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The extended conformal Einstein field equations with matter: The Einstein–Maxwell field

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ABSTRACT

A discussion is given of the conformal Einstein field equations coupled with matter whose energy–momentum tensor is trace-free. These resulting equations are expressed in terms of a generic Weyl connection. The article shows how in the presence of matter it is possible to construct a conformal gauge which allows to know *a priori* the location of the conformal boundary. In vacuum this gauge reduces to the so-called conformal Gaussian gauge. These ideas are applied to obtain (i) a new proof of the stability of Einstein–Maxwell de Sitter-like spacetimes; (ii) a proof of the semi-global stability of purely radiative Einstein–Maxwell spacetimes.

1. Introduction

The Einstein conformal field equations are a powerful tool to prove statements concerning the stability of vacuum spacetimes—see e.g. [1]. These methods have been extended to deal with the case of the gravitational field coupled to the Maxwell and Yang–Mills fields [2]. In [3] a more general version of the vacuum conformal field equations has been developed. These extended conformal field equations are written in terms of a Weyl connection. The extra gauge freedom incorporated in this representation of the equations allows the construction of gauge systems based on conformal structures of the spacetime. As it so often happens in this type of considerations, a judicious gauge choice based on geometrical considerations can greatly simplify the analysis in question. An example of the gauge choices that can be employed are the conformal Gaussian gauge systems introduced in [3]—see as well [4,5]. The extended conformal field equations in conjunction with conformal Gaussian systems have been used, among other things: to provide an existence proof of anti-de Sitter spacetimes [3]; to construct a representation of spatial infinity allowing for a regular finite initial value problem at spatial infinity [6]; to provide a new proof of the global stability of the de Sitter spacetime and the semi-global stability of Minkowski spacetime [7]; and to provide a semi-global stability result of purely radiative vacuum spacetimes [8].

The common feature in the applications described in the previous paragraph is that one is, ultimately, concerned with solutions to the vacuum Einstein field equations (with or without a cosmological constant). A key feature of conformal Gaussian systems in vacuum spacetimes is that the conformal geodesics upon which they are constructed render a canonical conformal factor which provides *a priori* knowledge about the location of the conformal boundary of the spacetime. This property is, however, lost if one considers conformal geodesics on non-vacuum spacetimes. In this article we show that it is possible to get around this difficulty if one considers a more general class of *conformal curves* to construct gauge systems. As in the case of conformal geodesics in vacuum spacetimes, the conformal curves provide again a canonical conformal factor which is known prior to evolution.

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As an application of the ideas described in the previous paragraph, in this article we will consider initial value problems for spacetimes $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ with cosmological constant λ satisfying the Einstein–Maxwell field equations

$$\tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R} = \lambda\tilde{g}_{\mu\nu} + \tilde{T}_{\mu\nu}, \tag{1a}$$

$$\tilde{T}_{\mu\nu} = \tilde{F}_{\mu\lambda}\tilde{F}^{\lambda}{}_{\nu} - \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{F}_{\lambda\rho}\tilde{F}^{\lambda\rho} \tag{1b}$$

$$\tilde{\nabla}^{\mu}\tilde{F}_{\mu\nu} = 0, \qquad \tilde{\nabla}_{[\mu}\tilde{F}_{\nu\lambda]} = 0, \tag{1c}$$

where $\tilde{\nabla}$ denotes the Levi-Civita connection of the metric $\tilde{g}_{\mu\nu}$, $\tilde{R}_{\mu\nu}$, $\tilde{R}$ are the associated Ricci tensor and Ricci scalar, and $\tilde{F}_{\mu\nu}$ denotes the Maxwell tensor—the conventions for the geometric quantities used above, will be set out in detail in Section 2. The discussion of the solutions to Eqs. (1a)–(1b) will be carried out in terms of a conformally rescaled, unphysical metric $g_{\mu\nu}$ related to the physical metric $\tilde{g}_{\mu\nu}$ according to

$$g_{\mu\nu} = \Theta^2\tilde{g}_{\mu\nu}. \tag{2}$$

The gauge systems based on the new class of conformal curves are used to provide a new (simpler) proof of the existence and stability of Einstein–Maxwell de Sitter-like spacetimes. We also provide a stability proof of purely radiative Einstein–Maxwell spacetimes. These particular applications lead us to consider the *extended Einstein conformal field equations with matter*. To the best of our knowledge, this is the first time these equations are considered.

The approach presented here differs from that in [2] in the following way. In [2] the conformal field equations are treated in terms of the Levi-Civita connection of the unphysical metric $g_{\mu\nu}$ and generalised harmonic coordinates are used, and the gauge choice for the metric in the conformal class is fixed by setting $R[g] = -6$, like in [1]. Instead in this article we adapt the ideas of [3] to use congruences to fix the coordinates and the conformal factor. The extended field equations for the matter case are similar to the conformal field equations in [2] in some form. However they differ in details since the use of Weyl connections introduce additional terms. For example some of the symmetries of the standard spinorial Maxwell equations are broken when expressing them in a Weyl connection—see (53). Therefore the full set of extended field equations is derived to assure completeness and consistency.

As a simplifying technical assumption, our general considerations will be restricted to matter models with a trace-free energy–momentum tensor—a property satisfied by the electromagnetic field. Although the particular examples to be considered are only concerned with the Einstein–Maxwell equations, most of our discussion can be adapted to trace-free perfect fluids (sometimes also called conformal fluids)—this will be discussed in future work.

Outline of the article

We start by summarising our conventions and the basic ideas behind the notion of conformal rescaling in Section 2. The conventions follow closely those used in Refs. [7,8]. Section 3 presents a brief review of the notion of Weyl connection and the transformation formulae for the connection and the Schouten tensor. Section 4 gives the formulation of the extended conformal field equations with matter in both a frame and a spinorial formalism. Section 5 introduces the concept of conformal curves and the associated generalised conformal Gaussian systems. These gauge systems are instrumental in our subsequent analysis as combined with the extended conformal field equations, they render hyperbolic reductions for which the location of the conformal boundary is know *a priori*. Section 6 provides a discussion of the procedure of hyperbolic reduction for the geometric part of the extended conformal field equations in generalised Gaussian systems. Section 7 is concerned with the matter part of the field equations, which in the case under consideration is given by the Maxwell field. Section 8 summarises the key structural properties of the evolution equations implied by the conformal field equations with a view to applications involving existence and stability results. Section 9 discusses the so-called propagation of the constraints. Section 10 is concerned with the first application of the methods developed in the article: a new proof of the stability of Einstein–Maxwell spacetimes which have a global structure similar to that of the de Sitter spacetime. Finally, Section 11 provides a second application: a stability result for Einstein–Maxwell radiative spacetimes. This result generalises the analysis for the purely vacuum case carried out in [8].

2. Basics and conventions

2.1. The curvature of the physical spacetime manifold

Throughout this article we work with a spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$, where $\tilde{g}_{\mu\nu}$, $(\mu, \nu = 0, 1, 2, 3)$ is a Lorentzian metric with signature $(+, -, -, -)$. We will denote by $\tilde{\nabla}$ the Levi-Civita connection of $\tilde{g}_{\mu\nu}$—that is, the unique torsion-free connection that preserves the metric $\tilde{g}_{\mu\nu}$. As in the introduction, let $\tilde{R}_{\mu\nu\lambda\rho}$, $\tilde{R}_{\mu\nu}$ and $\tilde{R}$ denote, respectively, the Riemann curvature tensor, the Ricci tensor and the Ricci scalar of the Levi-Civita connection $\tilde{\nabla}$. The conventions for the curvature used in this article are such that

$$\tilde{R}^{\mu}{}_{\nu\lambda\rho}\xi^{\nu} = \left(\tilde{\nabla}_{\lambda}\tilde{\nabla}_{\rho} - \tilde{\nabla}_{\rho}\tilde{\nabla}_{\lambda}\right)\xi^{\mu}, \qquad \tilde{R}_{\mu\nu} = \tilde{R}^{\alpha}{}_{\nu\alpha\mu}, \quad \tilde{R} = \tilde{R}_{\mu\nu}\tilde{g}^{\mu\nu}. \tag{3}$$

For the Riemann tensor one has the decomposition

$$\tilde{R}^{\mu}{}_{\nu\lambda\rho} = \tilde{C}^{\mu}{}_{\nu\lambda\rho} + 2\left(\delta^{\mu}{}_{[\lambda}\tilde{P}_{\rho]\nu} - \tilde{g}_{\nu[\lambda}\tilde{P}_{\rho]\sigma}\tilde{g}^{\sigma\mu}\right), \tag{4}$$

where $\tilde{C}^{\mu}{}_{\nu\lambda\rho}$ denotes the *conformal Weyl tensor* of $\tilde{g}_{\mu\nu}$, while the trace parts are given in terms of the *Schouten tensor* defined by

$$\tilde{P}_{\mu\nu} \equiv \frac{1}{2}\left(\tilde{R}_{\mu\nu} - \frac{1}{6}\tilde{R}\tilde{g}_{\mu\nu}\right).$$

If the Einstein equation (1a) holds, then one expresses the Schouten tensor in terms of the energy–momentum tensor $\tilde{T}_{\mu\nu}$ as

$$\tilde{P}_{\mu\nu} = \frac{1}{2}\tilde{T}_{\mu\nu} - \frac{1}{6}\tilde{T}_{\rho\sigma}\tilde{g}^{\rho\sigma}\tilde{g}_{\mu\nu} + \tilde{\lambda}\tilde{g}_{\mu\nu}, \quad \tilde{\lambda} \equiv \frac{1}{6}\lambda. \tag{5}$$

In the sequel, $i, j, \ldots$ will denote spacetime frame indices ranging $0, \ldots, 3$. Further frame conventions (including that of the connection coefficients) will be given in Section 4.1. Capital latin indices $A, B, \ldots$ will denote spinorial indices. In this respect, we follow the general conventions of Penrose and Rindler [9]. Further spinorial conventions will be discussed in Section 4.2 as they are required.

2.2. Conformal rescalings

Let $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ be two Lorentzian metrics which are conformally related according to Eq. (2). Let $[\tilde{g}]$ denote the conformal class of $\tilde{g}_{\mu\nu}$. Two invariants of the conformal class are the tensor

$$S_{\mu\nu}{}^{\lambda\rho} = \delta_{\mu}{}^{\lambda}\delta_{\nu}{}^{\rho} + \delta_{\nu}{}^{\lambda}\delta_{\mu}{}^{\rho} - \tilde{g}_{\mu\nu}\tilde{g}^{\lambda\rho} \tag{6}$$

and the conformal Weyl tensor

$$\tilde{C}^{\mu}{}_{\nu\lambda\rho} = C^{\mu}{}_{\nu\lambda\rho}.$$

Note that an analogous decomposition to Eq. (4) holds for the Riemann tensor $R^{\mu}{}_{\nu\lambda\rho}$.

For the metrics $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$, let ∇, $\tilde{\nabla}$ denote their respective Levi-Civita covariant derivative and let $\Gamma_{\mu}{}^{\lambda}{}_{\nu}$, $\tilde{\Gamma}_{\mu}{}^{\lambda}{}_{\nu}$ denote the corresponding Christoffel symbols. One has that

$$\Gamma_{\mu}{}^{\lambda}{}_{\nu} - \tilde{\Gamma}_{\mu}{}^{\lambda}{}_{\nu} = S_{\mu\nu}{}^{\lambda\rho}\Upsilon_{\rho}$$

where $\Upsilon_{\mu} \equiv \Theta^{-1}\nabla_{\mu}\Theta$.

3. Weyl connections

In this article, we will also consider connections $\hat{\nabla}$ (not necessarily Levi-Civita) which respect the conformal structure of the conformal class $[\tilde{g}]$, in the sense that

$$\hat{\nabla}_{\lambda}\tilde{g}_{\mu\nu} = -2b_{\lambda}\tilde{g}_{\mu\nu}, \qquad \hat{\nabla}_{\lambda}g_{\mu\nu} = -2f_{\lambda}g_{\mu\nu}, \tag{7}$$

for some 1-forms b_{μ} and f_{μ}. One has that

$$\hat{\Gamma}_{\mu}{}^{\lambda}{}_{\nu} - \tilde{\Gamma}_{\mu}{}^{\lambda}{}_{\nu} = S_{\mu\nu}{}^{\lambda\rho}b_{\rho}, \tag{8a}$$

$$\hat{\Gamma}_{\mu}{}^{\lambda}{}_{\nu} - \Gamma_{\mu}{}^{\lambda}{}_{\nu} = S_{\mu\nu}{}^{\lambda\rho}f_{\rho}. \tag{8b}$$

We shall write the above equations as

$$\hat{\nabla} - \tilde{\nabla} = S(b), \qquad \hat{\nabla} - \nabla = S(f).$$

The fact that $\tilde{g}_{\mu\nu}$ and $g_{\mu\nu}$ are assumed to be conformally related implies

$$b_{\mu} = \Upsilon_{\mu} + f_{\mu}.$$

In the following the 1-form b_{μ} and the conformal factor Θ will arise from solving conformal curves in $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$—see Section 5 for more details. The 1-form f_{μ} will be the 1-form related to the unphysical metric $g_{\mu\nu}$ and features in the hyperbolic reduction of the conformal field equations—see Section 6.

The Riemann and Ricci tensors of the Weyl connection $\hat{\nabla}$ are defined in an analogous way to (3) and will be denoted by $\hat{R}_{\mu\nu\lambda\rho}$ and $\hat{R}_{\mu\nu}$ respectively. The analogue of the decomposition (4) is given by

$$\hat{R}^{\mu}{}_{\nu\lambda\rho} = C^{\mu}{}_{\nu\lambda\rho} + 2\left(\delta^{\mu}{}_{[\lambda}\hat{P}_{\rho]\nu} - \delta^{\mu}{}_{\nu}\hat{P}_{[\lambda\rho]} - g_{\nu[\lambda}\hat{P}_{\rho]\sigma}g^{\sigma\mu}\right),$$
$$= C^{\mu}{}_{\nu\lambda\rho} + 2S_{\nu[\lambda}{}^{\mu\sigma}\hat{P}_{\rho]\sigma},$$

where the Schouten tensor of $\hat{\nabla}$, denoted by $\hat{P}_{\mu\nu}$, is given by

$$\hat{P}_{\mu\nu} \equiv \frac{1}{2}\left(\hat{R}_{\mu\nu} - \frac{1}{2}\hat{R}_{[\mu\nu]} - \frac{1}{6}g_{\mu\nu}\hat{R}_{\rho\lambda}g^{\rho\lambda}\right).$$

Alternatively, the latter decompositions could have been written using the physical metric $\tilde{g}_{\mu\nu}$.

Transformation rules between the curvature tensors of the Weyl connection $\hat{\nabla}$ and the Levi-Civita connections $\tilde{\nabla}$, ∇ can be found in [4]. Important for the subsequent discussion is the transformation rule for the Schouten tensor. This is given by

$$\tilde{P}_{\mu\nu} - \hat{P}_{\mu\nu} = \tilde{\nabla}_{\mu}b_{\nu} - \frac{1}{2}S_{\mu\nu}{}^{\rho\lambda}b_{\rho}b_{\lambda} \tag{9a}$$

$$= \hat{\nabla}_{\mu}b_{\nu} + \frac{1}{2}S_{\mu\nu}{}^{\rho\lambda}b_{\rho}b_{\lambda}. \tag{9b}$$

A similar expression holds between the tensors $P_{\mu\nu}$ and $\hat{P}_{\mu\nu}$ by replacing b_{μ} with f_{μ}, namely:

$$P_{\mu\nu} - \hat{P}_{\mu\nu} = \nabla_{\mu}f_{\nu} - \frac{1}{2}S_{\mu\nu}{}^{\rho\lambda}f_{\rho}f_{\lambda} \tag{10a}$$

$$= \hat{\nabla}_{\mu}f_{\nu} + \frac{1}{2}S_{\mu\nu}{}^{\rho\lambda}f_{\rho}f_{\lambda}. \tag{10b}$$

Finally, we introduce the Cotton–York tensor associated to the connection $\hat{\nabla}$

$$\hat{Y}_{\mu\nu\lambda} \equiv \hat{\nabla}_{\mu}\hat{P}_{\nu\lambda} - \hat{\nabla}_{\nu}\hat{P}_{\mu\lambda}.$$

The physical Cotton–York tensor can be expressed in terms of $\tilde{T}_{\mu\nu}$ and $\tilde{T}_{\mu\nu}\tilde{g}^{\mu\nu}$

$$\tilde{Y}_{\mu\nu\lambda} \equiv \tilde{\nabla}_{\mu}\tilde{P}_{\nu\lambda} - \tilde{\nabla}_{\nu}\tilde{P}_{\mu\lambda} = \tilde{\nabla}_{[\mu}\tilde{T}_{\nu]\lambda} - \frac{1}{3}\tilde{g}_{\lambda[\nu}\tilde{\nabla}_{\mu]}\tilde{T}.$$

Remark. It should be noted that above the definitions and decompositions are invariant under conformal rescalings. Nevertheless, when raising indices or applying contractions we have explicitly written out the metric to avoid ambiguity. In the sequel, a frame formalism will be used throughout. This choice will remove the ambiguity as the frame metric will always be η_{ij}. Consistent with Eq. (11) the metric $g_{\mu\nu}$ and its inverse will be used throughout for raising and lowering tensorial indices.

