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Sets of Efficient Points in a Normed Space*

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Sets of efficient points in a normed space with respect to the distances to the points of a given compact set are geometrically characterized; hull and closure properties are obtained. These results are relevant to geometry of normed spaces and are mostly useful in the context of location theory.

INTRODUCTION

A nonempty subset A of a normed space X being given, a natural multi-objective programming problem may be considered. The distances, associated with the norm, to every point of A are to be simultaneously minimized.

Such a topic occurs for instance in approximation theory, in geometry of normed spaces and also in location theory where it provides a general model applicable to different economic fields.

The basic concept of efficiency has been introduced under various shapes: among them, the set of strictly efficient points or set of minimal points or metric hull of A denoted by $c(A)$, the set of efficient points or Pareto optima denoted by $E(A)$, and the set of weakly efficient points or closest points to A denoted by $C(A)$. These sets, associated with A in a normed space, are formally defined further and have been studied in different frameworks:

(i) The metric hull or set of minimal points $c(A)$ by Kuhn [6] in optimization theory, by Beauzamy and Maurey [1] and Papini [9] in the geometry of Banach spaces.

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(ii) The set of Pareto optima by many authors working in multi-objective programming.

(iii) The set $C(A)$ of closest points to A by Fejer [5] in approximation theory and by Phelps [10] in geometry of normed spaces.

When the space X is a Hilbert space, then for every subset A, $c(A)$, $E(A)$ and $C(A)$ are actually the closed convex hull of A [5–6].

The aim of this paper is to give a geometrical descrition of the above-mentioned sets of efficient points. The idea to get such a description is to define subsets of X, able to play the role of halfspaces in the hilbertian case; in that case a halfspace appears naturally as the subset of points closer to 0 than to all other points of a halfline starting from 0 with a direction δ. Of course such subsets denoted by Q_δ are not halfspaces in any normed space. However, the analogy can be specified, and making use of these sets, a description of $c(A)$, $E(A)$, and $C(A)$ can be obtained.

It is then natural to study the so-called sets Q_δ; a geometrical description is given when the normed space is strictly convex or when the unit ball of X is a polytope; more precise results are obtained when X is two dimensional and when X is $\mathbb{R}^n$ with the l^1-norm and with the l^∞-norm. In the general case it is only possible to confine Q_δ between known sets called U_δ and V_δ.

Thanks to a characterization of $c(A)$, $E(A)$, and $C(A)$ which makes use of a subfamily of the whole family $(Q_\delta)_{\delta \neq 0}$, hull and closure properties are also obtained.

The paper is divided into four sections: Section 1 is devoted to definitions and first properties. Section 2 gives the main theorem relating the descrition of $c(A)$, $E(A)$, and $C(A)$ with the sets Q_δ. Section 3 studies the sets Q_δ in a general normed space. Section 4 deals with hull and closure properties.

The first motivation of this paper is location theory. Two previous papers [3, 4] study connections between the sets of efficient points and the set of solutions to a Fermat–Weber problem, for a finite subset A of a n-dimensional space.

The results obtained here are complementary of those of [3] and [4], but their links with geometry of normed spaces give to them another interest.

1. Definition and First Properties

1.1. *Notations*

Throughout this paper X is a real normed space where the norm is denoted by $|\cdot|$. The unit ball is $B = \{x \in X / |x| \leqslant 1\}$ and the unit sphere is

$S = \{x \in X / |x| = 1\}$. The ball centered at a with radius r will be denoted by $B(a, r)$.

For a subset D of X, $\text{int}(D)$, $\bar{D}$, $\text{co}(D)$, $\overline{\text{co}}(D)$, $D^c = X \backslash D$, and $-D$ are respectively the interior, the closure, the convex hull, the closed convex hull, the complementary, and the symmetric with respect to 0 of D. When D and D' are subsets of X and α is real, $D + D'$ is the set $\{x \in X / \exists y \in D, \exists y' \in D', x = y + y'\}$, and αD is the set $\{\alpha x / x \in D\}$.

A cone Γ will always be a cone with vertex at 0; a cone with vertex at x_0 will be denoted by $x_0 + \Gamma$. A segment $[u, v]$ is the set $\{\theta u + (1 - \theta) v, 0 \leqslant \theta \leqslant 1\}$.

For a convex subset D of X, $\text{Ext}(D)$ and $\text{Exp}(D)$ are the sets of respectively extreme and exposed points of D; a p-face of D is a p-dimensional face of D.

The subdifferential of the norm at x will be denoted by $\partial\gamma(x)$. If B^0 (resp. S^0) is the unit ball (resp. unit shere) of the dual X' of X, $\partial\gamma(0)$ is equal to B^0 and $\partial\gamma(x)$ is a weakly compact face of S^0 for $x \neq 0$. The correspondance $x \to \partial\gamma(x)$ is upper semi-continuous when X is equipped with the topology of the norm and X' with the weak topology. The directional derivative of the norm at x in the direction $\delta \neq 0$ is related with the subdifferential by $\gamma'(x, \delta) = \text{Max}\{(p, \delta) / p \in \partial\gamma(x)\}$.

If p belongs to B^0, $N(p)$ is the convex cone generated by the exposed face of B defined by $\{x \in B / (p, x) = 1\}$.

1.2. *Sets of Efficient Points*

First, we recall various kinds of efficiency.

DEFINITION 1.1. Let A be a nonempty subset of X. We call

(i) Set of strictly efficient points (with respect to A) the set

$$c(A) = \{x \in X / \forall y \neq x, \exists a \in A, |a - x| < |a - y|\},$$

(ii) Set of efficient points (with respect to A) the set

$$E(A) = \{x \in X / \forall y \neq x, (\exists a \in A, |a - x| < |a - y|) \quad \text{or} \quad (\forall a \in A, |a - x| \leqslant |a - y|)\}$$

(iii) Set of weakly efficient points (with respect to A) the set

$$C(A) = \{x \in X / \forall y \neq x, \exists a \in A, |a - x| \leqslant |a - y|\}.$$

An alternative definition (see [11]) is given by

PROPOSITION 1.1. *We have the following statements*

(1) $x \in c(A)$ *iff* $\bigcap_{a \in A} B(a, |x-a|) = \{x\}$,

(2) $x \in E(A)$ *iff* $\bigcap_{a \in A} B(a, |x-a|) \cap (\bigcup_{a \in A} \operatorname{int} B(a, |x-a|)) = \emptyset$,

(3) $x \in C(A)$ *iff* $\bigcap_{a \in A} \operatorname{int} B(a, |x-a|) = \emptyset$.

We now state some general elementary properties, which hold whatever the norm is.

PROPOSITION 1.2. *Let A and B be nonempty subsets of X.*

(1) $A \subset c(A) \subset E(A) \subset C(A)$.

(2) *If A is bounded then $c(A)$, $E(A)$, and $C(A)$ are bounded.*

(3) *If $A \subset B$, then $c(A) \subset c(B)$ and $C(A) \subset C(B)$, but not necessarily $E(A) \subset E(B)$.*

(4) $c(A) = c(\bar{A})$, $E(A) = E(\bar{A})$, *but not necessarily* $C(A) = C(\bar{A})$.

Proof. (1) It is a direct consequence of the definitions.

(2) If A is included in $B(0, r)$ then $C(A)$ is included in $B(0, 2r)$.

(3) If $A \subset B$, the inclusions $c(A) \subset c(B)$ and $C(A) \subset C(B)$ result at once from the definitions. An example where $A \subset B$ and $E(A) \not\subset E(B)$ is given below (cf. Examples 1 and 2).

(4) Let $x \notin c(A)$; then there exists $y \neq x$ such that $|a-x| \geqslant |a-y|$ for every $a \in A$ and consequently for every $a \in \bar{A}$. Thus $c(\bar{A}) \subset c(A)$, and we have $c(A) = c(\bar{A})$ thanks to (3). An analogous argument leads to $E(A) = E(\bar{A})$. An example where $C(A) \neq C(\bar{A})$ is given below (cf. Example 4). ∎

Examples in $\mathbb{R}^2$ with the l^1-Norm

Let $a = (1, 0)$, $b = (0, 1)$, $a' = (2, 0)$, $b' = (0, 2)$, $c = (1, 1)$, and $c' = (2, 2)$.

(1) If $A = \{a, b\}$, then $c(A) = A$, $E(A)$ is the closed set bounded by the square *oacb* and $C(A) = E(A)$.

(2) If $A = \{o, a, b\}$, then $c(A)$ is the union of the two segments of line $[o, a]$ and $[o, b]$, $E(A) = c(A)$ and $C(A)$ is the closed set bounded by the square *oacb'*.

