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Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation

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ABSTRACT

In this paper, the initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation over an open bounded domain $G \times (0, T)$, $G \in \mathbb{R}^n$ are considered. Based on an appropriate maximum principle that is formulated and proved in the paper, too, some a priory estimates for the solution and then its uniqueness are established. To show the existence of the solution, first a formal solution is constructed using the Fourier method of the separation of the variables. The time-dependent components of the solution are given in terms of the multinomial Mittag-Leffler function. Under certain conditions, the formal solution is shown to be a generalized solution of the initial-boundary-value problem for the generalized time-fractional multi-term diffusion equation that turns out to be a classical solution under some additional conditions. Another important consequence from the maximum principle is a continuously dependence of the solution on the problem data (initial and boundary conditions and a source function) that – together with the uniqueness and existence results – makes the problem under consideration to a well-posed problem in the Hadamard sense.

1. Introduction

The partial differential equations of fractional order begun to play an important role in particular in modeling of the so-called anomalous phenomena and in the theory of the complex systems (see e.g. [4,8,10,12–14,23,24,29,33] and references therein) during the last few decades. The recent book [37] is completely devoted to different applications of the fractional differential equations in physics, chemistry, technique, astrophysics, etc. and contains several dozens of interesting case studies. In this connection, the so-called time-fractional diffusion equation that is obtained from the classical diffusion equation by replacing the first-order time derivative by a fractional derivative of order α with $0 < \alpha \leqslant 1$ has to be especially mentioned. In the paper [35], the Green function for the time-fractional diffusion equation was shown to be a probability density with the mean square displacement proportional to t^α. As a consequence, the time-fractional diffusion equation appeared to be a suitable mathematical model for the so-called sub-diffusion processes and thus became important and useful for different applications.

By the attempts to describe some real processes with the equations of the fractional order, several researches confronted with the situation that the order α of the fractional derivative from the corresponding model equations did not remain constant and changed, say, in the interval from 0 to 1, from 1 to 2 or even from 0 to 2. To manage these phenomena, several approaches were suggested. One of them introduces the fractional derivatives of the variable order, i.e., the derivatives with the order that can change with the time or/and depending on the spatial coordinates (for the definitions and applications see e.g. [6,17,32]).

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On the other hand, in some recent publications (see e.g. [2–4,31,36] and references therein) the sub-diffusion processes with the mean square displacement with a logarithmic growth have been introduced. One of the approaches for modeling of such processes is to employ the time-fractional diffusion equations of distributed order. A derivative of the distributed order is introduced as a mean value of the fractional derivatives with the orders from an interval (say, $[0, 1]$) weighted with a positive weight function $\omega(\alpha)$. One important particular case of the time-fractional diffusion equation of distributed order is the multi-term time-fractional diffusion equation that is considered in this paper. In this case the weight function is taken in form of a finite linear combination of the Dirac δ-functions with the positive weight coefficients (see e.g. [3,22]).

As to the mathematical theory of the partial differential equations of fractional order in general and of the time-fractional diffusion equation in particular, it is still far away from to be at least nearly so complete as the one of the PDEs. In the literature, mainly the initial-value problems for these equations were considered until now (see e.g. [9,11,14,16,25,33,40]). As to the boundary-value or initial-boundary-value problems, they were mainly investigated for the equations with the constant coefficients and in the one-dimensional case (see e.g. [1,11,30,34]).

In the recent papers [20,21] by the author, the case of the generalized time-fractional diffusion equation with the variable coefficients and over an open bounded n-dimensional domain was considered. This equation is obtained from the classical diffusion equation by replacing the first-order time derivative by a fractional derivative of order α $(0 < \alpha \leqslant 1)$ and the Laplace operator by a linear second-order differential operator $\operatorname{div}(p(x) \operatorname{grad} u) - q(x)u$, $x \in G \subset R^n$ of the elliptic type with the variable coefficients. In the paper [41], some more general linear and quasilinear evolutionary partial integro-differential equations of second order were investigated. In particular, the global boundedness of appropriately defined weak solutions and a maximum principle for the weak solutions of such equations were established by employing a different technique compared to the one used in the papers [20,21]. In the paper [28], the stochastic analogues for the fractional Cauchy problems in a bounded domain with the Dirichlet boundary conditions were introduced and investigated. A connection between these problems and an iterated Brownian motion in a bounded domain was established in the paper, too. Differential equations of the distributed order and some of their applications were considered e.g. in [15,26,38]. In these papers, mainly the case of the initial-value problems for the equations with the constant coefficients was discussed. However, in the applications one has often to deal with the initial-boundary-problems for the equations with the variable coefficients.

The case of the multi-term time-fractional diffusion-wave equation with the constant coefficients was recently considered in [7]. In the paper [7], a solution of the corresponding initial-boundary-value problem was formally represented in form of the Fourier series via the multivariate Mittag-Leffler function introduced in [18]. Unfortunately, no proofs for the convergence of the series (i.e. no proofs that the obtained formal solutions are in fact solutions) and for the uniqueness of the solution were given in [7]. A proof of the convergence of the series defining the solution of the more general distributed order fractional Cauchy problems on bounded domains can be found in the paper [27].