4. The extended conformal field equations with matter

The idea of *vacuum conformal Einstein field equations* expressed in terms of the Levi-Civita connection ∇ of a conformally rescaled metric $g_{\mu\nu}$ and associated objects was originally introduced in [10–12]. The generalisation of these conformal equations to physical spacetimes containing matter was discussed in [2]. More recently, a more general type of vacuum conformal equations—the *extended conformal Einstein field equations*—expressed in terms of a Weyl connection $\hat{\nabla}$ has been introduced—see [3]. In this section we discuss how these extended conformal field equations can be modified to discuss spacetimes with matter.

4.1. Frame formulation

In the sequel, it will be convenient to consider a frame e_k, $k = 0, 1, 2, 3$ which is orthonormal with respect to the metric $g_{\mu\nu}$. That is,

$$g(e_i, e_j) = \eta_{ij}. \tag{11}$$

In what follows frame components are always taken with respect to the frame e_k. In particular, ∇_i, $\tilde{\nabla}_i$ will denote the covariant derivatives in the direction of e_i. The frame connection coefficients $\Gamma_i{}^k{}_j$ of ∇ with respect to the frame e_k are defined by the relation

$$\nabla_i e_j = \Gamma_i{}^k{}_j e_k.$$

In the sequel we shall be only use the frame connection coefficients $\Gamma_i{}^k{}_j$, so that no confusion should arise with the Christoffel symbols $\Gamma_{\mu}{}^{\lambda}{}_{\nu}$.

In order to discuss the extended conformal Einstein field equations, it will be convenient to depart slightly from the point of view taken in the previous section and regard, for the moment, the connection ∇ only as a metric connection with respect to $g_{\mu\nu}$—i.e. $\nabla_{\lambda}g_{\mu\nu} = 0$. Under this assumption, the connection ∇ could have torsion, and thus it would not be a Levi-Civita connection. As a consequence of having a metric connection and an orthonormal frame e_k the connection coefficients satisfy

$$\Gamma_i{}^k{}_j\eta_{kl} + \Gamma_i{}^k{}_l\eta_{kj} = 0.$$

Now, given the connection coefficients $\Gamma_i{}^k{}_j$ of a metric connection as above and a 1-form f_μ, one can define further connection coefficients $\hat{\Gamma}_i{}^k{}_j$ using the relation

$$\hat{\Gamma}_i{}^k{}_j = \Gamma_i{}^k{}_j + S_{ij}{}^{kl} f_l, \tag{12}$$

—cf. (8b). The torsion $\Sigma_i{}^k{}_j$ of the connection ∇ is defined by

$$\Sigma_i{}^k{}_j e_k \equiv \left(\Gamma_i{}^k{}_j - \Gamma_j{}^k{}_i\right) e_k - [e_i, e_j].$$

Let $\hat{\Sigma}_i{}^j{}_k$ denote the torsion of the connection $\hat{\nabla}$. It follows directly that

$$\Sigma_i{}^k{}_j = \hat{\Sigma}_i{}^k{}_j \tag{13}$$

so that $\hat{\nabla}$ will not be a Weyl connection of $[\tilde{g}]$ unless $\hat{\Sigma}_i{}^k{}_j = 0$.

In our subsequent discussion it will be convenient to distinguish between the *geometric curvature* $\hat{r}^k{}_{lij}$—i.e. the expression of the curvature related to the connection coefficients $\hat{\Gamma}_i{}^j{}_k$—and the *algebraic curvature* $\hat{R}^k{}_{lij}$—i.e. the decomposition of the curvature in terms of $C^k{}_{lij}$ and $\hat{P}_{ij}$. One has that

$$\hat{r}^k{}_{lij} \equiv e_i\left(\hat{\Gamma}_j{}^k{}_l\right) - e_j\left(\hat{\Gamma}_i{}^k{}_l\right) - \hat{\Gamma}_m{}^k{}_l\left(\hat{\Gamma}_i{}^m{}_j - \hat{\Gamma}_j{}^m{}_i\right) + \hat{\Gamma}_i{}^k{}_m\hat{\Gamma}_j{}^m{}_l - \hat{\Gamma}_j{}^k{}_m\hat{\Gamma}_i{}^m{}_l + \hat{\Sigma}_i{}^m{}_j\hat{\Gamma}_m{}^k{}_l,$$

$$\hat{R}^k{}_{lij} \equiv C^k{}_{lij} + 2\left(\delta^k{}_{[i}\hat{P}_{j]l} - \delta^k{}_l\hat{P}_{[ij]} - \eta_{l[i}\hat{P}_{j]}{}^k\right) = C^k{}_{lij} + 2S_{l[i}{}^{km}\hat{P}_{j]m}.$$

The distinction is necessary as $\hat{\Gamma}_i{}^k{}_j$, $C^k{}_{lij}$ and $\hat{P}_{ij}$ will be treated as independent unknowns so that potentially $\hat{r}^k{}_{lij} \neq \hat{R}^k{}_{lij}$. The requirement that the algebraic and geometric curvature are equal will be used as one of our field equations.

For ease of the subsequent discussion we introduce the following *zero quantities*:

$$\hat{\Sigma}_i{}^k{}_j e_k \equiv \left(\hat{\Gamma}_i{}^k{}_j - \hat{\Gamma}_j{}^k{}_i\right) e_k - [e_i, e_j], \tag{14a}$$

$$\hat{\Xi}^k{}_{lij} \equiv \hat{r}^k{}_{lij} - \hat{R}^k{}_{lij} \tag{14b}$$

$$\hat{\Delta}_{ij} \equiv \hat{\nabla}_i f_j - \hat{\nabla}_j f_i - \hat{P}_{ij} + \hat{P}_{ji}, \tag{14c}$$

$$\hat{\Delta}_{lij} \equiv \hat{\nabla}_i\hat{P}_{jl} - \hat{\nabla}_j\hat{P}_{il} - b_k C^k{}_{lij} - \tilde{Y}_{ijl} \tag{14d}$$

$$\hat{\Lambda}'_{lij} \equiv \hat{\nabla}_k C^k{}_{lij} - b_k C^k{}_{lij} - \tilde{Y}_{ijl}. \tag{14e}$$

The interpretation of the zero quantities (14a)–(14e) is as follows: the zero quantity given by (14a) measures the torsion of the connection $\hat{\nabla}$; that of (14b) relates the expression of the curvature of $\hat{\nabla}$ with its decomposition in terms of irreducible components; Eq. (14c) is contraction of (14b) over the first two indices. It is included here for later convenience. Eqs. (14d) and (14e) measure the deviation from the fulfilment of the Bianchi identity.

The *extended conformal Einstein field equations with matter* are then given by

$$\hat{\Sigma}_i{}^k{}_j e_k = 0, \qquad \hat{\Xi}^k{}_{lij} = 0, \qquad \hat{\Delta}_{lij} = 0, \qquad \hat{\Lambda}_{lij} = 0. \tag{15}$$

These equations yield differential conditions for the frame coefficients e_i, the spin coefficients $\hat{\Gamma}_i{}^j{}_k$, the components of the 1-form f_i, the components of the Schouten tensor $\hat{P}_{ij}$, and the Weyl tensor $C^k{}_{lij}$, respectively. The latter need to be complemented with the energy–momentum conservation equation

$$\tilde{\nabla}^i\tilde{T}_{ij} = 0,$$

whose particular details will depend on the matter model under consideration. These equations (and their spinorial version) are independent of any gauge choice. Thus if they hold in some particular gauge, they hold in any gauge.

Note that in Eqs. (15), the 1-form b_k relating $\hat{\nabla}$ and $\tilde{\nabla}$ remains unspecified. In the sequel it will be convenient to introduce the variables

$$d^k{}_{lij} \equiv \Theta^{-1}C^k{}_{lij}, \tag{16a}$$

$$d_i \equiv \Theta b_i = \nabla_i\Theta + \Theta f_i. \tag{16b}$$

In terms of the latter, the last two conformal field equations then read:

$$\hat{\Delta}_{lij} = \hat{\nabla}_i\hat{P}_{jl} - \hat{\nabla}_j\hat{P}_{il} - d_k d^k{}_{lij} - \tilde{Y}_{ijl} = 0, \tag{17a}$$

$$\Theta^{-1}\hat{\Lambda}'_{lij} = \hat{\Lambda}_{lij} = \hat{\nabla}_k d^k{}_{lij} - \Theta^{-1}\tilde{Y}_{ijl} - f_k d^k{}_{lij} = 0. \tag{17b}$$

As in the case of b_k, the newly introduced function Θ and the 1-form d_k remain unspecified at this stage. They will latter be fixed by the choice of a suitable conformal gauge.

Remark 1. If the extended conformal field equations (15) are satisfied then the frame e_k can be used to construct a metric $g_{\mu\nu}$ via the relation (11). The connection coefficients $\hat{\Gamma}_i{}^j{}_k$ give rise to a torsion-free connection, so that the connection $\Gamma_i{}^j{}_k$

given by (12) is the Levi-Civita connection of $g_{\mu\nu}$. Consequently, $\hat{\Gamma}_i{}^j{}_k$ defines a Weyl connection with conformal Weyl tensor given by $C^k{}_{lij}$ and Schouten tensor $\hat{P}_{ij}$. Showing that the solution so obtained implies a solution to the Einstein field equations requires bringing into consideration gauge conditions. This will be discussed together with the propagation of the constraints in Section 9.

Remark 2. As a consequence of the transformation rules for the Schouten tensor (9a)–(10a) and (9b)–(10b), the zero quantities (14a)–(14e) involved in the extended conformal field equations (15) transform covariantly (i.e. homogeneously) under a change in the conformal gauge. Thus, if they are satisfied in one gauge, then they are satisfied in all gauges.

4.2. *Spinorial formulation*

In the sequel we will make use of a spinorial version of the extended conformal field equations (15). We follow the general conventions of [9]. The use of this type of representation leads to simplifications, in particular, when obtaining a reduced system of propagation equations. However, we will switch to a frame representation whenever it is more convenient for the discussion.

The connection between the components of a tensor with respect to an orthonormal basis and its spinorial counterpart is realised by the constant Infeld–van der Waerden symbols. In particular, let $e_{AA'}$, $\hat{\nabla}_{AA'}$, $d_{AA'}$ denote, respectively, the spinorial counterparts of e_i, $\hat{\nabla}_i$, d_i. Furthermore, let $\Gamma_{AA'}{}^{BB'}{}_{CC'}$, $\hat{\Gamma}_{AA'}{}^{BB'}{}_{CC'}$ denote, respectively, the spinorial counterpart of the connection coefficients $\Gamma_i{}^j{}_k$, $\hat{\Gamma}_i{}^j{}_k$. As the connection defined by $\Gamma_i{}^j{}_k$ is assumed to be metric, it follows that one can write

$$\Gamma_{AA'}{}^{BB'}{}_{CC'} = \Gamma_{AA'}{}^B{}_C\epsilon_{C'}{}^{B'} + \bar{\Gamma}_{A'A}{}^{B'}{}_{C'}\epsilon_C{}^B, \qquad \Gamma_{AA'BC} = \Gamma_{AA'(BC)}.$$

The symmetry condition on the last pair of indices of the spin connection coefficient encodes the assumption of having a metric connection. For the spin Weyl connection coefficients one has

$$\hat{\Gamma}_{AA'}{}^{BB'}{}_{CC'} = \hat{\Gamma}_{AA'}{}^B{}_C\epsilon_{C'}{}^{B'} + \bar{\hat{\Gamma}}_{A'A}{}^{B'}{}_{C'}\epsilon_C{}^B,$$

with

$$\hat{\Gamma}_{AA'}{}^B{}_C = \Gamma_{AA'}{}^B{}_C + \epsilon_A{}^Bf_{CA'}, \qquad \hat{\Gamma}_{AA'CB} = \hat{\Gamma}_{AA'(CB)}.$$

Let $\hat{r}^{AA'}{}_{BB'CC'DD'}$ denote the spinorial counterpart of the geometric curvature $\hat{r}^k{}_{lij}$. For future use we note the Ricci identity:

$$\left(\hat{\nabla}_{CC'}\hat{\nabla}_{DD'} - \hat{\nabla}_{DD'}\hat{\nabla}_{CC'}\right)\mu^{AA'} = \hat{r}^{AA'}{}_{BB'CC'DD'}\mu^{BB'} - \hat{\Sigma}_{CC'}{}^{EE'}{}_{DD'}\hat{\nabla}_{EE'}\mu^{AA'}, \tag{18}$$

valid for any spinor $\mu^{AA'}$ and where $\hat{\Sigma}_{CC'}{}^{EE'}{}_{DD'}$ is the spinorial counterpart of the torsion.

In our conventions, the geometric and algebraic curvatures of a general Weyl connection satisfy

$$\hat{r}_{klij} = \hat{r}_{[kl]ij} + 2\eta_{kl}\hat{\nabla}_{[i}f_{j]}, \qquad \hat{R}_{klij} = \hat{R}_{[kl]ij} - 2\eta_{kl}\hat{P}_{[ij]}.$$

Their spinorial counterpart can be decomposed (recall the torsion vanishes in this connection) as

$$\hat{r}_{AA'BB'CC'DD'} = \epsilon_{A'B'}\hat{r}_{ABCC'DD'} + \epsilon_{AB}\bar{\hat{r}}_{A'B'CC'DD'},$$
$$\hat{R}_{AA'BB'CC'DD'} = \epsilon_{A'B'}\hat{R}_{ABCC'DD'} + \epsilon_{AB}\bar{\hat{R}}_{A'B'CC'DD'}$$

where

$$\hat{r}_{ABCC'DD'} = \hat{r}_{(AB)CC'DD'} + \frac{1}{2}\epsilon_{AB}(\hat{\nabla}_{CC'}f_{DD'} - \hat{\nabla}_{DD'}f_{CC'}),$$
$$\hat{R}_{ABCC'DD'} = \hat{R}_{(AB)CC'DD'} - \frac{1}{2}\epsilon_{AB}(\hat{P}_{CC'DD'} - \hat{P}_{DD'CC'}).$$

In terms of the reduced spin coefficients $\hat{\Gamma}_{AA'BC}$ one writes the geometric curvature as

$$\begin{aligned} r^C{}_{DAA'BB'} = {} & e_{AA'}\left(\hat{\Gamma}_{BB'}{}^C{}_D\right) - e_{BB'}\left(\hat{\Gamma}_{AA'}{}^C{}_D\right) - \hat{\Gamma}_{AA'}{}^F{}_B\hat{\Gamma}_{FB'}{}^C{}_D + \hat{\Gamma}_{BB'}{}^F{}_A\hat{\Gamma}_{FA'}{}^C{}_D \\ & - \overline{\hat{\Gamma}}_{AA'}{}^{F'}{}_{B'}\hat{\Gamma}_{BF'}{}^C{}_D + \overline{\hat{\Gamma}}_{BB'}{}^{F'}{}_{A'}\hat{\Gamma}_{AF'}{}^C{}_D + \hat{\Gamma}_{AA'}{}^C{}_F\hat{\Gamma}_{BB'}{}^F{}_D - \hat{\Gamma}_{BB'}{}^C{}_F\hat{\Gamma}_{AA'}{}^F{}_D. \end{aligned}$$

