papers on Lie homomorphisms, Lie derivations and some related maps [1,6,7,9,11–14,25,26,45] and so in [8] the final solutions to all Herstein's Lie map conjectures were obtained. Commuting traces of biadditive maps appear also in some linear preserver problems [5,18,21,22], automatic continuity problems [15,16,46] and some other Lie algebra problems [10]. Therefore we may expect that commuting traces of bilinear maps on triangular algebras shall also turn out to be useful. The results on Lie isomorphisms and commutativity preserving maps in the last two sections already indicate this.

A Lie isomorphism of an algebra $\mathcal{A}$ onto an algebra $\mathcal{B}$ is a linear bijective map θ which preserves commutators, i.e.,

$$\theta\big([x, y]\big) = \big[\theta(x), \theta(y)\big] \quad \text{for all } x, y \in \mathcal{A}.$$

Note that if φ is an isomorphism or the negative of an antiisomorphism from $\mathcal{A}$ onto $\mathcal{B}$ and τ is a linear map from $\mathcal{A}$ into the center of $\mathcal{B}$, sending commutators to zero, then $\varphi + \tau$ is a Lie homomorphism. In [32] Hua proved that each Lie automorphism of the algebra of all $n \times n$ matrices, $n \geqslant 3$, over a division ring is of such form. Somewhat later, in the series of papers [36,38,39] Martindale has extended Hua's theorem to more general rings. Let us also mention that similar result for von Neumann factors (i.e., prime von Neumann algebras) was obtained by Miers [40]. As we have already stated, it was Brešar [18] who solved the problem of describing the form of Lie isomorphisms between prime rings, using his own result on commuting traces. In 1994 Đoković [31] showed that every Lie automorphism of upper triangular matrix algebras $\mathcal{T}_n(R)$ over a commutative ring R without nontrivial idempotents has the standard form as well. A few years later Marcoux and Sourour [35] obtained a similar characterization for Lie isomorphisms between nest algebras. Using our main result (Theorem 3.1) we shall be able to describe the form of an arbitrary Lie isomorphism of a certain class of triangular algebras (Theorem 4.3). As corollaries to Theorem 4.3, characterizations of Lie isomorphisms of $n \times n$ upper triangular matrix algebras, and on nest algebras are obtained.

A commutativity preserving map is a map $\theta : \mathcal{A} \to \mathcal{B}$ satisfying $[\theta(x), \theta(y)] = 0$ whenever $[x, y] = 0$. The obvious examples are maps of the form

$$\theta(x) = \alpha\varphi(x) + \gamma(x) \quad \text{for all } x \in \mathcal{A}, \tag{1}$$

where α is a nonzero central element in $\mathcal{B}$, $\varphi : \mathcal{A} \to \mathcal{B}$ is an isomorphism or an antiisomorphism, and $\gamma : \mathcal{A} \to Z(\mathcal{B})$ is a linear map. Clearly, each Lie isomorphism preserves commutativity. Commutativity preserving maps have been studied for almost 30 years. The usual goal is to show that in certain cases maps of the form (1) are in fact the only examples of commutativity preserving maps. Probably the first result of this kind was obtained by Watkins [47] for the case where θ is a linear bijection and $\mathcal{A} = \mathcal{B}$ is the algebra of all $n \times n$ matrices, $n \geqslant 4$, over a field. Afterwards the series of papers [2,24,29,43,44] on commutativity preserving maps followed, refining Watkins's result in several ways. In particular, Choi, Jafarian, and Radjavi [29] also obtained some extensions of these results to the algebra of bounded linear operators on an infinite dimensional Hilbert space. Similar problems were solved for the algebra of all bounded linear operators of a nontrivial Banach space [42] and also for von Neumann factors [41]. Using his result on commuting traces, Brešar [18, Theorem 2] described the form of linear bijective commutativity preserving maps on a rather general class of prime algebras. Later Marcoux and Sourour [34] obtained the characterization of linear maps preserving commutativity in both directions (i.e., $[x, y] = 0$ if and only if $[\theta(x), \theta(y)] = 0$) on upper triangular matrix algebras $\mathcal{T}_n(F)$ over a field F. In the last section of the present paper we consider linear bijective maps θ satisfying

$$\left[\theta\left(x^2\right), \theta(x)\right] = 0 \quad \text{for all } x \in \mathcal{A}, \tag{2}$$

which is weaker than assuming that θ preserves commutativity. Applying our main result we describe the form of such maps on certain triangular algebras (see Theorem 5.2). Consequently, we are able to characterize linear bijective maps satisfying (2) of $n \times n$ upper triangular matrix algebras with $n > 2$, which has already been done in [3]. Using our main results we also obtain the characterization of linear bijective maps between nest algebras satisfying (2), which generalizes the above mentioned characterization of Lie isomorphisms [35].

Finally, it should be mentioned that there is a close connection between Lie derivations and Lie isomorphisms. Recall that a Lie derivation d on an algebra $\mathcal{A}$ is a linear map satisfying $d([x, y]) = [d(x), y] + [x, d(y)]$ for all $x, y \in \mathcal{A}$. In several cases it turns out that any Lie derivation is the sum of a derivation and a linear map whose image is central (see, e.g., [18,28,37]). Using our main theorem and the same techniques as in the sequel we could obtain such result for a certain class of triangular algebras. However, since Cheung [28] has recently solved this problem for a rather general class of triangular algebras, using a more direct approach, we omit stating it.

2. Triangular algebras

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. An $(\mathcal{A},\mathcal{B})$-bimodule $\mathcal{M}$ is *loyal* if $a\mathcal{M}b=0$ implies $a=0$ or $b=0$ for any $a\in\mathcal{A}$, $b\in\mathcal{B}$.

Obviously, each loyal $(\mathcal{A},\mathcal{B})$-bimodule $\mathcal{M}$ is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. Following [27] we state

Definition 2.2. Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a commutative ring R, and let $\mathcal{M}$ be a unital $(\mathcal{A},\mathcal{B})$-bimodule, which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. The R-algebra

$$\operatorname{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})=\left\{\begin{pmatrix} a & m\\ & b\end{pmatrix};\ a\in\mathcal{A},\ m\in\mathcal{M},\ b\in\mathcal{B}\right\}$$

under the usual matrix operations will be called a *triangular algebra*.

Consider a triangular algebra $\mathfrak{A}=\operatorname{Tri}(\mathcal{A},\mathcal{M},\mathcal{B})$. Any element of the form

$$\begin{pmatrix} a & 0\\ & b\end{pmatrix}\in\mathfrak{A}$$

will be denoted by $a\oplus b$. Let us define projections $\pi_{\mathcal{A}}:\mathfrak{A}\to\mathcal{A}$ and $\pi_{\mathcal{B}}:\mathfrak{A}\to\mathcal{B}$ by

$$\pi_{\mathcal{A}}:\begin{pmatrix} a & m\\ & b\end{pmatrix}\mapsto a\quad\text{and}\quad \pi_{\mathcal{B}}:\begin{pmatrix} a & m\\ & b\end{pmatrix}\mapsto b.$$

By [27, Proposition 3] we know that the center $Z(\mathfrak{A})$ of $\mathfrak{A}$ coincides with

$$\{a\oplus b\mid am=mb\text{ for all }m\in\mathcal{M}\}.$$

Moreover, $\pi_{\mathcal{A}}(Z(\mathfrak{A}))\subseteq Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathfrak{A}))\subseteq Z(\mathcal{B})$, and there exists a unique algebra isomorphism $\tau:\pi_{\mathcal{A}}(Z(\mathfrak{A}))\to\pi_{\mathcal{B}}(Z(\mathfrak{A}))$ such that $am=m\tau(a)$ for all $m\in\mathcal{M}$.

Lemma 2.3. *Let $\mathcal{M}$ be a loyal $(\mathcal{A},\mathcal{B})$-bimodule and let $f,g:\mathcal{M}\to\mathcal{A}$ be arbitrary maps. Suppose $f(m)n+g(n)m=0$ for all $m,n\in\mathcal{M}$. If $\mathcal{B}$ is noncommutative, then $f=g=0$.*

Proof. Using $f(m)n+g(n)m=0$ for all $m,n\in\mathcal{M}$, we see that

$$\big(f(m)nb_1\big)b_2=-g(nb_1)mb_2=\big(f(mb_2)n\big)b_1=-\big(g(n)m\big)b_2b_1=f(m)nb_2b_1$$

for all $m,n\in\mathcal{M}$ and $b_1,b_2\in\mathcal{B}$. Therefore, $f(\mathcal{M})\mathcal{M}[\mathcal{B},\mathcal{B}]=0$. Since $\mathcal{M}$ is loyal and $\mathcal{B}$ is noncommutative it follows that $f=0$. Clearly, $f=0$ yields $g=0$. □

Lemma 2.4. *Let $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ with a loyal $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$. Let $\alpha \in \pi_{\mathcal{A}}(Z(\mathfrak{A}))$ and let $a \in \mathcal{A}$ be a nonzero element. If $\alpha a = 0$, then $\alpha = 0$.*

Proof. We have $0 = \alpha am = am\tau(\alpha)$ for all $m \in \mathcal{M}$. Since $\mathcal{M}$ is loyal it follows that $\tau(\alpha) = 0$. Therefore, $\alpha = 0$. □

Lemma 2.5. *Let $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ with a loyal $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$. Then the center $Z(\mathfrak{A})$ of $\mathfrak{A}$ is a domain.*

Proof. Let $\lambda = \alpha \oplus \tau(\alpha)$, $\mu = \beta \oplus \tau(\beta) \in Z(\mathfrak{A})$. Suppose $\lambda\mu = 0$. Then $\alpha\beta = 0$. By Lemma 2.4 it follows that either $\alpha = 0$ or $\beta = 0$. Therefore, $\lambda = 0$ or $\mu = 0$. □

Lemma 2.6. $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *does not contain nonzero central ideals.*

Proof. Let I be a central ideal of $\mathfrak{A}$. Suppose $\alpha \oplus \tau(\alpha) \in I$. Hence,

$$\begin{pmatrix} \alpha & 0 \\ & \tau(\alpha) \end{pmatrix} \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha m \\ & 0 \end{pmatrix} \in I$$

for all $m \in \mathcal{M}$. This yields $\alpha\mathcal{M} = 0$ and so $\alpha = 0 = \alpha \oplus \tau(\alpha)$. □

Lemma 2.7. *Let R be 2-torsionfree. Then $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ satisfies the polynomial identity $[[x^2, y], [x, y]]$ if and only if both $\mathcal{A}$ and $\mathcal{B}$ are commutative.*

Proof. If $\mathcal{A}$ and $\mathcal{B}$ are commutative it follows easily that $\mathfrak{A}$ satisfies the polynomial identity $[[x^2, y], [x, y]]$.

Next, suppose that $[[x^2, y], [x, y]] = 0$ for all $x, y \in \mathfrak{A}$. Assume that, e.g., $\mathcal{A}$ is noncommutative. Let $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$ be arbitrary elements and let

$$x = \begin{pmatrix} a_1 & 0 \\ & 0 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} a_2 & m \\ & 0 \end{pmatrix}.$$

Then $[[x^2, y], [x, y]] = 0$ yields $a_1[a_1, a_2]a_1 m = 0$ for all $a_1, a_2 \in \mathcal{A}$ and $m \in \mathcal{M}$. Since $\mathcal{M}$ is faithful as a left $\mathcal{A}$-module we have $a_1[a_1, a_2]a_1 = 0$ for all $a_1, a_2 \in \mathcal{A}$. Replacing a_1 by $a_1 \pm 1_{\mathcal{A}}$ and comparing both identities, so obtained, it follows that $2[a_1, a_2] = 0$ for all $a_1, a_2 \in \mathcal{A}$. However, this contradicts our assumption. □

Recently, Cheung [27, Theorem 2] proved that each commuting linear map of $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is proper if $\pi_{\mathcal{A}}(Z(\mathfrak{A})) = Z(\mathcal{A})$ (or $\mathcal{A} = [\mathcal{A}, \mathcal{A}]$), $\pi_{\mathcal{B}}(Z(\mathfrak{A})) = Z(\mathcal{B})$ (or $\mathcal{B} = [\mathcal{B}, \mathcal{B}]$), and

$$Z(\mathfrak{A}) = \left\{a \oplus b \mid a \in Z(\mathcal{A}),\ b \in Z(\mathcal{B}),\ am_0 = m_0 b\right\}$$

for some $m_0 \in \mathcal{M}$. A similar result can be proved in the case $\mathcal{M}$ is loyal:

Remark 2.8. Every commuting linear map of $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is proper if the following conditions hold:

(i) $\pi_{\mathcal{A}}(Z(\mathfrak{A})) = Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathfrak{A})) = Z(\mathcal{B})$,
(ii) either $\mathcal{A}$ or $\mathcal{B}$ is noncommutative,
(iii) $\mathcal{M}$ is loyal.