(3) If $A = \{a, a', b, b'\}$, then $c(A)$ is the union of the two segments of line $[a, a']$ and $[b, b']$, $E(A)$ is the union of $c(A)$ and of the closed set bounded by the square *oacb* and $C(A)$ is the closed set bounded by the square *oa'c'b'*.

(4) If A is the open set bounded by the trapezoid *aa'b'b*, then $o \notin C(A)$ and $o \in C(\bar{A})$.

With some assumptions on the norm of X, a close relationship exists between different sets of efficiency and also with the closed convex hull.

PROPOSITION 1.3. *Let A be a nonempty subset of X.*

(1) *If X is strictly convex, then*

$$c(A) = E(A) = C(A) = C(\bar{A}).$$

(2) *If X is a Hilbert space, then*

$$c(A) = E(A) = C(A) = \overline{\mathrm{co}}(A).$$

(3) *If X is two-dimensional, strictly convex, and if A is bounded, then*

$$c(A) = E(A) = C(A) = \overline{\mathrm{co}}(A).$$

Proof. (1) Let $x \in C(A)$ and let $y \neq x$. With $z = (x + y)/2$, we associate $a \in A$ such that $|a - x| \leqslant |a - z|$. If $a - x$ and $a - y$ are colinear vectors, then we have $|a - x| < |a - z|$. If they are not colinear, then the strict convexity of the function $t \to |a - (1 - t)x - ty|$ on $[0, 1]$ involves $|a - x| < |a - y|$. Thus $x \in c(A)$. We conclude with (1) and (4) of Proposition 1.2.

(2) Let $x \notin \overline{\mathrm{co}}(A)$ and let y be the projection of x on $\overline{\mathrm{co}}(A)$. Then $|a - x| > |a - y|$ for every $a \in A$; therefore $x \notin C(A)$. Thus we have $C(A) \subset \overline{\mathrm{co}}(A)$.

Let $x \notin c(A)$ and let $y \neq x$ such that $|a - x| \geqslant |a - y|$ for every $a \in A$. The subset $\{u \in X / |u - y| = |u - x|\}$ is a hyperplane separating x and $\overline{\mathrm{co}}(A)$; therefore $x \notin \overline{\mathrm{co}}(A)$. Thus we have $\overline{\mathrm{co}}(A) \subset c(A)$. We conclude again with (1) of Proposition 1.2.

The proof of $C(A) = \overline{\mathrm{co}}(A)$ may be found in [5] and in [10] and the proof of $c(A) = \overline{\mathrm{co}}(A)$ in [6].

(3) A proof of this assertion is given in [10]; another one may be found in [11]. See also Corollary 4.1. ∎

Remark 1.1. It is advisible to note that Phelps [10] shows that the condition $C(A) \subset \overline{\mathrm{co}}(A)$ for every subset A of the normed space X is equivalent to say that X is a Hilbert space when its dimension is at least three and X is strictly convex when it is two dimensional. In dimension two, another characterization of strict convexity is given in Corollary 4.2.

Without any assumption on the norm of X, $c(A)$ is not necessarily included in $\overline{\mathrm{co}}(A)$, even when A is finite (see [1]). However when E is two dimensional, then $c(A)$ is always included in $\overline{\mathrm{co}}(A)$. The proof given in [4] when A is finite, making use of [13], is also valid when A is bounded. See also [11] and Corollary 4.1.

1.3. *Q_δ sets and P_δ sets*

Let δ be a vector of X with $\delta \neq 0$. The following complementary subsets of X will be used to obtain a descrition of the sets of efficient points:

$$Q_\delta = \{x \in X/\forall \lambda > 0,\ |x - \lambda\delta| > |x|\},$$
$$P_\delta = \{x \in X/\exists \lambda > 0,\ |x - \lambda\delta| \leqslant |x|\}.$$

When X is a Hilbert space with the inner product denoted by $(\cdot, \cdot)$, then Q_δ is the closed halfspace $\{x \in X/(x, \delta) \leqslant 0\}$ and P_δ is the open halfspace $\{x \in X/(x, \delta) > 0\}$.

To specify some properties of these sets we introduce another notation; let $x \in X$ and $y \in X$, we denote by

$$Q_{x,y} = \{z \in X/|z - x| > |z - y|\},$$
$$P_{x,y} = \{z \in X/|z - x| \leqslant |z - y|\}.$$

The next two propositions are straightforward consequences of the definitions and properties of the norm. We only state them for the sets of type "Q;" corresponding results for the sets of type "P" are easily obtained by complementarity.

PROPOSITION 1.3. *Let $\delta \neq 0$ and $\lambda \neq 0$. Then*

(1) $Q_{x+\lambda\delta,0} = x + Q_{\lambda\delta,0}$,

(2) $Q_{\lambda\delta,0} = \lambda Q_{\delta,0}$,

(3) $Q_{0,\lambda\delta} = \lambda Q_{0,\delta}$.

PROPOSITION 1.4. *Let $\delta \neq 0$ and $0 < \mu < \lambda$. Then*

(1) $Q_{\mu\delta,0} \subset Q_{\lambda\delta,0}$,

(2) $Q_{0,\mu\delta} \supset Q_{0,\lambda\delta}$,

(3) *Q_δ is a cone containing 0, which is the intersection of a decreasing family of open sets; more precisely*

$$Q_\delta = \bigcap \{Q_{\lambda\delta,0}/\lambda > 0\}.$$

In general, without assumptions on the space X, Q_δ sets and P_δ sets have no special topological properties. However, the next result gives a direct characterization of $\operatorname{int}(Q_{-\delta})$, the formulation of which may be compared with the definition of Q_δ.

PROPOSITION 1.5. *Let $\delta \neq 0$. Then*

$$\mathrm{int}(Q_{-\delta}) = \{x \in X / \exists \lambda > 0, |x - \lambda\delta| < |x|\} = \bigcup \{Q_{0,\lambda\delta}/\lambda > 0\}.$$

Proof. Let $x \in X$ be such that there exists $\lambda > 0$ with $|x - \lambda\delta| < |x|$. Then the convex function $t \to |x - t\delta|$ is necessarily strictly decreasing on $]-\infty, 0]$; whence we have, for every $\mu > 0$, $|x + \mu\delta| > |x|$, which means $x \in Q_{-\delta}$. As the set $\{x \in X / \exists \lambda > 0, |x - \lambda\delta| < |x|\}$ is open, it is included in $\mathrm{int}(Q_{-\delta})$.

Conversely, let $x \in \mathrm{int}(Q_{-\delta})$; then there exists a ball with radius $r > 0$, centered at x, included in $Q_{-\delta}$. Suppose $|\delta| = 1$, what is not restrictive; for $\rho \in [0, r]$, we have $x - \rho\delta \in Q_{-\delta}$ and therefore, for every $\mu > 0$, $|x - \rho\delta + \mu\delta| > |x - \rho\delta|$. In particular for $\mu = \rho$, we have $|x| > |x - \rho\delta|$ and the result follows. ▮

COROLLARY 1.1. *Let $\delta \neq 0$. Then* $\mathrm{int}(Q_\delta)$ *is dense in* Q_δ.

Proof. Let $x \in Q_\delta$; we choose $y_\alpha = x - \alpha\delta$. When α tends to 0, y_α converges to x, and y_α belongs to $\mathrm{int}(Q_\delta)$; indeed $|y + (\alpha\delta/2)| = |x - (\alpha\delta/2)| < |y_\alpha|$, i.e., $y \in \mathrm{int}(Q_\delta)$. ▮

COROLLARY 1.2. *Let $\delta \neq 0$. Then we have*

(1) $\mathrm{int}(Q_{-\delta}) \subset P_\delta$.

(2) *If S contains no segment parallel to δ, then $Q_\delta = \bar{P}_{-\delta}$.*

Proof. The first inclusion is a direct consequence of the definition of P_δ and of Proposition 1.5. Let $x \in P_\delta$; there exists $\lambda > 0$ such that $|x - \lambda\delta| \leqslant |x|$. As the convex function $t \to |x - t\delta|$ cannot be constant on any segment, we have $|x - \mu\delta| < |x|$ for each μ, $0 < \mu < \lambda$, i.e., $x \in \mathrm{int}(Q_{-\delta})$. The result follows by complementarity. ▮

The next definition has been introduced by Brown [2] in the context of approximation theory; it arose in studying the continuity of approximation. The property to be considered gives some topological properties of the sets and will be an essential assumption in Theorem 2.1.