4.2.1. *The uncontracted spinorial conformal field equations*

The spinorial version of the zero quantities (14a)–(14e) is given by

$$\hat{\Sigma}_{AA'}{}^{PP'}{}_{BB'}e_{PP'} \equiv [e_{AA'}, e_{BB'}] - \hat{\Gamma}_{AA'}{}^C{}_Be_{CB'} - \overline{\hat{\Gamma}}_{AA'}{}^{C'}{}_{B'}e_{BC'} + \hat{\Gamma}_{BB'}{}^C{}_Ae_{CA'} + \overline{\hat{\Gamma}}_{BB'}{}^{C'}{}_{A'}e_{AC'}, \tag{19a}$$

$$\hat{\Xi}_{ABCC'DD'} \equiv \hat{r}_{ABCC'DD'} - \hat{R}_{ABCC'DD'}, \tag{19b}$$

$$\hat{\Delta}_{CC'AA'BB'} \equiv \hat{\nabla}_{AA'}\hat{P}_{BB'CC'} - \hat{\nabla}_{BB'}\hat{P}_{AA'CC'} - d^{PP'}d_{PP'AA'BB'CC'} - \tilde{Y}_{AA'BB'CC'}, \tag{19c}$$

$$\hat{\Lambda}_{CC'AA'BB'} \equiv \hat{\nabla}^{PP'}d_{PP'CC'AA'BB'} - f^{PP'}d_{PP'CC'AA'BB'} - \Theta^{-1}\tilde{Y}_{AA'BB'CC'}. \tag{19d}$$

In terms of these spinorial zero quantities, the extended conformal field equations are given by

$$\hat{\Sigma}_{AA'}{}^{PP'}{}_{BB'}e_{PP'} = 0, \qquad \hat{\Delta}_{ABCC'DD'} = 0, \qquad \hat{\Delta}_{AA'BB'CC'} = 0, \qquad \hat{\Lambda}_{AA'BB'CC'} = 0. \tag{20}$$

4.2.2. The contracted spinorial conformal field equations

Eqs. (19a)–(19c) are antisymmetric upon interchange of a pair of indices. This structural property will be used to obtain a contracted version of the equations which will be systematically used in the sequel. The associated zero quantities are given by

$$\frac{1}{2}\hat{\Sigma}_{(A|Q|}{}^{PP'}{}_{B)}{}^{Q}e_{PP'} \equiv \hat{\nabla}_{(A|Q'|}e_{B)}{}^{Q'} - \hat{\Gamma}_{(A|Q'|}{}^{P}{}_{B)}e_{P}{}^{Q'} - \bar{\hat{\Gamma}}_{Q'(A}{}^{P'Q'}e_{B)P'}, \tag{21a}$$

$$\frac{1}{2}\hat{\Xi}_{(AB)CQ'D}{}^{Q'} \equiv \frac{1}{2}\left(\hat{r}_{ABCQ'D}{}^{Q'} - \hat{R}_{ABCQ'D}{}^{Q'}\right), \tag{21b}$$

$$\frac{1}{2}\hat{\Delta}_{CC'(A|Q'|B)}{}^{Q'} = \hat{\nabla}_{(A|Q'|}\hat{P}_{B)}{}^{Q'}{}_{CC'} + d^{Q}{}_{C'}\phi_{ABCQ} - \tilde{Y}_{ABCC'}, \tag{21c}$$

and their complex conjugate versions. Above the symmetries of the spinorial counterparts of d_{klij} and $\tilde{Y}_{ijk}$ have been exploited by writing

$$d_{AA'BB'CC'DD'} = \phi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \overline{\phi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD},$$

$$\tilde{Y}_{AA'BB'CC'} = \tilde{Y}_{ABCC'}\epsilon_{A'B'} + \overline{\tilde{Y}}_{A'B'C'C}\epsilon_{AB},$$

with

$$\phi_{ABCD} = \phi_{(ABCD)}, \qquad \tilde{Y}_{ABCC'} = \tilde{Y}_{(ABC)C'}.$$

Using the latter formulae Eq. (19d) reduces to its more usual form:

$$\hat{\Lambda}_{A'ABC} \equiv \hat{\nabla}^{Q}{}_{A'}\phi_{ABCQ} - f^{Q}{}_{A'}\phi_{ABCQ} - \Theta^{-1}\tilde{Y}_{ABCA'}.$$

The extended conformal field equations can be expressed in terms of these contracted zero quantities as:

$$\hat{\Sigma}_{(A|Q|}{}^{PP'}{}_{B)}{}^{Q}e_{PP'} = 0, \qquad \hat{\Xi}_{(AB)CQ'D}{}^{Q'} = 0, \qquad \hat{\Delta}_{CC'(A|Q'|B)}{}^{Q'}, \qquad \hat{\Lambda}_{A'ABC} = 0. \tag{22}$$

It is important to remark that the contracted Eqs. (22) are fully equivalent to (20).

5. Conformal curves and generalised conformal Gaussian gauge systems

The advantage of considering extended conformal equations in terms of Weyl connections is that they allow to use gauge systems based on conformally invariant objects. An example of these gauge systems are the *conformal Gaussian systems* introduced in [3,5]. These gauge systems are based on conformal geodesics. Conformal Gaussian systems are of great utility in the discussion of evolution problems for the conformal field equations as they provide a canonical conformal factor as well as structural simplifications in the form of the evolution equations.

It was shown in [3,5] that for vacuum spacetimes the conformal factor is quadratic in the conformal time and can be read of from the initial data of the evolution system. Accordingly, the location of the conformal boundary is know *a priori*. This predetermined character of the conformal factor hinges crucially on the fact that the physical spacetime is vacuum. In the sequel we show that the conformal geodesic equations in the presence of matter can be modified in such a way that one has again a conformal factor known *a priori*.

5.1. A class of conformal curves

Let $I \in \mathbb{R}$ be an open interval. We will consider a class of *conformal curves*, $x^{\mu}(\tau)$, whose tangent vector $v^{\mu} \equiv \dot{x}^{\mu}$ is coupled to a 1-form b_{ν} via the equations

$$\tilde{\nabla}_{v}v^{\rho} + v^{\mu}v^{\nu}S_{\mu\nu}{}^{\rho\lambda}b_{\lambda} = 0, \tag{23a}$$

$$\tilde{\nabla}_{v}b_{\nu} - \frac{1}{2}v^{\mu}S_{\mu\nu}{}^{\rho\lambda}b_{\rho}b_{\lambda} = \tilde{H}_{\mu\nu}v^{\mu}. \tag{23b}$$

where $\tilde{H}_{\mu\nu}$ transforms under $\nabla = \tilde{\nabla} + S(\Upsilon)$ as

$$\tilde{H}_{\mu\nu} - H_{\mu\nu} = \tilde{\nabla}_{\mu}\Upsilon_{\nu} - \frac{1}{2}S_{\mu\nu}{}^{\rho\lambda}\Upsilon_{\rho}\Upsilon_{\lambda} = \nabla_{\mu}\Upsilon_{\nu} + \frac{1}{2}S_{\mu\nu}{}^{\rho\lambda}\Upsilon_{\rho}\Upsilon_{\lambda}. \tag{24}$$

The tensor $\tilde{H}_{\mu\nu}$ is, in principle, arbitrary. This freedom will be exploited in the sequel by fixing $\tilde{H}_{\mu\nu}$ such that one obtains an *a priori* known conformal factor associated to the conformal curves.

Eqs. (23a)–(23b) will be supplemented with a frame propagation equation via

$$\tilde{\nabla}_v e_k^\rho + v^\mu e_k^\nu S_{\mu\nu}{}^{\rho\lambda} b_\lambda = 0. \quad (25)$$

In a slight abuse of terminology, we will call a triple (v^μ, b_ν, e_k^μ) solving Eqs. (23a)–(23b) and (25) a *conformal curve*, since these curves exhibit the conformally invariant behaviour described in the following lemma. Following the work in [5] one can prove

Lemma 1. *Let $(x^\mu(\tau), b_\nu(\tau), e_k^\mu(\tau))$ be a conformal curve. Then $(x^\mu(\tau), (b_\nu - h_\nu)(\tau), e_k^\mu(\tau))$ satisfies (25), (23a) and (23b) expressed in the connection $\check{\nabla} = \tilde{\nabla} + S(h)$. In particular, in terms of the Weyl connection given by $\hat{\nabla} = \tilde{\nabla} + S(b)$ the conformal curve equations take the form*

$$\hat{\nabla}_v v^\mu = 0, \qquad \hat{H}_{\mu\nu} v^\mu = 0, \qquad \hat{\nabla}_v e_k^\mu = 0. \quad (26)$$

Moreover, conformal curves are preserved as point sets under reparametrisations of τ by fractional linear transformations.

Remark. Comparing (9a), (9b), (10a), (10b) and (24) we can see that

$$\begin{aligned} \tilde{J}_{\mu\nu} &\equiv \tilde{P}_{\mu\nu} - \tilde{H}_{\mu\nu} \\ &= \hat{P}_{\mu\nu} - \hat{H}_{\mu\nu} \\ &= P_{\mu\nu} - H_{\mu\nu}. \end{aligned}$$

Hence, $\tilde{J}_{\mu\nu}$ as defined above is a conformal invariant. Furthermore, one has that $\hat{H}_{\mu\nu} v^\mu = 0$ implies

$$\hat{P}_{\mu\nu} v^\mu = \tilde{J}_{\mu\nu} v^\mu. \quad (27)$$

A vector frame satisfying (25) is called *Weyl propagated*. It is noted that the velocity can be chosen as one of the frame vectors due to (23a). Suppose along a conformal curve we define

$$\Theta(\tau) \equiv |\tilde{g}(v, v)|^{-1/2}.$$

Then $\Theta(\tau)$ satisfies

$$\dot{\Theta} = \Theta \langle b, v \rangle. \quad (28)$$

Letting $g_{\mu\nu}$ be given as in (2), one finds that $g(v, v) = 1$. In fact, for a Weyl propagated frame e_k^μ the frame metric

$$\eta_{jk} \equiv g(e_j, e_k)$$

is constant along the curve. Hence, a g-orthonormal frame evolves into a g-orthonormal frame along the curve. Following an analogous discussion for conformal geodesics given in [5] one differentiates (28) twice along the curves and substitutes (23a) and (23b) to obtain

$$\dddot{\Theta} = \left(\tilde{\nabla}_v(\tilde{H}(v, v)) + \tilde{H}(v, b^\sharp)\tilde{g}(v, v) + \langle b, v \rangle \tilde{H}(v, v) \right) \Theta. \quad (29)$$

Note that for $\tilde{H}_{\mu\nu} = \tilde{\lambda}\tilde{g}_{\mu\nu}$ the right hand side vanishes exactly. Thus, the following result holds:

Lemma 2. *Suppose that $(v^\mu(\tau), b_\nu(\tau), e_k(\tau))$ is a solution to the conformal curve equations (23a), (23b) and (25) with respect to the metric $\tilde{g}_{\mu\nu}$ such that $x(\tau)$ is a timelike curve in $\tilde{\mathcal{M}}$ defined on some open interval I. If $\tilde{g}_{\mu\nu}$ satisfies the Einstein equations with matter and the tensor $\tilde{H}_{\mu\nu}$ is chosen such that*

$$\tilde{H}_{\mu\nu} = \tilde{\lambda}\tilde{g}_{\mu\nu}, \quad (30)$$

then:

(i)

$$\tilde{J}_{\mu\nu} = \frac{1}{2}\left(\tilde{T}_{\mu\nu} - \frac{1}{3}\tilde{T}_{\rho\sigma}\tilde{g}^{\rho\sigma}\tilde{g}_{\mu\nu} \right); \quad (31)$$

(ii) *the conformal factor Θ is given for $\tau \in I$ by*

$$\Theta = \Theta_* + \dot{\Theta}_*(\tau - \tau_*) + \frac{1}{2}\ddot{\Theta}_*(\tau - \tau_*)^2, \quad (32)$$

where a quantity with a subscript $*$ is constant along $x(\tau)$. The coefficients $\dot{\Theta}_*$ and $\ddot{\Theta}_*$ satisfy the constraints:

$$\dot{\Theta}_* = \langle b_*, v_* \rangle \Theta_*, \qquad \ddot{\Theta}_* = \Theta_*^{-1}\left(\tilde{\lambda} + \frac{1}{2} g^\sharp(d_*, d_*) \right). \quad (33)$$

Remark. Thus, the choice given by Eq. (30) for the tensor $\tilde{H}_{\mu\nu}$ gives rise to a *canonical* conformal factor quadratic in τ which can be determined *a priori* from the initial data. Furthermore, as a consequence of Eqs. (27) and (31), one of the contractions of the Schouten tensors with the tangent to the conformal curve can be expressed in terms of the physical energy–momentum tensor. This observation is of utility in the construction of evolution equations for the components of the Schouten tensor—there is no need of an evolution equation for this contraction. For vacuum spacetimes one has $\tilde{J}_{\mu\nu} = 0$ and one recovers known results for conformal geodesics—see e.g. [3,5]. In the presence of matter the conformal curves given by Eqs. (23a)–(23b) are no longer conformal geodesics. However, our choice of $\tilde{H}_{\mu\nu}$ gives the same behaviour for the *canonical* conformal factor as in the vacuum case.

We will also require the following result:

Lemma 3. *Suppose that* $(v^\mu(\tau), b_\nu(\tau), e_k(\tau))$ *is a conformal curve as in Lemma 2. Let* $\tilde{g}_{\mu\nu}$ *satisfy the Einstein field equations with matter and* $\tilde{H}_{\mu\nu} = \tilde{\lambda}\tilde{g}_{\mu\nu}$*. If* $v^\mu = e_0^\mu$ *at* $\tau = \tau_0 \in I$*, then*

$$b_k(\tau) = b_\mu e_k^\mu = \Theta^{-1}\left(\dot{\Theta}, d_{a*}\right)$$

for $\tau \in I$*, where* $d_{a*} = \Theta b_a(\tau_0)$*, for* $a = 1, 2, 3$*.*

The proof of this result is a calculation analogous to the one described in the proof of Lemma 3.2 in [3].

Remark. This lemma shows that the components of the 1-form d_μ with respect to the Weyl-propagated frame e_k are, in the same way it happens with the conformal factor Θ, known *a priori* in terms of initial data. Notice, however, that unless one knows e_k explicitly, the 1-form d_μ is, strictly speaking, not known.

5.2. Jacobi fields for conformal curves

Suppose we are given a congruence of conformal curves with velocity v. The separation vector ζ satisfies $[v, \zeta] = 0$ along each conformal curve and will be referred to as a Jacobi field. Recall that Weyl connections are torsion-free, so that $[v, \zeta] - \hat{\nabla}_v \zeta + \hat{\nabla}_\zeta v = 0$. Thus we get an evolution equation for ζ

$$\partial_\tau \zeta_k = \hat{\nabla}_v \zeta^k = \zeta^j (\hat{\nabla}_j v^k).$$

When the Jacobi field becomes tangent to the curve at a point p we say that p is a conjugate point. These points are of interest to us for the following reason. If we create Gaussian coordinate system by dragging spatial coordinates along the congruence beyond a conjugate point then these are ill-defined. For this reason we measure $|\zeta - g(v, \zeta)v|^2 = (\eta^{ab}\zeta_a\zeta_b)$ with $a, b = 1, 2, 3$. As long as this quantity does not vanish our coordinate system will be well-defined.

5.3. Generalised conformal Gaussian systems

In analogy to the way conformal geodesics have been used in [3,6–8], to construct conformal Gaussian gauge systems, here we will use the conformal curves solving Eqs. (23a)–(25) and (23b) to construct what we will call *generalised conformal Gaussian systems*.

Let $\tilde{\mathcal{S}}$ be a space-like hypersurface in the spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$. On $\tilde{\mathcal{S}}$ we choose an initial conformal factor $\Theta_* > 0$, a frame field e_{k*}, and a 1-form b_* such that

$$\Theta_*^2 \tilde{g}(e_{i*}, e_{j*}) = \eta_{ij}, \tag{34}$$

and e_{0*} is orthogonal to $\tilde{\mathcal{S}}$. For fixed $\tilde{H}_{\mu\nu}$ and given $x_* \in \tilde{\mathcal{S}}$ there exists a unique conformal curve $(x^\mu(\tau), b_\nu(\tau))$, which for $\tau = 0$ passes through x_* and which satisfies the initial conditions

$$\dot{x}^\mu = e_{0*}^\mu, \qquad b_\nu = b_{\nu *}. \tag{35}$$

If all data are smooth, then in some neighbourhood $\mathcal{U} \in \tilde{\mathcal{S}}$ these curves define a smooth caustic free congruence covering $\mathcal{U}$. Furthermore, b_ν defines a smooth 1-form on $\mathcal{U}$ which allows to construct a Weyl connection $\hat{\nabla}$. A smooth frame field e_k^μ and the related conformal factor Θ are obtained in $\mathcal{U}$ by solving the propagation equations (25) and (28) for given initial data

$$e_k^\mu = e_{k*}^\mu, \qquad \Theta = \Theta_* \tag{36}$$

on $\tilde{\mathcal{S}}$. Then $e_0^\mu(\tau) = \dot{x}^\mu(\tau)$ on $\mathcal{U}$ and we define $\hat{\chi}_i{}^j \equiv \hat{\nabla}_i v^j = \hat{\Gamma}_i{}^j{}_0$. The frame one obtains from solving the propagation equation is orthonormal for the metric $g_{\mu\nu} = \Theta^2 \tilde{g}_{\mu\nu}$, while $\hat{\Gamma}_0{}^j{}_k = 0$. Dragging along local coordinates x^a on $\tilde{\mathcal{S}}$ with the congruence and setting $x^0 = \tau$, one obtains a coordinate system. A coordinate system, a frame field and a conformal factor constructed with the above procedure will be known as a *generalised conformal Gaussian system*.