Proof. Let $F : \mathfrak{A} \to \mathfrak{A}$ be a commuting linear map. Without loss of generality we may assume that $\mathcal{B}$ is noncommutative. By [27, Proposition 4] F is of the form

$$F : \begin{pmatrix} a & m \\ & b \end{pmatrix} \mapsto \begin{pmatrix} f_1(a) + f_2(b) + f_3(m) & f_1(1)m - mg_1(1) \\ & g_1(a) + g_2(b) + g_3(m) \end{pmatrix},$$

where $f_1 : \mathcal{A} \to \mathcal{A}$, $f_2 : \mathcal{B} \to Z(\mathcal{A})$, $f_3 : \mathcal{M} \to Z(\mathcal{A})$, $g_1 : \mathcal{A} \to Z(\mathcal{B})$, $g_2 : \mathcal{B} \to \mathcal{B}$ and $g_3 : \mathcal{M} \to Z(\mathcal{B})$ are linear maps. Moreover,

$$f_3(m)m = mg_3(m)$$

for all $m \in \mathcal{M}$. Since $\pi_{\mathcal{A}}(Z(\mathfrak{A})) = Z(\mathcal{A})$ and $\pi_{\mathcal{B}}(Z(\mathfrak{A})) = Z(\mathcal{B})$, by [27, Theorem 1(ii)] it suffices to prove that $f_3(m) \oplus g_3(m) \in Z(\mathfrak{A})$ for all $m \in \mathcal{M}$. Linearizing $(f_3(m) - \tau^{-1}(g_3(m)))m = 0$ and using Lemma 2.3 we get $f_3(m) = \tau^{-1}(g_3(m))$ and thus $f_3(m) \oplus g_3(m) \in Z(\mathfrak{A})$ for all $m \in \mathcal{M}$. □

We close this section with the following two standard examples of triangular algebras, i.e., upper triangular matrix algebras and nest algebras.

Upper triangular matrix algebras

Let $\mathcal{M}_{l\times m}(R)$ denote the set of all $l \times m$ matrices and let $\mathcal{T}_n(R)$ denote the algebra of all $n \times n$ upper triangular matrices over R. For $n \geqslant 2$ and each $1 \leqslant l \leqslant n-1$ the algebra $\mathcal{T}_n(R)$ can be represented as a triangular algebra of the form

$$\mathcal{T}_n(R) = \begin{pmatrix} \mathcal{T}_l(R) & \mathcal{M}_{l\times(n-l)}(R) \\ & \mathcal{T}_{n-l}(R) \end{pmatrix}.$$

Remark 2.9. If R is a commutative domain then $\mathcal{M} = \mathcal{M}_{l\times(n-l)}(R)$ is a loyal $(\mathcal{T}_l(R), \mathcal{T}_{n-l}(R))$-bimodule.

Proof. Suppose $A \in \mathcal{T}_l(R)$ and $B \in \mathcal{T}_{n-l}(R)$ are nonzero. Hence there exists a nonzero (i, j)th entry a_{ij} of A and a nonzero (s, t)-entry b_{st} of B for some $1 \leqslant i \leqslant j \leqslant l$, $1 \leqslant s \leqslant t \leqslant n-l$. Pick $M \in \mathcal{M}$ such that $m_{js} = 1$ and all its other entries are zero. Then $AMB \neq 0$, since its (i, t)th entry equals $a_{ij}b_{st} \neq 0$. Thus, $\mathcal{M}$ is loyal. □

Remark 2.10. Let $n > 2$ be an integer and R be 2-torsionfree. Then $\mathcal{T}_n(R)$ does not satisfy the polynomial identity $[[x^2, y], [x, y]]$.

Proof. Since $\mathcal{T}_n(R) = \mathrm{Tri}(R, \mathcal{M}_{1\times(n-1)}(R), \mathcal{T}_{n-1}(R))$ and $\mathcal{T}_{n-1}(R)$ is noncommutative it follows by Lemma 2.7 that $\mathcal{T}_n(R)$ does not satisfy the polynomial identity $[[x^2, y], [x, y]]$. □

Nest algebras

A *nest* is a chain $\mathcal{N}$ of closed subspaces of a complex Hilbert space H containing $\{0\}$ and H which is closed under arbitrary intersections and closed linear span. The *nest algebra* associated to $\mathcal{N}$ is the algebra

$$\mathcal{T}(\mathcal{N}) = \{T \in \mathcal{B}(H) \mid T(N) \subseteq N \text{ for all } N \in \mathcal{N}\}.$$

A nest $\mathcal{N}$ is called trivial if $\mathcal{N} = \{0, H\}$. The reader is referred to [30] for the general theory of nest algebras. We will make use of a standard result (see [27, Proposition 5] and [30, Chapter 2]) which allows one to consider a nontrivial nest algebra as a triangular algebra. Namely, if $N \in \mathcal{N}\backslash\{0, H\}$ and E is the orthonormal projection onto N, then $\mathcal{N}_1 = E(\mathcal{N})$ and $\mathcal{N}_2 = (1 - E)(\mathcal{N})$ are nests of N and $N^\perp$, respectively. Moreover, $\mathcal{T}(\mathcal{N}_1) = E\mathcal{T}(\mathcal{N})E$, $\mathcal{T}(\mathcal{N}_2) = (1 - E)\mathcal{T}(\mathcal{N})(1 - E)$ and

$$\mathcal{T}(\mathcal{N}) = \begin{pmatrix} \mathcal{T}(\mathcal{N}_1) & E\mathcal{T}(\mathcal{N})(1-E) \\ & \mathcal{T}(\mathcal{N}_2) \end{pmatrix}.$$

Remark 2.11. $\mathcal{M} = E\mathcal{T}(\mathcal{N})(1 - E)$ is a loyal $(\mathcal{T}(\mathcal{N}_1), \mathcal{T}(\mathcal{N}_2))$-bimodule.

Proof. Suppose $A \in \mathcal{T}(\mathcal{N}_1)$ and $B \in \mathcal{T}(\mathcal{N}_2)$ are nonzero operators. Clearly, there exist $u \in N$ and $v \in N^\perp$ such that $Au \neq 0$ and $Bv = w \neq 0$. Let $M : x \mapsto \langle x, w\rangle u$. Note that $M \in E\mathcal{T}(\mathcal{N})(1 - E)$ and $AMBv \neq 0$. This means that $\mathcal{M}$ is loyal. □

Recall that the center of each nest algebra coincides with $\mathbb{C}1$ [30, Corollary 19.5]. Using this the following assertion follows almost immediately.

Remark 2.12. Let $\mathcal{N}$ be a nest on a Hilbert space H with $\dim_{\mathbb{C}} H > 1$. Then $\mathcal{T}(\mathcal{N})$ is noncommutative.

Remark 2.13. Let $\mathcal{N}$ be a nest on a Hilbert space H with $\dim_{\mathbb{C}} H > 2$. Then $[[x^2, y], [x, y]]$ and $[x, [y, [z, w]]]$ are not polynomial identities on $\mathcal{T}(\mathcal{N})$.

Proof. If $\mathcal{N}$ is trivial, then $\mathcal{T}(\mathcal{N}) = \mathcal{B}(H)$ does not satisfy neither $[[x^2, y], [x, y]]$ nor $[x, [y, [z, w]]]$ provided that $\dim_{\mathbb{C}} H > 2$. This can be easily deduced from the standard PI theory, and on the other hand one can easily check this directly. Now, assume that there is $N \in \mathcal{N}\backslash\{0, H\}$. Let E be the orthonormal projection onto N. Then $\mathcal{T}(\mathcal{N}) = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$, where $\mathcal{A} = \mathcal{T}(\mathcal{N}_1) = E\mathcal{T}(\mathcal{N})E$, $\mathcal{B} = \mathcal{T}(\mathcal{N}_2) = (1 - E)\mathcal{T}(\mathcal{N})(1 - E)$ are nest algebras and $\mathcal{M} = E\mathcal{T}(\mathcal{N})(1 - E)$. By Remark 2.12 either $\mathcal{A}$ or $\mathcal{B}$ is noncommutative, since

$\dim_{\mathbb{C}} N > 1$ or $\dim_{\mathbb{C}} N^{\perp} > 1$. Hence by Lemma 2.7, $\mathcal{T}(\mathcal{N})$ does not satisfy $[[x^2, y], [x, y]]$. On the other hand, setting

$$x = y = z = \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ & 0 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix}$$

for some nonzero $m \in \mathcal{M}$ we see that $[x, [y, [z, w]]] = w \neq 0$. □

3. Commuting traces of bilinear maps

Theorem 3.1. *Let* $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ *be a triangular algebra over a* 2*-torsionfree commutative ring* R*. If*

(i) *each commuting linear map on* $\mathcal{A}$ *or* $\mathcal{B}$ *is proper,*
(ii) $\pi_{\mathcal{A}}(Z(\mathfrak{A})) = Z(\mathcal{A}) \neq \mathcal{A}$ *and* $\pi_{\mathcal{B}}(Z(\mathfrak{A})) = Z(\mathcal{B}) \neq \mathcal{B}$,
(iii) $\mathcal{M}$ *is loyal,*

then each commuting trace $q : \mathfrak{A} \to \mathfrak{A}$ *of a bilinear map is proper.*

Proof. For convenience we set $\mathcal{A}_1 = \mathcal{A}$, $\mathcal{A}_2 = \mathcal{B}$ and $\mathcal{A}_3 = \mathcal{M}$. We denote the unity of $\mathcal{A}_1$ by 1 and the unity of $\mathcal{A}_2$ by $1'$. Suppose that q is a trace of a bilinear map $B : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$. Hence there exist bilinear maps $f_{ij} : \mathcal{A}_i \times \mathcal{A}_j \to \mathcal{A}_1$, $g_{ij} : \mathcal{A}_i \times \mathcal{A}_j \to \mathcal{A}_2$ and $h_{ij} : \mathcal{A}_i \times \mathcal{A}_j \to \mathcal{A}_3$, $1 \leqslant i \leqslant j \leqslant 3$, such that

$$q : \begin{pmatrix} a_1 & a_3 \\ & a_2 \end{pmatrix} \mapsto \begin{pmatrix} F(a_1, a_2, a_3) & H(a_1, a_2, a_3) \\ & G(a_1, a_2, a_3) \end{pmatrix},$$

where

$$F(a_1, a_2, a_3) = \sum_{1 \leqslant i \leqslant j \leqslant 3} f_{ij}(a_i, a_j),$$

$$G(a_1, a_2, a_3) = \sum_{1 \leqslant i \leqslant j \leqslant 3} g_{ij}(a_i, a_j),$$

$$H(a_1, a_2, a_3) = \sum_{1 \leqslant i \leqslant j \leqslant 3} h_{ij}(a_i, a_j).$$

Since q is commuting it follows that

$$0 = \left[\begin{pmatrix} F & H \\ & G \end{pmatrix}, \begin{pmatrix} a_1 & a_3 \\ & a_2 \end{pmatrix} \right] = \begin{pmatrix} [F, a_1] & Fa_3 + Ha_2 - a_1 H - a_3 G \\ & [G, a_2] \end{pmatrix}.$$

Let us first consider the identity

$$0=[F,a_1]=\sum_{1\leqslant i\leqslant j\leqslant 3}\big[f_{ij}(a_i,a_j),a_1\big]\quad \text{for all } a_i\in\mathcal{A}_i,\ i=1,2,3. \tag{3}$$

Setting $a_2=0$, $a_3=0$ we see that $[f_{11}(a_1,a_1),a_1]=0$ for each $a_1\in\mathcal{A}_1$. Next, putting $a_3=0$ in (3) we get $[f_{12}(a_1,a_2),a_1]+[f_{22}(a_2,a_2),a_1]=0$. Replacing a_1 by $-a_1$ and comparing both identities we obtain that $2[f_{12}(a_1,a_2),a_1]=0$. Since R is 2-torsionfree we have $[f_{12}(a_1,a_2),a_1]=0$ and hence $f_{22}(a_2,a_2)\in Z(\mathcal{A}_1)$ for all $a_1\in\mathcal{A}_1$, $a_2\in\mathcal{A}_2$. Similarly, setting $a_2=0$ in (3) we obtain $[f_{13}(a_1,a_3),a_1]=0$ and $f_{33}(a_3,a_3)\in Z(\mathcal{A}_1)$ for all $a_1\in\mathcal{A}_1$, $a_3\in\mathcal{A}_3$. It now follows from (3) that also f_{23} maps into $Z(\mathcal{A}_1)$. Summarizing the above conclusions we see that

$$\begin{aligned}
&a_1\mapsto f_{11}(a_1,a_1) &&\text{is a commuting trace,}\\
&a_1\mapsto f_{12}(a_1,a_2) &&\text{is a commuting linear map for each } a_2\in\mathcal{A}_2,\\
&a_1\mapsto f_{13}(a_1,a_3) &&\text{is a commuting linear map for each } a_3\in\mathcal{A}_3,\\
&\quad f_{22},\ f_{23},\ f_{33} &&\text{map into } Z(\mathcal{A}_1).
\end{aligned}$$