DEFINITION 1.2. Let $\delta \neq 0$. The normed space X is said to possess the property (B_δ) if for each x with $|x - \delta| \leqslant |x|$, there exists $\alpha > 0$ and a neighbourhood W of x such that, for every $y \in W$, $|y - \alpha\delta| \leqslant |y|$.

It is not restrictive to assume $\alpha < 1$ in this definition. With the above-mentioned notations, we have an equivalent formulation:

PROPOSITION 1.6. *Let $\delta \neq 0$. The space X has the property (B_δ) if and only if, when x belongs to $P_{\lambda\delta,0}$ for $\lambda > 0$, then there exists μ (which a priori depends on x), $0 < \mu < \lambda$, such that x belongs to* $\text{int}(P_{\mu\delta,0})$.

The next two propositions deal with properties of Q_δ and P_δ sets of a topological nature.

PROPOSITION 1.7. *Let $\delta \neq 0$. If X has the property (B_δ), then Q_δ is closed.*

Proof. We prove that P_δ is open. Let $x \in P_\delta$; then there exists $\lambda > 0$ such that $x \in P_{\lambda\delta,0}$. Property (B_δ) implies that there exists μ, $0 < \mu < \lambda$, such that $x \in \text{int}(P_{\mu\delta,0})$, and $\text{int}(P_{\mu\delta,0})$ is an open set included in P_δ. ∎

PROPOSITION 1.8. *Let $\delta \neq 0$ and suppose that X has the property (B_δ). If a compact A is included in P_δ, then A is included in $P_{\lambda\delta,0}$ for some $\lambda > 0$.*

Proof. Let $x \in A$; as A is included in P_δ, there exists $\lambda > 0$, which depends on x, such that $x \in P_{\lambda\delta,0}$. Property (B_δ) implies that there exists $\mu(x)$ and a neighbourhood $W(x)$ of x such that $W(x)$ is included in $P_{\mu(x)\delta,0}$. As A is compact, the covering $(W(x))_{x \in A}$ of A contains a finite subset which is also a covering. Thus, there exists $\{x_1, ..., x_n\} \subset A$ such that A is included in $W(x_1) \cup W(x_2) \cup \cdots \cup W(x_n)$. Let $\mu = \min\{\mu(x_1), \mu(x_2), ..., \mu(x_n)\}$. For $i = 1, 2, ..., n$, we have $W(x_i) \subset P_{\mu(x_i)\delta,0} \subset P_{\mu\delta,0}$ and then $A \subset P_{\mu\delta,0}$. ∎

It should be noted that, in general, if a compact K is included in the union of an increasing family of closed sets, K is not necessarily included in one of these closed sets.

To give examples of spaces which possess the property (B_δ) we need

LEMMA 1.1. *If S contains no segment parallel to $\delta \neq 0$, then we have $P_{\lambda\delta,0} \subset \text{int}(P_{\mu\delta,0})$ for $0 < \mu < \lambda$, whence X has the property (B_δ).*

Proof. Let $x \in P_{\lambda\delta,0}$. We have $x \neq 0$ and $|x - \lambda\delta| \leqslant |x|$. As the function $t \to |x - t\delta|$ is strictly convex on $\mathbb{R}$, we have $|x - \mu\delta| < |x|$ for every μ with $0 < \mu < \lambda$. A continuity argument implies the existence of a neighbourhood W of x such that, for every $y \in W$, $|y - \mu\delta| < |y|$ and then $y \in P_{\mu\delta,0}$. Thus we have $x \in \text{int}(P_{\mu\delta,0})$. ∎

We also introduce a definition (see [3]).

DEFINITION 1.3. A normed space is said to possess the Diff–Max property if for every $x \in X$, there exists a neighbourhood W of x such that for every $y \in W$, $\partial\gamma(y)$ is included in $\partial\gamma(x)$.

Now, the next theorem gives conditions under which X has property (B_δ).

THEOREM 1.1. *The space X possesses the property (B_δ) for each $\delta \neq 0$ in the following cases*:

(1) *X is strictly convex,*

(2) *X is two dimensional,*

(3) *X has the Diff–Max property,*

(4) *X is polyhedral.*

Proof. (1) is a direct consequence of Lemma 1.1. This result may also be found in [3].

(2) We prove the stronger result: for each $\delta \neq 0$, we have $P_{\lambda\delta,0} \subset \text{int}(P_{\mu\delta,0})$ for $0 < \mu < \lambda$. Without restriction suppose that $\lambda = 1$ and $\frac{1}{2} < \mu < 1$. Let $x \in P_{\delta,0}$; thanks to Proposition 1.5, it is sufficient to consider the case where $|x| = |x - t\delta|$ for every $t \in [0, 1]$.

Let k such that $\mu < k < 1$ and let $\alpha > 0$ such that $k < 1 - \alpha$. The sphere centered at x, with radius $|x|$ contains the segment $[0, \delta]$. Hence, with a translation, the sphere centered at $x + \alpha\delta$, with the same radius contains the segment $[\alpha\delta, (1+\alpha)\delta]$; as 0 is not an interior point of the corresponding ball, we have $|x + \alpha\delta| \geqslant |x|$. Besides, for $\rho \in [\alpha, 1+\alpha]$, we have $|x| = |x + \alpha\delta - \rho\delta|$, which gives $|x + \alpha\delta - \rho\delta| \leqslant |x + \alpha\delta|$, i.e., $x + \alpha\delta \in P_{\rho\delta,0}$.

Similarly, with the opposite translation, we have for $\sigma \in [-\alpha, 1-\alpha]$, $x - \alpha\delta \in P_{\sigma\delta,0}$.

Consequently, $x + \alpha\delta$ and $x - \alpha\delta$ belong to $P_{k\delta,0}$ when α belongs to $[0, 1-k]$. Thus the segment $I = [x - (1-k)\delta,\ x + (1-k)\delta]$ is included in $P_{k\delta,0}$. Making use of Proposition 1.3 the segment $(\mu/k)\, I$ is included in $P_{\mu\delta,0}$ and the open cone, which contains x, generated by this segment is also included in $P_{\mu\delta,0}$.

It must be noted that the assumption of two dimensionality is fundamental for the validity of the proof.

(3) Without restriction suppose that $|\delta| = 1$. Let $\delta \neq 0$ and let $x \in X$ such that $|x - \delta| \leqslant |x|$. By the Diff–Max property there exists $r > 0$ such that, for $|y - x| < r$, we have $\partial\gamma(y) \subset \partial\gamma(x)$. Let $\rho = r/2$, and let $\mu < r/2$, $\mu < 1$. We prove that if $|x - y| < \rho$, then $|y - \mu\delta| \leqslant |y|$.

Indeed, we have $|y - \mu\delta - x| < r$, and therefore $\partial\gamma(y - \mu\delta) \subset \partial\gamma(x)$. Let $p \in \partial\gamma(y - \mu\delta)$; then $(p, x - \mu\delta) = |x| - \mu(p, \delta) \leqslant |x - \mu\delta|$. As $|x - \mu\delta| \leqslant |x|$, we have $(p, \delta) \geqslant 0$ and we deduce $|y - \mu\delta| = (p, y - \mu\delta) \leqslant |y|$.

(4) It is a consequence of (3) and of a result given in [3]: a finite-dimensional space has the Diff–Max property if and only if it is polyhedral. Another proof may be found in [2]. ∎

2. Efficient Points and Q_δ Sets

This section is devoted to a characterization of the sets of efficiency $c(A)$, $E(A)$, and $C(A)$, associated with a compact set A, making use of the Q_δ sets.

THEOREM 2.1. *Let A be a compact set. If X has the property (B_δ) for each $\delta \neq 0$, then*

(1) $x \in c(A)$ iff for every $\delta \neq 0$, $A \cap (x + Q_\delta) \neq \emptyset$,

(2) $x \in E(A)$ iff for every $\delta \neq 0$, $A \cap (x + Q_\delta) \neq \emptyset$ or $A \cap (x + \operatorname{int}(Q_{-\delta})) = \emptyset$.

Without any assumption on X, we have

(3) $x \in C(A)$ iff for every $\delta \neq 0$, $A \cap (x + \bar{P}_\delta) \neq \emptyset$.

Proof. (1) The following statements are equivalent

(i) $x \notin c(A)$,

(ii) $\exists \delta \neq 0$, $\exists \lambda > 0$, $\forall a \in A$, $|a - x - \lambda\delta| \leqslant |a - x|$,

(iii) $\exists \delta \neq 0$, $\exists \lambda > 0$, $A \subset x + P_{\lambda\delta,0}$.

Moreover, by Proposition 1.8, (iii) is equivalent to the inclusion $A \subset x + P_\delta$ for some $\delta \neq 0$; in other words there exists $\delta \neq 0$ such that $A \cap (x + Q_\delta) = \emptyset$.

