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Asymptotic properties of isometries

Soon-Mo Jung [1]

Mathematics Section, College of Science and Technology, Hong-Ik University, 339-701 Chochiwon, South Korea

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Abstract

In this paper, we prove two theorems on the local stability of isometries in connection with (ε, p)-isometries. These theorems reveal that a large class of (ε, p)-isometries, defined on various restricted domains, are stable.
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1. Introduction

Let (E, d_E) and (F, d_F) be metric spaces. A mapping $I : E \to F$ is called an isometry if I satisfies the equation

$$d_F\big(I(x), I(y)\big) = d_E(x, y)$$

for all $x, y \in E$.

Extending the definition by Hyers and Ulam [11], we may call a mapping $f : E \to F$ an ε-isometry if f satisfies the inequality

$$\big|d_F\big(f(x), f(y)\big) - d_E(x, y)\big| \leqslant \varepsilon \tag{$*$}$$

E-mail address: smjung@wow.hongik.ac.kr.
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for all $x, y \in E$. If in this case there exist an isometry $I : E \to F$ and a constant $k \geqslant 0$ such that $d_F(f(x), I(x)) \leqslant k\varepsilon$ for all $x \in E$, then we may say that the isometry from E to F is stable (in the sense of Hyers and Ulam).

Hyers and Ulam proved in the same paper [11] the stability of isometries between real Hilbert spaces. In fact, they proved that if a surjective mapping $f : E \to E$, where E is a real Hilbert space, satisfies $f(0) = 0$ as well as the inequality $(*)$ for some $\varepsilon \geqslant 0$ and for all $x, y \in E$, then there exists a surjective isometry $I : E \to E$ such that $\|f(x) - I(x)\| \leqslant 10\varepsilon$ for all $x \in E$.

This result of Hyers and Ulam was the first one concerning the stability of isometries and was further generalized by Bourgin [4]. Indeed, Bourgin proved the following theorem: Assume that E is a Banach space and that F belongs to a class of uniformly convex spaces. If a mapping $f : E \to F$ satisfies $f(0) = 0$ as well as the inequality $(*)$ for some $\varepsilon \geqslant 0$ and for all $x, y \in E$, then there exists a linear isometry $I : E \to F$ such that $\|f(x) - I(x)\| \leqslant 12\varepsilon$ for each $x \in E$.

Subsequently, Hyers and Ulam [12] studied a stability problem for spaces of continuous mappings: Let S_1 and S_2 be compact metric spaces and $C(S_i)$ denote the space of real-valued continuous mappings on S_i equipped with the metric topology with $\|\cdot\|_\infty$. If a homeomorphism $T : C(S_1) \to C(S_2)$ satisfies the inequality

$$\left| \left\| T(f) - T(g) \right\|_\infty - \|f - g\|_\infty \right| \leqslant \varepsilon \tag{$**$}$$

for some $\varepsilon \geqslant 0$ and for all $f, g \in C(S_1)$, then there exists an isometry $U : C(S_1) \to C(S_2)$ such that $\|T(f) - U(f)\|_\infty \leqslant 21\varepsilon$ for every $f \in C(S_1)$.

This result of Hyers and Ulam was significantly generalized by Bourgin again (see [5]): Let S_1 and S_2 be completely regular Hausdorff spaces and let $T : C(S_1) \to C(S_2)$ be a surjective mapping satisfying the inequality $(**)$ for some $\varepsilon \geqslant 0$ and for all $f, g \in C(S_1)$. Then there exists a linear isometry $U : C(S_1) \to C(S_2)$ such that $\|T(f) - U(f)\|_\infty \leqslant 10\varepsilon$ for any $f \in C(S_1)$.

The study of stability problems for isometries on finite-dimensional Banach spaces was continued by Bourgin [6].