Analogously, the identity

$$0=[G,a_2]=\sum_{1\leqslant i\leqslant j\leqslant 3}\big[g_{ij}(a_i,a_j),a_2\big]$$

for all $a_i\in\mathcal{A}_i$, $i=1,2,3$, implies

$$\begin{aligned}
&a_2\mapsto g_{22}(a_2,a_2) &&\text{is a commuting trace,}\\
&a_2\mapsto g_{12}(a_1,a_2) &&\text{is a commuting linear map for each } a_1\in\mathcal{A}_1,\\
&a_2\mapsto g_{23}(a_2,a_3) &&\text{is a commuting linear map for each } a_3\in\mathcal{A}_3,\\
&\quad g_{11},\ g_{13},\ g_{33} &&\text{map into } Z(\mathcal{A}_2).
\end{aligned}$$

It remains to consider

$$Fa_3+Ha_2-a_1H-a_3G=0. \tag{4}$$

Let $a_1=0$, $a_2=0$. Then

$$f_{33}(a_3,a_3)a_3=a_3g_{33}(a_3,a_3) \tag{5}$$

for all $a_3\in\mathcal{A}_3$. Next, setting $a_1=0$, $a_3=0$ in (4) it follows $0=Ha_2=h_{22}(a_2,a_2)a_2$ for all $a_2\in\mathcal{A}_2$. Clearly, $h_{22}(1',1')=0$. Replacing a_2 by $a_2\pm 1'$ we get

$$h_{22}(a_2, a_2) + \bigl(h_{22}(a_2, 1') + h_{22}(1', a_2)\bigr)(a_2 + 1') = 0,$$
$$-h_{22}(a_2, a_2) - \bigl(h_{22}(a_2, 1') + h_{22}(1', a_2)\bigr)(a_2 - 1') = 0.$$

Comparing both identities we get $2(h_{22}(a_2, 1') + h_{22}(1', a_2)) = 0$, which further implies $h_{22}(a_2, a_2) = 0$ for all $a_2 \in \mathcal{A}_2$. Analogously, setting $a_2 = 0$, $a_3 = 0$ in (4) yields $h_{11}(a_1, a_1) = 0$ for all $a_1 \in \mathcal{A}_1$. Further, letting $a_3 = 0$ in (4) we see that $h_{12}(a_1, a_2)a_2 - a_1h_{12}(a_1, a_2) = 0$ for all $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$. Replacing a_1 by $-a_1$ and comparing both identities yields $a_1h_{12}(a_1, a_2) = 0$ for all $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$. Since $h_{12}(1, a_2) = 0$, the substitution $a_1 \mapsto a_1 + 1$ implies $h_{12}(a_1, a_2) = 0$ for all $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$. Thus $H(a_1, a_2, a_3) = h_{13}(a_1, a_3) + h_{23}(a_2, a_3) + h_{33}(a_3, a_3)$. Our next aim is to prove that

$$h_{23}(a_2, a_3)a_2 = a_3g_{22}(a_2, a_2) - f_{22}(a_2, a_2)a_3 \tag{6}$$

for all $a_2 \in \mathcal{A}_2$, $a_3 \in \mathcal{A}_3$. Setting $a_1 = 0$ in (4) and using (5) we obtain

$$\begin{aligned}&\bigl(f_{22}(a_2, a_2) + f_{23}(a_2, a_3)\bigr)a_3 + \bigl(h_{33}(a_3, a_3) + h_{23}(a_2, a_3)\bigr)a_2\\&\quad - a_3\bigl(g_{22}(a_2, a_2) + g_{23}(a_2, a_3)\bigr) = 0.\end{aligned} \tag{7}$$

Replacing a_2 by $-a_2$ we get

$$2f_{22}(a_2, a_2)a_3 + 2h_{23}(a_2, a_3)a_2 - 2a_3g_{22}(a_2, a_2) = 0$$

and hence (6) follows. Now, using (6) together with (7) one gets

$$h_{33}(a_3, a_3)a_2 = a_3g_{23}(a_2, a_3) - f_{23}(a_2, a_3)a_3 \tag{8}$$

for all $a_2 \in \mathcal{A}_2$, $a_3 \in \mathcal{A}_3$. In a similar manner, taking $a_2 = 0$ in (4) and using (5), it follows that

$$a_1h_{13}(a_1, a_3) = f_{11}(a_1, a_1)a_3 - a_3g_{11}(a_1, a_1), \tag{9}$$

$$a_1h_{33}(a_3, a_3) = f_{13}(a_1, a_3)a_3 - a_3g_{13}(a_1, a_3) \tag{10}$$

for all $a_1 \in \mathcal{A}_1$, $a_3 \in \mathcal{A}_3$. Using (5), (6), (8), (9), (10) together with (4) we obtain

$$a_1h_{23}(a_2, a_3) + a_3g_{12}(a_1, a_2) = h_{13}(a_1, a_3)a_2 + f_{12}(a_1, a_2)a_3 \tag{11}$$

for all $a_i \in A_i$, $i = 1, 2, 3$.

Recall that $[f_{13}(a_1, a_3), a_1] = 0$ for all $a_1 \in \mathcal{A}_1$, $a_3 \in \mathcal{A}_3$. Hence, replacing a_1 by $a_1 + 1$ implies that $f_{13}(1, a_3) \in Z(\mathcal{A}_1)$ for each $a_3 \in \mathcal{A}_3$. Thus, using (ii) we see that the identity (10) yields

$$h_{33}(a_3, a_3) = \alpha(a_3)a_3 \tag{12}$$

for all $a_3 \in \mathcal{A}_3$, where $\alpha(a_3) = f_{13}(1, a_3) - \tau^{-1}(g_{13}(1, a_3)) \in Z(\mathcal{A}_1)$. Next, we claim that

$$f_{33}(a_3, a_3) \oplus g_{33}(a_3, a_3) \in Z(\mathfrak{A}) \tag{13}$$

for each $a_3 \in \mathcal{A}_3$. Namely, by the complete linearization of (5) we obtain

$$\beta(l, m)n + \beta(n, l)m + \beta(m, n)l = 0 \tag{14}$$

for all $l, m, n \in \mathcal{A}_3$, where

$$\beta(m, n) = f_{33}(m, n) - \tau^{-1}\big(g_{33}(m, n)\big) + f_{33}(n, m) - \tau^{-1}\big(g_{33}(n, m)\big).$$

Obviously, the map $\beta : \mathcal{A}_3 \times \mathcal{A}_3 \to Z(\mathcal{A}_1)$ is bilinear and symmetric. Pick $a, b \in \mathcal{A}_1$ such that $[a, b] \neq 0$. Replacing l by al in (14) and subtracting (14) multiplied by a we get

$$\big(\beta(al, m) - \beta(l, m)a\big)n + \big(\beta(n, al) - \beta(n, l)a\big)m = 0$$

for all $l, m, n \in \mathcal{A}_3$. According to Lemma 2.3, $\beta(al, m) = \beta(l, m)a$ and hence $\beta(l, m)[a, b] = 0$ for all $l, m \in \mathcal{A}_3$. Now, since $[a, b] \neq 0$ Lemma 2.4 yields $\beta = 0$ and so, in particular, $\beta(m, m) = 0$ for all $m \in \mathcal{A}_3$. Thus, (13) holds. Our next aim is to prove that

$$\begin{aligned} f_{13}(a_1, a_3) &= \alpha(a_3)a_1 + \tau^{-1}\big(g_{13}(a_1, a_3)\big), \\ g_{23}(a_2, a_3) &= \tau\big(\alpha(a_3)\big)a_2 + \tau\big(f_{23}(a_2, a_3)\big) \end{aligned} \tag{15}$$

for all $a_i \in \mathcal{A}_i$, $i = 1, 2, 3$. Let $E(a_1, a_3) = f_{13}(a_1, a_3) - \alpha(a_3)a_1 - \tau^{-1}(g_{13}(a_1, a_3))$. Using (10) and (12) we get $E(a_1, a_3)a_3 = 0$, which further yields $E(a_1, a_3)b_3 + E(a_1, b_3)a_3 = 0$ for all $a_1 \in \mathcal{A}_1$ and $a_3, b_3 \in \mathcal{A}_3$. Using Lemma 2.3 we see that $E = 0$. Thus, f_{13} is as in (15). Analogously, using (8) one proves that g_{23} has the desired form as well.

Next, we consider maps f_{12} and g_{12}. By (i) we may assume that each commuting linear map on $\mathcal{A}_1$ is proper. Since $a_1 \mapsto f_{12}(a_1, a_2)$ is a commuting linear map on $\mathcal{A}_1$ for each $a_2 \in \mathcal{A}_2$, there exist maps $\gamma : \mathcal{A}_2 \to Z(\mathcal{A}_1)$ and $\delta : \mathcal{A}_1 \times \mathcal{A}_2 \to Z(\mathcal{A}_1)$ such that

$$f_{12}(a_1, a_2) = \gamma(a_2)a_1 + \delta(a_1, a_2), \tag{16}$$

where δ is R-linear in the first argument. Let us show that γ is R-linear and δ is R-bilinear. Clearly

$$\begin{aligned} f_{12}(a_1, a_2 + b_2) &= \gamma(a_2 + b_2)a_1 + \delta(a_1, a_2 + b_2), \\ f_{12}(a_1, a_2) + f_{12}(a_1, b_2) &= \gamma(a_2)a_1 + \delta(a_1, a_2) + \gamma(b_2)a_1 + \delta(a_1, b_2) \end{aligned}$$

and so

$$\big(\gamma(a_2 + b_2) - \gamma(a_2) - \gamma(b_2)\big)a_1 + \delta(a_1, a_2 + b_2) - \delta(a_1, a_2) - \delta(a_1, b_2) = 0$$

for all $a_1 \in \mathcal{A}_1$, $a_2, b_2 \in \mathcal{A}_2$. Commuting with $b_1 \in \mathcal{A}_1$ we get $(\gamma(a_2 + b_2) - \gamma(a_2) - \gamma(b_2))[a_1, b_1] = 0$ for all $a_1, b_1 \in \mathcal{A}_1$, $a_2, b_2 \in \mathcal{A}_2$ and $a_3 \in \mathcal{A}_3$. Now, Lemma 2.4 yields that γ is R-linear. Consequently, δ is R-linear in the second argument. Let $\gamma'(a_1) = g_{12}(a_1, 1') - \tau(\delta(a_1, 1'))$ for all $a_1 \in \mathcal{A}_1$. Since τ is R-linear and since g_{12} and δ are both R-bilinear, it follows that γ' is R-linear as well. We claim that

$$g_{12}(a_1, a_2) = \gamma'(a_1)a_2 + \tau\big(\delta(a_1, a_2)\big) \tag{17}$$

for all $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$. Namely by (9), $h_{13}(1, a_3) = f_{11}(1, 1)a_3 - a_3 g_{11}(1, 1)$ for all $a_3 \in \mathcal{A}_3$. Hence setting $a_1 = 1$ in (11) we get

$$h_{23}(a_2, a_3) = a_3\big\{\eta a_2 + \tau\big(f_{12}(1, a_2)\big) - g_{12}(1, a_2)\big\} \tag{18}$$

for all $a_2 \in \mathcal{A}_2$ and $a_3 \in \mathcal{A}_3$, where $\eta = \tau(f_{11}(1, 1)) - g_{11}(1, 1)$. Similarly, using (6) and (11) we obtain

$$h_{13}(a_1, a_3) = \big\{\theta a_1 + \tau^{-1}\big(g_{12}\big(a_1, 1'\big)\big) - f_{12}\big(a_1, 1'\big)\big\}a_3 \tag{19}$$

for all $a_1 \in \mathcal{A}_1$ and $a_3 \in \mathcal{A}_3$, where $\theta = \tau^{-1}(g_{22}(1', 1')) - f_{22}(1', 1')$. Now (16), (18) and (19) together with (11) imply

$$\begin{aligned} &a_1 a_3\big(\eta a_2 + \tau\big(f_{12}(1, a_2)\big) - g_{12}(1, a_2)\big) + a_3 g_{12}(a_1, a_2) \\ &\quad = \big(\theta a_1 + \tau^{-1}\big(g_{12}\big(a_1, 1'\big)\big) - f_{12}\big(a_1, 1'\big)\big)a_3 a_2 + \big(\gamma(a_2)a_1 + \delta(a_1, a_2)\big)a_3 \end{aligned}$$

and so

$$\begin{aligned} &a_1 a_3\big\{\big(\eta + \tau\big(\gamma\big(1'\big) - \theta\big)\big)a_2 + \tau\big(\delta(1, a_2)\big) - g_{12}(1, a_2)\big\} \\ &\quad = a_3\big\{\gamma'(a_1)a_2 + \tau\big(\delta(a_1, a_2)\big) - g_{12}(a_1, a_2)\big\} \end{aligned} \tag{20}$$

for all $a_i \in \mathcal{A}_i$, $i = 1, 2, 3$. Pick $a_1, b_1 \in \mathcal{A}_1$, such that $[a_1, b_1] \neq 0$. Replacing a_3 by $b_1 a_3$ in (20) and subtracting (20) multiplied by b_1 we get

$$[a_1, b_1]\mathcal{A}_3\big\{\big(\eta + \tau\big(\gamma(1) - \theta\big)\big)a_2 + \tau\big(\delta(1, a_2)\big) - g_{12}(1, a_2)\big\} = 0$$

for all $a_2 \in \mathcal{A}_2$. Since $\mathcal{A}_3$ is loyal it follows that

$$g_{12}(1, a_2) = \big(\eta + \tau\big(\gamma(1) - \theta\big)\big)a_2 + \tau\big(\delta(1, a_2)\big)$$