(2) The following are equivalent

(i) $x \notin E(A)$,

(ii) $\exists \delta \neq 0$, $\exists \lambda > 0$, $(\forall a \in A,\ |a - x - \lambda\delta| \leqslant |a - x|)$ and $(\exists a \in A,\ |a - x - \lambda\delta| < |a - x|)$,

(iii) $\exists \delta \neq 0$, $(\exists \lambda > 0,\ A \subset x + P_{\lambda\delta,0})$ and $(\exists \mu > 0,\ A \cap (x + Q_{0,\mu\delta}) \neq \emptyset)$.

Indeed the equivalence between (i) and (ii) is a direct consequence of the definition of $E(A)$ and the equivalence between (ii) and (iii) follows by Proposition 1.4. Then, by Propositions 1.8 and 1.5, (iii) is equivalent to the relations $A \subset x + P_\delta$ and $A \cap x + \operatorname{int}(Q_{-\delta}) \neq \emptyset$ for some $\delta \neq 0$, and the result follows.

(3) In the same manner, we have the equivalences

(i) $x \notin C(A)$,

(ii) $\exists \delta \neq 0$, $\exists \lambda > 0$, $A \subset x + Q_{0,\lambda\delta}$.

According to Proposition 1.5, $\operatorname{int}(Q_{-\delta})$ is an increasing union of open sets $Q_{0,\lambda\delta}$; then, as the set A is compact, (ii) is equivalent to the inclusion $A \subset x + \operatorname{int}(Q_{-\delta})$ for some $\delta \neq 0$ and the result follows. ∎

Remark 2.1. If A is a finite subset of X, Theorem 2.1 is always valid, even if property (B_δ) is not satisfied. It is the case in the framework of location theory where X is a finite dimensional normed space, and where A is the set of existing facilities.

Remark 2.2. The preceding theorem allows us to give a nice geometric descrition of $c(A)$, $E(A)$, and $C(A)$, when A is a compact set, to the extent that the sets Q_δ are themselves well described.

Particularly when X is a Hilbert space, then each Q_δ is a closed halfspace and each closed halfspace is a Q_δ set; therefore, when A is a compact set, we obtain again the equality

$$c(A) = E(A) = C(A) = \overline{\text{co}}(A).$$

Moreover, this very particular case shows that the assumption on the compacity of A is necessary; take as a counterexample the open unit ball of X.

When X is strictly convex, then, for each $\delta \neq 0$, X has the property (B_δ); we have $Q_\delta = \bar{P}_\delta$ (see Corollary 1.2) and by complementarity $\text{int}(Q_{-\delta}) = P_\delta$. Therefore Theorem 2.1 implies, for each compact set A, $c(A) = E(A) = C(A)$.

COROLLARY 2.1. *Let A be a compact set.*

(1) *If X has the property (B_δ) for each $\delta \neq 0$, $c(A)$ and $E(A)$ are closed.*

(2) *$C(A)$ is closed.*

Proof. We only prove that $E(A)$ is closed; the other proofs are built on the same pattern.

Let $x \notin E(A)$. From the assumptions and Theorem 2.1, there exists $\delta \neq 0$ such that $A \subset x + P_\delta$ and $A \cap (x + \text{int}(Q_{-\delta})) \neq \varnothing$. Since A is compact and P_δ is open, there exists a neighbourhood W_1 of x such that, for every $y \in W_1$, we have $A \subset y + P_\delta$. If $\bar{a}$ is a point of A, such that $\bar{a} - x \in \text{int}(Q_{-\delta})$, there exists a neighbourhood W_2 of x such that, for every $y \in W_2$, we have $\bar{a} - y \in \text{int}(Q_{-\delta})$. Hence $W_1 \cap W_2$ is a neighbourhood of x, which is included in the complementary set of $E(A)$. ∎

As a straightforward consequence, we get a result of [1].

COROLLARY 2.2. *If X is an infinite-dimensional reflexive normed and strictly convex space and if A is compact, then $c(A)$, $E(A)$, and $C(A)$ are weakly compact.*

Proof. The result follows from Proposition 1.2 and from Corollary 2.1. ▮

Without property (B_δ) assertions (1) and (2) of Theorem 2.1 fail. For instance, we give in dimension three an example as follows.

Counterexample

In $\mathbb{R}^2$, let D be the half-disk defined by $D = \{(x, y)/x^2 + y^2 \leqslant 1, x \geqslant 0\}$, and let T be the triangle defined by $T = \{(x, y)/-x+1 \leqslant y \leqslant x+1, x \leqslant 0\}$. In $\mathbb{R}^3$, we put $\Gamma = \{(x, y, z)/(x, y) \in D \cup T, z = 1\}$ and $B = \text{co}\{\Gamma \cup -\Gamma\}$. B is a convex compact set which is symmetric and contains 0 in its interior. We choose on $\mathbb{R}^3$ the norm for which the unit ball is B.

Let $A = \{(x, y, z)/(x-\frac{1}{2})^2 + y^2 \leqslant \frac{1}{4}, x \geqslant \frac{1}{2}, z = 1\} \cup \{(-1, 0, 1)\}$. With the vector $\delta = (0, 0, 1)$, we have $A \subset P_\delta$, i.e., $A \cap Q_\delta = \varnothing$, and $A \cap \text{int}(Q_{-\delta}) \neq \varnothing$. However, $0 \in c(A)$, whence $0 \in E(A)$; indeed we have $|a| = 1$ for each $a \in A$, and $\bigcap (B(a, 1), a \in A) = \{0\}$.

Property (B_δ) is not satisfied since Q_δ is not closed. Indeed, let $u = (1, 0, 1)$. We have $|u| = |u - \lambda\delta| = 1$ for $\lambda \in [0, 2]$, which means that $u \in P_\delta$; i.e., $u \notin Q_\delta$. For $0 < \varphi < \pi/2$, put $u_\varphi = (\cos\varphi, \sin\varphi, 1)$. When φ converges to zero, u_φ converges to u. We show that $u_\varphi \in Q_\delta$: from the convexity of the norm it is sufficient to prove that $|u_\varphi - \lambda\delta| > |u_\varphi|$, i.e., $u_\varphi - \lambda\delta \notin B$ for $\lambda \in]0, 2[$. We prove the result by contradiction. Suppose that $u_\varphi - \lambda\delta \in B$. Then there exists $v \in \Gamma$, $w \in -\Gamma$, $\theta \in [0, 1]$ such that $u_\varphi - \lambda\delta = \theta v + (1-\theta) w$. In particular, we have $1 - \lambda = 2\theta - 1$. Hence the point $(\cos\varphi, \sin\varphi)$ is a convex combination with positive weights θ and $(1-\theta)$ of a point of the half-disk D and of a point of the triangle $-T$, which is impossible.

It is also to be noted that the conclusion of Proposition 1.8 fails: A is included in P_δ, but A is not included in $P_{\lambda\delta,0}$ for some $\lambda > 0$; indeed, for $a \in A$, put $\lambda(a) = \text{Sup}\{\lambda/\lambda > 0, a \in P_{\lambda\delta,0}\}$; when a tends to u, $\lambda(a)$ tends to zero.

3. Geometrical Description of the Q_δ Sets

3.1. *General Results*

Two cones of X, associated with a vector $\delta \neq 0$, play a prominent part in the description of the Q_δ sets. They are defined by

$$U_\delta = \bigcup \{N(p)/p \in \text{Ext}(B^0), (p, \delta) < 0\},$$

$$V_\delta = \bigcup \{N(p)/p \in \text{Ext}(B^0), (p, \delta) \leqslant 0\}.$$

A characterization of U_δ and V_δ, making use explicitely of the subdifferential $\partial\gamma$ or making use of the directional derivative $\gamma'(\cdot, \cdot)$ of the norm, is useful.

PROPOSITION 3.1. *Let $\delta \neq 0$ and let $x \in X$. We have the equivalences*

(1) $x \in U_\delta$ *iff* $\exists p \in \partial\gamma(x)$ *with* $(p, \delta) < 0$ *iff* $\gamma'(x, -\delta) > 0$,

(2) $x \in V_\delta$ *iff* $\exists p \in \partial\gamma(x)$ *with* $(p, \delta) \leqslant 0$ *iff* $\gamma'(x, -\delta) \geqslant 0$.

Proof. The case $x = 0$ being obvious, we assume $x \neq 0$.