In this paper, the generalized multi-term time-fractional diffusion equation with the variable coefficients is considered. The employed technique follows the lines of the recent papers [20,21] by the author, where the case of the initial-boundary-value problems for the generalized time-fractional diffusion equation over an open bounded n-dimensional domain was investigated. The rest of the paper is organized as follows. In the second section, the problem we deal with in this paper is formulated and the notion of its solution is introduced in the appropriate spaces of functions. Then the maximum principle formulated and proved earlier by the author for the generalized time-fractional diffusion equation is extended for the case of the multi-term equations. In the proof of the maximum principle, an appropriate extremum principle for the Caputo fractional derivative plays a very important role. In the third section, the maximum principle is applied to show that the initial-boundary-value problem under consideration possesses at most one classical solution. This solution – if it exists – depends continuously on the data given in the problem. In the fourth section, the notion of the generalized solution of the initial-boundary-value problem for the generalized multi-term time-fractional diffusion equation is introduced and some existence results for the generalized solution are given. Using the Fourier method of the separation of the variables, a formal solution of the problem is first constructed. This solution contains in particular the multinomial Mittag-Leffler function that is defined and investigated in the section, too. Based on the properties of the multinomial Mittag-Leffler function, the formal solution is shown to be a generalized solution of the initial-boundary-value problem that turns out to be a classical solution under some additional restrictions, for example, in the case of one spatial variable.

2. Maximum principle

In this section, the maximum principle well known for the PDEs of the elliptic and parabolic type is extended for the case of the generalized multi-term time-fractional diffusion equation over an open bounded domain $G \times (0, T)$, $G \subset R^n$. The generalized multi-term time-fractional diffusion equation is obtained from the classical diffusion equation by replacing the first-order time derivative by a linear combination of the fractional derivatives with different orders less than or equal to one and the second-order spatial derivative by a more general liner second-order differential operator:

$$P(D_t)u(x,t) = L_x\big(u(x,t)\big) + F(x,t), \quad (x,t) \in \Omega_T := G \times (0,T),\ G \subset R^n, \tag{1}$$

where

$$L_x(u) := \operatorname{div}\big(p(x) \operatorname{grad} u\big) - q(x)u, \tag{2}$$

$$p \in C^1(\bar{G}), \quad q \in C(\bar{G}), \quad 0 < p(x), \quad 0 \leqslant q(x), \quad x \in \bar{G}, \tag{3}$$

$$P(D_t) = D_t^{\alpha} + \sum_{i=1}^{m} \lambda_i D_t^{\alpha_i},$$
$$0 < \alpha_m < \cdots < \alpha_1 < \alpha \leqslant 1, \quad 0 \leqslant \lambda_i, \quad i = 1, \ldots, m, \ m \in \mathbb{N}_0 \tag{4}$$

with the fractional derivatives

$$\left(D_t^{\alpha} f\right)(t) := \left(I^{1-\alpha} f'\right)(t), \quad 0 < \alpha \leqslant 1, \tag{5}$$

defined in the Caputo sense, I^{α} being the fractional Riemann–Liouville integral

$$\left(I^{\alpha} f\right)(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, & 0 < \alpha < 1, \\ f(t), & \alpha = 0, \end{cases}$$

and the domain G with the boundary S being open and bounded in R^n.

The operator L_x is the well-known linear elliptic differential operator of the second order

$$L_x(u) = \sum_{k=1}^{n} \left(p(x) \frac{\partial^2 u}{\partial x_k^2} + \frac{\partial p}{\partial x_k} \frac{\partial u}{\partial x_k} \right) - q(x) u,$$

that can be represented in the form

$$L_x(u) = p(x) \Delta u + (\operatorname{grad} p, \operatorname{grad} u) - q(x) u, \tag{6}$$

Δ being the Laplace operator.

In general, Eq. (1) has an infinite number of solutions. In the real world situations that are modeled with Eq. (1), certain conditions that describe an initial state of the corresponding process and the observations of its visible parts ensure the deterministic character of the process. In the paper, the initial-boundary-value problem

$$u|_{t=0} = u_0(x), \quad x \in \bar{G}, \tag{7}$$

$$u|_S = v(x,t), \quad (x,t) \in S \times [0,T] \tag{8}$$

for Eq. (1) is considered. Here S denotes the boundary of the domain G and $\bar{G}$ as usual its closure.

Following [20,21], the notion of the classical solution of the problem (1), (7), (8) is first introduced.

Definition 1. A classical solution of the problem (1), (7), (8) is called a function u defined in the domain $\bar{\Omega}_T := \bar{G} \times [0,T]$ that belongs to the space $CW_T(G) := C(\bar{\Omega}_T) \cap W_t^1((0,T)) \cap C_x^2(G)$ and satisfies both Eq. (1) and the initial and boundary conditions (7)–(8). By $W_t^1((0,T))$ the space of the functions $f \in C^1((0,T])$ such that $f' \in L((0,T))$ is denoted.

If a classical solution to the initial-boundary-value problem (7), (8) for Eq. (1) exists, then the functions F, u_0 and v given in the problem have to belong to the spaces $C(\Omega_T)$, $C(\bar{G})$ and $C(S \times [0,T])$, respectively. In the further discussions, we always suppose these inclusions to be valid.

In the next section, the uniqueness of the solution of the problem (1), (7), (8) is proved. The method used for doing this is based on an appropriate maximum principle for Eq. (1). In the proof of the maximum principle the following extremum principle for the Caputo fractional derivative (5) plays an essential role:

Theorem 1. *Let a function $f \in W_t^1((0,T)) \cap C([0,T])$ attain its maximum over the interval $[0,T]$ at the point $\tau = t_0$, $t_0 \in (0,T]$. Then the Caputo fractional derivative of the function f is non-negative at the point t_0 for any α, $0 < \alpha \leqslant 1$:*

$$0 \leqslant \left(D_t^{\alpha} f\right)(t_0), \quad 0 < \alpha \leqslant 1. \tag{9}$$

For the proof of the theorem we refer the reader to the paper [20]. To illustrate the extremum principle for the Caputo fractional derivative, a simple example is now presented.