We will follow the setup used in [7,8], where we use a global frame field $c_{\underline{s}}$ ($\underline{s} = 0, 1, 2, 3$) constructed from the coordinate vectors in such a way that $c_0 = \partial_\tau$ and $c_{\underline{r}}$ ($\underline{r} = 1, 2, 3$) is a constant linear combination of the spatial coordinate vectors. The frame e_k then is written in terms of its expansion $e_k = e_k^{\underline{s}} c_{\underline{s}}$. The same is done for the spinorial version.

Remark. Above we have set up a generalised conformal Gaussian system in the physical spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$. However, due to the conformal invariance of conformal curves proven in Lemma 1 the same gauge can also be constructed starting from an spacelike hypersurface $\check{\mathcal{S}}$ in a conformally related spacetime $(\check{\mathcal{M}}, \check{g}_{\mu\nu})$ such that $\tilde{\mathcal{S}} \subset \check{\mathcal{S}}$, $\tilde{\mathcal{M}} \subset \check{\mathcal{M}}$. The initial data will be related in the obvious way, with Θ_* and b_* changing accordingly while the frame remain the same. We will make us of this fact later on when constructing a generalised conformal Gaussian system from hyperboloidal data given on $\mathcal{S}$ in the unphysical spacetime $(\mathcal{M}, g_{\mu\nu})$.

Due to Lemma 1 a generalised conformal Gaussian system is characterised on $\mathcal{U}$ by the explicit conditions

$$v^\mu = \partial_\tau, \qquad \hat{\Gamma}_0{}^j{}_k = 0, \qquad \hat{P}_{0k} = \tilde{J}_{0k}. \tag{37}$$

Setting

$$\tilde{J}_{\mu\nu} = \frac{1}{2}\left(\tilde{T}_{\mu\nu} - \frac{1}{3}\tilde{T}_{\rho\sigma}\tilde{g}^{\rho\sigma}\tilde{g}_{\mu\nu}\right)$$

one has, by virtue of Lemma 2, that the solution of the evolution of the conformal factor is known *a priori*. In the sequel *we will only consider trace-free matter* (the Maxwell field). Consequently,

$$\tilde{J}_{\mu\nu} = \frac{1}{2}\tilde{T}_{\mu\nu}, \qquad \tilde{\nabla}^\mu \tilde{J}_{\mu\nu} = 0.$$

The latter implies $\nabla^\mu J_{\mu\nu} = 0$ if one defines $J_{\mu\nu} \equiv \Theta^{-2}\tilde{J}_{\mu\nu}$.

6. Hyperbolic reductions of the conformal field equations

In this section we discuss how to extract a symmetric hyperbolic system of propagation equations. For this, we resort to a space-spinor formalism – see e.g. [13] – based on the spinorial counterpart, $\tau^{AA'}$, of a timelike vector τ^μ in the conformally rescaled spacetime $(\mathcal{M}, g_{\mu\nu})$. More precisely, the vector τ^μ will be taken to be parallel to the tangent vector v^μ to the conformal curves described in Section 5. The normalisation condition $\tau_\mu \tau^\mu = 2$ will be used.

The reduced symmetric hyperbolic system of evolution equations is to be deduced from the following contractions of the conformal field equations

$$\tau^{AA'}\hat{\Sigma}_{AA'}{}^{PP'}{}_{BB'}e_{PP'} = 0, \qquad \tau^{CC'}\hat{\Xi}_{ABCC'DD'} = 0, \qquad \tau^{AA'}\hat{\Delta}_{AA'BB'CC'} = 0, \tag{38}$$

together with

$$\tau_{(A}{}^{A'}\hat{\Lambda}_{|A'|B)CD} = 0. \tag{39}$$

6.1. The space spinor formalism in brief

In what follows, we will consider spin dyads $\{\delta_A\}$ for which the spinor $\tau^{AA'}$ admits the decomposition

$$\tau^{AA'} = \epsilon_0{}^A \epsilon_{0'}{}^{A'} + \epsilon_1{}^A \epsilon_{1'}{}^{A'}.$$

In particular, one has that

$$\tau_{AA'}\tau^{BA'} = \epsilon_A{}^B. \tag{40}$$

Using the spinor $\tau^{AA'}$, the gauge conditions (37) can be rewritten as

$$\tau^{AA'}e_{AA'} = \sqrt{2}\partial_\tau, \qquad \tau^{AA'}\hat{\Gamma}_{AA'}{}^B{}_C = 0, \qquad \tau^{AA'}\hat{P}_{AA'BB'} = \tilde{J}_{BB'} \tag{41}$$

where $\tilde{J}_{BB'}$ denotes the spinorial counterpart of $\tilde{J}_{ij}\tau^i$.

The spinor $\tau^{AA'}$ can also be used to obtain an unprimed version of the spinorial Weyl connection covariant derivative $\hat{\nabla}_{AA'}$. More precisely, one has that:

$$\hat{\nabla}_{AB} \equiv \tau_B{}^{A'}\hat{\nabla}_{AA'}.$$

The latter, in turn, can be decomposed in its irreducible parts:

$$\hat{\nabla}_{AB} = \frac{1}{2}\epsilon_{AB}\hat{\mathcal{P}} + \hat{\mathcal{D}}_{AB}, \tag{42}$$

where

$$\hat{\mathcal{P}} \equiv \tau^{AA'}\hat{\nabla}_{AA'}, \qquad \hat{\mathcal{D}}_{AB} \equiv \tau_{(B}{}^{A'}\hat{\nabla}_{A)A'}.$$

The differential operator $\hat{\mathcal{D}}_{AB}$ is the so-called *Sen connection* of $\hat{\nabla}$ relative to the vector field $\tau^{AA'}$—see e.g. [13].

6.2. Hyperbolic reduction of a first model equation

The procedure and subtleties of deriving hyperbolic equations from the extended conformal equations (20) will be illustrated with a model equation.

Let $M_{i\mathcal{K}}$ and $N_{ij\mathcal{K}} = N_{[ij]\mathcal{K}}$ be two tensorial quantities, where $\mathcal{K}$ stands for any set of tensor or bundle indices, satisfying the equation

$$\hat{\nabla}_i M_{j\mathcal{K}} - \hat{\nabla}_j M_{i\mathcal{K}} = N_{ij\mathcal{K}}. \tag{43}$$

Two derive an evolution equation we contract with τ^i:

$$\hat{\nabla}_\tau M_{j\mathcal{K}} - \hat{\nabla}_j (M_{i\mathcal{K}}\tau^i) = N_{ij\mathcal{K}} - M_{i\mathcal{K}}(\hat{\nabla}_j \tau^i)$$

from where it follows that

$$\sqrt{2}\partial_\tau M_{j\mathcal{K}} - \hat{\nabla}_j (M_{i\mathcal{K}}\tau^i) = N'_{ij\mathcal{K}} \equiv N_{ij\mathcal{K}} + \tau^i (\hat{\Gamma}_i{}^l{}_j M_{l\mathcal{K}} + \hat{\Gamma}_i{}^{\mathcal{L}}{}_{\mathcal{K}} M_{j\mathcal{L}}) - M_{i\mathcal{K}}(\hat{\nabla}_j \tau^i).$$

Note that in our setup the connection coefficients in the expression vanish due to our gauge choice. However, the following analysis is valid without this condition and it should be observed that these terms have a polynomial form. We note that $M_{0\mathcal{K}}$ appears inside the second term. However, we cannot obtain an evolution equation for $M_{0\mathcal{K}}$ from Eq. (43) since setting $j = 0$ in order to get the evolution equation for $M_{0\mathcal{K}}$ makes both sides reduce trivially to zero due to the skew symmetry in ij. Instead, $M_{0\mathcal{K}}$ must be determined from the symmetries of $M_{i\mathcal{K}}$ (if any) or if not, then it must be regarded as free data.

The subsequent discussion of the properties of the model equation (43) will be carried out in the spinor formalism. From the spinorial version of (43)

$$\hat{\nabla}_{AA'} M_{BB'\mathcal{K}} - \hat{\nabla}_{BB'} M_{AA'\mathcal{K}} = N_{AA'BB'\mathcal{K}}$$

one obtains the contracted versions

$$\hat{\nabla}_{(A|P'|} M_{B)}{}^{P'}{}_{\mathcal{K}} = \frac{1}{2} N_{(A|P'|B)}{}^{P'}{}_{\mathcal{K}}, \tag{44a}$$

$$\hat{\nabla}_{P(A'} M_{B')}{}^{P'}{}_{\mathcal{K}} = \frac{1}{2} N_{P(A'}{}^{P'}{}_{B')\mathcal{K}}. \tag{44b}$$

In order to change to space spinor components it is observed that one has to contract with $\tau_A{}^{A'}$ inside the derivative. One uses that $M_{BB'\mathcal{K}} = -M_{BP\mathcal{K}}\tau^P{}_{B'}$ so that

$$\begin{aligned}\tau_P{}^{A'}\tau_Q{}^{B'}\hat{\nabla}_{AA'} M_{BB'\mathcal{K}} &= \tau_P{}^{A'}\tau_Q{}^{B'}\left(-\tau^R{}_{B'}\hat{\nabla}_{AA'} M_{BR\mathcal{K}} - M_{BR\mathcal{K}}\hat{\nabla}_{AA'}\tau^R{}_{B'}\right)\\ &= \hat{\nabla}_{AP} M_{BQ\mathcal{K}} - M_{BR\mathcal{K}}\,\hat{\chi}_{AP}{}^R{}_Q.\end{aligned}$$

Thus, from (44a)–(44b) one obtains

$$\hat{\nabla}_{(A|P|} M_{B)}{}^P{}_{\mathcal{K}} = \frac{1}{2} N''{}_{(A|P|B)}{}^P{}_{\mathcal{K}} \equiv \frac{1}{2} N'{}_{(A|P|B)}{}^P{}_{\mathcal{K}} - M_{(A}{}^R{}_{|\mathcal{K}|}\,\hat{\chi}_{B)PR}{}^P$$

$$\hat{\nabla}_{A(P} M^A{}_{Q)\mathcal{K}} = \frac{1}{2} N''{}_{A(P}{}^A{}_{Q)\mathcal{K}} \equiv \frac{1}{2} N'{}_{A(P}{}^A{}_{Q)\mathcal{K}} + M^A{}_{R\mathcal{K}}\,\hat{\chi}_{A(P}{}^R{}_{Q)}.$$

Using the decomposition (42) for $\hat{\nabla}_{AB}$ and writing $M_{AB\mathcal{K}}$ as

$$M_{AB\mathcal{K}} = \frac{1}{2}\epsilon_{AP} m_{\mathcal{K}} + m_{AB\mathcal{K}}$$

one obtains:

$$-\frac{1}{2}\hat{\mathcal{P}} m_{(AB)\mathcal{K}} + \frac{1}{2}\hat{\mathcal{D}}_{AB} m_{\mathcal{K}} + \hat{\mathcal{D}}_{P(A} m_{B)}{}^P{}_{\mathcal{K}} = \frac{1}{2} N''{}_{(A|P|B)}{}^P{}_{\mathcal{K}},$$

$$\frac{1}{2}\hat{\mathcal{P}} m_{(PQ)\mathcal{K}} - \frac{1}{2}\hat{\mathcal{D}}_{PQ} m_{\mathcal{K}} + \hat{\mathcal{D}}_{A(P} m^A{}_{Q)\mathcal{K}} = \frac{1}{2} N''{}_{A(P}{}^A{}_{Q)\mathcal{K}}.$$

Making linear combinations of the latter equations one finally arrives at:

$$\hat{\mathcal{P}} m_{AB\mathcal{K}} - \hat{\mathcal{D}}_{AB} m_{\mathcal{K}} = E^{[M]}_{AB\mathcal{K}} \equiv \frac{1}{2}\left(N''{}_{P(A}{}^P{}_{B)\mathcal{K}} - N''{}_{(A|P|B)}{}^P{}_{\mathcal{K}}\right), \tag{45a}$$

$$\hat{\mathcal{D}}_{P(A} m^P{}_{B)\mathcal{K}} = C^{[M]}_{AB\mathcal{K}} \equiv \frac{1}{2}\left(N''{}_{P(A}{}^P{}_{B)\mathcal{K}} + N''{}_{(A|P|B)}{}^P{}_{\mathcal{K}}\right). \tag{45b}$$

The terms $E^{[M]}_{AB\mathcal{K}}$ and $C^{[M]}_{AB\mathcal{K}}$ introduced on the right hand side are formed from the original term $N_{AB\mathcal{K}}$ and connection coefficients. It will be seen that in their explicit form they are polynomial in the variables of our system. We will write the original variable, here M, in square brackets and the variables E and C are used to indicate whether the term is for the evolution or the constraint equation.

Eq. (45a) is an evolution equation for the spinorial components $m_{AB\mathcal{K}}$. In what concerns the "timelike" components $m_{\mathcal{K}}$ (i.e. $M_{0\mathcal{K}}$) one will have two possible situations:

(i) there exists an external equation that relates $m_{\mathcal{K}}$ and $m_{AB\mathcal{K}}$ and possibly some other variables to each other. Then (45a) may lead to a symmetric hyperbolic system of equations for $m_{AB\mathcal{K}}$. A special case of the above mentioned equation arises when $M_{AB\mathcal{K}}$ has a symmetry relating the two components;

(ii) $m_{\mathcal{K}}$ cannot be reexpressed in terms of the $m_{AB\mathcal{K}}$ in which case the former is regarded as free data which has to be specified by means of a gauge choice. This may lead to a transport equation for $m_{AB\mathcal{K}}$.

On the other hand, (45b) is a constraint equation for the components $m_{AB\mathcal{K}}$ which one expects to hold at latter times if satisfied initially—the so-called propagation of the constraints. Note that, *a priori*, there are no constraints for $m_{\mathcal{K}}$.

6.3. Hyperbolic reduction of a second model equation

In the discussion of the propagation of the constraints a different type of model equation will be considered. In what follows we will briefly discuss its hyperbolic reduction.

Let $M_{ij\mathcal{K}} = M_{[ij]\mathcal{K}}$ and $N_{kij\mathcal{K}} = N_{[kij]\mathcal{K}}$ denote two tensorial quantities, where again $\mathcal{K}$ stands for any set of spinor indices. The model equation to be considered is given by

$$\hat{\nabla}_{[k}M_{ij]\mathcal{K}} = N_{kij\mathcal{K}}. \tag{46}$$

This type of equation is motivated by the observation that if $\boldsymbol{\omega}$ is a 2-form, then its Lie derivative with respect to a vector field τ^{μ} is given by

$$\mathcal{L}_{\tau}\boldsymbol{\omega} = (i_{\tau}\mathrm{d} + \mathrm{d}i_{\tau})\boldsymbol{\omega},$$

where $i_{\tau}\boldsymbol{\omega}$ denotes the contraction of the 2-form $\boldsymbol{\omega}$ with τ^{μ}. Now, if $i_{\tau}\boldsymbol{\omega} = 0$ (as it is the case with the zero quantities associated with the extended conformal field equations), one finds that

$$\mathcal{L}_{\tau}\omega_{ij} = \tau^{k}\hat{\nabla}_{[k}\omega_{ij]}.$$

In what follows, let ϵ_{ijkl} denote the components with respect to the frame e_k of the volume form of the metric $g_{\mu\nu}$. Now,

$$\begin{aligned}\hat{\nabla}_{[k}M_{ij]\mathcal{K}} &= \delta_{[k}{}^{l}\delta_{i}{}^{m}\delta_{j]}{}^{n}N_{lmn\mathcal{K}},\\ &= -\frac{1}{6}\epsilon_{kijp}\epsilon^{lmnp}N_{lmn\mathcal{K}},\\ &= -\frac{1}{3}\epsilon_{kijp}\,{}^{*}N^{rp}{}_{r\mathcal{K}}.\end{aligned}$$

Because of the connection with the Lie derivative, it follows then that

$$\tau^{k}\hat{\nabla}_{[k}M_{ij]\mathcal{K}} = -\frac{1}{3}\tau^{k}\epsilon_{kijp}\,{}^{*}N^{rp}{}_{r\mathcal{K}},$$

implies an hyperbolic equation for the tensorial field $M_{ij\mathcal{K}}$. The relevance of this equations to prove the propagation of the constraints depends on whether its right hand side can be casted as an homogeneous expression of other zero quantities—see Section 9.