(1) Let x be such that $(p_0, \delta) < 0$ for some $p_0 \in \partial\gamma(x)$. Suppose that $(p, \delta) \geqslant 0$ for every $p \in \mathrm{Ext}(\partial\gamma(x))$. Making use of Krein–Milman theorem, the hyperplane defined by $\{p/(p, \delta) = \frac{1}{2}(p_0, \delta)\}$ strictly separates p_0 and $\partial\gamma(x)$ what is impossible. As $\mathrm{Ext}(\partial\gamma(x)) \subset \mathrm{Ext}(B^0)$ we have $(p, \delta) < 0$ for some $p \in \mathrm{Ext}(B^0)$; then $x \in U_\delta$. The converse is immediate.

By the result $\gamma'(x, \delta) = \mathrm{Max}\{(p, \delta)/p \in \partial\gamma(x)\}$, we obtain $x \in U_\delta$ if and only if $\gamma'(x, -\delta) > 0$.

(2) Let x be such that $(p_0, \delta) \leqslant 0$ for some $p_0 \in \partial\gamma(x)$. By (1) we can suppose that $(p_0, \delta) = 0$. The hyperplane defined by $\{p/(p, \delta) = 0\}$ is a supporting hyperplane of $\partial\gamma(x)$ and such a hyperplane contains at least an extreme point of $\partial\gamma(x)$ whence contains an extreme point of B^0; then $x \in V_\delta$.

The converse is immediate and we terminate the proof as in (1). ∎

COROLLARY 3.1. *Let $\delta \neq 0$. Then*

(1) *V_δ is closed.*

(2) *If the unit ball B is smooth, then $U_\delta \backslash \{0\}$ is open.*

Proof. (1) V_δ is closed since $\gamma'(x, \delta)$ is an upper semi-continuous function of x.

(2) If B is smooth, then $\partial\gamma(x)$ is a singleton at $x \neq 0$, and $x \to \partial\gamma(x)$ is continuous on $X \backslash \{0\}$. Hence $U_\delta \backslash \{0\}$ is open. ∎

The next theorem emphasizes the relationship between Q_δ, U_δ, and V_δ.

THEOREM 3.1. *Let $\delta \neq 0$. We have*

(i) $\mathrm{int}(U_\delta) = \mathrm{int}(Q_\delta) \subset U_\delta \subset Q_\delta \subset \overline{\mathrm{int}(U_\delta)} = \overline{\mathrm{int}(Q_\delta)} = \bar{U}_\delta = \bar{Q}_\delta$,

(ii) $\bar{Q}_\delta \subset V_\delta = \bar{P}_{-\delta}$.

Proof. By Proposition 3.1 we have $x \in U_\delta$ (resp. $x \in V_\delta$) if and only if $-\delta$ is a direction along which the directional derivative of the norm at x is positive (resp. nonnegative).

By the definition of Q_δ and Proposition 1.4 we have $x \in Q_\delta$ (resp. $x \in \bar{P}_{-\delta}$) if and only if $-\delta$ is a strict ascent (resp. ascent) direction from x for the norm.

By Proposition 1.5 we have $x \in \operatorname{int}(Q_\delta)$ if and only if δ is a strict descent direction from x for the norm.

These properties imply at once, $U_\delta \subset Q_\delta \subset V_\delta$ and making use of the convexity of the function $t \to |x - t\delta|$, $\operatorname{int}(Q_\delta) \subset U_\delta$ and $V_\delta = \bar{P}_{-\delta}$.

As $\operatorname{int}(Q_\delta)$ is an open set included in U_δ and which contains $\operatorname{int}(U_\delta)$, we have $\operatorname{int}(U_\delta) = \operatorname{int}(Q_\delta)$. As V_δ is closed, we have $\bar{Q}_\delta \subset V_\delta$.

Finally by Corollary 1.1, we have $\overline{\operatorname{int}(Q_\delta)} = \bar{Q}_\delta$. ∎

The equality $\bar{P}_{-\delta} = V_\delta$, with Theorem 2.1, gives a good description of $C(A)$ when A is compact. To use effectively the characterization of $c(A)$ and $E(A)$ when A is compact and when X satisfies (B_δ) for every $\delta \neq 0$ (Theorem 2.1), some complementary results are needed.

THEOREM 3.2. *Let $\delta \neq 0$*

(1) *If X is strictly convex, then $Q_\delta = V_\delta = \bar{P}_{-\delta}$.*

(2) *If X is two dimensional, then $Q_\delta = \bar{U}_\delta$.*

(3) *If X has the* Diff–Max *property—and therefore if X is polyhedral—then $Q_\delta = U_\delta$.*

Proof. (1) is a straightforward consequence of Corollary 1.2 and Theorem 3.1.

(2) is a consequence of the closedness of Q_δ (Proposition 1.7 and Theorem 1.1) and of Theorem 3.1.

(3) is a consequence of Theorem 3.1 and of the following lemma:

LEMMA 3.1. *Let $\delta \neq 0$. If X has the* Diff–Max *property, then U_δ is closed.*

Proof. Let $x \notin U_\delta$; from Proposition 3.1 for every $p \in \partial\gamma(x)$ we have $(p, \delta) \geqslant 0$. Thanks to the Diff–Max property, $\partial\gamma(y)$ is included in $\partial\gamma(x)$ for every y belonging to some neighbourhood W of x. Hence for every $y \in W$, we have $(q, \delta) \geqslant 0$ when q is in $\partial\gamma(y)$, which means $W \cap U_\delta = \varnothing$. ∎

3.2. *Example of Q_δ Sets in a Strictly Convex Space*

When X is strictly convex, by Theorem 3.2 we have $Q_\delta = V_\delta = \bar{P}_{-\delta}$ for every $\delta \neq 0$, and by Theorems 2.1 and 1.1, the equality $c(A) = C(A)$ holds for every compact set A. Thus we find again a result yet given in Section 1 (Proposition 1.3).

As an example, we study the real vector space $L^r(T, \mu)$ where (T, μ) is a positive measure space and $1 < r < +\infty$. Let s be such that $(1/r) + (1/s) = 1$ and let us denote $\operatorname{sgn}(g)$ the function which takes the value $+1$ (resp. -1

and 0) when g is positive (resp. negative and null). Then for $\delta \in L^r(T, \mu)$, $\delta \neq 0$, we have

$$Q_\delta = \left\{ k \operatorname{sgn}(p) |p|^{r/s} / k \geqslant 0,\ p \in L^s(T, \mu),\ \int_T p(t)\, \delta(t)\, d\mu(t) \leqslant 0 \right\}.$$

Indeed each $p \in L^s(T, \mu)$ of norm 1 is an extreme point of the unit ball of $L^s(T, \mu)$ and, as a consequence of the equality case in the Hölder inequality, we have $N(p) = \{k \operatorname{sgn}(p) |p|^{r/s} / k \geqslant 0\}$.

When A is compact, thanks to Theorem 2.1 we obtain $x \in c(A)$ if and only if, for every $\delta \in L^r(T, \mu)$, there exists $a \in A$ such that

$$\int_T \operatorname{sgn}(a(t) - x(t)) |a(t) - x(t)|^{r/s}\, \delta(t)\, d\mu(t) \leqslant 0.$$

3.3. *Examples of Q_δ Sets in a Polyhedral Space*

In a polyhedral space X, by Theorem 3.2 we have $Q_\delta = U_\delta$ for each $\delta \neq 0$.

Each $N(p)$ with $p \in \operatorname{Ext}(B^0)$ is a convex cone generated by a face of B whose dimension is $n - 1$, and Q_δ is a finite union of such cones.