for all $a_2 \in \mathcal{A}_2$. Consequently, (20) implies

$$\mathcal{A}_3\big(\gamma'(a_1)a_2 + \tau\big(\delta(a_1, a_2)\big) - g_{12}(a_1, a_2)\big) = 0$$

for all $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$, and so we see that (17) holds. Let $\varepsilon = \theta - \gamma(1')$ and $\varepsilon' = \eta - \gamma'(1)$. Hence using (18) and (19) together with (16) and (17) we obtain

$$h_{23}(a_2, a_3) = a_3\bigl(\varepsilon' a_2 + \tau\bigl(\gamma(a_2)\bigr)\bigr),$$
$$h_{13}(a_1, a_3) = \bigl(\varepsilon a_1 + \tau^{-1}\bigl(\gamma'(a_1)\bigr)\bigr)a_3 \tag{21}$$

for all $a_i \in \mathcal{A}_i$, $i = 1, 2, 3$. Next, let us prove that

$$f_{11}(a_1, a_1) = \varepsilon a_1^2 + \tau^{-1}\bigl(\gamma'(a_1)\bigr)a_1 + \tau^{-1}\bigl(g_{11}(a_1, a_1)\bigr),$$
$$g_{22}(a_2, a_2) = \varepsilon' a_2^2 + \tau\bigl(\gamma(a_2)\bigr)a_2 + \tau\bigl(f_{22}(a_2, a_2)\bigr) \tag{22}$$

for all $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$. Using (9) together with (21) we get

$$\bigl(f_{11}(a_1, a_1) - \varepsilon a_1^2 - \tau^{-1}\bigl(\gamma'(a_1)\bigr)a_1 - \tau^{-1}\bigl(g_{11}(a_1, a_1)\bigr)\bigr)a_3 = 0$$

for all $a_1 \in \mathcal{A}_1$ and $a_3 \in \mathcal{A}_3$. Now, since $\mathcal{A}_3$ is faithful as a left $\mathcal{A}_1$-module it follows that f_{11} has the desired form. Analogously, we see that g_{22} has the form described in (22). Setting $a_1 = 1$, $a_2 = 1'$ in (11) and using (16), (17) and (21) we see that $\varepsilon a_3 = a_3\varepsilon'$ for all $a_3 \in \mathcal{A}_3$. This means that $\varepsilon \oplus \varepsilon' \in Z(\mathfrak{A})$. We are now able to make the final step of the proof. Let us define $\lambda = \varepsilon \oplus \varepsilon'$ and the map $\mu : \mathfrak{A} \to Z(\mathfrak{A})$ by

$$\begin{pmatrix} a_1 & a_3 \\ & a_2 \end{pmatrix} \mapsto \begin{pmatrix} \tau^{-1}(\gamma'(a_1)) + \gamma(a_2) + \alpha(a_3) & 0 \\ & \gamma'(a_1) + \tau(\gamma(a_2) + \alpha(a_3)) \end{pmatrix}.$$

Obviously, μ is linear. Using all conclusions derived above we see that $\nu(x) = q(x) - \lambda x^2 - \mu(x)x$ belongs to $Z(\mathfrak{A})$ for each $x \in \mathfrak{A}$. □

Recall that an algebra $\mathfrak{A}$ over a commutative ring R is said to be *central* over R if $Z(\mathfrak{A}) = R1$. We continue with a technical lemma, which will be used to cover some special situations where the theorem above does not work.

Lemma 3.2. *Let $\mathfrak{A} = \mathrm{Tri}(R, \mathcal{M}, \mathcal{B})$, where $\mathcal{B}$ is noncommutative and both $\mathfrak{A}$ and $\mathcal{B}$ are central over a commutative 2-torsionfree ring R. If*

- (i) *each commuting linear map on $\mathcal{B}$ is proper,*
- (ii) *for any $r \in R$ and $m \in \mathcal{M}$, $rm = 0$ implies $r = 0$ or $m = 0$,*
- (iii) *there exist $m_0 \in \mathcal{M}$ and $b_0 \in \mathcal{B}$ such that m_0b_0 and m_0 are linearly independent over R,*

then each commuting trace $q : \mathfrak{A} \to \mathfrak{A}$ of a bilinear map is proper.

Proof. We shall follow the proof of Theorem 3.1; therefore we will use the same notation. The proof is almost the same except at the following three places.

The first one concerns the proof of (13):

$$f_{33}(a_3, a_3) \oplus g_{33}(a_3, a_3) \in R1 \quad \text{for all } a_3 \in \mathcal{A}_3.$$

Namely, by (5) we have $(f_{33}(a_3,a_3) - \tau^{-1}(g_{33}(a_3,a_3)))a_3 = 0$. Since $f_{33}(a_3,a_3) - \tau^{-1}(g_{33}(a_3,a_3)) \in R1$ it follows according to the assumption (ii) that $f_{33}(a_3,a_3) = \tau^{-1}(g_{33}(a_3,a_3))$ for all $a_3 \in \mathcal{A}_3$.

The second place concerns the proof of (15):

$$f_{13}(a_1,a_3) = \alpha(a_3)a_1 + \tau^{-1}\big(g_{13}(a_1,a_3)\big),$$
$$g_{23}(a_2,a_3) = \tau\big(\alpha(a_3)\big)a_2 + \tau\big(f_{23}(a_2,a_3)\big)$$

for all $a_i \in \mathcal{A}_i$, $i = 1,2,3$. Namely, by (10), (8) and (12) we see that

$$\big(f_{13}(a_1,a_3) - \alpha(a_3)a_1 - \tau^{-1}\big(g_{13}(a_1,a_3)\big)\big)a_3 = 0,$$
$$a_3\big(g_{23}(a_2,a_3) - \tau\big(\alpha(a_3)\big)a_2 - \tau\big(f_{23}(a_2,a_3)\big)\big) = 0 \tag{23}$$

for all $a_i \in \mathcal{A}_i$, $i = 1,2,3$. Since $f_{13}(a_1,a_3) - \alpha(a_3)a_1 - \tau^{-1}(g_{13}(a_1,a_3)) \in R1$ it follows easily from (ii) that f_{13} has the desired form. Since $a_2 \mapsto g_{23}(a_2,a_3)$ is a commuting linear map on $\mathcal{A}_2$ there exist maps $\psi : \mathcal{A}_3 \to R1'$ and $\omega : \mathcal{A}_2 \times \mathcal{A}_3 \to R1'$ such that

$$g_{23}(a_2,a_3) = \psi(a_3)a_2 + \omega(a_2,a_3),$$

where ω is linear in the first argument. Let us prove that ψ is linear and ω is bilinear. Clearly,

$$g_{23}(a_2,a_3+b_3) = \psi(a_3+b_3)a_2 + \omega(a_2,a_3+b_3),$$
$$g_{23}(a_2,a_3) + g_{23}(a_2,b_3) = \psi(a_3)a_2 + \omega(a_2,a_3) + \psi(b_3)a_2 + \omega(a_2,b_3)$$

and so

$$\big(\psi(a_3+b_3) - \psi(a_3) - \psi(b_3)\big)a_2 + \omega(a_2,a_3+b_3) - \omega(a_2,a_3) - \omega(a_2,b_3) = 0$$

for all $a_2 \in \mathcal{A}_2$, $a_3, b_3 \in \mathcal{A}_3$. Commuting with $b_2 \in \mathcal{A}_2$ we get

$$\big(\psi(a_3+b_3) - \psi(a_3) - \psi(b_3)\big)[a_2,b_2] = 0$$

for all $a_2, b_2 \in \mathcal{A}_2$, $a_3, b_3 \in \mathcal{A}_3$. Pick $a_2, b_2 \in \mathcal{A}_2$ such that $[a_2,b_2] \neq 0$. Since $\mathcal{A}_3$ is faithful as a right $\mathcal{A}_2$-module there exists $c_3 \in \mathcal{A}_3$ such that $c_3[a_2,b_2] \neq 0$. Thus,

$$\tau^{-1}\big(\psi(a_3+b_3) - \psi(a_3) - \psi(b_3)\big)c_3[a_2,b_2] = 0$$

for all $a_3, b_3 \in \mathcal{A}_3$. Now (ii) yields that ψ is linear. Consequently, ω is linear in the second argument. Now, (23) can be rewritten as

$$a_3\big(\big(\psi(a_3) - \tau\big(\alpha(a_3)\big)\big)a_2 + \omega(a_2,a_3) - \tau\big(f_{23}(a_2,a_3)\big)\big) = 0 \tag{24}$$

for all $a_2 \in \mathcal{A}_2$ and $a_3 \in \mathcal{A}_3$. Setting $a_2 = b_0$ and $a_3 = m_0$ we get

$$\big(\tau^{-1}\big(\psi(m_0)\big) - \alpha(m_0)\big)m_0 b_0 + \big(\tau^{-1}\big(\omega(b_0, m_0)\big) - f_{23}(b_0, m_0)\big)m_0 = 0.$$

According to (iii) this implies that $\alpha(m_0) = \tau^{-1}(\psi(m_0))$, $f_{23}(b_0, m_0) = \tau^{-1}(\omega(b_0, m_0))$. Replacing a_3 by $a_3 + m_0$ and a_2 by b_0 in (24) we obtain

$$\big(\tau^{-1}\big(\psi(a_3)\big) - \alpha(a_3)\big)m_0 b_0 + \big(\tau^{-1}\big(\omega(b_0, a_3)\big) - f_{23}(b_0, a_3)\big)m_0 = 0.$$

Now, assumption (iii) yields $\alpha(a_3) = \tau^{-1}(\psi(a_3))$ for all $a_3 \in \mathcal{A}_3$. Consequently, (24) can be rewritten as

$$\big(\tau^{-1}\big(\omega(a_2, a_3)\big) - f_{23}(a_2, a_3)\big)a_3 = 0,$$

which further implies that $\omega(a_2, a_3) = \tau(f_{23}(a_2, a_3))$ for all $a_2 \in \mathcal{A}_2$ and $a_3 \in \mathcal{A}_3$. Thus, g_{23} has the desired form as well.

The final place that must be changed is the one concerning the form of the maps f_{12} and g_{12}. Since $a_2 \mapsto g_{12}(a_1, a_2)$ is a commuting R-linear map on $\mathcal{A}_2$ there exist maps $\gamma' : \mathcal{A}_1 \to R1'$ and $\delta' : \mathcal{A}_1 \times \mathcal{A}_2 \to R1'$ such that

$$g_{12}(a_1, a_2) = \gamma'(a_1)a_2 + \delta'(a_1, a_2), \tag{25}$$

where γ' is R-linear and δ' is R-bilinear. Note that since $\tau : \mathcal{A}_1 \to \mathcal{A}_2$ is R-linear and $\mathcal{A}_1 = R1$, we have $rm = mr$ for all $m \in \mathcal{A}_3$ and $r \in R$. We also point out that here each of the maps f_{ij} takes values in $R1$. Now (25), (18), (19) together with (11) yield

$$\begin{aligned} &a_1 a_3\big(\eta a_2 + \tau\big(f_{12}(1, a_2)\big) - \gamma'(1)a_2 - \delta'(1, a_2)\big) + a_3\big(\gamma'(a_1)a_2 + \delta'(a_1, a_2)\big) \\ &\quad = \big(\theta a_1 + \tau^{-1}\big(g_{12}\big(a_1, 1'\big)\big) - f_{12}\big(a_1, 1'\big)\big)a_3 a_2 + f_{12}(a_1, a_2)a_3 \end{aligned}$$

and hence

$$\begin{aligned} &a_3\big\{\tau\big(a_1\tau^{-1}(\eta) + \tau^{-1}\big(\gamma'(a_1) - \gamma'(1) - g_{12}\big(a_1, 1'\big)\big) - \theta a_1 + f_{12}\big(a_1, 1'\big)\big)a_2 \\ &\quad + \tau\big(\big(f_{12}(1, a_2) - \tau^{-1}\big(\delta'(1, a_2)\big)\big)a_1 + \tau^{-1}\big(\delta'(a_1, a_2)\big) - f_{12}(a_1, a_2)\big)\big\} = 0 \end{aligned} \tag{26}$$

for all $a_i \in \mathcal{A}_i$, $i = 1, 2, 3$. Pick $a_2, b_2 \in \mathcal{A}_2$ such that $[a_2, b_2] \neq 0$. Since $\mathcal{A}_3$ is faithful as a right $\mathcal{A}_2$-module the last identity yields

$$\tau\big(a_1\tau^{-1}(\eta) + \tau^{-1}\big(\gamma'(a_1) - g_{12}\big(a_1, 1'\big)\big) - \theta a_1 + f_{12}\big(a_1, 1'\big)\big)[a_2, b_2] = 0$$

and hence

$$\big(a_1\tau^{-1}(\eta) + \tau^{-1}\big(\gamma'(a_1) - g_{12}\big(a_1, 1'\big)\big) - \theta a_1 + f_{12}\big(a_1, 1'\big)\big)a_3[a_2, b_2] = 0$$

for all $a_1 \in \mathcal{A}_1$, where $a_3 \in \mathcal{A}_3$ is such that $a_3[a_2, b_2] \neq 0$. Therefore, (ii) implies that $a_1\tau^{-1}(\eta) + \tau^{-1}(\gamma'(a_1) - g_{12}(a_1, 1')) - \theta a_1 + f_{12}(a_1, 1') = 0$ for all $a_1 \in \mathcal{A}_1$. Accordingly, (26) implies

$$f_{12}(a_1, a_2) = \left(f_{12}(1, a_2) - \tau^{-1}\left(\delta'(1, a_2)\right)\right)a_1 + \tau^{-1}\left(\delta'(a_1, a_2)\right)$$

for all $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$. Let $\gamma(a_2) = f_{12}(1, a_2) - \tau^{-1}(\delta'(1, a_2))$ and $\delta(a_1, a_2) = \tau^{-1}(\delta'(a_1, a_2))$. Hence $f_{12}(a_1, a_2) = \gamma(a_2)a_1 + \delta(a_1, a_2)$.