6.4. The reduced geometric equations

Following the discussion of the model equation (43) in the previous section one introduces the unprimed spinorial fields e^{s}_{AB}, $\hat{\Gamma}_{ABCD}$ and $\hat{P}_{ABCD}$ defined by

$$\begin{aligned}&e^{s}_{AB} \equiv \tau_{A}{}^{A'}e^{s}_{BA'},\\ &f_{AB} \equiv \tau_{B}{}^{A'}f_{AA'},\\ &\hat{\Gamma}_{ABCD} \equiv \tau_{B}{}^{B'}\hat{\Gamma}_{AB'CD},\\ &\hat{P}_{ABCD} \equiv \tau_{B}{}^{A'}\tau_{D}{}^{C'}\hat{P}_{AA'CC'},\\ &\tilde{Y}_{ABCD} \equiv \tau_{D}{}^{C'}\tilde{Y}_{ABCC'}\end{aligned}$$

from which the original spacetime spinors $e^{s}_{AA'}$ (the spinorial version of e^{s}_{k}), $f_{AA'}$, $\hat{\Gamma}_{AA'BC}$, $\hat{P}_{AA'BB'}$, $\tilde{Y}_{ABCC'}$ can be recovered using the identity (40). Following the discussion from the previous section, the conformal equations (19a)–(19c) can only yield evolution equations for the components

$$e_{(AB)}, \qquad f_{(AB)}, \qquad \hat{\Gamma}_{(AB)CD}, \qquad \hat{P}_{(AB)CD}.$$

Due to the absence of further symmetries in the fields $e_{AB}, f_{AB}, \hat{\Gamma}_{ABCD}, \hat{P}_{ABCD}$ the components

$$e_Q{}^Q, \qquad f_Q{}^Q, \qquad \hat{\Gamma}_Q{}^Q{}_{CD}, \qquad \hat{P}_Q{}^Q{}_{CD},$$

are regarded as freely specifiable, and will be fixed by means of the gauge conditions (41) so that in particular:

$$f_Q{}^Q = 0, \qquad \hat{\Gamma}_Q{}^Q{}_{CD} = 0, \qquad \hat{P}_Q{}^Q{}_{CD} = \tilde{J}_{CD}.$$

Consequently one obtains evolution equations of the form

$$\partial_\tau e^{s}_{(AB)} = H^{[e]}_{AB}, \tag{47a}$$

$$\partial_\tau f_{(AB)} = H^{[f]}_{AB}, \tag{47b}$$

$$\partial_\tau \hat{\Gamma}_{(AB)CD} = H^{[\Gamma]}_{ABCD}, \tag{47c}$$

$$\partial_\tau \hat{P}_{(AB)CD} = \hat{\mathcal{D}}_{AB}\tilde{J}_{CD} + \tilde{Y}_{ABCD} + H^{[P]}_{(AB)CD}. \tag{47d}$$

where $H^{[e]}_{AB}$ is a quadratic expression involving $e^{s}_{(AB)}$ and $\hat{\Gamma}_{(AB)CD}$; $H^{[f]}_{AB}$ contains a quadratic expression involving $f_{(AB)}$ and $\hat{\Gamma}_{(AB)CD}$ and a linear term involving $\hat{P}_{(AB)CD}$; $H^{[\Gamma]}_{ABCD}$ contains terms quadratic in $\hat{\Gamma}_{(AB)CD}$, a quadratic expression involving Θ and ϕ_{ABCD} and a linear expression in $\hat{P}_{(AB)CD}$; finally, $H^{[P]}_{(AB)CD}$ contains quadratic terms in $\hat{\Gamma}_{(AB)CD}$ and $\hat{P}_{(AB)CD}$ and in d_{AB} and ϕ_{ABCD}. Their explicit form will not be important for our subsequent discussion.

As it will be discussed in Section 7.5, the spinor $\tilde{Y}_{ABCD}$ for the case of the Einstein–Maxwell system contains derivatives of the Maxwell field. These terms enter in the principal of Eq. (47d). This feature requires us to introduce new field equations—essentially, the covariant derivative of the Maxwell spinor. The term $\hat{\mathcal{D}}_{AB}\tilde{J}_{CD}$ in Eq. (47d) will lead to similar problems, for it will be seen that $\tilde{J}_{CD}$ is quadratic in the Maxwell spinor.

For the evolution of the Jacobi field we split its space spinor into irreducible components

$$\tau_B{}^{B'}\zeta_{AB'} = \frac{1}{2}\epsilon_{AB}\zeta + \zeta_{AB}, \quad \text{with } \zeta = \zeta_A{}^A, \; \zeta_{AB} = \tau_{(B}{}^{B'}\zeta_{A)B'}.$$

Then we get the evolution equations

$$\partial_\tau \zeta = \frac{1}{2}\zeta\,\hat{\chi}_P{}^P{}_Q{}^Q + \zeta^{CD}\hat{\chi}_{(CD)Q}{}^Q, \tag{48a}$$

$$\partial_\tau \zeta_{AB} = \frac{1}{2}\zeta\,\hat{\chi}_P{}^P{}_{AB} + \zeta^{CD}\hat{\chi}_{(CD)AB}. \tag{48b}$$

6.5. The reduced Bianchi equation

In order to construct an evolution equation for the Weyl spinor ϕ_{ABCD} we consider the zero quantity

$$\hat{\Lambda}_{ABCD} \equiv \tau_A{}^{A'}\hat{\Lambda}_{A'BCD},$$
$$= \hat{\nabla}^Q{}_A\phi_{BCDQ} - f^Q{}_A\phi_{BCDQ} - \Theta^{-1}\tilde{Y}_{BCDA}.$$

Again, using the decomposition (42) one obtains

$$\hat{\Lambda}_{ABCD} = -\frac{1}{2}\hat{\mathcal{P}}\phi_{ABCD} + \hat{\mathcal{D}}^Q{}_A\phi_{BCDQ} - f^Q{}_A\phi_{BCDQ} - \Theta^{-1}\tilde{Y}_{BCDA}.$$

Accordingly,

$$-2\hat{\Lambda}_{ABCD} = \hat{\mathcal{P}}\phi_{ABCD} - 2\hat{\mathcal{D}}^Q{}_{(A}\phi_{BCD)Q} + 2f^Q{}_{(A}\phi_{BCD)Q} + 2\Theta^{-1}\tilde{Y}_{(ABCD)}. \tag{49}$$

renders the desired reduced equation. In Eq. (49) we notice again the presence of the term $\tilde{Y}_{(ABCD)}$ so that the same problem arises as for the reduced equation (47d). We will thus treat this term in the same way as outline for Eq. (47d), in order to ensure the symmetric hyperbolicity of the system. We observe the presence of the potentially singular term Θ^{-1}. However, as will be seen in the sequel, this term is cancelled out by a similar term appearing in the explicit form of $\tilde{Y}_{ABCD}$.

Finally, it is noticed that the remaining content of the zero quantity Λ_{ABCD} is contained in

$$\hat{\Lambda}_P{}^P{}_{CD} = \hat{\mathcal{D}}^{PQ}\phi_{PQCD} - f^{PQ}\phi_{CDPQ} - \Theta^{-1}\tilde{Y}_{CD}{}^P{}_P,$$

corresponding to the constraints associated to the Bianchi identity (19d).

7. The spinorial Maxwell equations

Up to this point our discussion has been completely general and irrespective of the trace-free matter models under consideration. In order to proceed further, explicit information about the matter model has to be provided—in our case the Maxwell field.

7.1. The Maxwell equations in the physical spacetime

The physical spacetime Maxwell equations (1c) are equivalent to the spinorial equation

$$\tilde{\nabla}^{AA'}\tilde{\phi}_{AB} = 0, \tag{50}$$

where the antisymmetric Maxwell tensor $\tilde{F}_{\mu\nu}$ and the totally symmetric spinor $\tilde{\phi}_{AB}$ are related to each other by the correspondence

$$\tilde{F}_{\mu\nu} \leftrightarrow \tilde{F}_{AA'BB'} \equiv \tilde{\phi}_{AB}\epsilon_{A'B'} + \bar{\tilde{\phi}}_{A'B'}\epsilon_{AB}, \quad \tilde{\phi}_{AB} = \frac{1}{2}\tilde{F}_{AQ'B}{}^{Q'}.$$

The energy–momentum tensor (1b) is given in spinorial terms by

$$\tilde{T}_{AA'BB'} = \tilde{\phi}_{AB}\bar{\tilde{\phi}}_{A'B'}, \qquad \tilde{\nabla}^{AA'}\tilde{T}_{AA'BB'} = 0.$$

7.2. The Maxwell equations in the unphysical spacetime

If upon the conformal rescaling (2) one imposes the transformation rule

$$\phi_{AB} = \Theta^{-1}\tilde{\phi}_{AB}, \tag{51}$$

then one obtains that

$$\nabla^{Q}{}_{A'}\phi_{BQ} = 0. \tag{52}$$

In terms of the Weyl connection $\hat{\nabla}$ one has that

$$\hat{\nabla}^{Q}{}_{A'}\phi_{BQ} = f^{Q}{}_{A'}\phi_{BQ}. \tag{53}$$

For later use we define the Hermitian conjugate of Maxwell spinor

$$\phi^{\dagger}_{AB} = \tau_A{}^{A'}\tau_B{}^{B'}\bar{\phi}_{A'B'}. \tag{54}$$

For more details on the Hermitian conjugation map for space spinors, see e.g. [3].

With regards to the energy–momentum tensor one has that

$$T_{AA'BB'} \equiv \phi_{AB}\bar{\phi}_{A'B'} = \Theta^{-2}\tilde{T}_{AA'BB'}, \qquad \nabla^{AA'}T_{AA'BB'} = 0. \tag{55}$$

The last property follows from the trace-freeness property of $T_{AA'BB'}$ in four dimensions [2]. As a consequence of this discussion, the following zero quantity is introduced:

$$\hat{\omega}_{A'B} \equiv \hat{\nabla}^{Q}{}_{A'}\phi_{BQ} - f^{Q}{}_{A'}\phi_{BQ}.$$

In the sequel it will be seen that in order to obtain a symmetric hyperbolic reduction of the conformal Einstein–Maxwell equations, it is necessary to introduce the derivatives of the Maxwell field as a variable. For this, one considers for a given gauge choice a spinorial field $\hat{\psi}_{AA'BC} = \hat{\psi}_{AA'(BC)}$ and an associated zero quantity $\hat{\omega}_{AA'BC}$. These two quantities are related by

$$\hat{\omega}_{AA'BC} \equiv \hat{\psi}_{AA'BC} - \hat{\nabla}_{AA'}\phi_{BC}$$

and under a connection change (8b) the spinorial field $\hat{\psi}_{AA'BC}$ is adapted as

$$\hat{\psi}_{AA'BC} = \psi_{AA'BC} - 2\phi_{A(B}f_{C)A'}. \tag{56}$$

The zero quantity $\hat{\omega}_{AA'BC}$ will be handled in the sequel as a constraint. In order to obtain an equation for $\hat{\psi}_{AA'BC}$ we adopt the strategy used in [2] and make use of the Ricci identity – cf. Eq. (18) – for the Weyl connection $\hat{\nabla}$ applied to the spinor ϕ_{AB}:

$$\hat{\nabla}_{AA'}\hat{\nabla}_{BB'}\phi_{EF} - \hat{\nabla}_{BB'}\hat{\nabla}_{AA'}\phi_{EF} = -2\phi_{P(E}\hat{r}^{P}{}_{F)AA'BB'} - \hat{\Sigma}_{AA'}{}^{PP'}{}_{BB'}\hat{\nabla}_{PP'}\phi_{EF}.$$

Replacing the derivatives of ϕ_{AB} by $\hat{\psi}_{AA'BC}$ and assuming that the conformal equations (20) are satisfied one obtains the required equation:

$$\hat{\nabla}_{AA'}\hat{\psi}_{BB'EF} - \hat{\nabla}_{BB'}\hat{\psi}_{AA'EF} = -2\phi_{P(E}\hat{R}^{P}{}_{F)AA'BB'}. \tag{57}$$

To the latter we associate the following zero quantity:

$$\hat{\omega}_{AA'BB'EF} \equiv \hat{\nabla}_{AA'}\hat{\psi}_{BB'EF} - \hat{\nabla}_{BB'}\hat{\psi}_{AA'EF} + 2\phi_{P(E}\hat{R}^{P}{}_{F)AA'BB'}. \tag{58}$$

Remark. It can be verified from (51) and (56) that the zero quantities $\hat{\omega}_{A'B}$, $\hat{\omega}_{A'ABC}$, $\hat{\omega}_{AA'BB'CC'}$ transform homogeneously upon changes of conformal gauge. This observation is of relevance for the propagation of the constraints.

7.3. The reduced Maxwell equations

The reduced equations implied by the Maxwell equations (52) is handled in a similar way to the Bianchi identity (19d)—one considers the unprimed zero quantity

$$\begin{aligned}\hat{\omega}_{AB} &\equiv \tau_A{}^{A'}\hat{\omega}_{A'B},\\ &= \hat{\nabla}^Q{}_A\phi_{BQ} - f^Q{}_A\phi_{BQ},\\ &= -\frac{1}{2}\hat{\mathcal{P}}\phi_{AB} + \hat{\mathcal{D}}^Q{}_A\phi_{BQ} - f^Q{}_A\phi_{BQ}\end{aligned}$$

from where one obtains the propagation equation

$$-2\hat{\omega}_{(AB)} = \hat{\mathcal{P}}\phi_{AB} - 2\hat{\mathcal{D}}^Q{}_{(A}\phi_{B)Q} + 2f^Q{}_{(A}\phi_{B)Q} = 0, \tag{59}$$

and the constraint

$$\omega_Q{}^Q = \hat{\mathcal{D}}^{PQ}\phi_{PQ} - f^{PQ}\phi_{PQ} = 0.$$

7.4. The reduced equations for the derivatives of the Maxwell spinor

The treatment of the equation associated with the zero quantity $\omega_{AA'BB'EF}$ follows the model discussed in Section 6.2. In particular, one has to consider the contracted zero quantity

$$\frac{1}{2}\omega_{(A|Q'|B)}{}^{Q'}{}_{EF} \equiv \hat{\nabla}_{(A|Q'|}\hat{\psi}_{B)}{}^{Q'}{}_{EF} + 2\phi_{P(E}\hat{R}^P{}_{F)(A|Q'|B)}{}^{Q'}$$

and its complex conjugate. The procedure described in Section 6.2 then leads to

$$\hat{\mathcal{P}}\hat{\psi}_{(AB)CD} - \hat{\mathcal{D}}_{AB}\hat{\psi}_P{}^P{}_{CD} = H^{[\psi]}_{ABCD}, \tag{60}$$

where $\hat{\psi}_{ABCD}$ is the space spinor version of $\hat{\psi}_{AA'BC}$ given by $\hat{\psi}_{ABCD} \equiv \tau_B{}^{A'}\hat{\psi}_{AA'CD}$. The source term $H^{[\psi]}_{ABCD}$ contains quadratic terms involving ϕ_{ABCD} and ψ_{ABCD}, ϕ_{AB} and $\hat{P}_{ABCD}$ and ϕ_{AB} and ϕ_{ABCD}. Some of parts of the quadratic expression involving ϕ_{AB} and $\hat{P}_{ABCD}$ lead to terms cubic in ϕ_{AB}. As in the case of the reduced equations (47a)–(47d), the explicit form of the source $H^{[\psi]}_{ABCD}$ will not be required.