We denote by I the set $I = \{1, 2, \ldots, n\}$ and as examples we study:

(a) *$\mathbb{R}^n$ with the l^1-norm*

PROPOSITION 3.2. *Let $\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{R}^n$, $\delta \neq 0$ and let $I^0 = \{i \in I / \delta_i = 0\}$. For $\mathscr{K}$ and $\mathscr{J}$, complementary subsets of $I \backslash I^0$, let $\Gamma_{\mathscr{J}\mathscr{K}} = \{x \in \mathbb{R}^n / x_i \geqslant 0$ if $i \in \mathscr{J},\ x_i \leqslant 0$ if $i \in \mathscr{K}\}$. Then, we have*

$$Q_\delta = \bigcup \left\{ \Gamma_{\mathscr{J}\mathscr{K}} / (\mathscr{J}, \mathscr{K}) \text{ such that } \sum_{i \in \mathscr{J}} \delta_i < \sum_{i \in \mathscr{K}} \delta_i \right\}.$$

Proof. The extreme points of B^0 are the 2^n vectors defined by $p = (p_1, p_2, \ldots, p_n)$ with $p_i = +1$ or $p_i = -1$. Let p be such a vector; then

$$N(p) = \{x \in \mathbb{R}^n / x_i \geqslant 0 \text{ if } p_i = +1 \text{ and } x_i \leqslant 0 \text{ if } p_i = -1\}.$$

Let $\mathscr{J}$ and $\mathscr{K}$ be two complementary subsets of $I \backslash I^0$ and let $\pi_{\mathscr{J}\mathscr{K}}$ be the following subset of $\operatorname{Ext}(B^0)$:

$$\pi_{\mathscr{J}\mathscr{K}} = \{q = (q_1, q_2, \ldots, q_n) / q_i = +1 \text{ if } i \in \mathscr{J}, q_i = -1 \text{ if } i \in \mathscr{K} \text{ and } q_i = +1 \text{ or } -1 \text{ if } i \in I^0\}.$$

Then $p \in \pi_{\mathscr{J}\mathscr{K}}$ satisfies $(p, \delta) < 0$ if and only if $\sum_{i \in \mathscr{J}} \delta_i < \sum_{i \in \mathscr{K}} \delta_i$, and the result follows since we have $Q_\delta = U_\delta$ and $\Gamma_{\mathscr{J}\mathscr{K}} = \bigcup \{N(p),\ p \in \pi_{\mathscr{J}\mathscr{K}}\}$. ∎

Remark 3.1. If $n=2$, thanks to Proposition 3.2 we obtain only eight Q_δ sets, the four quarters of plane: $\mathbb{R}^+ \times \mathbb{R}^+$, $\mathbb{R}^+ \times \mathbb{R}^-$, $\mathbb{R}^- \times \mathbb{R}^+$, $\mathbb{R}^- \times \mathbb{R}^-$ and the four halfplanes: $\mathbb{R} \times \mathbb{R}^+$, $\mathbb{R} \times \mathbb{R}^-$, $\mathbb{R}^- \times \mathbb{R}$, $\mathbb{R}^+ \times \mathbb{R}$. As every one of these four quarters of plane is included in one of these four halfplanes, according to Theorem 2.1, the set of strictly efficient points can be defined by

$$c(A) = \{x \in \mathbb{R}^2 / A \cap (x + Q_\delta) \neq \emptyset, \delta = (\varepsilon/2, \varepsilon'/2), \varepsilon = \pm 1, \varepsilon' = \pm 1\}$$

and we obtain exactly the description of the rectangular hull defined by Juel [7] in $\mathbb{R}^2$.

(a) *$\mathbb{R}^n$ with the l^∞-norm.*

PROPOSITION 3.3. *Let $\delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}^n$, $\delta \neq 0$ and let $I^0 = \{i \in I / \delta_i = 0\}$, $I^+ = \{i \in I / \delta_i > 0\}$, $I^- = \{i \in I / \delta_i < 0\}$. For each $k \in I$, let*

$$N_k^+ = \{x \in \mathbb{R}^n / x_k = |x|_\infty\},$$
$$N_k^- = \{x \in \mathbb{R}^n / -x_k = |x|_\infty\}.$$

Then, we have

$$Q_\delta = \bigcup (N_k^+ / k \in I^-) \cup \bigcup (N_k^- / k \in I^+).$$

Proof. The extreme points of B^0 are the $2n$ vectors of $\mathbb{R}^n$: e_i and $-e_i$, with $i \in I$, where e_i has all its coordinates null except the ith which is 1. Then $N_k^+ = N(p_k^+)$ and $N_k^- = N(p_k^-)$ and as $Q_\delta = U_\delta$, the result follows since the condition $(p_k^+, \delta) < 0$ (resp. $(p_k^-, \delta) < 0$) is equivalent to $k \in I^-$ (resp. $k \in I^+$). ∎

The cone N_k^+ (resp. N_k^-) is the cone generated by the face of the unit cube $[-1, +1]^n$ the equation of which is $x_k = 1$ (resp. $x_k = -1$).

4. Hull and Closure Properties

In some cases (two-dimensional space, $\mathbb{R}^n$ with l^∞-norm, $\mathbb{R}^3$ with l^1-norm) it is possible to improve Theorem 2.1 by giving a characterization of $c(A)$, $E(A)$, and $C(A)$ making use of a subset $\mathcal{Q}$ of the family $(Q_\delta)_{\delta \neq 0}$, instead of the whole family.

This improvement is useful practically and theoretically:

(i) The explicit determination of the sets of efficient points will be easier if it requires only a little number of cones Q; it happens especially in a two-dimensional space, a situation which is relevant to location theory (see, e.g., [8, 12, 14]).

(ii) It will be shown that the cones $Q \in \mathcal{Q}$, in the three aforementioned cases, are all closed and convex, and then, that the operators c, E, and C, working on the compact sets of X have a hull property and a closure property (these words are explained in Remark 4.1).

The first subsection gives general results on the hull and the closure properties. The examples $\mathbb{R}^n$ with l^∞-norm and $\mathbb{R}^3$ with l^1-norm are developed in the second subsection and the results peculiar to $\mathbb{R}^2$ are exposed in the third subsection.

4.1. *General Results*

The following notations will be used in this subsection. Let $\mathcal{Q}$ be a set of cones Q with vertex at zero and whose interior is nonempty. Let $\mathcal{T}$ be the set of cones $\mathcal{T} = \{\operatorname{int}(Q)^c / Q \in \mathcal{Q}\}$.

When A is a subset of X, we define

$$\mathcal{Q}(A) = \{x \in X / \forall Q \in \mathcal{Q}, A \cap (x + Q) \neq \varnothing\},$$

$$\mathcal{T}(A) = \bigcap (y + T / y \in X, T \in \mathcal{T}, A \subset y + T),$$

$$\mathcal{Q}'(A) = \{x \in X / \forall Q \in \mathcal{Q}, A \cap (x + Q) \neq \varnothing \text{ or } A \cap (x - \operatorname{int} Q) = \varnothing\},$$

$$\mathcal{T}'(A) = \bigcap (y + T / y \in X, T \in \mathcal{T}, A \subset y + T \text{ and } A \cap (y - T^c) \neq \varnothing.$$

THEOREM 4.1. *If each $Q \in \mathcal{Q}$ is convex and closed, then we have for each compact subset A of X,*

$$\mathcal{Q}(\mathcal{Q}(A)) = \mathcal{Q}(A) = \mathcal{T}(A) = \mathcal{T}(\mathcal{T}(A)),$$

$$\mathcal{Q}'(\mathcal{Q}'(A)) = \mathcal{Q}'(A) = \mathcal{T}'(A) = \mathcal{T}'(\mathcal{T}'(A)).$$

Remark 4.1. The sets $\mathcal{Q}(A)$ and $\mathcal{Q}'(A)$ will be identified later with $c(A)$ and $E(A)$, respectively, under appropriate assumptions on A and X.

The statement $\mathcal{Q}(A) = \mathcal{T}(A)$ (resp. $\mathcal{Q}'(A) = \mathcal{T}'(A)$) expresses a hull property to the extent that $\mathcal{T}(A)$ (resp. $\mathcal{T}'(A)$) is the intersection of subsets containing A and with a particular other property. The analogy with the closed convex hull of A (defined as the intersection of closed halspaces containing A) allows this formulation.

The statement $\mathcal{Q}(\mathcal{Q}(A)) = \mathcal{Q}(A)$ (resp. $\mathcal{Q}'(\mathcal{Q}'(A)) = \mathcal{Q}'(A)$) expresses a closure property.

The proof of the theorem will be a consequence of the following lemmas, in which only the first equality or inclusion is proved, the proof of the second being omitted.

LEMMA 4.1. *If each* $Q \in \mathcal{Q}$ *is convex then for each* $A \subset X$, $\mathcal{Q}(A) \subset \mathcal{T}(A)$ *and* $\mathcal{Q}'(A) \subset \mathcal{T}'(A)$.

Proof. Let $x \notin \mathcal{T}(A)$; then there exist $Q \in \mathcal{Q}$ and $x \in X$ such that $A \cap (y + \text{int}(Q)) = \emptyset$ and $x \in y + \text{int}(Q)$. As Q is a convex cone and $\text{int}(Q) \neq \emptyset$ we have $Q + \text{int}(Q) = \text{int}(Q)$. Hence $x + Q \subset y + \text{int}(Q)$ and then $A \cap (x + Q) = \emptyset$, i.e., $x \notin \mathcal{Q}(A)$. ▮

LEMMA 4.2. *If each* $Q \in \mathcal{Q}$ *is closed and if* A *is a compact set, then* $\mathcal{T}(A) \subset \mathcal{Q}(A)$ *and* $\mathcal{T}'(A) \subset \mathcal{Q}'(A)$.