Let us consider a family of functions in the form

$$f(t) := -at^2 + bt + c, \quad 0 < a, \ 0 < b \leqslant 2a, \ c \in \mathbb{R} \tag{10}$$

on the closed interval $[0,1]$. The conditions on the parameters of the function f ensure the existence of the maximum point $t = b/(2a)$ that belongs to the interval $[0,1]$. The function f is evidently a $C^1([0,1])$ function and thus fulfills all conditions of the theorem.

Let us now evaluate the Caputo fractional derivative of the function f at the maximum point $t = b/(2a)$. Simple calculations lead to the following formulae for $0 < \alpha \leqslant 1$:

$$\left(D_t^\alpha f\right)(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\left(-\frac{2at}{2-\alpha}+b\right), \qquad \left(D_t^\alpha f\right)(t)\Big|_{t=\frac{b}{2a}} = \frac{(\frac{b}{2a})^{1-\alpha}b}{\Gamma(2-\alpha)}\frac{1-\alpha}{2-\alpha}.$$

To get the expression above, the well-known formulae

$$\left(D_t^\alpha \tau^\beta\right)(t) = \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)}t^{\beta-\alpha}, \quad 0<\beta,\ 0<\alpha\leqslant 1, \tag{11}$$

$$\left(D_t^\alpha C\right)(t) \equiv 0, \quad C \text{ being a constant} \tag{12}$$

for the Caputo fractional derivative were used. Thus

$$\left(D_t^\alpha f\right)(t)\Big|_{t=\frac{b}{2a}} > 0, \quad 0<\alpha<1, \qquad \left(D^\alpha f\right)(t)\Big|_{t=\frac{b}{2a}} = 0, \quad \alpha=1,$$

that is in accordance with the statement of the theorem.

The maximum principle for the generalized multi-term time-fractional diffusion equation (1) is given by the following theorem:

Theorem 2. *Let a function $u \in CW_T(G)$ be a solution of the generalized multi-term time-fractional diffusion equation* (1) *in the domain Ω_T and $F(x,t) \leqslant 0$, $(x,t) \in \Omega_T$.*

Then either $u(x,t) \leqslant 0$, $(x,t) \in \bar{\Omega}_T$ or the function u attains its positive maximum on the bottom or back-side parts $S_G^T := (\bar{G} \times \{0\}) \cup (S \times [0,T])$ of the boundary of the domain Ω_T, i.e.,

$$u(x,t) \leqslant \max\left\{0, \max_{(x,t)\in S_G^T} u(x,t)\right\}, \quad \forall (x,t) \in \bar{\Omega}_T. \tag{13}$$

Proof. The proof follows the method of the contradiction. Let us first suppose that the statement of the theorem does not hold true, i.e., $\exists (x_0,t_0)$, $x_0 \in G$, $0 < t_0 \leqslant T$ with the property

$$u(x_0,t_0) > \max_{(x,t)\in S_G^T}\left\{0, u(x,t)\right\} = M > 0. \tag{14}$$

Denoting the number $u(x_0,t_0) - M > 0$ by ϵ, an auxiliary function

$$w(x,t) := u(x,t) + \frac{\epsilon}{2}\frac{T-t}{T}, \quad (x,t) \in \bar{\Omega}_T$$

is now introduced. The inequalities

$$\begin{aligned} &w(x,t) \leqslant u(x,t) + \frac{\epsilon}{2}, \quad (x,t) \in \bar{\Omega}_T, \\ &w(x_0,t_0) \geqslant u(x_0,t_0) = \epsilon + M \geqslant \epsilon + u(x,t) \\ &\qquad \geqslant \epsilon + w(x,t) - \frac{\epsilon}{2} \geqslant \frac{\epsilon}{2} + w(x,t), \quad (x,t) \in S_G^T \end{aligned}$$

follow now from the conditions of the theorem and the assertion made at the beginning of the proof. The last inequality means that the function w cannot attain its maximum on the part S_G^T of the boundary of the domain Ω_T. Let us denote the maximum point of the continuous function w over the domain $\bar{\Omega}_T$ by (x_1,t_1). Then $x_1 \in G$, $0 < t_1 \leqslant T$. Of course, the inequality

$$w(x_1,t_1) \geqslant w(x_0,t_0) \geqslant \epsilon + M > \epsilon \tag{15}$$

holds true. On the other hand, Theorem 1 and the necessary conditions for the existence of the maximum of the function w over the domain $\bar{\Omega}_T$ lead to the relations

$$\begin{cases} (D_t^\alpha w)(t_1) \geqslant 0, \quad (D_t^{\alpha_i} w)(t_1) \geqslant 0, \quad i = 0,\dots,m, \\ \operatorname{grad} w|_{(x_1,t_1)} = 0, \\ \Delta w|_{(x_1,t_1)} \leqslant 0. \end{cases} \tag{16}$$