The reduction procedure described in the previous lines does not provide an evolution equation for the components $\hat{\psi}_P{}^P{}_{CD}$. To get around this, we write

$$\hat{\psi}_{ABCD} = \nu_{ABCD} + \frac{1}{2}\epsilon_{AB}\nu_{CD} \tag{61}$$

where

$$\nu_{ABCD} \equiv \hat{\psi}_{(AB)CD}, \qquad \nu_{CD} \equiv \hat{\psi}_P{}^P{}_{CD}.$$

Let also

$$\nu \equiv \hat{\psi}_{PQ}{}^{PQ}.$$

It follows then that

$$\hat{\psi}_{(ABCD)} = \nu_{(ABCD)} \qquad \hat{\psi}_{P(BC)}{}^P = \nu_{P(BC)}{}^P + \frac{1}{2}\nu_{BC}, \qquad \hat{\psi}_{PQ}{}^{PQ} = \nu_{PQ}{}^{PQ} = \nu.$$

Now, assuming that

$$\hat{\omega}_{AB} = 0, \qquad \hat{\omega}_{AA'BC} = 0$$

so that $\hat{\psi}_{ABCD} = \hat{\nabla}_{AB}\phi_{CD}$ and the Maxwell equations hold, one finds that

$$\nu_{ABCD} = \hat{\mathcal{D}}_{AB}\phi_{CD} \tag{62a}$$

$$\begin{aligned}\nu_{CD} &= \hat{\mathcal{P}}\phi_{CD} = -2\hat{\mathcal{D}}_{Q(C}\phi_{D)}{}^Q - 2f^Q{}_{(C}\phi_{D)Q}\\ &= -2\nu_{Q(CD)}{}^Q - 2f^Q{}_{(C}\phi_{D)Q}\end{aligned} \tag{62b}$$

$$\nu_{AB}{}^{AB} = \hat{\mathcal{D}}_{AB}\phi^{AB}. \tag{62c}$$

In particular, the relation (62b) allows us to express the full content of the field $\hat{\psi}_{ABCD}$ in terms of $\nu_{(ABCD)}$, ν_{AB} and $\nu_{PQ}{}^{PQ}$—the term $\nu_{P(AB)}{}^{P}$ being redundant. Substituting the decomposition (61) into Eq. (60) one obtains:

$$\hat{\mathcal{P}}\nu_{(ABCD)} - \hat{\mathcal{D}}_{(AB}\nu_{CD)} = H^{[\psi]}_{(ABCD)}, \tag{63a}$$

$$\hat{\mathcal{P}}\nu_{Q(BC)}{}^{Q} - \hat{\mathcal{D}}_{Q(B}\nu_{C)}{}^{Q} = H^{[\psi]}_{Q(BC)}{}^{Q}. \tag{63b}$$

$$\hat{\mathcal{P}}\nu - \hat{\mathcal{D}}_{PQ}\nu^{PQ} = H^{[\psi]PQ}_{PQ}. \tag{63c}$$

Using Eq. (62b) one finds that

$$\hat{\mathcal{P}}\nu_{Q(BC)}{}^{Q} = -\frac{1}{2}\hat{\mathcal{P}}\nu_{BC} - \hat{\mathcal{P}}f^{Q}{}_{(B}\phi_{C)Q} - f^{Q}{}_{(B}\hat{\mathcal{P}}\phi_{C)Q}.$$

Using Eq. (47b) to replace $\hat{\mathcal{P}}f_{QB}$ by $H^{[\psi]}_{AB}$ and Eq. (59) to express $\hat{\mathcal{P}}\phi_{CQ}$ in terms of $\hat{\psi}_{ABCD}$ and a quadratic expression in f_{AB} and ϕ_{AB} it follows from (63b) that

$$\hat{\mathcal{P}}\nu_{BC} + 2\hat{\mathcal{D}}_{Q(B}\nu_{C)}{}^{Q} = H'^{[\psi]}_{BC}, \tag{64}$$

where $H'^{[\psi]}_{BC}$ contains the terms appearing in $H^{[\psi]}_{BC}$ as well as a linear combination of the terms appearing in $H^{[f]}_{AB}$ with terms quadratic in f_{AB} and $\hat{\psi}_{ABCD}$ and terms cubic in f_{AB} and ϕ_{AB}.

Eqs. (63a), (63c) together with (64) constitute a symmetric hyperbolic system of equations for the components $\nu_{(ABCD)}$, ν_{AB} and ν of the spinorial field $\hat{\psi}_{ABCD}$.

7.5. The decomposition of the physical Cotton–York tensor

In vacuum spacetimes the physical Cotton–York tensor $\tilde{Y}_{ijk}$ vanishes. Thus, it does not appear in the Bianchi equations (15), (14e), (16a), (16b), (17a) and (17b). In the case of trace-free matter $\tilde{Y}_{ijk}$ is, in general, non-vanishing and carries information about the physical fields in both (14e) and (17b). The field $\tilde{Y}_{ijk}$ can be written in terms of *unphysical variables* as:

$$\begin{aligned}\tilde{Y}_{ijk} &\equiv \tilde{\nabla}_{[i}\tilde{T}_{j]k}\\ &= \Theta^2\left(\hat{\nabla}_{[i}T_{j]k} + 3b_{[i}T_{j]k} - 2f_{[i}T_{j]k} - g_{k[i}T_{j]m}g^{mn}b_n\right).\end{aligned}$$

Substituting Eqs. (52) and (55), recalling that $\eta_{AA'BB'} \equiv \epsilon_{AB}\epsilon_{A'B'}$ is the spinorial counterpart of η_{ij} and assuming that $\hat{\psi}_{AA'BC} = \hat{\nabla}_{AA'}\phi_{BC}$ i.e. $\hat{\omega}_{AA'BC} = 0$ one obtains

$$\begin{aligned}\tilde{Y}_{AA'BB'CC'} = \frac{1}{2}\Theta^2\big(&\hat{\psi}_{AA'BC}\bar{\phi}_{B'C'} + \bar{\hat{\psi}}_{AA'B'C'}\phi_{BC} - \hat{\psi}_{BB'AC}\bar{\phi}_{A'C'} - \bar{\hat{\psi}}_{BB'A'C'}\phi_{AC}\\ &+ (3b_{AA'} - 2f_{AA'})\phi_{BC}\bar{\phi}_{B'C'} - (3b_{BB'} - 2f_{BB'})\phi_{AC}\bar{\phi}_{A'C'} - \epsilon_{CA}\epsilon_{C'A'}\phi_{BE}\bar{\phi}_{B'E'}b^{EE'} + \epsilon_{CB}\epsilon_{C'B'}\phi_{AE}\bar{\phi}_{A'E'}b^{EE'}\big).\end{aligned}$$

From the latter one readily finds that

$$\begin{aligned}\tilde{Y}_{ABCC'} &\equiv \frac{1}{2}\tilde{Y}_{AQ'B}{}^{Q'}{}_{CC'},\\ &= \frac{1}{2}\Theta^2\big(\hat{\psi}_{(A|Q'|B)C}\bar{\phi}^{Q'}{}_{C'} + \phi_{C(A}\bar{\psi}_{B)Q'}{}^{Q'}{}_{C'} + 3\Theta^{-1}\phi_{C(A}d_{B)Q'}\bar{\phi}^{Q'}{}_{C'} - 2\phi_{C(A}f_{B)Q'}\bar{\phi}^{Q'}{}_{C'} - \epsilon_{C(A}\phi_{B)E}\bar{\phi}_{C'E'}b^{EE'}\big).\end{aligned}$$

The corresponding unprimed version $\tilde{Y}_{ABCD}$ can be written entirely in terms of ϕ_{AB}, $\phi^{\dagger}_{AB}$, Θ, $\dot{\Theta}$ and d_{AB}, by recalling that from Eq. (54) it follows that

$$\bar{\phi}_{A'B'} = \tau^{A}{}_{A'}\tau^{B}{}_{B'}\phi^{\dagger}_{AB}.$$

These explicit expressions will not be required in the subsequent discussion. Important to note is that due to the presence of an overall factor of Θ^2 in $\tilde{Y}_{ABCC'}$, the term $\Theta^{-1}\tilde{Y}_{ABCD}$ in Eq. (49) is formally regular at the points where $\Theta = 0$.

7.6. Behaviour of the field variables and the zero quantities under gauge changes

Before discussing the structural properties of the reduced conformal Einstein–Maxwell equations in Section 8 and the propagation of the constraints in Section 9 we would like to briefly highlight the topic of gauge choice and gauge invariance. The field variables and the zero quantities that have been introduced in previous chapters have all been defined for a specific choice of Weyl connection $\hat{\nabla}$, metric $g_{\mu\nu}$, frame $\{e_k\}$ and spinor dyad δ_A related by (11), (7), (8b). Implied in their definitions are the transformation rules under gauge change. These rules have either been explicitly given – e.g. (10b), (51), (56) – or can be derived directly from these rules and the extended conformal field equations (15). Therefore we refrain from listing them again.

However we would like to highlight that as a particular consequence of these transformation rules, it follows that the various zero quantities defined in earlier chapters are conformally covariant. Thus if they vanish in one gauge they will also vanish in another. This will be used in the discussion of the propagation of the constraint in Section 9.

8. Structural properties of the reduced conformal Einstein–Maxwell equations

We summarise the analysis of Sections 6 and 7 in a form suitable for the applications that will be given in the sequel. We introduce the notation

$$\boldsymbol{\upsilon} \equiv \left(e^{\underline{s}}_{AB}, \hat{\Gamma}_{ABCD}, \hat{P}_{ABCD}, \zeta, \zeta_{AB}\right),$$
$$\boldsymbol{\phi} \equiv (\phi_{ABCD}),$$
$$\boldsymbol{\varphi} \equiv (\phi_{AB}),$$
$$\boldsymbol{\psi} \equiv (\psi_{ABCD}),$$

where it is understood that $\boldsymbol{\upsilon}$, $\boldsymbol{\phi}$, $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ contain only the independent irreducible components of the respective spinors. Let also

$$\mathbf{u} \equiv (\boldsymbol{\upsilon}, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\psi}).$$

In terms of these quantities the propagation equations (47a)–(47d) can be written as:

$$\partial_\tau \boldsymbol{\upsilon} = \mathbf{K}\boldsymbol{\upsilon} + \mathbf{Q}(\boldsymbol{\upsilon}, \boldsymbol{\upsilon}) + \mathbf{R}(\boldsymbol{\varphi}, \boldsymbol{\psi}) + \mathbf{T}(\boldsymbol{\phi}, \boldsymbol{\psi}, \boldsymbol{\upsilon}) + \mathbf{L}\boldsymbol{\phi}, \tag{65}$$

where $\mathbf{K}$ denotes a matrix with constant coefficients, $\mathbf{Q}(\boldsymbol{\upsilon}, \boldsymbol{\upsilon})$, $\mathbf{R}(\boldsymbol{\varphi}, \boldsymbol{\psi})$ bilinear vector value functions with constant coefficients and $\mathbf{T}(\boldsymbol{\phi}, \boldsymbol{\psi}, \boldsymbol{\upsilon})$ a trilinear vector valued function with constant coefficients. On the other hand, $\mathbf{L}$ is a linear matrix-valued function with coefficients depending on the coordinates. Eqs. (49), (59) and (63a)–(63c) can be written in the form

$$\left(\sqrt{2}\mathbf{E}_{5\times5} + \mathbf{A}^0_{5\times5}\right)\partial_\tau\boldsymbol{\phi} + \mathbf{A}^{\underline{r}}_{5\times5}\partial_{\underline{r}}\boldsymbol{\phi} = \mathbf{B}(\boldsymbol{\upsilon})\boldsymbol{\phi} + \mathbf{M}(\boldsymbol{\psi}, \boldsymbol{\varphi}) + \mathbf{N}(\boldsymbol{\varphi}, \boldsymbol{\varphi}), \tag{66a}$$

$$\left(\sqrt{2}\mathbf{E}_{3\times3} + \mathbf{A}^0_{3\times3}\right)\partial_\tau\boldsymbol{\varphi} + \mathbf{A}^{\underline{r}}_{3\times3}\partial_{\underline{r}}\boldsymbol{\varphi} = \mathbf{C}(\boldsymbol{\upsilon})\boldsymbol{\varphi}, \tag{66b}$$

$$\left(\sqrt{2}\mathbf{E}_{9\times9} + \mathbf{A}^0_{9\times9}\right)\partial_\tau\boldsymbol{\psi} + \mathbf{A}^{\underline{r}}_{9\times9}\partial_{\underline{r}}\boldsymbol{\psi} = \mathbf{D}(\boldsymbol{\upsilon})\boldsymbol{\psi} + \mathbf{U}(\boldsymbol{\upsilon}, \boldsymbol{\varphi}) + \mathbf{V}(\boldsymbol{\upsilon}, \boldsymbol{\phi}) + \mathbf{W}(\boldsymbol{\upsilon}, \boldsymbol{\upsilon}, \boldsymbol{\phi}), \tag{66c}$$

where $E_{3\times3}, E_{5\times5}, E_{8\times8}$ denote, respectively, the $3\times3, 5\times5$ and 8×8 identity matrices, while $A^{\underline{s}}_{3\times3}, A^{\underline{s}}_{5\times5}, A^{\underline{s}}_{9\times9}$, $\underline{s} = 0, \ldots, 3$ are 3×3, 5×5 and 8×8 Hermitian matrices depending on the coordinates. On the other hand $\mathbf{B}(\boldsymbol{\upsilon})$, $\mathbf{C}(\boldsymbol{\upsilon})$, $\mathbf{D}(\boldsymbol{\upsilon})$ denote constant matrix-valued linear function of the entries of $\boldsymbol{\upsilon}$, while $\mathbf{M}(\boldsymbol{\psi}, \boldsymbol{\varphi})$, $\mathbf{N}(\boldsymbol{\varphi}, \boldsymbol{\varphi})$, $\mathbf{U}(\boldsymbol{\upsilon}, \boldsymbol{\varphi})$, $\mathbf{V}(\boldsymbol{\upsilon}, \boldsymbol{\phi})$ denote bilinear functions with coordinate dependent coefficients. Finally, $\mathbf{W}(\boldsymbol{\upsilon}, \boldsymbol{\upsilon}, \boldsymbol{\phi})$ is a trilinear function. The Hermitian matrices

$$\sqrt{2}\mathbf{E}_{3\times3} + \mathbf{A}^0_{3\times3}, \qquad \sqrt{2}\mathbf{E}_{5\times5} + \mathbf{A}^0_{5\times5}, \qquad \sqrt{2}\mathbf{E}_{9\times9} + \mathbf{A}^0_{9\times9}$$

imply real symmetric matrices if one decomposes the entries of $\boldsymbol{\phi}$, $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ into real and imaginary parts. Hence, (65) and (66a)–(66c) give rise to a symmetric hyperbolic system for $\mathbf{u}$.

9. Propagation of the constraints

In this section we show that the conformal constraint equations propagate by virtue of the conformal evolution equations, thus implying a solution to the whole conformal field equations. More precisely,

Lemma 4. *Let $\mathcal{V}$ be an open subset of $\mathcal{S}$ and let $\mathcal{U}$ be an open neighbourhood in $\mathcal{S} \times [0, \infty)$. Assume that the unknowns $(\boldsymbol{\upsilon}, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\psi})$ given on $\mathcal{U}$ represent a smooth solution of the reduced equations (65), (66a)–(66c) for data on $\mathcal{V}$ satisfying the Einstein–Maxwell conformal constraint equations. Let $g_{\mu\nu}$ be the metric for which the frame obtained from the unknowns $\boldsymbol{\upsilon}$ is orthonormal and let $\mathcal{D}^+(\mathcal{V}) \subset \mathcal{U}$ be the future domain of dependence of $\mathcal{V}$ with respect to $g_{\mu\nu}$. Then the conformal Einstein–Maxwell field equations*

$$\hat{\Sigma}_{AA'}{}^{BB'}{}_{CC'} = 0, \qquad \hat{\Xi}_{ABCC'DD'} = 0, \qquad \hat{\Delta}_{AA'BB'CC'} = 0, \qquad \hat{\Lambda}_{AA'BBCC'} = 0,$$
$$\hat{\omega}_{A'A} = 0, \qquad \hat{\omega}_{AA'BC} = 0, \qquad \hat{\omega}_{AA'BB'CD} = 0,$$

are satisfied on $\mathcal{D}^+(\mathcal{V})$ by the fields $(\boldsymbol{\upsilon}, \boldsymbol{\phi}, \boldsymbol{\varphi}, \boldsymbol{\psi})$. Furthermore, the metric

$$\tilde{g}_{\mu\nu} = \Theta^{-2} g_{\mu\nu}$$

is a solution to the (physical) Einstein–Maxwell field equations on

$$\{p \in \mathcal{D}^+(\mathcal{V}) \mid \Theta(p) \neq 0\}.$$

The proof of this result follows a combination of the techniques discussed in [2] and in [3]. We divide the proof in several steps. In order to ease the presentation, in the subsequent discussion we will use tensorial notation whenever possible. The discussion of the propagation of the constraints follows the lines of the arguments given in [2,3]. This argument requires long computations to obtain a complicated system of subsidiary equations for the various zero quantities involved in the extended conformal field equations. Since the argument is not particularly illuminating and for the sake of the presentation, we follow the spirit of previous sections of the article and present a schematic description of the procedure.