Following the rest of the proof of Theorem 3.1 we obtain the conclusion of the lemma. □

Next, we give an example of a triangular algebra with an improper commuting trace. The example was constructed using the improper linear commuting map from [27, Example 1].

Example 3.3. For a field F let

$$\mathcal{A} = \left\{ \begin{pmatrix} t & a & x & y \\ & t & 0 & z \\ & & s & b \\ & & & s \end{pmatrix};\ a, b, s, t, x, y, z \in F \right\}$$

and let $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, F^4, F)$. Note that by Remark 2.8 each commuting linear map on $\mathfrak{A}$ is proper. However, there exists an improper commuting linear map on $\mathcal{A}$ [27, Example 1]. Thus, $\mathfrak{A}$ does not satisfy the condition (i) from Lemma 3.2 and the condition (ii) from Theorem 3.1. We claim that $q : \mathfrak{A} \to \mathfrak{A}$ defined by

$$\begin{pmatrix} t & a & x & y & m_1 \\ & t & 0 & z & m_2 \\ & & s & b & m_3 \\ & & & s & m_4 \\ & & & & r \end{pmatrix} \mapsto \begin{pmatrix} 0 & (t-r)x & 0 & xz & xm_2 \\ & 0 & 0 & 0 & 0 \\ & & 0 & (s-r)z & zm_4 \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

is an improper commuting trace of a bilinear map. Namely, pick $u \in \mathfrak{A}$ such that $x = m_2 = 1$ and all its other entries are 0. Obviously, $q(u) \notin Fu^2 + Fu + F1$.

Corollary 3.4. *Let $n \geqslant 2$ and let R be a 2-torsionfree commutative domain. Then each commuting trace $q : \mathcal{T}_n(R) \to \mathcal{T}_n(R)$ of a bilinear map is proper.*

Proof. First, let $n > 3$. Note that $\mathcal{T}_n(R) = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ for $\mathcal{A} = \mathcal{T}_2(R)$, $\mathcal{B} = \mathcal{T}_{n-2}(R)$ and $\mathcal{M} = \mathcal{M}_{2\times(n-2)}(R)$. By [27, Corollary 6] each commuting linear map on $\mathcal{A}$ and $\mathcal{B}$ is proper. The assumption (ii) of Theorem 3.1 clearly holds in our case and by Remark 2.9, $\mathcal{M}$ is a loyal $(\mathcal{A}, \mathcal{B})$-bimodule. Thus, Theorem 3.1 yields the conclusion.

Further, if $n = 3$ we may write $\mathcal{T}_3(R) = \mathrm{Tri}(R, \mathcal{M}_{1\times 2}(R), \mathcal{T}_2(R))$. Now Lemma 3.2 yields the desired conclusion. Since the assumptions (i) and (ii) of Lemma 3.2 obviously hold true, let us just verify (iii). Set

$$M = [\,1 \quad 0\,] \in \mathcal{M}_{1\times 2}(R) \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} \in \mathcal{T}_2(R).$$

Then MB and M are linearly independent over R.

In the case $n = 2$ one can obtain the conclusion by a direct but tedious computation, so we omit details in this case. □

Corollary 3.5. *Let $\mathcal{N}$ be a nest of a Hilbert space H. Then each commuting trace $q : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N})$ of a bilinear map is proper.*

Proof. Note that the corollary trivially holds in the case $\dim_{\mathbb{C}} H = 1$. If $\dim_{\mathbb{C}} H = 2$ we have either $\mathcal{T}(\mathcal{N}) \cong \mathcal{T}_2(\mathbb{C})$ or $\mathcal{T}(\mathcal{N}) \cong \mathcal{M}_2(\mathbb{C})$. Corollary 3.4 implies the conclusion in the first case, while [23, Theorem 3.1] implies it in the second one.

Further, suppose that $\dim_{\mathbb{C}} H > 2$. We consider the following three cases.

Case 1. Assume that $\mathcal{N}$ is trivial. Then $\mathcal{T}(\mathcal{N}) = \mathcal{B}(H)$ is a centrally closed prime algebra and hence the result follows from [18, Theorem 1].

Case 2. Suppose that there exists $N \in \mathcal{N}\backslash\{0, H\}$ such that $\dim N > 1$ and $\dim N^{\perp} > 1$. Let E be an orthonormal projection onto N. Note that

$$\mathcal{T}(\mathcal{N}) = \begin{pmatrix} \mathcal{A} & \mathcal{M} \\ & \mathcal{B} \end{pmatrix},$$

where $\mathcal{A} = \mathcal{T}(\mathcal{N}_1) = E\mathcal{T}(\mathcal{N})E$ and $\mathcal{B} = \mathcal{T}(\mathcal{N}_2) = (1 - E)\mathcal{T}(\mathcal{N})(1 - E)$ are nest algebras and $\mathcal{M} = E\mathcal{T}(\mathcal{N})(1 - E)$. By Cheung's result [27, Corollary 7] each commuting linear map on $\mathcal{A}$ and $\mathcal{B}$ is proper. Since the center of each nest algebra coincides with $\mathbb{C}1$, we have $\pi_{\mathcal{A}}(Z(\mathcal{T}(\mathcal{N}))) = Z(\mathcal{A})$ and also $\pi_{\mathcal{B}}(Z(\mathcal{T}(\mathcal{N}))) = Z(\mathcal{B})$. By Remark 2.12, $\mathcal{A}$ and $\mathcal{B}$ are noncommutative, since $\dim_{\mathbb{C}} N > 1$ and $\dim_{\mathbb{C}} N^{\perp} > 1$. By Remark 2.10, $\mathcal{M}$ is a loyal $(\mathcal{A}, \mathcal{B})$-bimodule. Thus, we may apply Theorem 3.1, which completes the proof in this case.

Case 3. Finally, assume that for each $N \in \mathcal{N}\backslash\{0, H\}$ we have either $\dim N = 1$ or $\dim N^{\perp} = 1$. Without loss of generality we may assume that there exists $N \in \mathcal{N}\backslash\{0, H\}$ with $\dim N = 1$. Consequently, we have either $\mathcal{N} = \{0, N, H\}$ or $\mathcal{N} = \{0, N, L, H\}$, where $N \subset L$ and $\dim L^{\perp} = 1$. Let E be the orthonormal projection onto N. Hence we have $\mathcal{T}(\mathcal{N}) = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$, where $\mathcal{A} = E\mathcal{T}(\mathcal{N})E = \mathbb{C}E$ and $\mathcal{B} = (1 - E)\mathcal{T}(\mathcal{N})(1 - E)$ are nest algebras and $\mathcal{M} = E\mathcal{T}(\mathcal{N})(1 - E)$. Note that our nest algebra $\mathcal{T}(\mathcal{N})$ satisfies the assumptions (i) and (ii) of Lemma 3.2. We claim that (iii) also holds true. First, suppose $\mathcal{N} = \{0, N, H\}$. Take nonzero vectors $u \in N$ and $v, w \in N^{\perp}$ such that $\langle v, w\rangle = 0$. We define $Bx = \langle x, w\rangle v$ and $Mx = \langle x, v\rangle u$. Note that $B \in \mathcal{B}$ and $M \in \mathcal{M}$. One can easily verify that MB and M are linearly independent over $\mathbb{C}$. Next, assume that $\mathcal{N} = \{0, N, L, H\}$. Pick nonzero vectors $u \in N$, $v \in L \cap N^{\perp}$ and $w \in L^{\perp}$ and define $Bx = \langle x, w\rangle v$ and $Mx = \langle x, v\rangle u$. Again, note that $B \in \mathcal{B}$, $M \in \mathcal{M}$, and MB, M are linearly independent over $\mathbb{C}$. □

4. Lie isomorphisms

Lemma 4.1. *Let $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ and $\mathfrak{A}' = \mathrm{Tri}(\mathcal{A}', \mathcal{M}', \mathcal{B}')$ be triangular algebras over a commutative ring R with $1/2 \in R$ and let $\theta : \mathfrak{A} \to \mathfrak{A}'$ be a Lie isomorphism. If*

(i) *each commuting trace of a bilinear map on $\mathfrak{A}'$ is proper,*
(ii) *at least one of $\mathcal{A}$, $\mathcal{B}$ and at least one of $\mathcal{A}'$, $\mathcal{B}'$ are noncommutative,*
(iii) *$\mathcal{M}'$ is loyal,*

then $\theta = \varphi + \tau$, where $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is a homomorphism or the negative of an antihomomorphism, φ is one-to-one, and $\tau : \mathfrak{A} \to Z(\mathfrak{A}')$ is a linear map sending commutators to zero. Moreover, if $\mathfrak{A}'$ is central over R, then φ is onto.

Proof. It is clear that θ satisfies $[\theta(x), \theta(x^2)] = 0$ for all $x \in \mathfrak{A}$. Replacing x by $\theta^{-1}(y)$, $y \in \mathfrak{A}'$, we get $[y, \theta(\theta^{-1}(y)^2)] = 0$ for all $y \in \mathfrak{A}'$. This means that the map $q(y) = \theta(\theta^{-1}(y)^2)$ is commuting. Since q is also the trace of a bilinear map $B : \mathfrak{A}' \times \mathfrak{A}' \to \mathfrak{A}'$, $B(y, z) = \theta(\theta^{-1}(y)\theta^{-1}(z))$, there exist $\lambda \in Z(\mathfrak{A}')$, a linear map $\mu_1 : \mathfrak{A}' \to Z(\mathfrak{A}')$, and a trace $\nu_1 : \mathfrak{A}' \to Z(\mathfrak{A}')$ of a bilinear map such that

$$\theta\big(\theta^{-1}(y)^2\big) = \lambda y^2 + \mu_1(y)y + \nu_1(y) \tag{27}$$

for $y \in \mathfrak{A}'$. Let $\mu = \mu_1\theta$ and $\nu = \nu_1\theta$. Hence μ and ν are mappings of $\mathfrak{A}$ into $Z(\mathfrak{A}')$ and μ is linear. Note that (27) can be rewritten as

$$\theta\big(x^2\big) = \lambda\theta(x)^2 + \mu(x)\theta(x) + \nu(x) \tag{28}$$

for all $x \in \mathfrak{A}$. We claim that $\lambda \neq 0$. Assume $\lambda = 0$. Then by (28) we have $\theta(x^2) - \mu(x)\theta(x) \in Z(\mathfrak{A}')$, and so

$$\begin{aligned}\theta\big(\big[[x^2, y], [x, y]\big]\big) &= \big[\big[\theta\big(x^2\big), \theta(y)\big], \big[\theta(x), \theta(y)\big]\big] = \mu(x)\big[\big[\theta(x), \theta(y)\big], \big[\theta(x), \theta(y)\big]\big] \\ &= 0\end{aligned}$$

for all $x, y \in \mathfrak{A}$. Consequently, $[[x^2, y], [x, y]] = 0$ for all $x, y \in \mathfrak{A}$. According to our assumptions this contradicts Lemma 2.7. Thus, $\lambda \neq 0$. Next, we define $\varphi : \mathfrak{A} \to \mathfrak{A}'$ by

$$\varphi(x) = \lambda\theta(x) + \frac{1}{2}\mu(x). \tag{29}$$

According to (28) we have

$$\varphi(x^2) = \lambda\theta(x^2) + \frac{1}{2}\mu(x^2) = \lambda^2\theta(x)^2 + \lambda\mu(x)\theta(x) + \lambda\nu(x) + \frac{1}{2}\mu(x^2),$$

while on the other hand

$$\varphi(x)^2 = \left(\lambda\theta(x) + \frac{1}{2}\mu(x)\right)^2 = \lambda^2\theta(x)^2 + \lambda\mu(x)\theta(x) + \frac{1}{4}\mu(x)^2.$$