Proof. Let $x \notin \mathcal{Q}(A)$; then there exists $Q \in \mathcal{Q}$ such that $A \cap (x + Q) = \emptyset$. As A is compact and Q is closed, there exists a ball $B(x, r)$ $(r > 0)$ such that for every $u \in B(x, r)$, $A \cap (u + Q) = \emptyset$. As Q is a cone, whose interior is non empty, there exists $z \in (x + \text{int}(Q)) \cap B(x, r)$. Then $y = 2x - z$ satisfies $x \in y + \text{int}(Q)$ and $A \cap (y + \text{int}(Q)) = \emptyset$; this means $x \notin \mathcal{T}(A)$. ▮

LEMMA 4.3. *If each* $Q \in \mathcal{Q}$ *is convex, then for each* $A \subset X$, $\mathcal{Q}(\mathcal{Q}(A)) = \mathcal{Q}(A)$ *and* $\mathcal{Q}'(\mathcal{Q}'(A)) = \mathcal{Q}'(A)$.

Proof. We have $\mathcal{Q}(A) \subset \mathcal{Q}(\mathcal{Q}(A))$ since $B \subset \mathcal{Q}(B)$ for each subset B of X.

Let $x \notin \mathcal{Q}(A)$. Then there exists $Q \in \mathcal{Q}$ such that $(x + Q) \cap A = \emptyset$. Let $y \in x + Q$. As Q is a convex cone the interior of which is non empty, we have $y + Q \subset x + Q$, whence $A \cap (y + Q) = \emptyset$, i.e., $y \notin \mathcal{Q}(A)$. Thus $(x + Q) \cap \mathcal{Q}(A) = \emptyset$ which means that $x \notin \mathcal{Q}(\mathcal{Q}(A))$. ▮

LEMMA 4.4. *For each* $A \subset X$, *we have* $\mathcal{T}(\mathcal{T}(A)) = \mathcal{T}(A)$ *and* $\mathcal{T}'(\mathcal{T}'(A)) = \mathcal{T}'(A)$.

Proof. Let $x \in X$ and $T \in \mathcal{T}$. Then $A \subset x + T$ if and only if $\mathcal{T}(A) \subset x + T$. The equality $\mathcal{T}(\mathcal{T}(A)) = \mathcal{T}(A)$ follows from definition. ▮

4.2. *Examples of Hull and Closure Properties for the Set of Strictly Efficient Points*

FIRST EXAMPLE. $\mathbb{R}^n$ with l^∞-norm.

We denote by $\mathcal{Q}$ the set of $2n$ cones $\{N_k^+\}_{1 \leqslant k \leqslant n} \cup \{N_k^-\}_{1 \leqslant k \leqslant n}$ as defined in Proposition 3.3; thus $\mathcal{Q}$ is a subset of the family $(Q_\delta)_{\delta \neq 0}$. Moreover each Q_δ contains at least a cone $Q \in \mathcal{Q}$. Then if A is a compact set, Theorems 1.1 and 2.1 imply the equality $c(A) = \mathcal{Q}(A)$.

As each cone $Q \in \mathcal{Q}$ is closed and convex, Theorem 4.1 implies the closure property $c(c(A)) = c(A)$.

Denoting by T_k^+ (resp. T_k^-) the cone generated by the 2^{n-1} $(n-1)$-faces of the unit cube, different from the one whose equation is $x_k = 1$ (resp.

$x_k = -1$), we obtain $\mathscr{T} = \{T_k^+\}_{1 \leqslant k \leqslant n} \cup \{T_k^-\}_{1 \leqslant k \leqslant n}$ and we have the hull property $c(A) = \mathscr{T}(A)$.

SECOND EXAMPLE. $\mathbb{R}^3$ with l^1-norm.

Let us denote by $\mathscr{Q}$ the set of the twelve cones Q_δ of $\mathbb{R}^3$ associated, according to Proposition 3.2, to the vectors $(\varepsilon, \varepsilon', 0)$, $(\varepsilon, 0, \varepsilon')$, $(0, \varepsilon, \varepsilon')$ with $\varepsilon = \pm 1$ and $\varepsilon' = \pm 1$. For instance, if $\delta = (\varepsilon, \varepsilon', 0)$ then $Q_\delta = \varepsilon\mathbb{R}^- \times \varepsilon'\mathbb{R}^- \times \mathbb{R}$. Thus each $Q \in \mathscr{Q}$ is a "quarter of space" and therefore is convex and closed.

Moreover we have the following property: each cone Q_δ, $\delta \neq 0$, contains a cone $Q \in \mathscr{Q}$. Indeed, let $Q_\delta \notin \mathscr{Q}$; then δ is colinear with none of the twelve above-mentioned vectors. This means that the plane orthogonal to δ and passing through 0 contains zero vertex or two symmetric vertices of the ball B, which is the unit cube $[-1, +1]^3$.

In the first case (resp. in the second case) the set $\{p \in \operatorname{Ext}(B^0)/(p, \delta) < 0\}$ has four (resp. three) elements which are the vertices of a square which is a 2-face of B^0. At least two such vertices define a 1-face F_0 of B^0. Let δ_0 be a vector orthogonal to the plane passing through zero and containing F_0, such that $(p, \delta_0) \leqslant 0$ for $p \in \operatorname{Ext}(B^0)$ provided that $(p, \delta) < 0$. Then δ_0 is colinear with one of the twelves vectors defining $\mathscr{Q}$ and whence there exists $Q \in \mathscr{Q}$ such that $Q \in Q_{\delta_0}$.

The family $\mathscr{T}$ contains twelve elements which are "three quarters of space" of type $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$.

Now, if A is a compact set, Theorems 1.1 and 2.1 imply the equality $c(A) = \mathscr{Q}(A)$. As each $Q \in \mathscr{Q}$ is closed and convex, Theorem 4.1 gives the closure property $c(c(A)) = c(A)$ and the hull property $c(A) = \mathscr{T}(A)$.

Remark 4.2. In the two cases studied above there exist cones Q which are not convex. However, considering the subfamily $\mathscr{Q}$, whose each element is a closed convex cone, we obtain for the set of strictly efficient points a closure property and a hull property.

4.3. *Hull and Closure Properties in a Two-Dimensional Space*

In this subsection if $u \neq 0$ and $v \neq 0$ are two noncolinear vectors we denote by $K(u, v)$ the closed convex cone, generated by the segment $[u, v]$ and by $H^+(u)$ and $H^-(u)$ the two closed halfplanes bounded by the line through 0 and u.

A geometrical description of the Q_δ sets is easily obtained in dimension two.

Let $\delta \neq 0$ and let D and D' the two supporting lines to B parallel to δ. The set $F = D \cap B$ (resp. $D' \cap B$) is a face of B. If F is a singleton $\{u\}$, Q_δ is the closed halfplane $H^+(u)$ or $H^-(u)$ which does not contain δ; otherwise F is a segment $[u, v]$, with $v = u - k\delta$, $k > 0$ and Q_δ is the cone $K(v, -u)$.

Hence, when δ is parallel to a 1-face of B, Q_δ is a closed cone limited by a salient angle and $\bar{P}_\delta$ is a closed cone limited by a reentrant angle. When δ is not parallel to a 1-face of B, then Q_δ and $\bar{P}_\delta$ are closed halfplanes.

Consequently if B is a polytope which has $2p$ extreme points, and if δ is parallel to a 1-face of B, then Q_δ is the closed convex cone generated by 0 and $(p-1)$ consecutive 1-faces of B.

The two following theorems deal respectively with the set of strictly efficient or efficient points and with the set of weakly efficient points.

THEOREM 4.2. *In a two-dimensional normed space let $\mathscr{Q}$ be the set of cones defined as follows*:

(i) *with each 1-face $[u, v]$ of B, the two cones $K(u, -v)$ and $K(v, -u)$ are associated,*

(ii) *with each exposed point u of B which does not belong to a 1-face of B, the two closed halfplanes $H^+(u)$ and $H^-(u)$ are associated.*

Then if A is a compact set

(1) $c(A) = \mathscr{Q}(A) = \mathscr{T}(A)$,

(2) $E(A) = \mathscr{Q}'(A) = \mathscr{T}'(A)$ *provided that B contains at least six extreme points.*

Proof. First, we prove that every Q_δ with $\delta \neq 0$, contains a cone $Q \in \mathscr{Q}$. We have only to consider the case $Q_\delta = H^+(u)$ (resp. $Q_\delta = H^-(u)$) where $u \in B$ belongs to a 1-face $[u, v]$ of B. But in this case Q_δ contains the cone $K(-u, v)$ (resp. $K(u, -v)$).