(a) *Propagation of the constraints as a consequence of the subsidiary equations.*

In the subsequent discussion it will be assumed that the reduced conformal field equations (38)–(39) are satisfied. Furthermore, it will be assumed that the gauge conditions (37) hold.

We define the following zero quantities associated to the conformal gauge:

$$\delta_k \equiv b_k - f_k - \Upsilon_k,$$
$$\gamma_{ij} \equiv \frac{1}{2}\tilde{T}_{ij} - \hat{P}_{ij} - \hat{\nabla}_i b_j - \frac{1}{2}S_{ij}{}^{kl} b_k b_l,$$
$$\varsigma_{ij} \equiv \hat{P}_{[ij]} - \hat{\nabla}_{[i} f_{j]} - \frac{1}{2}S_{[ij]}{}^{kl} f_k f_l,$$

where $\tilde{T}_{ij}$ is given by the matter model under consideration. Under the assumption that Eqs. (38)–(39) and (37) are satisfied, a computation along the lines discussed in [3] shows that

$$\partial_\tau \delta_k = H^{[\delta]}_k,$$
$$\partial_\tau \gamma_{ij} = H^{[\gamma]}_{ij},$$
$$\partial_\tau \varsigma_{ij} = H^{[\varsigma]}_{ij},$$

where $H^{[\delta]}_k$ is an homogeneous expression in the zero-quantities δ_k, γ_{ij}, ς_{ij} and $\hat{\Sigma}_{i}{}_k$; $H^{[\gamma]}_{ij}$ is an homogeneous expression in γ_{ij}; finally $H^{[\varsigma]}_{ij}$ is homogeneous in $\hat{\Xi}^k{}_{lij}$. The explicit form of the *source terms* in the above equations and the evolution equations for the other zero quantities to be discussed in the sequel will not be required in the following discussion.

The discussion of the propagation equations for the zero quantities $\hat{\Sigma}_i{}^j{}_k$, $\hat{\Xi}^k{}_{lij}$ and $\hat{\Delta}_{kij}$ follow the model of Eq. (46) discussed in Section 6.3. In this case a lengthy computation shows that

$$\mathcal{L}_\tau \hat{\Sigma}_i{}^j{}_k = H^{[\Sigma]}{}_i{}^j{}_k,$$
$$\mathcal{L}_\tau \hat{\Xi}^k{}_{lij} = H^{[\Xi]k}{}_{lij},$$
$$\mathcal{L}_\tau \hat{\Delta}_{kij} = H^{[\Delta]}{}_{kij},$$

where $H^{[\Sigma]}{}_i{}^j{}_k$ is a homogeneous expression in the zero-quantities $\hat{\Sigma}_{i}{}_k$ and $\hat{\Xi}^k{}_{lij}$; $H^{[\Xi]k}{}_{lij}$ is an homogeneous expression of the geometrical zero-quantities $\hat{\Xi}^k{}_{lij}$, $\hat{\Delta}_{kij}$, $\hat{\Sigma}_i{}^j{}_k$, $\hat{\Lambda}_{kij}$ and the gauge zero-quantity δ_k; finally $H^{[\Delta]}{}_{kij}$ is homogeneous in the geometrical zero-quantities $\hat{\Delta}_{kij}$, $\hat{\Sigma}_{i}{}_k$, $\hat{\Lambda}_{kij}$, the gauge zero-quantities γ_{ij} and δ_k, and the tensorial counterpart of the spinorial matter zero-quantities $\hat{\omega}_{A'A}$, $\hat{\omega}_{AA'BC}$, $\hat{\omega}_{AA'BB'CD}$.

The construction of a propagation equation for the Bianchi equation $\hat{\Lambda}_{kij}$ is slightly different. Following the discussion in [3] one considers the quantity $\hat{\nabla}^k \hat{\Lambda}_{kij}$. A lengthy manipulation using the definition of $\hat{\Lambda}_{kij}$ and symmetries of the Weyl tensor shows that

$$\hat{\nabla}^k \hat{\Lambda}_{kij} = H^{[\Lambda]}_{ij},$$

where $H^{[\Lambda]}_{ij}$ depends homogeneously on the geometrical zero-quantities $\hat{\Xi}^k{}_{lij}$, $\hat{\Sigma}_i{}^j{}_k$, the gauge zero-quantity ς_{ij}, and the tensorial counterpart of the matter zero-quantities $\hat{\omega}_{A'A}$, $\hat{\omega}_{AA'BC}$, $\hat{\omega}_{AA'BB'CD}$. Now, the spinorial counterpart of $\hat{\nabla}^k \hat{\Lambda}_{kij}$ is given by $\hat{\nabla}^{PP'} \hat{\Lambda}_{P'PBC}$. A space-spinor decomposition shows that the components of $\hat{\Lambda}_{A'ABC}$ satisfy a symmetric hyperbolic equation. In particular, for the Bianchi constraint one has that

$$\hat{\mathcal{P}} \hat{\Lambda}_P{}^P{}_{AB} - \hat{\mathcal{D}}^Q{}_{(A} \hat{\Lambda}_P{}^P{}_{B)Q} = H^{[\Lambda]}_{AB},$$

where $H^{[\Lambda]}_{AB}$ has the same dependence on zero-quantities as $H^{[\Lambda]}_{ij}$.

Finally, for the constraints associated to the matter equations (the Maxwell field) one has that an analogous procedure to the one described in [2] renders also symmetric hyperbolic equations for the components of $\hat{\omega}_{A'A}$, $\hat{\omega}_{AA'BC}$, $\hat{\omega}_{AA'BB'CD}$ which are homogeneous in the matter zero-quantities themselves and in the geometric zero-quantities.

Summarising: in the gauge (41) and as a consequence of the reduced equations the geometrical zero-quantities

$$\hat{\Sigma}_{AA'}{}^{BB'}{}_{CC'}, \qquad \hat{\Xi}_{ABCC'DD'}, \qquad \hat{\Delta}_{ABCC'}, \qquad \hat{\Lambda}_{A'ABC},$$

together with the Maxwell zero-quantities

$$\hat{\omega}_{A'A}, \qquad \hat{\omega}_{AA'BC}, \qquad \hat{\omega}_{AA'BB'CD}$$

and the gauge zero-quantities

$$\delta_{AA'}, \qquad \gamma_{AA'BB'}, \qquad \varsigma_{AA'BB'}$$

form a symmetric hyperbolic system which is homogeneous in the zero quantities themselves. Accordingly, if the zero-quantities vanish on $\mathcal{V} \subset \mathcal{S}$, then the zero-quantities vanish on $\mathcal{D}^+(\mathcal{V})$. Hence one has a solution to the conformal Einstein–Maxwell field equations of $\mathcal{D}^+(\mathcal{V})$.

(b) *A solution to the conformal Einstein–Maxwell field equations implies a solution to the physical Einstein–Maxwell field equations.*

Assume now that on $\mathcal{U}$ one has a solution to the extended Einstein–Maxwell conformal field equations—that is,

$$\hat{\Sigma}_{AA'}{}^{BB'}{}_{CC'} = 0, \qquad \hat{\Xi}_{ABCC'DD'} = 0, \qquad \hat{\Delta}_{AA'BB'CC'} = 0, \qquad \hat{\Lambda}_{A'BCD} = 0,$$
$$\hat{\omega}_{A'A} = 0, \qquad \hat{\omega}_{AA'BB'CD} = 0.$$

Assume also that the additional zero quantities satisfy

$$\hat{\omega}_{AA'BC} = 0, \qquad \delta_{AA'} = 0, \qquad \gamma_{AA'BB'} = 0, \qquad \varsigma_{AA'BB'} = 0.$$

The solution to the reduced conformal field equations provides, in particular, fields

$$e_{AA'}, \qquad f_{AA'} \qquad \hat{\Gamma}_{AA'}{}^{BB'}{}_{CC'} \quad \text{on } \mathcal{U},$$

where in this discussion the 1-form $f_{AA'}$ is defined via

$$f_{AA'} \equiv \hat{\Gamma}_{AA'}{}^{Q}{}_{Q}$$

as $\hat{\Sigma}_{AA'}{}^{BB'}{}_{CC'} = 0$. The connection coefficients $\hat{\Gamma}_{AA'}{}^{BB'}{}_{CC'}$ give rise to a torsion-free connection $\hat{\nabla}$. Motivated by the relation (11) one can use the frame $e_{AA'}$ and the frame metric $\eta_{AA'BB'} \equiv \epsilon_{AB}\epsilon_{A'B'}$ to construct a metric $g_{\mu\nu}$. By construction $e_{AA'}(\epsilon_{CD}) = 0$ so that

$$\hat{\nabla}_{AA'}\epsilon_{CD} = -\hat{\Gamma}_{AA'}{}^{Q}{}_{Q}\epsilon_{CD} = -f_{AA'}\epsilon_{CD},$$

which is the spinorial counterpart of

$$\hat{\nabla}_{\mu} g_{\nu\lambda} = -2f_{\mu} g_{\nu\lambda}.$$

Thus, $\hat{\nabla}$ is a Weyl connection for the metric $g_{\mu\nu}$. Motivated by (12), one defines the connection ∇ with connection coefficients

$$\Gamma_{AA'}{}^{CC'}{}_{BB'} \equiv \hat{\Gamma}_{AA'}{}^{CC'}{}_{BB'} - S_{AA'BB'}{}^{CC'PP'} f_{PP'}$$

then

$$\nabla_{AA'}\epsilon_{CD} = 0,$$

so that $\nabla_{\mu} g_{\nu\lambda} = 0$—that is, ∇ is a metric connection. Using the invariance of the torsion under change of connection—cf. Eq. (13)—it follows that ∇ is torsion-free. Thus, because of uniqueness, ∇ must be the Levi-Civita connection of $g_{\mu\nu}$.

Now, from

$$\hat{\Xi}_{ABCC'DD'} \equiv \hat{r}_{ABCC'DD'} - \hat{R}_{ABCC'DD'} = 0,$$

the fields $\hat{P}_{AA'BB'}$ and ϕ_{ABCD} on $\mathcal{U}$ obtained as a solution of the reduced conformal field equations can be identified, respectively, with the Schouten and Weyl spinors of the Weyl connection $\hat{\nabla}$—recall that the decomposition in terms of irreducible components is unique. Due to conformal invariance, the Weyl tensor of the Weyl connection $\hat{\nabla}$ is also the Weyl tensor of the Levi-Civita connection ∇.

Motivated by the rescaling (2) we use the transformation rule (8a) to define a *physical* connection $\tilde{\nabla}$. From $\delta_{AA'} = 0$ one has that

$$b_{AA'} = \Upsilon_{AA'} + f_{AA'},$$

and accordingly $\tilde{\nabla}$ is the Levi-Civita connection of the metric $\tilde{g}_{\mu} \equiv \Theta^{-2} g_{\mu\nu}$. Using the transformation rule (9b) one finds that the physical Schouten spinor is given by

$$\tilde{P}_{AA'BB'} \equiv \hat{P}_{AA'BB'} + \hat{\nabla}_{AA'}\left(f_{BB'} + \Upsilon_{BB'}\right) + \frac{1}{2} S^{PP'QQ'}{}_{AA'BB'}\left(f_{PP'} + \Upsilon_{PP'}\right)\left(f_{QQ'} + \Upsilon_{QQ'}\right).$$

Note that since $\varsigma_{AA'BB'} = 0$, one has that

$$\tilde{P}_{AA'BB'} - \tilde{P}_{BB'AA'} = 0, \qquad \tilde{P}_{[ij]} = 0.$$

Furthermore, from $\gamma_{AA'BB'} = 0$ one finds that

$$\tilde{P}_{AA'BB'} = \frac{1}{2}\tilde{T}_{AA'BB'}, \qquad \tilde{P}_{ij} = \frac{1}{2}\tilde{T}_{ij}. \tag{67}$$

From the field equations $\hat{\omega}_{A'A} = 0$, $\hat{\omega}_{AA'BB'CC'} = 0$ and the constraint $\hat{\omega}_{AA'BC} = 0$, one has that $\tilde{\phi}_{AB}$ satisfies the physical Maxwell equations. Thus, $\tilde{T}_{AA'BB'}$ defined by

$$\tilde{T}_{AA'BB'} = \tilde{\phi}_{AB}\tilde{\phi}_{A'B'},$$

is the energy–momentum tensor of the Maxwell field and the equations given in (67) are equivalent to the Einstein–Maxwell field equations.

10. A first application: stability of Einstein–Maxwell de Sitter-like spacetimes

The use of a gauge based on the conformal curves described in Section 5 allows to directly transcribe the analysis of the conformal boundary for vacuum de Sitter-like spacetimes to the case of Einstein–Maxwell de Sitter-like spacetimes.

For the de Sitter-like spacetimes one can formulate two slightly different Cauchy initial value problems: one where initial data is prescribed on a standard Cauchy hypersurface, and a second one where the data is prescribed on one portion of the conformal boundary—say, past null infinity. The de Sitter-like spacetimes that will be considered have Cauchy slices with the topology of $\mathbb{S}^3$. The construction of suitable coordinate systems and a frame vectors this type of configurations has been discussed in detail in [7,8].

10.1. Structure of the conformal boundary

Following the general ideas of [7], here we present a brief discussion of the structure of the conformal boundary of de Sitter-like Einstein–Maxwell spacetimes.

10.1.1. Standard Cauchy problem

If the initial hypersurface $\mathcal{S}$ is a standard Cauchy hypersurface one has that

$$\Theta = \Theta_* \left(1 + \tau\langle b_*, v_*\rangle + \frac{1}{2}\tau^2\left(\tilde{\lambda}\Theta_*^{-2} + \frac{1}{2}g^\sharp(b_*, b*)\right)\right),$$

for some $\Theta_* \neq 0$—cf. Eqs. (32)–(33) of Lemma 2. The conformal factor vanishes at

$$\tau_\pm = \frac{-2\Theta_*\langle d, v\rangle_* \pm 2\Theta_*\sqrt{|2\tilde{\lambda} + h^\sharp(d, d)_*|}}{2\tilde{\lambda} + g^\sharp(d, d)_*}.$$

One has then that

$$\mathscr{I}^\pm = \{\tau_\pm\} \times \mathcal{S}. \tag{68}$$

Furthermore, $\nabla_k\Theta\nabla^k\Theta = -2\tilde{\lambda}$, so that both components of null infinity are space-like.

10.1.2. Cauchy problem on past null infinity

In the case of an initial value problem prescribed on null infinity, one has that $\Theta_* = 0$ so that

$$\Theta = \langle d, v\rangle_*\tau + \frac{1}{2}\ddot{\Theta}_*\tau^2.$$

Using Lemma 3 one finds that $g^\sharp(d, d)_* = -2\tilde{\lambda}$ and, if one sets $d_* = (\nabla\Theta)_*$, that

$$d_k(\tau) = \left(\sqrt{-2\tilde{\lambda}} + \ddot{\Theta}_*\tau, 0, 0, 0\right).$$

The conformal factor vanishes at

$$\mathscr{I}^- = \{\tau = 0\} \times \mathcal{S}, \qquad \mathscr{I}^+ = \left\{\tau = -2\dot{\Theta}_*/\ddot{\Theta}_*\right\} \times \mathcal{S}. \tag{69}$$

Note that the location of $\mathscr{I}^+$ is determined by the free data $\ddot{\Theta}_*$.