Comparing these two relations we get

$$\varphi(x^2) - \varphi(x)^2 \in Z(\mathfrak{A}') \tag{30}$$

for all $x \in \mathfrak{A}$. Linearizing (30) we obtain

$$\varphi(x \circ y) - \varphi(x) \circ \varphi(y) \in Z(\mathfrak{A}') \tag{31}$$

for all $x, y \in \mathfrak{A}$, where $x \circ y$ denotes $xy + yx$. By (29) we have

$$\begin{aligned}\lambda\varphi([x, y]) &= \lambda^2\theta([x, y]) + \frac{1}{2}\lambda\mu([x, y]) = [\lambda\theta(x), \lambda\theta(y)] + \frac{1}{2}\lambda\mu([x, y])\\ &= \left[\varphi(x) - \frac{1}{2}\mu(x), \varphi(y) - \frac{1}{2}\mu(y)\right] + \frac{1}{2}\lambda\mu([x, y])\\ &= [\varphi(x), \varphi(y)] + \frac{1}{2}\lambda\mu([x, y])\end{aligned}$$

and hence

$$\lambda\varphi([x, y]) - [\varphi(x), \varphi(y)] \in Z(\mathfrak{A}') \tag{32}$$

for all $x, y \in \mathfrak{A}$. Multiplying (31) by λ and comparing with (32) we get

$$2\lambda\varphi(xy) - (\lambda + 1)\varphi(x)\varphi(y) - (\lambda - 1)\varphi(y)\varphi(x) \in Z(\mathfrak{A}')$$

for all $x, y \in \mathfrak{A}$. Consequently, the map

$$\varepsilon(x, y) = \lambda\varphi(xy) - \frac{1}{2}(\lambda + 1)\varphi(x)\varphi(y) - \frac{1}{2}(\lambda - 1)\varphi(y)\varphi(x)$$

maps from $\mathfrak{A} \times \mathfrak{A}$ into $Z(\mathfrak{A}')$. Denote $\frac{1}{2}(\lambda + 1)$ by α. Therefore

$$\lambda\varphi(xy) = \alpha\varphi(x)\varphi(y) + (\alpha - 1)\varphi(y)\varphi(x) + \varepsilon(x, y) \tag{33}$$

for all $x, y \in \mathfrak{A}$. Our aim is to show that $\varepsilon = 0$ and that either $\alpha = 0$ or $\alpha = 1$. According to (33) we have

$$\begin{aligned}\lambda^2\varphi(xyz) &= \lambda^2\varphi\big(x(yz)\big) = \lambda\alpha\varphi(x)\varphi(yz) + \lambda(\alpha-1)\varphi(yz)\varphi(x) + \lambda\varepsilon(x, yz)\\ &= \alpha\varphi(x)\big(\alpha\varphi(y)\varphi(z) + (\alpha-1)\varphi(z)\varphi(y) + \varepsilon(y,z)\big)\\ &\quad + (\alpha-1)\big(\alpha\varphi(y)\varphi(z) + (\alpha-1)\varphi(z)\varphi(y) + \varepsilon(y,z)\big)\varphi(x) + \lambda\varepsilon(x, yz)\\ &= \alpha^2\varphi(x)\varphi(y)\varphi(z) + \alpha(\alpha-1)\varphi(x)\varphi(z)\varphi(y) + \alpha(\alpha-1)\varphi(y)\varphi(z)\varphi(x)\\ &\quad + (\alpha-1)^2\varphi(z)\varphi(y)\varphi(x) + \lambda\varepsilon(x, yz) + \lambda\varepsilon(y,z)\varphi(x).\end{aligned}$$

On the other hand,

$$\begin{aligned}\lambda^2\varphi(xyz) &= \lambda^2\varphi\big((xy)z\big) = \lambda\alpha\varphi(xy)\varphi(z) + \lambda(\alpha-1)\varphi(z)\varphi(xy) + \lambda\varepsilon(xy, z)\\ &= \alpha^2\varphi(x)\varphi(y)\varphi(z) + \alpha(\alpha-1)\varphi(y)\varphi(x)\varphi(z) + \alpha(\alpha-1)\varphi(z)\varphi(x)\varphi(y)\\ &\quad + (\alpha-1)^2\varphi(z)\varphi(y)\varphi(x) + \lambda\varepsilon(xy, z) + \lambda\varepsilon(x,y)\varphi(z).\end{aligned}$$

Comparing these two identities we obtain

$$\alpha(\alpha-1)\big[\varphi(y), \big[\varphi(z), \varphi(x)\big]\big] + \lambda\varepsilon(y,z)\varphi(x) - \lambda\varepsilon(x,y)\varphi(z) \in Z\big(\mathfrak{A}'\big) \tag{34}$$

for all $x, y, z \in \mathfrak{A}$. Replacing z by x^2 in (34) and using (30) we get

$$\lambda\varepsilon\big(y, x^2\big)\varphi(x) - \lambda\varepsilon(x,y)\varphi(x)^2 \in Z\big(\mathfrak{A}'\big) \tag{35}$$

for all $x, y \in \mathfrak{A}$, which can be in view of (29) written as

$$-\lambda^3\varepsilon(x,y)\theta(x)^2 + \lambda^2\big(\varepsilon\big(y, x^2\big) + \mu(x)\varepsilon(x,y)\big)\theta(x) \in Z\big(\mathfrak{A}'\big) \tag{36}$$

for all $x, y \in \mathfrak{A}$. Commuting with arbitrary $u \in \mathfrak{A}'$ and then with $[\theta(x), u]$ we get

$$\lambda^3\varepsilon(x,y)\big[\big[\theta(x)^2, u\big], \big[\theta(x), u\big]\big] = 0 \tag{37}$$

for all $x, y \in \mathfrak{A}$. We may assume that $\mathcal{A}'$ is noncommutative. Pick $a_1, a_2 \in \mathcal{A}'$ such that $a_1[a_1, a_2]a_1 \neq 0$ (see the proof of Lemma 2.7). Setting

$$\theta(x_0) = \begin{pmatrix} a_1 & 0 \\ & 0 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} a_2 & m \\ & 0 \end{pmatrix}$$

for some $x_0 \in \mathfrak{A}$ and an arbitrary $m \in \mathcal{M}'$ in (37) we obtain

$$\pi_{\mathcal{A}'}\big(\lambda^3\varepsilon(x_0, y)\big)a_1[a_1, a_2]a_1 m = 0$$

for all $m \in \mathcal{M}'$. By the loyality of $\mathcal{M}'$ we have $\pi_{\mathcal{A}'}(\lambda^3\varepsilon(x_0, y))a_1[a_1, a_2]a_1 = 0$ and hence by Lemma 2.4 $\pi_{\mathcal{A}'}(\lambda^3\varepsilon(x_0, y)) = 0$, since $a_1[a_1, a_2]a_1 \neq 0$. Therefore $\lambda^3\varepsilon(x_0, \mathfrak{A}) = 0$. Since $\lambda \neq 0$, Lemma 2.5 implies $\varepsilon(x_0, \mathfrak{A}) = 0$. According to (36) we now have

$\lambda^2\varepsilon(y, x_0^2)\theta(x_0) \in Z(\mathfrak{A}')$ for all $y \in \mathfrak{A}$, which further yields that $\varepsilon(\mathfrak{A}, x_0^2) = 0$. We claim that ε is symmetric. Namely, setting $z = x$ in (34) and using (29) we get

$$\lambda^2\big(\varepsilon(y, x) - \varepsilon(x, y)\big)\theta(x) \in Z\big(\mathfrak{A}'\big) \tag{38}$$

for all $x, y \in \mathfrak{A}$. If $x = x_0$, then $\lambda^2\varepsilon(y, x_0)\theta(x_0) \in Z(\mathfrak{A}')$ for all $y \in \mathfrak{A}$. Thus, similarly as above, we see that $\varepsilon(\mathfrak{A}, x_0) = 0$. Next, replacing x by $x + x_0$ in (38) we obtain

$$\lambda^2\big(\varepsilon(y, x) - \varepsilon(x, y)\big)\theta(x_0) \in Z\big(\mathfrak{A}'\big)$$

for all $x, y \in \mathfrak{A}$. This, however, implies that ε is symmetric.

Replacing x by $x_0 + y$ in (35) we obtain

$$\lambda\big(\varepsilon\big(y, y^2\big) + \varepsilon(y, x_0 \circ y)\big)\varphi(x_0) + \lambda\varepsilon(y, x_0 \circ y)\varphi(y) \\ - \lambda\varepsilon(y, y)\big(\varphi(x_0) \circ \varphi(y)\big) - \lambda\varepsilon(y, y)\varphi(x_0)^2 \in Z\big(\mathfrak{A}'\big).$$

On the other hand, replacing x by $-x_0 + y$ in (35) we get

$$\lambda\big(-\varepsilon\big(y, y^2\big) + \varepsilon(y, x_0 \circ y)\big)\varphi(x_0) - \lambda\varepsilon(y, x_0 \circ y)\varphi(y) \\ + \lambda\varepsilon(y, y)\big(\varphi(x_0) \circ \varphi(y)\big) - \lambda\varepsilon(y, y)\varphi(x_0)^2 \in Z\big(\mathfrak{A}'\big).$$

Comparing these two relations it follows that

$$2\lambda\varepsilon(y, x_0 \circ y)\varphi(x_0) - 2\lambda\varepsilon(y, y)\varphi(x_0)^2 \in Z\big(\mathfrak{A}'\big)$$

for all $y \in \mathfrak{A}$, which can be in view of (29) written as

$$-2\lambda^3\varepsilon(y, y)\theta(x_0)^2 + 2\lambda^2\big(\varepsilon(y, x_0 \circ y) - \mu(x_0)\varepsilon(y, y)\big)\theta(x_0) \in Z\big(\mathfrak{A}'\big)$$

for all $y \in \mathfrak{A}$. Consequently,

$$2\lambda^3\varepsilon(y, y)\big[\big[\theta(x_0)^2, u\big], \big[\theta(x_0), u\big]\big] = 0$$

for all $y \in \mathfrak{A}$ and $u \in \mathfrak{A}'$. Similarly as above it follows that $2\lambda^3\varepsilon(y, y) = 0$ and so $\varepsilon(y, y) = 0$ for all $y \in \mathfrak{A}$. The linearization of $\varepsilon(y, y) = 0$ gives $0 = \varepsilon(x, y) + \varepsilon(y, x) = 2\varepsilon(x, y)$ for all $x, y \in \mathfrak{A}$. Whence it follows that $\varepsilon = 0$. Accordingly, (34) yields

$$\lambda^4\alpha(\alpha - 1)\big[\theta(x), \big[\theta(y), \big[\theta(z), \theta(w)\big]\big]\big] = 0$$

for all $x, y, z, w \in \mathfrak{A}$. Since θ is onto we have $\lambda^4\alpha(\alpha - 1)[x', [y', [z', w']]] = 0$ for all $x', y', z', w' \in \mathfrak{A}'$. Let us set

$$x' = y' = z' = \begin{pmatrix} 1_{\mathcal{A}'} & 0 \\ & 0 \end{pmatrix} \quad \text{and} \quad w' = \begin{pmatrix} 0 & m \\ & 0 \end{pmatrix},$$

where $m \in \mathcal{M}'$ is arbitrary. Hence $\pi_{\mathcal{A}'}(\lambda^4\alpha(\alpha - 1))m = 0$ for all $m \in \mathcal{M}'$. Therefore $\pi_{\mathcal{A}'}(\lambda^4\alpha(\alpha - 1)) = 0$ and hence $\lambda^4\alpha(\alpha - 1) = 0$. Using Lemma 2.5 we see that $\alpha = 0$ or $\alpha = 1$.

First, assume that $\alpha = 0$. Since $\alpha = (\lambda + 1)/2$ it follows $\lambda = -1$, which by (33) further implies that φ is an antihomomorphism. Let $\tau(x) = \mu(x)/2$. By (29) we see that $\theta = -\varphi + \tau$, which clearly yields that $\tau([x, y]) = 0$ for all $x, y \in \mathfrak{A}$. In an analogous manner we see that if $\alpha = 1$ then $\theta = \varphi + \tau$, φ is a homomorphism and $\tau(x) = -\mu(x)/2$ sends commutators to zero.

We also have to prove that φ is one-to-one. Suppose that $\varphi(w) = 0$ for some $w \in \mathfrak{A}$. Then $\theta(w) \in Z(\mathfrak{A}')$ and hence $w \in Z(\mathfrak{A})$. Thus, $\ker(\varphi) \subseteq Z(\mathfrak{A})$. However, by Lemma 2.6 our triangular algebra $\mathfrak{A}$ does not contain nonzero central ideals. Hence, $\ker(\varphi) = 0$.

It remains to prove that φ is onto in the case $\mathfrak{A}'$ is central over R. First, we show that $\varphi(1) = 1'$. Namely, since θ is a Lie isomorphism we have $\theta(1) \in Z(\mathfrak{A}')$ and hence $\varphi(1) = \theta(1) - \tau(1) \in Z(\mathfrak{A}')$. Further, since φ is a homomorphism or the negative of an antihomomorphism we see that $\varphi(x) = \varphi(x1) = \varphi(1)\varphi(x)$ for all $x \in \mathfrak{A}$. Using $\varphi(x) = \theta(x) - \tau(x)$ we get $(\varphi(1) - 1')\theta(x) - (\varphi(1) - 1')\tau(x) = 0$ for all $x \in \mathfrak{A}$. Hence we see that $(\varphi(1) - 1')[\mathfrak{A}', \mathfrak{A}'] = 0$. Consequently, $\pi_{\mathcal{A}'}(\varphi(1) - 1')[\mathcal{A}', \mathcal{A}'] = 0$. Then Lemma 2.4 implies $\pi_{\mathcal{A}'}(\varphi(1) - 1') = 0$ and so $\varphi(1) = 1'$. Obviously, we may write $\tau(x) = f(x)1'$ for some linear map $f : \mathfrak{A} \to R$. Since φ is R-linear we have $\theta(x) = \varphi(x) + f(x)1' = \varphi(x + f(x)1)$ for all $x \in \mathfrak{A}$. Consequently, φ is onto, since θ is bijective. The proof of the lemma is thus completed. □

Let us point out that the proof just given is in its first part only a modification of that of [18, Theorem 3]. By a careful inspection of this proof one could easily verify that the following result holds true.