Let us now estimate the value of the operator $P(D_t)u - L_x(u) - F$ for the solution u of the generalized multi-term time-fractional diffusion equation (1) at the point (x_1,t_1) defined above. The function u satisfies the relation

$$u(x,t) = w(x,t) - \frac{\epsilon}{2}\frac{T-t}{T}, \quad (x,t) \in \bar{\Omega}_T \tag{17}$$

according to the definition of the function w. Then the formulae (11)–(12) can be applied to get the expression

$$P(D_t)u = P(D_t)w + \frac{\epsilon}{2T}\left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \sum_{i=1}^{m}\lambda_i \frac{t^{1-\alpha_i}}{\Gamma(2-\alpha_i)}\right). \tag{18}$$

The inequalities $0 \leqslant \lambda_i$, $i = 1, \ldots, m$ and the formulae (3), (6), (15)–(18) lead now to the following chain of the equalities and inequalities:

$$\begin{aligned}
&\big(P(D_t)u - \operatorname{div}(p \operatorname{grad} u) + qu - F\big)\big|_{(x_1,t_1)} \\
&\quad = P(D_t)w|_{(x_1,t_1)} + \frac{\epsilon}{2T}\left(\frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)} + \sum_{i=1}^{m}\lambda_i \frac{t_1^{1-\alpha_i}}{\Gamma(2-\alpha_i)}\right) - p\Delta w|_{(x_1,t_1)} \\
&\qquad - (\operatorname{grad} p|_{x_1}, \operatorname{grad} w|_{(x_1,t_1)}) + q(x_1)\left(w(x_1,t_1) - \frac{\epsilon}{2}\frac{T-t_1}{T}\right) - F(x_1,t_1) \\
&\quad \geqslant \frac{\epsilon}{2T}\left(\frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)} + \sum_{i=1}^{m}\lambda_i \frac{t_1^{1-\alpha_i}}{\Gamma(2-\alpha_i)}\right) + q(x_1)\epsilon\left(1 - \frac{T-t_1}{2T}\right) > 0,
\end{aligned}$$

that contradicts the condition of the theorem saying that the function u is a solution of Eq. (1). The obtained contradiction shows that the assumption made at the beginning of the theorem proof cannot be true that completes the proof of the theorem. □

Following the lines of the proof of the maximum principle, the minimum principle can be established if we substitute $-u$ instead of u in the reasoning above.

Theorem 3. *Let a function $u \in CW_T(G)$ be a solution of the generalized multi-term time-fractional diffusion equation* (1) *in the domain Ω_T and $F(x,t) \geqslant 0$, $(x,t) \in \Omega_T$.*

Then either $u(x,t) \geqslant 0$, $(x,t) \in \bar{\Omega}_T$ or the function u attains its negative minimum on the bottom or back-side parts $S_G^T = (\bar{G} \times \{0\}) \cup (S \times [0,T])$ of the boundary of the domain Ω_T, i.e.,

$$u(x,t) \geqslant \min\Big\{0, \min_{(x,t)\in S_G^T} u(x,t)\Big\}, \quad \forall (x,t) \in \bar{\Omega}_T. \tag{19}$$

Let us mention here that the corresponding maximum and minimum principles hold true for the distributed order time-fractional diffusion equation, too. This case has been considered in [22].

3. Uniqueness of the solution

In this section, the maximum principle is employed to prove the uniqueness of the solution of the initial-boundary-value problem (7)–(8) for the generalized multi-term time-fractional diffusion equation (1). Moreover, this solution – if it exists – continuously depends on the problem data.

The main result of the section is given in the following theorem:

Theorem 4. *The initial-boundary-value problem* (7)–(8) *for Eq.* (1) *possesses at most one solution in the sense of Definition* 1. *This solution continuously depends on the data given in the problem, i.e.*

$$\|u - \tilde{u}\|_{C(\bar{\Omega}_T)} \leqslant \max\{\epsilon_0, \epsilon_1\} + \frac{T^\alpha}{\Gamma(1+\alpha)}\epsilon \tag{20}$$

for the solutions u and $\tilde{u}$ of the problem (1), (7)–(8) *with the data F, u_0, v and $\tilde{F}$, $\tilde{u_0}$, $\tilde{v}$, respectively, that satisfy the norm estimates*

$$\begin{aligned}
&\|F - \tilde{F}\|_{C(\bar{\Omega}_T)} \leqslant \epsilon, \\
&\|u_0 - \tilde{u}_0\|_{C(\bar{G})} \leqslant \epsilon_0, \qquad \|v - \tilde{v}\|_{C(S\times[0,T])} \leqslant \epsilon_1.
\end{aligned}$$

Proof. We start with the proof of the following a priori estimate for the norm of a solution u of the problem (1), (7)–(8):

$$\|u\|_{C(\bar{\Omega}_T)} \leqslant \max\{M_0, M_1\} + \frac{T^\alpha}{\Gamma(1+\alpha)}M, \tag{21}$$

where

$$M := \|F\|_{C(\bar{\Omega}_T)}, \qquad M_0 := \|u_0\|_{C(\bar{G})}, \qquad M_1 := \|v\|_{C(S\times[0,T])}. \tag{22}$$

To prove the estimate, we first introduce an auxiliary function w:

$$w(x,t) := u(x,t) - \frac{M}{\Gamma(1+\alpha)} t^{\alpha}, \quad (x,t) \in \bar{\Omega}_T.$$