Then from Theorem 2.1 we obtain $c(A) = \mathscr{Q}(A)$ and as each $Q \in \mathscr{Q}$ is closed and convex, Theorem 4.1 implies $\mathscr{Q}(A) = \mathscr{T}(A)$.

We prove now that $\mathscr{Q}'(A) \subset E(A)$, the converse being immediate. Suppose that there exists $x \in \mathscr{Q}'(A)$ and $x \notin E(A)$. Then there exists $\delta \neq 0$ such that $A \cap (x + Q_\delta) = \varnothing$ and $A \cap (x - \operatorname{int}(Q_\delta)) \neq \varnothing$ and for every $Q \in \mathscr{Q}$: $A \cap (x + Q) \neq \varnothing$ or $A \cap (x - \operatorname{int}(Q)) = \varnothing$. As Q_δ does not belong to $\mathscr{Q}$, Q_δ is a closed halfplane defined by $Q_\delta = H^+(u)$ where u belongs to a 1-face $[u, v]$ of B and then Q_δ contains the cone $Q_0 = K(-u, v)$. Hence we have necessarily $A \cap (x - \operatorname{int}(Q_0)) = \varnothing$.

Now we have two cases since B contains at least six extreme points:

(a) B contains at least an exposed point which does not belong to a 1-face of B.

(b) At least a 1-face of B does not contain v.

In the two cases we obtain a cone $Q \in \mathscr{Q}$ such that $A \cap (x + Q) = \varnothing$ and $A \cap (x - \operatorname{int}(Q)) \neq 0$ and the result follows by contradiction. ∎

Remark 4.3. If B has only four extreme points, the assertion (2) of Theorem 4.2 fails as it is shown by the following example: let A be the set $\{0\} \times [0, 1]$ in $\mathbb{R}^2$ with the l^1-norm; then $E(A) = A$ and $\mathcal{Q}(A) = \{0\} \times \mathbb{R}$.

However, in this case we have [14] $E(A) = \mathcal{Q}(A) \cap C(A)$ for every compact set A.

THEOREM 4.3. *In a two-dimensional normed space let $\mathcal{P}$ be the set of halfplanes $H^+(u)$ or $H^-(u)$ where $u \in \operatorname{Ext}(B)$. Then if A is a compact set*

(1) $C(A) = \{x \in X / \forall P \in \mathcal{P}, A \cap (x + P) \neq \emptyset\}$,

(2) $C(A) = \bigcap (y + P / y \in X, P \in \mathcal{P}, A \subset y + P)$,

(3) $C(C(A)) = C(A)$.

Proof. It is sufficient to prove (1). Assertions (2) and (3) are easy consequences of Theorem 4.1.

Assertion (2) gives a very useful characterization of $C(A)$ as intersection of a well-defined family of halfplanes.

Let $x \in X$ such that for each $P \in \mathcal{P}$, $A \cap (x + P) \neq \emptyset$ and let $\delta \neq 0$. If δ is parallel (resp. nonparallel) to a 1-face of B, $\bar{P}_\delta$ contains (resp. is equal to) a halfplane $H^+(u)$ or $H^-(u)$ with $u \in \operatorname{Ext}(B)$. Then $A \cap (x + \bar{P}_\delta) \neq \emptyset$ which means that $x \in C(A)$.

Conversely let $x \in C(A)$ and let $P \in \mathcal{P}$. We can supose that $P = H^+(u)$ with $u \in \operatorname{Ext}(B)$. If u is exposed, there exists $\delta \neq 0$ such that $P = \bar{P}_\delta$ and then $A \cap (x + P) \neq \emptyset$. If u is not exposed, there exists a sequence $\{u_n\}_{n \in \mathbb{N}}$ of exposed points of B which converges to u, and for each integer $n \in \mathbb{N}$, there exists $a_n \in A$ such that $a_n - x \in H^+(u_n)$. As A is compact, there exists a subsequence of $\{a_n\}_{n \in \mathbb{N}}$ converging to a point $a \in A$ and we have $a - x \in H^+(u)$, i.e., $A \cap (x + P) \neq \emptyset$. ∎

Remark 4.4. If B is a polytope, the set $\mathcal{Q}$ contains only a finite number of cones, which are all of type $K(u, -v)$ and the set $\mathcal{P}$ contains only a finite number of halfplanes. The results of Theorems 4.2 and 4.3 are then doubtless useful to find good algorithms to build the sets $c(A)$, $E(A)$, and $C(A)$ with respect to a finite subset A. On this topic, see [12] and [14].

COROLLARY 4.1. *Let X be a two-dimensional space and let A be a compact set in X.*

(1) *We have $c(A) \subset \operatorname{co}(A) \subset C(A)$.*

(2) *If X is strictly convex, then $c(A) = \operatorname{co}(A) = C(A)$.*

Proof. (1) Let $x \notin \operatorname{co}(A)$; then there exists a closed halfplane H such that $A \cap (x + H) = \emptyset$. It is clear that H contains a Q_δ set, with $\delta \neq 0$; then

$A \cap (x + Q_\delta) = \varnothing$ which means $x \notin c(A)$. The inclusion $\text{co}(A) \subset C(A)$ is an immediate consequence of Proposition 4.2.

Note that no general inclusion relation holds between $\text{co}(A)$ and $E(A)$ (see examples in Sect. 1.2).

(2) By the strict convexity assumption, the set of closed halfplanes of X is equal to the family of $\bar{P}_\delta$ sets and the result follows. ∎

Remark 4.5. If two normes γ_1 and γ_2 have the same set $\mathscr{Q}$ according to Proposition 4.1 (resp. $\mathscr{P}$ according to Proposition 4.2) then for every compact set A, we have $c_1(A) = c_2(A)$ and $E_1(A) = E_2(A)$ (resp. $C_1(A) = C_2(A)$).

This happens, for instance, when the norms are polyhedral and when the two unit balls have the same direction of extreme points. The corresponding result for $E(A)$ with a finite subset A is given in [12].

In the same way if the sets $\mathscr{Q}_1$ and $\mathscr{Q}_2$ associated with γ_1 and γ_2 satisfy $\mathscr{Q}_1 \subset \mathscr{Q}_2$, then, for every compact set A, $c_1(A) \subset c_2(A)$ and $E_1(A) \subset E_2(A)$. Some results of [12] are relevant to this remark.

Remark 4.6. Some characterizations of strict convexity of X related to this topic has been yet given.

(i) A two-dimensional normed space is strictly convex if and only if for every bounded subset A of X we have $C(A) = \overline{\text{co}}(A)$ (Phelps [10]).

(ii) A two-dimensional normed space is strictly convex if and only if for every bounded subset A of X we have $E(A) = \overline{\text{co}}(A)$ (Plastria [11]).

(iii) A normed space is strictly convex if and only if, for every two-points set A, we have $\text{co}(A) \subset c(A)$ (Beauzamy and Maurey [1]).

Thanks to Corollary 4.1 and a characterization of Beauzamy and Maurey another result is obtained:

COROLLARY 4.2. *A two-dimensional normed space is strictly convex if and only if, for every compact subset A of X we have* $\text{co}(A) \subset c(A)$.

Note that the same wording appears in [9] without the condition of two dimensionality. However, this restriction is necessary; the following example shows that in some strictly convex three-dimensional space, the inclusion $\text{co}(A) \subset c(A)$ does not hold for some three-points set.

COUNTEREXAMPLE. The space $\mathbb{R}^3$ is equipped with orthogonal axes $0x$, $0y$, $0z$.

Let C be the cylinder whose basis is the circle centered at 0 with radius 1 in the plane $x0y$ and whose generating lines are parallel to $0z$. Let Γ be the curve given by $x = \cos\theta$, $y = \sin\theta$, $z = k\sin(3\theta)$ where k is positive and small enough.

The existence of a strictly convex ball B, symmetric with respect to 0 and tangent to C along Γ can be proved.

Let $A = \{a_1, a_2, a_3\}$ be the three-points set with $a_1 = (\sqrt{3}/2,\ 1/2,\ 0)$, $a_2 = (-\sqrt{3}/2,\ 1/2,\ 0)$, $a_3 = (0, -1, 0)$ and let $\delta = (-1, 0, 0)$.

The ball B is the unit ball for a norm in $\mathbb{R}^3$. The set Q_δ is the cone with vertex at 0, generated by a "halfball" limited by the curve Γ and which does not contain δ. Then $Q_\delta \cap A = \emptyset$, i.e., $0 \notin c(A)$. However, $0 \in \operatorname{co}(A)$.

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