Remark 4.2. Let $\mathfrak{A}$ and $\mathfrak{A}'$ be unital algebras central over a field F with $\mathrm{char}(F) \neq 2$ and let $\theta : \mathfrak{A} \to \mathfrak{A}'$ be a Lie isomorphism. If

(i) each commuting trace of a bilinear map on $\mathfrak{A}'$ is proper,
(ii) $\mathfrak{A}$ and $\mathfrak{A}'$ do not satisfy the polynomial identity $[[x^2, y], [x, y]]$,
(iii) $\mathfrak{A}'$ does not satisfy the polynomial identity $[x, [y, [z, w]]]$,

then $\theta = \varphi + \tau$, where $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is an isomorphism or the negative of an antiisomorphism, and $\tau : \mathfrak{A} \to F1'$ is a linear map sending commutators to zero.

Theorem 4.3. *Let $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ and $\mathfrak{A}' = \mathrm{Tri}(\mathcal{A}', \mathcal{M}', \mathcal{B}')$ be triangular algebras over a commutative ring R with $1/2 \in R$ and let $\theta : \mathfrak{A} \to \mathfrak{A}'$ be a Lie isomorphism. If*

(i) *each commuting linear map on $\mathcal{A}'$ or $\mathcal{B}'$ is proper,*
(ii) $\pi_{\mathcal{A}'}(Z(\mathfrak{A}')) = Z(\mathcal{A}') \neq \mathcal{A}'$ *and* $\pi_{\mathcal{B}'}(Z(\mathfrak{A}')) = Z(\mathcal{B}') \neq \mathcal{B}'$,
(iii) *either $\mathcal{A}$ or $\mathcal{B}$ is noncommutative,*
(iv) *$\mathcal{M}'$ is loyal,*

then $\theta = \varphi + \tau$, where $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is a homomorphism or the negative of an antihomomorphism, φ is one-to-one, and $\tau : \mathfrak{A} \to Z(\mathfrak{A}')$ is a linear map sending commutators to zero. Moreover, if $\mathfrak{A}'$ is central over R, then φ is onto.

Proof. Using Theorem 3.1 we see that each commuting trace of a bilinear map on $\mathfrak{A}'$ is proper. Thus, we may apply Lemma 4.1, which yields the conclusion. □

Corollary 4.4. *Let $n \geqslant 2$ and let R be a commutative domain with $1/2 \in R$. If $\theta : \mathcal{T}_n(R) \to \mathcal{T}_n(R)$ is a Lie isomorphism, then $\theta = \varphi + \tau$, where $\varphi : \mathcal{T}_n(R) \to \mathcal{T}_n(R)$ is an isomorphism or the negative of an antiisomorphism and $\tau : \mathcal{T}_n(R) \to R1$ is a linear map sending commutators to zero.*

Proof. In the case $n = 2$ we refer to [31, Theorem 6]. Next, suppose $n > 2$. We may write

$$\mathcal{T}_n(R) = \operatorname{Tri}\big(R, \mathcal{M}_{1\times(n-1)}(R), \mathcal{T}_{n-1}(R)\big).$$

By Corollary 3.4 each commuting trace of a bilinear map on $\mathcal{T}_n(R)$ is proper. Moreover, $\mathcal{T}_{n-1}(R)$ is noncommutative and $\mathcal{M}_{1\times(n-1)}(R)$ is a loyal $(R, \mathcal{T}_{n-1}(R))$-bimodule. Thus, Lemma 4.1 yields the conclusion. □

Corollary 4.5. *Let $\mathcal{N}$ and $\mathcal{N}'$ be nests on a Hilbert space H. If $\theta : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N}')$ is a Lie isomorphism, then $\theta = \varphi + \tau$, where $\varphi : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N}')$ is an isomorphism or the negative of an antiisomorphism, and $\tau : \mathcal{T}(\mathcal{N}) \to \mathbb{C}1'$ is a linear map sending commutators to zero.*

Proof. Note that the corollary trivially holds in case $\dim_{\mathbb{C}} H = 1$ (namely, $\theta = \mathrm{id} + (\theta - \mathrm{id})$). If $\dim_{\mathbb{C}} H = 2$ we have either $\mathcal{T}(\mathcal{N}) = \mathcal{T}(\mathcal{N}') \cong \mathcal{T}_2(\mathbb{C})$ or $\mathcal{T}(\mathcal{N}) = \mathcal{T}(\mathcal{N}') \cong \mathcal{M}_2(\mathbb{C})$. Corollary 4.4 implies the conclusion in the first case, while [23, Proposition 4.1] implies it in the second one.

Further, suppose that $\dim_{\mathbb{C}} H > 2$. Obviously, each nest algebra is central over $\mathbb{C}$. We claim that assumptions (i)–(iii) of Remark 4.2 hold in this case. Namely, (i) follows from Corollary 3.5, while (ii) and (iii) follow from Remark 2.13. Thus, we may apply Remark 4.2, which concludes the proof. □

As mentioned in the introduction, the last two corollaries are similar to the main results from [31] and [35].

5. Commutativity preserving maps

Lemma 5.1. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be unital algebras central over a field F with $\mathrm{char}(F) \neq 2$. Suppose that $\theta : \mathfrak{A} \to \mathfrak{A}'$ is a bijective linear map satisfying $[\theta(x^2), \theta(x)] = 0$ for all $x \in \mathfrak{A}$. If*

- (i) *each commuting trace of a bilinear map on $\mathfrak{A}'$ is proper,*
- (ii) *$\mathfrak{A}$ and $\mathfrak{A}'$ do not satisfy the polynomial identity $[[x^2, y], [x, y]]$,*

then

$$\theta(x) = \alpha\varphi(x) + \gamma(x)1' \quad \textit{for all } x \in \mathfrak{A},$$

where $\alpha \in F$, $\alpha \neq 0$, $\varphi : \mathfrak{A} \to \mathfrak{A}'$ *is a Jordan isomorphism, and* $\gamma : \mathfrak{A} \to F$ *is a linear map.*

Proof. Since $[\theta(x^2), \theta(x)] = 0$ for all $x \in \mathfrak{A}$ we may argue as in the proof of Lemma 4.1. Noting that $[y, \theta(\theta^{-1}(y)^2)] = 0$ for all $y \in \mathfrak{A}'$ we see that there exist $\lambda \in F1'$, a linear map $\mu : \mathfrak{A} \to F$, and a map $\nu : \mathfrak{A} \to F$ such that

$$\theta(x^2) = \lambda\theta(x)^2 + \mu(x)\theta(x) + \nu(x)1'$$

for all $x \in \mathfrak{A}$. In order to prove that $\lambda \neq 0$, we first show that θ maps $F1$ onto $F1'$. Since θ is linear it suffices to prove that $\theta(1) \in F1'$. Taking $x \pm 1$ for x in $[\theta(x^2), \theta(x)] = 0$ we get $2[\theta(x), \theta(1)] = 0$ for any $x \in \mathfrak{A}$. Since θ is bijective and $\mathrm{char}(F) \neq 2$ it follows that $\theta(1)$ lies in the center of $\mathfrak{A}'$ which is by our assumption equal to $F1'$. Thus we have $\theta(F1) = F1'$. Suppose $\lambda = 0$. Then $\theta(x^2) - \mu(x)\theta(x) \in F1'$ for all $x \in \mathfrak{A}$. Hence $\theta(x^2 - \mu(x)x) \in F1'$, which further implies that $x^2 - \mu(x)x \in F1$ for all $x \in \mathfrak{A}$. Therefore $[[x^2, y], [x, y]] = 0$ for all $x \in \mathfrak{A}$, which contradicts (ii). Thus $\lambda \neq 0$. Next, define $\varphi : \mathfrak{A} \to \mathfrak{A}'$ by

$$\varphi(x) = \lambda\theta(x) + \frac{1}{2}\mu(x)1'. \tag{39}$$

Clearly, φ is linear. We claim that φ is a Jordan homomorphism. Namely, the same argument as in the proof of Lemma 4.1 gives us

$$\varphi(x^2) - \varphi(x)^2 \in F1'$$

for all $x \in \mathfrak{A}$. Whence it follows that the map $\varepsilon : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}'$ defined by

$$\varepsilon(x, y) = \varphi(x \circ y) - \varphi(x) \circ \varphi(y)$$

is a symmetric bilinear map with range in $F1'$; here $x \circ y = xy + yx$. Our aim is to show that $\varepsilon = 0$. Pick any $x, y \in \mathfrak{A}$. Note that

$$\begin{aligned}
\varphi\big(x^2 \circ (y \circ x)\big) &= \varphi\big(x^2\big) \circ \varphi(y \circ x) + \varepsilon\big(x^2, y \circ x\big) \\
&= \left(\varphi(x)^2 + \frac{1}{2}\varepsilon(x, x)\right) \circ \big(\varphi(x) \circ \varphi(y) + \varepsilon(x, y)\big) + \varepsilon\big(x^2, y \circ x\big) \\
&= \varphi(x)^2 \circ \big(\varphi(y) \circ \varphi(x)\big) + \varepsilon(x, x)\big(\varphi(y) \circ \varphi(x)\big) \\
&\quad + 2\varepsilon(x, y)\varphi(x)^2 + \varepsilon(x, x)\varepsilon(x, y) + \varepsilon\big(x^2, y \circ x\big)
\end{aligned}$$

and

$$\begin{aligned}\varphi\big((x^2\circ y)\circ x\big) &= \varphi\big(x^2\circ y\big)\circ\varphi(x)+\varepsilon\big(x^2\circ y,x\big)\\ &= \left(\left(\varphi(x)^2+\frac{1}{2}\varepsilon(x,x)\right)\circ\varphi(y)+\varepsilon\big(x^2,y\big)\right)\circ\varphi(x)+\varepsilon\big(x^2\circ y,x\big)\\ &= \big(\varphi(x)^2\circ\varphi(y)\big)\circ\varphi(x)+\varepsilon(x,x)\big(\varphi(y)\circ\varphi(x)\big)\\ &\quad+2\varepsilon\big(x^2,y\big)\varphi(x)+\varepsilon\big(x^2\circ y,x\big).\end{aligned}$$

However, $x^2\circ(y\circ x)=(x^2\circ y)\circ x$ and so it follows that

$$\varepsilon(x,y)\varphi(x)^2-\varepsilon\big(x^2,y\big)\varphi(x)\in F1'. \tag{40}$$

Commuting with arbitrary $u\in\mathfrak{A}'$ and then with $[\varphi(x),u]$ we get

$$\varepsilon(x,y)\big[\big[\varphi(x)^2,u\big],\big[\varphi(x),u\big]\big]=0.$$

Thus, using (39) we obtain

$$\lambda^3\varepsilon(x,y)\big[\big[\theta(x)^2,u\big],\big[\theta(x),u\big]\big]=0$$

for all $x,y\in\mathfrak{A}$ and $u\in\mathfrak{A}'$. Since θ is onto, by (ii) there exist $x_0\in\mathfrak{A}$ and $u_0\in\mathfrak{A}'$ such that $[[\theta(x_0)^2,u_0],[\theta(x_0),u_0]]\neq 0$. Hence $\lambda^3\varepsilon(x_0,\mathfrak{A})=0$, which in turn implies $\varepsilon(x_0,\mathfrak{A})=0$. Then according to (40) we also have $\varepsilon(x_0^2,\mathfrak{A})\varphi(x_0)\in F1'$ and hence $\lambda\varepsilon(x_0^2,\mathfrak{A})[\theta(x_0),u_0]=0$. Since $\lambda\neq 0$ and $[\theta(x_0),u_0]\neq 0$ it follows that $\varepsilon(x_0^2,\mathfrak{A})=0$. Replacing x by x_0+y in (40) we obtain

$$\begin{aligned}&\varepsilon(y,y)\varphi(x_0)^2+\varepsilon(y,y)\big(\varphi(x_0)\circ\varphi(y)\big)-\varepsilon(x_0\circ y,y)\varphi(x_0)\\ &\quad-\varepsilon(x_0\circ y,y)\varphi(y)-\varepsilon\big(y^2,y\big)\varphi(x_0)\in F1'.\end{aligned}$$

On the other hand, replacing x by $-x_0+y$ in (40) we get

$$\begin{aligned}&\varepsilon(y,y)\varphi(x_0)^2-\varepsilon(y,y)\big(\varphi(x_0)\circ\varphi(y)\big)-\varepsilon(x_0\circ y,y)\varphi(x_0)\\ &\quad+\varepsilon(x_0\circ y,y)\varphi(y)+\varepsilon\big(y^2,y\big)\varphi(x_0)\in F1'.\end{aligned}$$