If u is a solution of the problem (1), (7)–(8), then it can be easy checked that w is a solution of the same problem with the functions $F_1(x,t) := F(x,t) - M - M\sum_{i=1}^{m} \lambda_i \frac{t^{\alpha-\alpha_i}}{\Gamma(1+\alpha-\alpha_i)} - q(x)M\frac{t^{\alpha}}{\Gamma(1+\alpha)}$, $v_1(x,t) := v(x,t) - \frac{M}{\Gamma(1+\alpha)} t^{\alpha}$ instead of F and v, respectively. The function F_1 satisfies the estimate $F_1(x,t) \leqslant 0$, $(x,t) \in \bar{\Omega}_T$ because $|F(x,t)| \leqslant M$, M being defined as $\|F\|_{C(\bar{\Omega}_T)}$. We can apply now the maximum principle to the solution w and obtain thus the estimate

$$w(x,t) \leqslant \max\{M_0, M_1\}, \quad (x,t) \in \bar{\Omega}_T, \tag{23}$$

where the constants M_0, M_1 are defined as in (22). This estimate can be rewritten for the function u in the form

$$\begin{aligned} u(x,t) &= w(x,t) + \frac{M}{\Gamma(1+\alpha)} t^{\alpha} \\ &\leqslant \max\{M_0, M_1\} + \frac{T^{\alpha}}{\Gamma(1+\alpha)} M, \quad (x,t) \in \bar{\Omega}_T. \end{aligned} \tag{24}$$

The estimate

$$u(x,t) \geqslant -\max\{M_0, M_1\} - \frac{T^{\alpha}}{\Gamma(1+\alpha)} M, \quad (x,t) \in \bar{\Omega}_T$$

follows from the minimum principle given in Theorem 3 and the same argumentations as the ones presented above if the auxiliary function w is defined as

$$w(x,t) := u(x,t) + \frac{M}{\Gamma(1+\alpha)} t^{\alpha}, \quad (x,t) \in \bar{\Omega}_T.$$

The last two estimates lead to the inequality (21).

In its turn, the inequality (21) guarantees the uniqueness of solution u of the homogeneous problem (1), (7)–(8) with zero initial and boundary conditions, i.e. the problem with the data $F \equiv 0$, $u_0 \equiv 0$, and $v \equiv 0$. This problem has one trivial solution $u(x,t) \equiv 0$, $(x,t) \in \bar{\Omega}_T$ that is unique due to the estimate (21). Because the problem under consideration is a linear one, the uniqueness of the solution of the problem (1), (7)–(8) in the general case follows from the uniqueness of the homogeneous problem with zero initial and boundary conditions that can be proved following the standard scheme for the linear equations.

Finally, the inequality (20) is obtained from the estimate (21) rewritten for the function $u - \tilde{u}$ that is a solution of the problem (1), (7)–(8) with the functions $F - \tilde{F}$, $u_0 - \tilde{u}_0$, and $v - \tilde{v}$ instead of the functions F, u_0, and v, respectively. □

4. Generalized solution and some existence results

In the previous section, the uniqueness of the classical solution of the problem (1), (7)–(8) was established. To show the existence of the solution, the notion of the classical solution should be extended in one or another way. Of course, this extension has to be not too large in order to not lose the uniqueness property. In this paper, the notion of the generalized solution in the sense of Vladimirov (see [39]) is employed.

Definition 2. Let $F_k \in C(\bar{\Omega}_T)$, $u_{0k} \in C(\bar{G})$ and $v_k \in C(S \times [0,T])$, $k = 1, 2, \ldots$ be the sequences of functions that satisfy the following conditions:

1) there exist the functions F, u_0, and v, such that

$$\|F_k - F\|_{C(\bar{\Omega}_T)} \to 0 \quad \text{as } k \to \infty, \tag{25}$$

$$\|u_{0k} - u_0\|_{C(\bar{G})} \to 0 \quad \text{as } k \to \infty, \tag{26}$$

$$\|v_k - v\|_{C(S\times[0,T])} \to 0 \quad \text{as } k \to \infty, \tag{27}$$

2) for any $k = 1, 2, \ldots$ there exists the classical solution u_k of the initial-boundary-value problem

$$u_k|_{t=0} = u_{0k}(x), \quad x \in \bar{G}, \tag{28}$$

$$u_k|_S = v_k(x,t), \quad (x,t) \in S \times [0,T] \tag{29}$$

for the generalized multi-term time-fractional diffusion equation

$$P(D_t)u_k(x,t) = L_x\big(u_k(x,t)\big) + F_k(x,t). \tag{30}$$

Suppose, there exists a function $u \in C(\bar{\Omega}_T)$ such that

$$\|u_k - u\|_{C(\bar{G})} \to 0 \quad \text{as } k \to \infty. \tag{31}$$

The function u is called a generalized solution of the problem (1), (7)–(8).

The generalized solution of the problem (1), (7)–(8) in the sense of Vladimirov is a continuous function, not a distribution. Still, the generalized solution is not required to be from the functional space $C(\bar{\Omega}_T) \cap W_t^1((0,T]) \cap C_x^2(G)$, where the classical solution has to belong to.

It follows from Definition 2 that if the problem (1), (7)–(8) possesses a classical solution then this solution is a generalized solution of the problem, too. In this sense, Definition 2 extends the notion of the classical solution of the problem (1), (7)–(8). Let us consider some properties of the generalized solution including its uniqueness.

If the problem (1), (7), (8) possesses a generalized solution, then the functions F, u_0 and v given in the problem have to belong to the spaces $C(\bar{\Omega}_T)$, $C(\bar{G})$ and $C(S \times [0,T])$, respectively. In the further discussions, these inclusions are always supposed to be valid.