10.2. Stability of Einstein–Maxwell de Sitter-like spacetimes

Combining the *a priori* knowledge on the structure of the conformal boundary discussed in the previous sections with the structural properties of the reduced equations (65), (66a)–(66c) discussed in Section 8, Lemma 4 on the propagation of the constraints, and Kato's existence and stability theorems for symmetric hyperbolic systems [14–16] one obtains the following existence and stability result for de Sitter-like Einstein–Maxwell spacetimes. The proof is identical to that in [7,8] and it is omitted. Let in what follows $\mathring{\mathbf{u}}$ denote the solution to the reduced equations (65), (66a)–(66c) corresponding to the (vacuum) de Sitter spacetime.

Theorem 1. *Let $\mathbf{u}_0 = \mathring{\mathbf{u}}_0 + \breve{\mathbf{u}}_0$ be Einstein–Maxwell Cauchy (standard or at past null infinity) data for a de Sitter-like spacetime. There exists $\varepsilon > 0$ such that if $\breve{\mathbf{u}}_0$ is sufficiently small, then there exists on $[\tau_-, \tau_+] \times \mathcal{S}$ a unique smooth solution $\mathbf{u} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}$ to the conformal propagation equations (65), (66a)–(66c) such that the associated congruence of conformal curves contains no conjugate points in $[\tau_-, \tau_+]$. The field $\mathbf{u}$ implies a smooth solution to the Einstein–Maxwell field equations with positive cosmological constant for which the sets $\mathscr{I}^\pm$ defined by (68) – in the standard Cauchy problem – or by (69) – in the Cauchy problem with data at null infinity – represent past and future null infinity.*

Remark. Note that this stability result for Einstein–Maxwell spacetimes is given with respect to a vacuum reference spacetime.

11. A second application: stability of Einstein–Maxwell radiative spacetimes

As a second example of our approach, we obtain a generalisation of the stability results for purely radiative spacetimes discussed in [8]. In contrast to the stability proof for de Sitter-like Einstein–Maxwell spacetimes, in this case the reference solution has a non-vanishing electromagnetic field. For the sake of conciseness most of the technical details are omitted and we only remark on those aspects of the analysis that differ from the treatment for vacuum spacetimes given in [8].

11.1. Einstein–Maxwell initial data sets with vanishing mass

In what follows, a static solution to the Einstein–Maxwell solutions (an electrostatic solution) will be understood to be a triple $(\tilde{h}_{\alpha\beta}, \tilde{\Phi}, \tilde{\Psi})$, solving the *electrostatic field equations*. The (negative definite) Riemannian 3-metric $\tilde{h}_{\alpha\beta}$ is the metric of the quotient manifold, and Φ, Ψ denote, respectively, the gravitational and electric potentials. Any static, asymptotically flat solution to the Einstein–Maxwell equations admits an analytic compactification of a neighbourhood of spatial infinity i—see [17]. The triple $(\tilde{h}_{\alpha\beta}, \tilde{\Phi}, \tilde{\Psi})$ can be suitably rescaled to render another triple $(h_{\alpha\beta}, \Phi, \Psi)$ which is analytic in a neighbourhood $\mathcal{B}_a(i)$ and solves the *conformal electrostatic field equations*. Any such triple gives rise to a solution $(\bar{h}_{\alpha\beta}, \bar{\Omega}, \bar{E}_\alpha)$ of the (conformally rescaled) time symmetric Einstein–Maxwell constraints

$$\bar{r} = 2\bar{\Omega}^2 \bar{E}_\alpha \bar{E}^\alpha + 8\bar{\Omega}^{1/2} \bar{D}_\alpha \bar{D}^\alpha (\Omega^{-1/2}),$$
$$\bar{D}_\alpha \bar{E}^\alpha = 0$$

with vanishing mass and charge—here $\bar{D}$ and $\bar{r}$ denote, respectively, the Levi-Civita connection and Ricci scalar of the metric $\bar{h}_{\alpha\beta}$; the tensor $\bar{E}_\alpha$ is the electric field. From $(\bar{h}_{\alpha\beta}, \bar{\Omega}, \bar{E}_\alpha)$ one can construct initial data for the extended conformal field equations. In particular, data for the Schouten and Weyl tensors are given, respectively by the expressions

$$\bar{P}_{\alpha\beta} = -\bar{\Omega}^{-1}\mathcal{C}\left(\bar{D}_\alpha \bar{D}_\beta \bar{\Omega}\right) - \bar{\Omega}^2 \mathcal{C}\left(\bar{E}_\alpha \bar{E}_\beta\right) + \frac{1}{12}\left(\bar{r} - 2\bar{\Omega}^2 \bar{E}_\alpha \bar{E}^\alpha - 8\bar{\Omega}^{1/2} \bar{D}_\alpha \bar{D}^\alpha (\bar{\Omega}^{-1/2})\right) h_{\alpha\beta},$$
$$\bar{d}_{\alpha\beta} = \bar{\Omega}^{-2}\mathcal{C}(\bar{D}_\alpha \bar{D}_\beta \bar{\Omega}) + \bar{\Omega}^{-1}\mathcal{C}(\bar{r}_{\alpha\beta}) + \bar{\Omega}\mathcal{C}(\bar{E}_\alpha \bar{E}_\beta),$$

which can be shown to be analytic in $\mathcal{B}_a(i)$. In these last expressions, $\mathcal{C}$ denotes the trace-free part of the tensor in parenthesis.

11.2. Construction of a reference radiative Einstein–Maxwell spacetime

Let $(\bar{h}_{\alpha\beta}, \bar{\Omega}, \bar{E}_\alpha)$ on $\bar{\mathcal{S}}$ be one of the solutions to the time symmetric conformal constraint discussed in the previous subsection. For the present purposes it will be convenient to consider a conformal factor $\bar{\Omega}$ which is negative—it is obtained by making the obvious sign changes in the relevant equations. By construction $\bar{\Omega}$ satisfies the following *asymptotic flatness conditions*:

$$\bar{\Omega} < 0 \quad \text{on } \bar{\mathcal{S}} \setminus i, \qquad \bar{\Omega}(i) = 0, \qquad D_\mathcal{A}\bar{\Omega}(i) = 0, \qquad D_\mathcal{A} D_\mathcal{B} \bar{\Omega}(i) = 2h_{\mathcal{A}\mathcal{B}}(i). \tag{70}$$

We work in a suitably small neighbourhood, $\mathcal{B}_a(i) \subset \bar{\mathcal{S}}$ such that all the statements made in the sequel make sense. We use the coordinates $x^\mathcal{A}$, $\mathcal{A} = 1, 2, 3$ centred at i and consider the following initial data for a congruence of conformal curves:

$$\bar{\tau}_* = 0, \qquad \dot{x}^\mu = \bar{n}^\mu, \qquad \bar{\Theta}_* = \bar{\Omega}, \qquad \dot{\bar{\Theta}}_* = 0, \qquad \bar{d}_* \equiv \bar{\Theta}_* \bar{b}_* = (\mathrm{d}\bar{\Theta})_*. \tag{71}$$

The coordinates $x^\mathcal{A}$ are extended off $\bar{\mathcal{S}}$ by dragging along the congruence of conformal curves to obtain generalised conformal Gaussian coordinates. It can be readily verified that $\ddot{\bar{\Theta}} > 0$ on $\bar{\mathcal{S}}$. It follows that along each conformal curve the conformal factor $\bar{\Theta}$ is given by

$$\bar{\Theta}(\bar{\tau}) = \bar{\Omega} + \frac{1}{2}\ddot{\bar{\Theta}}_* \bar{\tau}^2 = \bar{\Omega}\left(1 - \frac{\bar{\tau}^2}{\bar{\omega}^2}\right), \quad \bar{\omega} \equiv \sqrt{\frac{2\bar{\Omega}}{-\ddot{\bar{\Theta}}_*}}, \qquad \bar{\omega}(i) = 0. \tag{72}$$

Define the conformal boundary, $\bar{\mathscr{I}}$, in a natural way as the locus of points in the development of the data on $\bar{\mathcal{S}}$ for which $\bar{\Theta} = 0$ and $\mathrm{d}\bar{\Theta} \neq 0$. It is easy to see that a conformal curve with data given by (71) passes through $\bar{\mathscr{I}}$ whenever $\bar{\tau} = \pm\bar{\omega}$.

Having located the conformal boundary for the evolution of data on $\mathcal{B}_a(i)$ for the Einstein–Maxwell system, one can discuss now the existence of solutions to the propagation system given by (65), (66a)–(66c). For this we extend the data on $\mathcal{B}_a(i)$ to data on the whole of $\bar{\mathcal{S}} \simeq \mathbb{S}^3$ in the way discussed in [1,7]. Using the same methods as in [8] and Lemma 4 one obtains the following local result:

Theorem 2. *Given radiative data for the conformal Einstein–Maxwell equations, there exist a $\bar{T}_0 > 0$ and on $\mathring{\mathcal{M}} \equiv [-\bar{T}_0, \bar{T}_0] \times \bar{\mathcal{S}}$ a unique smooth solution $\mathbf{u}$ to the propagation equations (65), (66a)–(66c). The solution $\mathbf{u}$ implies a solution to the conformal Einstein–Maxwell equations on*

$$\mathcal{M} \equiv \mathring{\mathcal{M}} \cap I^-(i) \subset \mathcal{D}(\mathcal{B}_a(i)).$$

The spacetime $(\mathcal{M} \setminus \bar{\mathscr{I}}, \mathring{g}_{\mu\nu})$ implied by the solution to the conformal Einstein–Maxwell field equations is conformally related to an Einstein–Maxwell spacetime, $(\mathcal{M} \setminus \bar{\mathscr{I}}, \bar{\Theta}^{-2}\mathring{g}_{\mu\nu})$, with vanishing cosmological constant. The spacetime $(\mathcal{M} \setminus \bar{\mathscr{I}}, \Theta^{-2}\mathring{g}_{\mu\nu})$ is a radiative spacetime for which the set $\bar{\mathscr{I}}^+$ corresponds to its future null infinity, while the point $i^+ = (0, i) \in \{0\} \times \bar{\mathcal{S}}$ is its future timelike infinity.

The conformal affine parameter $\bar{\tau}$ defines, in a natural way, a foliation of the manifold $\mathring{\mathcal{M}}$. Let $\mathcal{S}_{\bar{\tau}}$ denote the surfaces of constant $\bar{\tau}$. For fixed $\bar{\tau}$ one has that $\mathcal{S}_{\bar{\tau}}$ is diffeomorphic to $\mathbb{S}^3$. Let $\bar{\tau}_0 \in (0, \bar{T}_0)$ and define

$$\mathcal{S}_0 \equiv \{-\bar{\tau}_0\} \times \mathbb{S}^3, \qquad \mathcal{Z} \equiv \{p \in \mathcal{S}_0 | \bar{\Theta} = 0\}.$$

The set $\mathcal{S}_0$ intersects null infinity in a *hyperboloidal way*. Furthermore, let

$$\mathring{\mathcal{H}} \equiv \{p \in \mathcal{S}_0 | \bar{\Theta} > 0\}.$$

Define

$$\tau \equiv \bar{\tau} + \bar{\tau}_0, \qquad \mathring{\Theta}(\tau) \equiv \bar{\Theta}(\tau - \bar{\tau}_0)$$

so that $\tau = 0$ on $\mathcal{S}_0$ and

$$\mathring{\Theta}(\tau) = \bar{\Omega}\left(\left(1 - \frac{\bar{\tau}_0^2}{\bar{\omega}^2}\right) + 2\frac{\bar{\tau}_0}{\bar{\omega}^2}\tau - \frac{1}{\bar{\omega}^2}\tau^2\right).$$

The initial value of $\mathring{\Theta}$ on $\mathcal{S}_0$ will be denoted by $\mathring{\Omega}$. It can be verified that $\mathring{\Omega}$ is a boundary defining function. In what follows, let

$$\mathring{\mathbf{u}}(\tau, x) \equiv \mathbf{u}(\tau - \bar{\tau}_0, x).$$

The following is an obvious corollary of Theorem 2—for details of the proof see the analogous construction in [7].

Corollary 1. *The field $\mathring{\mathbf{u}}(\tau, x)$ implies hyperboloidal data on $\mathring{\mathcal{H}}$ for the conformal Einstein–Maxwell field equations.*

11.2.1. Structure of the conformal boundary

We consider now hyperboloidal data which is "close" in some suitable sense to the hyperboloidal data given by Corollary 1. Using analogous arguments to the ones used in [8] one can prove the following result.

Proposition 1. *Given a radiative electrovacuum hyperboloidal initial data set $(\mathcal{H}, h_{\alpha\beta}, K_{\alpha\beta}, \Omega)$ sufficiently close to a reference radiative electrovacuum data $(\mathring{\mathcal{H}}, \mathring{h}_{\alpha\beta}, \mathring{K}_{\alpha\beta}, \mathring{\Omega})$, there exists a choice of initial data for the congruence of conformal curves such that the conformal factor Θ is given by*

$$\Theta = \Theta_* + \dot{\Theta}_*\tau + \ddot{\Theta}_*\tau^2, \tag{73}$$

with

$$\Theta_* = \Omega, \qquad \dot{\Theta}_* = \langle d, e_0 \rangle, \qquad 2\Omega\ddot{\Theta} = g^\sharp(d, d)_*.$$

Furthermore, if the point $i^+ \equiv (-\Omega/\dot{\Theta}_*, 0, 0, 0)$ is contained in the development of the initial data, then it is the unique point at which the conformal factor Θ satisfies the (timelike infinity) conditions

$$\Theta(i^+) = 0, \qquad \mathrm{d}\Theta(i^+) = 0, \qquad \text{Hess}\ \Theta(i^+)\ \textit{non-degenerate}.$$

As it is customary, let $\mathscr{I}$ (null infinity) denote the set of points for which $\Theta = 0$ where the conformal factor is given by (73).

11.2.2. A stability result for purely radiative spacetimes

The information about the conformal boundary of a hypothetical radiative Einstein–Maxwell spacetime arising from hyperboloidal data which is contained in Proposition 1 allows to readily obtain a stability result for a spacetime belonging to the class arising from Theorem 2. The proof of the following result is similar to that in [8]—see also [18,7].

Theorem 3. *Let* $\mathbf{u}_0 = \mathring{\mathbf{u}}_0 + \breve{\mathbf{u}}_0$ *be hyperboloidal initial data for the Einstein–Maxwell conformal field equations. Given* $\tau_+ \equiv -\Omega/\dot{\Theta}_*$ *and if* $\mathring{u}_0$ *is sufficiently small, there exists on* $[0, \bar{\tau}_0] \times \mathcal{S}$ *a unique solution* $\mathbf{u} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}$ *to the (reduced) conformal propagation equations (65), (66a)–(66c) such that the associated congruence of conformal curves contains no conjugate points in* $[0, \tau_+]$. *The solution* $\mathbf{u} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}$ *on* $\mathcal{D}^+(\mathcal{S})$ *implies a smooth solution* $(\mathcal{M}, \tilde{g})$ *to the electrovacuum Einstein field equations with vanishing cosmological constant, where* $\tilde{g}_{\mu\nu} = \Theta^{-2} g_{\mu\nu}$ *with* Θ *given by (73). The spacetime* $(\mathcal{M}, \tilde{g})$ *has a conformal boundary given by the set of points for which* $\Theta = 0$. *The conformal boundary consists of the set* $\mathscr{I}$, *which represents future null infinity, and the point* $i^+ \equiv (\tau_+, 0, 0, 0)$, *which represents timelike infinity.*

Remark. The purely radiative spacetimes used as reference solutions in our analysis are not perturbations of the Minkowski spacetime. A way of seeing this is to consider the Newman–Penrose constants of the spacetime. The Newman–Penrose constants are a set of absolutely conserved quantities defined as integrals of certain components of the Weyl tensor and the Maxwell fields over cuts of null infinity—see [19–21] for the Einstein–Maxwell case. In [22] it has been shown that the value of the Newman–Penrose constants for a vacuum radiative spacetime coincides with the value of the rescaled Weyl spinor at i^+—this result can be extended to the electrovacuum case using the methods of this article. For the radiative spacetimes arising from the construction of [17] it can be seen that the value of the Weyl spinor at i^+ is essentially the mass quadrupole of the *seed* static spacetime. It follows, that the Newman–Penrose constants of the radiative spacetime can take arbitrary values. On the other hand, for the Minkowski spacetime, the Newman–Penrose constants are exactly zero, and those of perturbations thereof will be small. Thus, in this precise sense, our radiative spacetimes are, generically, not perturbations of the Minkowski spacetime, unless all the Newman–Penrose constants vanish.

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