Comparing these two relations it follows that

$$2\varepsilon(y,y)\varphi(x_0)^2-2\varepsilon(x_0\circ y,y)\varphi(x_0)\in F1'$$

for all $y\in\mathfrak{A}$. Consequently, $\varepsilon(y,y)[[\varphi(x_0)^2,u_0],[\varphi(x_0),u_0]]=0$, which further implies $\lambda^3\varepsilon(y,y)[[\theta(x_0)^2,u_0],[\theta(x_0),u_0]]=0$ for all $y\in\mathfrak{A}$. Hence $\varepsilon(y,y)=0$ for all $y\in\mathfrak{A}$. The linearization of $\varepsilon(y,y)=0$ gives $0=\varepsilon(x,y)+\varepsilon(y,x)=2\varepsilon(x,y)$ for all $x,y\in\mathfrak{A}$. Whence it follows that $\varepsilon=0$. Thus, we have just proved that φ is a Jordan homomorphism. Setting $\alpha=\lambda^{-1}$ and $\gamma(x)=-\lambda^{-1}\mu(x)/2$, we have $\theta(x)=\alpha\varphi(x)+\gamma(x)1'$. As $\theta(F1)=F1'$, we see that $\varphi(1)\in F1'$ which further yields $\varphi(1)=1'$ since φ is a Jordan homomorphism and since θ is surjective. Whence $\theta(x) = \varphi(\alpha x + \gamma(x)1)$, showing that φ is surjective. Finally, $\varphi(x) = 0$ implies $\theta(x) \in F1'$ and hence $x = \beta 1$ for some $\beta \in F$. Consequently, $\beta 1' = 0$ and so $\beta = 0$ proving that φ is one-to-one. □

It should be mentioned that the proof just given is actually a modification of the one of [18, Theorem 2]. However, the proof given here is somewhat shorter and also modified in such a way that the assumption $\mathrm{char}(F) \neq 3$ is not needed. This improved argument was suggested to us by our colleague Maja Fošner.

Theorem 5.2. *Let $\mathfrak{A} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ and $\mathfrak{A}' = \mathrm{Tri}(\mathcal{A}', \mathcal{M}', \mathcal{B}')$ be algebras central over a field F with $\mathrm{char}(F) \neq 2$, and let $\theta : \mathfrak{A} \to \mathfrak{A}'$ be a bijective linear map satisfying $[\theta(x^2), \theta(x)] = 0$ for all $x \in \mathfrak{A}$. If*

- (i) *each commuting linear map on $\mathcal{A}'$ or $\mathcal{B}'$ is proper,*
- (ii) *$Z(\mathcal{A}') = F1_{\mathcal{A}'} \neq \mathcal{A}'$ and $Z(\mathcal{B}') = F1_{\mathcal{B}'} \neq \mathcal{B}'$,*
- (iii) *either $\mathcal{A}$ or $\mathcal{B}$ is noncommutative,*
- (iv) *$\mathcal{M}'$ is loyal,*

then

$$\theta(x) = \alpha\varphi(x) + \gamma(x)1' \quad \text{for all } x \in \mathfrak{A},$$

where $\alpha \in F$, $\alpha \neq 0$, $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is a Jordan isomorphism, and $\gamma : \mathfrak{A} \to F$ is a linear map.

Proof. Using Theorem 3.1 we see that each commuting trace of a bilinear map on $\mathfrak{A}'$ is proper. According to Lemma 2.7, $\mathfrak{A}$ and $\mathfrak{A}'$ do not satisfy $[[x^2, y], [x, y]]$. Thus, we may apply Lemma 5.1, which concludes the proof. □

Recall that any Jordan isomorphism on a triangular matrix algebra $\mathcal{T}_n(F)$ over a field F with $\mathrm{char}(F) \neq 2$ is either an isomorphism or an antiisomorphism [4]. Using Corollary 3.4 and Remark 2.10 together with Lemma 5.1 we may conclude

Corollary 5.3. *Let $n > 2$ be an integer and let F be a field with $\mathrm{char}(F) \neq 2$. Suppose that $\theta : \mathcal{T}_n(F) \to \mathcal{T}_n(F)$ is a bijective linear map satisfying $[\theta(x^2), \theta(x)] = 0$ for all $x \in \mathcal{T}_n(F)$. Then*

$$\theta(x) = \alpha\varphi(x) + \gamma(x)1 \quad \text{for all } x \in \mathcal{T}_n(F),$$

where $\alpha \in F$, $\alpha \neq 0$, $\varphi : \mathcal{T}_n(F) \to \mathcal{T}_n(F)$ is either an isomorphism or an antiisomorphism, and $\gamma : \mathcal{T}_n(F) \to F$ is a linear map.

We remark that Corollary 5.3 is almost identical to [3, Theorem 1.2].

Recently, Zhang [48] and also Lu [33] proved that any Jordan isomorphism between nest algebras is either an isomorphism or an antiisomorphism. Using Corollary 3.5 and Remark 2.13 together with Lemma 5.1 we may therefore conclude

Corollary 5.4. *Let $\mathcal{N}$ and $\mathcal{N}'$ be nests on a Hilbert space H with $\dim_{\mathbb{C}} H > 2$. If $\theta : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N}')$ is a bijective linear map satisfying $[\theta(x^2), \theta(x)] = 0$ for all $x \in \mathcal{T}(\mathcal{N})$, then*

$$\theta(x) = \alpha\varphi(x) + \gamma(x)1' \quad \text{for all } x \in \mathcal{T}(\mathcal{N}),$$

where $\alpha \in \mathbb{C}$, $\alpha \neq 0$, $\varphi : \mathcal{T}(\mathcal{N}) \to \mathcal{T}(\mathcal{N}')$ is either an isomorphism or an antiisomorphism, and $\gamma : \mathcal{T}(\mathcal{N}) \to \mathbb{C}$ is a linear map.

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References

[1] R. Banning, M. Mathieu, Commutativity preserving mappings on semiprime rings, Comm. Algebra 25 (1997) 247–265.
[2] L.B. Beasley, Linear transformations on matrices: the invariance of commuting pairs of matrices, Linear Multilinear Algebra 6 (1978) 179–183.
[3] K.I. Beidar, M. Brešar, M.A. Chebotar, Functional identities on upper triangular matrix algebras, J. Math. Sci. 102 (2000) 4557–4565.
[4] K.I. Beidar, M. Brešar, M.A. Chebotar, Jordan isomorphisms of triangular matrix algebras over a connected commutative ring, Linear Algebra Appl. 312 (2000) 197–201.
[5] K.I. Beidar, M. Brešar, M.A. Chebotar, Y. Fong, Applying functional identities to some linear preserver problems, Pacific J. Math. 204 (2002) 257–271.
[6] K.I. Beidar, M. Brešar, M.A. Chebotar, W.S. Martindale 3rd, On Herstein's Lie map conjectures, I, Trans. Amer. Math. Soc. 353 (2001) 4235–4260.
[7] K.I. Beidar, M. Brešar, M.A. Chebotar, W.S. Martindale 3rd, On Herstein's Lie map conjectures, II, J. Algebra 238 (2001) 239–264.
[8] K.I. Beidar, M. Brešar, M.A. Chebotar, W.S. Martindale 3rd, On Herstein's Lie map conjectures, III, J. Algebra 249 (2002) 59–94.
[9] K.I. Beidar, S.-C. Chang, M.A. Chebotar, Y. Fong, On functional identities in left ideals of prime rings, Comm. Algebra 28 (2000) 3041–3058.
[10] K.I. Beidar, M.A. Chebotar, On Lie-admissible algebras whose commutator Lie algebras are Lie subalgebras of prime associative algebras, J. Algebra 233 (2000) 675–703.
[11] K.I. Beidar, M.A. Chebotar, On Lie derivations of Lie ideals of prime rings, Israel J. Math. 123 (2001) 131–148.
[12] K.I. Beidar, M.A. Chebotar, On surjective Lie homomorphisms onto Lie ideals of prime rings, Comm. Algebra 29 (2001) 4775–4793.
[13] K.I. Beidar, Y. Fong, On additive isomorphisms of prime rings preserving polynomials, J. Algebra 217 (1999) 650–667.
[14] K.I. Beidar, W.S. Martindale 3rd, A.V. Mikhalev, Lie isomorphisms in prime rings with involution, J. Algebra 169 (1994) 304–327.
[15] M.I. Berenguer, A.R. Villena, Continuity of Lie derivations on Banach algebras, Proc. Edinb. Math. Soc. 41 (1998) 625–630.

[16] M.I. Berenguer, A.R. Villena, Continuity of Lie isomorphisms of Banach algebras, Bull. London Math. Soc. 31 (1999) 6–10.
[17] M. Brešar, Centralizing mappings and derivations in prime rings, J. Algebra 156 (1993) 385–394.
[18] M. Brešar, Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings, Trans. Amer. Math. Soc. 335 (1993) 525–546.
[19] M. Brešar, Functional identities: a survey, Contemp. Math. 259 (2000) 93–109.
[20] M. Brešar, Commuting maps: a survey, Taiwanese J. Math., in press.
[21] M. Brešar, P. Šemrl, Normal-preserving linear mappings, Canad. Math. Bull. 37 (1994) 306–309.
[22] M. Brešar, P. Šemrl, Linear Preservers on $\mathcal{B}(X)$, in: Banach Center Publ., vol. 38, 1997, pp. 49–58.
[23] M. Brešar, P. Šemrl, Commuting traces of biadditive maps revisited, Comm. Algebra 31 (2003) 381–388.
[24] G.H. Chan, M.H. Lin, Linear transformations on symmetric matrices that preserve commutativity, Linear Algebra Appl. 47 (1982) 11–22.
[25] M.A. Chebotar, On Lie automorphisms of simple rings of characteristic 2, Fundam. Prikl. Mat. 2 (1996) 1257–1268.
[26] M.A. Chebotar, On Lie isomorphisms in prime rings with involution (in Russian), Comm. Algebra 27 (1999) 2767–2777.
[27] W.-S. Cheung, Commuting maps of triangular algebras, J. London Math. Soc. (2) 63 (2001) 117–127.
[28] W.-S. Cheung, Lie derivations of triangular algebras, Linear Multilinear Algebra 51 (2003) 299–310.
[29] M.D. Choi, A.A. Jafarian, H. Radjavi, Linear maps preserving commutativity, Linear Algebra Appl. 87 (1987) 227–242.
[30] K.R. Davidson, Nest Algebras, in: Pitman Res. Notes Math. Ser., vol. 191, Longmans, Harlow, 1988.
[31] D.Ž. Đoković, Automorphisms of the Lie algebra of upper triangular matrices over a connected commutative ring, J. Algebra 170 (1994) 101–110.
[32] L. Hua, A theorem on matrices over an s-field and its applications, J. Chinese Math. Soc. (N.S.) 1 (1951) 110–163.
[33] F. Lu, Jordan isomorphisms of nest algebras, Proc. Amer. Math. Soc. 131 (2003) 147–154.
[34] L.W. Marcoux, A.R. Sourour, Commutativity preserving linear maps and Lie automorphisms of triangular matrix algebras, Linear Algebra Appl. 288 (1999) 89–104.
[35] L.W. Marcoux, A.R. Sourour, Lie isomorphisms of nest algebras, J. Funct. Anal. 164 (1999) 163–180.
[36] W.S. Martindale 3rd, Lie isomorphisms of primitive rings, Proc. Amer. Math. Soc. 14 (1963) 909–916.
[37] W.S. Martindale 3rd, Lie derivations of primitive rings, Michigan J. Math. 11 (1964) 183–187.
[38] W.S. Martindale 3rd, Lie isomorphisms of simple rings, J. London Math. Soc. 44 (1969) 213–221.
[39] W.S. Martindale 3rd, Lie isomorphisms of prime rings, Trans. Amer. Math. Soc. 142 (1969) 437–455.
[40] C.R. Miers, Lie isomorphisms of factors, Trans. Amer. Math. Soc. 147 (1970) 55–63.
[41] C.R. Miers, Commutativity preserving maps of factors, Canad. J. Math. 40 (1988) 248–256.
[42] M. Omladič, On operators preserving commutativity, J. Funct. Anal. 66 (1986) 105–122.
[43] S. Pierce, W. Watkins, Invariants of linear maps on matrix algebras, Linear Multilinear Algebra 6 (1978) 185–200.
[44] H. Radjavi, Commutativity preserving operators on symmetric matrices, Linear Algebra Appl. 61 (1984) 219–224.
[45] G.A. Swain, Lie derivations of the skew elements of prime rings with involution, J. Algebra 184 (1996) 679–704.
[46] A.R. Villena, Lie derivations on Banach algebras, J. Algebra 226 (2000) 390–409.
[47] W. Watkins, Linear maps that preserve commuting pairs of matrices, Linear Algebra Appl. 14 (1976) 29–35.
[48] J.H. Zhang, Jordan isomorphisms of nest algebras, Acta Math. Sinica 45 (2002) 819–824.