Let us show that the sequence u_k, $k = 1, 2, \ldots$ defined by the relations (25)–(30) of Definition 2 is always a uniformly convergent one in $\bar{\Omega}_T$, i.e., there always exists a function $u \in C(\bar{\Omega}_T)$ that satisfies the property (31). Indeed, applying the estimate (20) from Theorem 4 to the functions u_k and u_p that are the classical solutions of the corresponding initial-boundary-value problems (28)–(29) for Eq. (30) one gets the inequality

$$\|u_k - u_p\|_{C(\bar{\Omega}_T)} \leqslant \max\big\{\|u_{0k} - u_{0p}\|_{C(\bar{G})}, \|v_k - v_p\|_{C(S\times[0,T])}\big\} + \frac{T^\alpha}{\Gamma(1+\alpha)}\|F_k - F_p\|_{C(\bar{\Omega}_T)}, \tag{32}$$

that, together with the relations (25)–(27), means that u_k, $k = 1, 2, \ldots$ is a Cauchy sequence in $C(\bar{\Omega}_T)$ that converges to a function $u \in C(\bar{\Omega}_T)$.

Moreover, the estimate (21) established in the proof of Theorem 4 for the classical solution of the problem (1), (7)–(8) remains valid for the generalized solution, too. To show this, the inequality

$$\begin{aligned} &\|u_k\|_{C(\bar{\Omega}_T)} \leqslant \max\{M_{0k}, M_{1k}\} + \frac{T^\alpha}{\Gamma(1+\alpha)} M_k, \\ &\quad M_{0k} := \|u_{0k}\|_{C(\bar{G})}, \quad M_{1k} := \|v_k\|_{C(S\times[0,T])}, \quad M_k := \|F\|_{C(\bar{\Omega}_T)} \end{aligned} \tag{33}$$

that is valid $\forall k = 1, 2, \ldots$ is considered as k tends to $+\infty$.

The estimate (21) for the generalized solution is a basis for the following important uniqueness theorem.

Theorem 5. *The problem* (1), (7)–(8) *possesses at most one generalized solution in the sense of Definition* 2. *The generalized solution – if it exists – continuously depends on the data given in the problem in the sense of the estimate* (20).

The proof of the theorem follows the lines of the proof of Theorem 4 and is omitted here.

In contrary to the situation with the classical solution of the problem (1), (7)–(8), the existence of the generalized solution can be shown in the general case under some standard restrictions on the problem data and the boundary S of the domain G. In this paper, the existence of the solution of the initial-boundary-value problem

$$u|_{t=0} = u_0(x), \quad x \in \bar{G}, \tag{34}$$

$$u|_S = 0, \quad (x,t) \in S \times [0,T] \tag{35}$$

for the equation

$$P(D_t)u(x,t) = L_x\big(u(x,t)\big) \tag{36}$$

is considered to demonstrate the technique that can be used with the appropriate standard modifications in the general case, too. The generalized solution of the problem (36)–(35) can be constructed by using the Fourier method of the separation of the variables. Let us look for a particular solution u of Eq. (36) in the form

$$u(x,t) = T(t)X(x), \quad (x,t) \in \bar{\Omega}_T, \tag{37}$$

that satisfies the boundary condition (35). Substitution of the function (37) into Eq. (36) and separation of the variables lead to the equation

$$\frac{(P(D_t)T)(t)}{T(t)} = \frac{L_x(X)}{X(x)} = -\mu, \tag{38}$$

μ being a constant not depending on the variables t and x. The last equation, together with the boundary condition (35), is equivalent to the fractional differential equation

$$(P(D_t)T)(t) + \mu T(t) = 0 \tag{39}$$

and the eigenvalue problem

$$-L_x(X) = \mu X, \tag{40}$$

$$X|_S = 0, \quad x \in S \tag{41}$$

for the operator $-L_x$. Due to the condition (3), the operator $-L_x$ is a positive definite and self-adjoint linear operator. The theory of the eigenvalue problems for such operators is well known (see e.g. [39]). In particular, the eigenvalue problem (40)–(41) has a counted number of the positive eigenvalues $0 < \mu_1 \leqslant \mu_2 \leqslant \cdots$ with the finite multiplicity and – if the boundary S of G is a smooth surface – any function $f \in \mathcal{M}_L$ can be represented through its Fourier series in the form

$$f(x) = \sum_{k=1}^{\infty} (f, X_k) X_k(x), \tag{42}$$

where $X_k \in \mathcal{M}_L$ are the eigenfunctions corresponding to the eigenvalues μ_k:

$$-L_x(X_k) = \mu_k X_k, \quad k = 1, 2, \ldots. \tag{43}$$

By $\mathcal{M}_L$, the space of the functions f that satisfy the boundary condition (41) and the inclusions $f \in C^1(\bar{\Omega}_T) \cap C^2(G)$, $L_x(f) \in L^2(G)$ is denoted.

Now we have to solve the fractional differential equation (39) with $\mu = \mu_k$, $k = 1, 2, \ldots$. To this end, the general results presented in [18,19] are rewritten for the case under consideration and given as a theorem.

Theorem 6. *The linear one-dimensional space of solutions of the fractional differential equation* (39) *with* $\mu = \mu_k$, $k = 1, 2, \ldots$ *is presented in the form* $C_k T_k(t)$, T_k *being defined by*

$$T_k(t) = \left(1 - \mu_k t^{\alpha} E_{(k)}(t)\right), \tag{44}$$

where

$$E_{(k)}(t) = E_{(\alpha, \alpha-\alpha_1, \ldots, \alpha-\alpha_m), 1+\alpha}\left(-\mu_k t^{\alpha}, -\lambda_1 t^{\alpha-\alpha_1}, \ldots, -\lambda_m t^{\alpha-\alpha_m}\right)$$

is a particular case of the multinomial Mittag-Leffler function

$$E_{(a_1, \ldots, a_m), b}(z_1, \ldots, z_m) := \sum_{k=0}^{\infty} \sum_{\substack{l_1+\cdots+l_m=k \\ l_1 \geqslant 0, \ldots, l_m \geqslant 0}} (k; l_1, \ldots, l_m) \frac{\prod_{i=1}^{m} z_i^{l_i}}{\Gamma(b + \sum_{i=1}^{m} a_i l_i)}$$

with the multinomial coefficients defined by

$$(k; l_1, \ldots, l_m) := \frac{k!}{l_1! \times \cdot \times l_m!}.$$

Let us mention here that in [15] the solution of the eigenvalue problem for the distributed order fractional derivative has been given.

It was shown in [18], that the function T_k defined by (44) belongs to the function space C^1_{-1} that consists of all functions f such that $f' \in C_{-1}$, C_{-1} being defined as follows:

Definition 3. A function $f(t)$, $t > 0$ is said to be in the space C_{-1} if there exists a real number p $(p > -1)$, such that $f(t) = t^p f_1(t)$ and $f_1(t) \in C([0, \infty))$.

Clearly the inclusion $C^1_{-1} \subset W^1_t((0, T))$ holds true. We recall that by $W^1_t((0, T))$ the space of the functions $f \in C^1((0, T])$ such that $f' \in L((0, T))$ is denoted.

Any of the functions

$$u_k(x, t) = C_k T_k(t) X_k(x), \quad k = 1, 2, \ldots \tag{45}$$

and thus the finite sums

$$u_l(x,t) = \sum_{k=1}^{l} C_k T_k(t) X_k(x), \quad l = 1, 2, \ldots \tag{46}$$

satisfy both Eq. (36) and the boundary condition (35). To construct a function that satisfies the initial condition (34), too, the notion of a formal solution is introduced.

Definition 4. A formal solution of the problem (34)–(36) is called the Fourier series in the form

$$u(x,t) = \sum_{k=1}^{\infty} (u_0, X_k) T_k(t) X_k(x), \tag{47}$$

X_k, $k = 1, 2, \ldots$ being the eigenfunctions corresponding to the eigenvalues μ_k of the eigenvalue problem (40)–(41) and T_k, $k = 1, 2, \ldots$ defined by (44).

Under certain conditions, the formal solution (47) can be proved to be the generalized solution of the problem (34)–(36).

Theorem 7. *Let the function u_0 in the initial condition* (34) *be from the space $\mathcal{M}_L$. Then the formal solution* (47) *of the problem* (34)–(36) *is its generalized solution.*

It can be easily verified that the functions u_l, $l = 1, 2, \ldots$ defined by (46) are the classical solutions of the problem (36)–(35) with the initial conditions

$$u_{0l}(x) = \sum_{k=1}^{l} (u_0, X_k) X_k(x) \tag{48}$$

instead of u_0. Because the function u_0 is from the functional space $\mathcal{M}_L$, its Fourier series converges uniformly to the function u_0, so that

$$\|u_{0l} - u_0\|_{C(\bar{G})} \to 0 \quad \text{as } l \to \infty.$$

To prove the theorem, one only needs to show that the sequence u_l, $l = 1, 2, \ldots$ of the partial sums (46) converges uniformly on $\bar{\Omega}_T$. To this end, an appropriate estimate for the functions T_k, $k = 1, 2, \ldots$ is needed. It was shown in [18] that the solution T_k, $k = 1, 2, \ldots$ of the fractional differential equation (39) with $\mu = \mu_k$, $k = 1, 2, \ldots$ has the following integral representation:

$$T_k(t) = 1 - \frac{1}{2\pi i} \int_{\gamma(\epsilon;\delta)} \frac{\mu_k e^{st}}{s^\alpha + \sum_{i=1}^m \lambda_i s^{\alpha_i} + \mu_k} \frac{ds}{s}, \quad \frac{\pi\alpha}{2} < \delta \leqslant \pi\alpha. \tag{49}$$

The contour $\gamma(\epsilon;\delta)$ consists of two rays $S_{-\delta}$ and S_δ ($\arg s = -\delta$, $|s| \geqslant \epsilon$ and $\arg s = \delta$, $|s| \geqslant \epsilon$) and a circular arc $C_\delta(0;\epsilon)$ ($|s| = \epsilon$, $-\delta \leqslant \arg s \leqslant \delta$). The parameter $\epsilon = \epsilon(k)$ is chosen in a way that all zeros of the function $s^\alpha + \sum_{i=1}^m \lambda_i s^{\alpha_i} + \mu_k$ are located inside of the circle $|s| = \epsilon$. In particular, it is the case when ϵ satisfies the inequality $\epsilon^\alpha > \sum_{i=1}^m \lambda_i \epsilon^{\alpha_i} + \mu_k$ that always has a solution in the form $\epsilon > \epsilon_k$ due to the condition $\alpha > \alpha_i$, $i = 1, \ldots, m$ and to the fact, that all coefficients λ_i, $i = 1, \ldots, m$ and the eigenvalues μ_k, $k = 1, 2, \ldots$ are positive numbers.

The inequality

$$\left| \frac{\mu_k}{s^\alpha + \sum_{i=1}^m \lambda_i s^{\alpha_i} + \mu_k} \right| \leqslant \frac{\mu_k}{|s|^\alpha - \sum_{i=1}^m \lambda_i |s|^{\alpha_i} - \mu_k}$$

is valid for an arbitrary s. If s belongs to the contour $\gamma(\epsilon;\delta)$ (this means in particular that $|s| \geqslant \epsilon$) then the estimate

$$|s|^\alpha - \sum_{i=1}^m \lambda_i |s|^{\alpha_i} - \mu_k = |s|^\alpha \left(1 - \sum_{i=1}^m \lambda_i |s|^{\alpha_i - \alpha} - \mu_k |s|^{-\alpha} \right) \geqslant \frac{|s|^\alpha}{2}$$

can be deduced by choosing an appropriate $\epsilon = \epsilon(k)$. As a consequence, for any fixed $M > 0$ not depending on k we can find an $\epsilon = \epsilon(k)$ such that

$$\frac{\mu_k}{|s|^\alpha - \sum_{i=1}^m \lambda_i |s|^{\alpha_i} - \mu_k} \leqslant \frac{2\mu_k}{|s|^\alpha} \leqslant M, \quad s \in \gamma(\epsilon;\delta).$$

Then we can estimate the functions T_k, $k = 1, 2, \ldots$:

$$\left|T_k(t)\right| = \left|1 - \frac{1}{2\pi i}\int\limits_{\gamma(\epsilon;\delta)} \frac{\mu_k e^{st}}{s^\alpha + \sum_{i=1}^{m-1}\lambda_i s^{\alpha_i} + \mu_k}\,\frac{ds}{s}\right|$$
$$\leqslant 1 + \frac{1}{2\pi}\left|\int\limits_{\gamma(\epsilon;\delta)} e^{st} M\,\frac{ds}{s}\right| = 1 + \frac{1}{2\pi}\left|\int\limits_{\gamma(\epsilon_1;\delta)} e^{\tau} M\,\frac{d\tau}{\tau}\right| = 1 + \frac{1}{2\pi}\frac{C_1}{\Gamma(1)} = C,$$

C being not depending on k. In the last formula, the well-known integral representation

$$\frac{1}{\Gamma(\zeta)} = \frac{1}{2\pi i}\int\limits_{\gamma(\epsilon;\delta)} e^{\tau}\tau^{-\zeta}\,d\tau, \quad \epsilon > 0,\ \frac{\pi}{2} < \delta \leqslant \pi$$

for the Gamma-function was used.

Combining this last estimate and the fact that the Fourier series

$$\sum_{k=1}^{\infty}(u_0, X_k)X_k(x)$$

of the function $u_0 \in \mathcal{M}_L$ uniformly converges on $\bar{\Omega}_T$ we arrive at the statement that the sequence u_l, $l = 1, 2, \ldots$ of the partial sums (46) converges uniformly on $\bar{\Omega}_T$, what we wanted to prove.

We mention here that in the paper [27] a probabilistic method similar to the one used in the proof of Theorem 7 was applied for the distributed order fractional Cauchy problems on bounded domains. In [27], some explicit bounds for the time derivative of the solution have been obtained, too. In particular, it has been shown there that $|T_k(t)| \leqslant 1$, $k = 1, 2, \ldots .$

In some cases, the generalized solution (47) can be shown to be the classical solution of the initial-value-problem for the generalized time-fractional diffusion equation, too. One important example is given in the following theorem.

Theorem 8. *Let $u_0 \in \mathcal{M}_L$ and the open domain G be a one-dimensional interval $(0, l)$. Then the classical solution of the initial-value-problem*

$$u|_{t=0} = u_0(x), \quad 0 \leqslant x \leqslant l,$$
$$u(0,t) = u(l,t) = 0, \quad 0 \leqslant t \leqslant T$$

for the generalized multi-term time-fractional diffusion equation

$$\left(D_t^\alpha + \sum_{i=1}^{m}\lambda_i D_t^{\alpha_i}\right)u = \frac{\partial}{\partial x}\left(p(x)\frac{\partial u}{\partial x}\right) - q(x)u \tag{50}$$

with

$$0 < \alpha_m < \cdots < \alpha_1 < \alpha \leqslant 1, \quad 0 \leqslant \lambda_i, \quad i = 1, \ldots, m,\ m \in \mathbb{N}_0$$

and

$$p \in C^1([0,l]), \quad q \in C([0,l]), \quad 0 < p(x), \quad 0 \leqslant q(x), \quad x \in [0,l]$$

exists and is given by the formula (47)*.*

The proof of the theorem follows mainly the lines of the proof of the same result for the one-dimensional parabolic PDEs (the case $\alpha = 1$) presented in [39] and is omitted here.

It is worth mentioning that in the recent paper [5] an inverse problem for the initial-boundary-value problem with the homogeneous Neumann boundary conditions for Eq. (50) with $n = 0$ and $q(x) \equiv 0$, $x \in [0, l]$ was considered. It was proved there that both the differentiation order α and the coefficient $p(x)$, $0 < x < l$ are uniquely determined by the data $u(0, t)$, $0 < t < T$ that can be measured for a concrete diffusion process. This result can be considered to be a theoretical background for experimentally determining the order α of the corresponding anomalous diffusion phenomena. Of course, similar problems should be considered for the more general anomalous diffusion equation (1) in the n-dimensional domains, the cases $n = 2$ and $n = 3$ being the most interesting from the viewpoint of the real world applications.

Finally we mention that the theory presented in the paper can be applied with some small modifications to the case of the infinite domain $\Omega = G \times (0, \infty)$, $G \subset R^n$, too.

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