In 1978, Gruber [10] obtained an elegant result as follows: Let E and F be real normed spaces. Suppose that $f : E \to F$ is a surjective mapping and it satisfies the inequality $(*)$ for some $\varepsilon \geqslant 0$ and for all $x, y \in E$. Furthermore, assume that $I : E \to F$ is an isometry with $f(p) = I(p)$ for some $p \in E$. If $\|f(x) - I(x)\| = o(\|x\|)$ as $\|x\| \to \infty$ uniformly, then I is a surjective linear isometry and $\|f(x) - I(x)\| \leqslant 5\varepsilon$ for all $x \in E$. If in addition f is continuous, then $\|f(x) - I(x)\| \leqslant 3\varepsilon$ for all $x \in E$.

Gevirtz [9] established the stability of isometries between arbitrary Banach spaces: Given real Banach spaces E and F, let $f : E \to F$ be a surjective mapping satisfying the inequality $(*)$ for some $\varepsilon \geqslant 0$ and for all $x, y \in E$. Then there exists a surjective isometry $I : E \to F$ such that $\|f(x) - I(x)\| \leqslant 5\varepsilon$ for each $x \in E$. Later, the upper bound 5ε could be improved to the sharp one, 2ε, by Omladič and Šemrl [15].

It is surprising that Dolinar [7] recently proved the superstability of isometries. Indeed, he proved that for $p > 1$ every surjective (ε, p)-isometry $f : E \to F$ between finite-dimensional real Banach spaces is an isometry, where a mapping $f : E \to F$ is called an (ε, p)-isometry if f satisfies the inequality

$$\big|\big\|f(x) - f(y)\big\| - \|x - y\|\big| \leqslant \varepsilon\|x - y\|^p$$

for some $\varepsilon \geqslant 0$ and for all $x, y \in E$. (One can further refer to [7,19] for more exact definition of (ε, p)-isometry.)

On the other hand, Swain [22] considered the stability of isometries on bounded metric spaces and proved the following result: Let M be a subset of a compact metric space (E, d) and let $\delta > 0$ be given. Then there exists an $\varepsilon > 0$ such that if $f : M \to E$ satisfies the inequality $(*)$ for all $x, y \in M$, then there exists an isometry $I : M \to E$ with $d(f(x), I(x)) \leqslant \delta$ for any $x \in M$.

The stability problem of isometries on bounded subsets of $\mathbf{R}^n$ was studied by Fickett [8]: For $t \geqslant 0$, let us define $K_0(t) = K_1(t) = t$, $K_2(t) = 3\sqrt{3t}$, $K_i(t) = 27t^{m(i)}$, where $m(i) = 2^{1-i}$ for $i \geqslant 3$. Let S be a bounded subset of $\mathbf{R}^n$ with diameter $d(S)$, and suppose that $3K_n(\varepsilon/d(S)) \leqslant 1$ for some $\varepsilon \geqslant 0$. If a mapping $f : S \to \mathbf{R}^n$ satisfies the inequality $(*)$ for all $x, y \in S$, then there exists an isometry $I : S \to \mathbf{R}^n$ such that $|f(x) - I(x)| \leqslant d(S)K_{n+1}(\varepsilon/d(S))$ for each $x \in S$.

Recently, the author and Kim [13] investigated the stability of isometries on restricted domains. For more general information on the stability of isometries and related topics, one can refer to [17,18] (see also [1,3,7,14,16,20,21]).

In this paper, we will prove the local stability of a class of asymptotic isometries. We refer the reader to the paper [21] of Skof for the exact definition of asymptotic isometries. Indeed, Skof [21] has investigated many interesting properties of a large class of asymptotic isometries. It may be interesting to compare our main results with those of [21].

2. Local stability of isometries on unbounded domains

Let $(G, +)$ be an abelian metric group with a metric $d(\cdot\,,\cdot)$ satisfying

$$d(x + z, y + z) = d(x, y) \quad \text{and} \quad d(2x, 2y) = 2d(x, y) \tag{1}$$

for all $x, y, z \in G$. Furthermore, we assume that for each given $y \in G$ the equation

$$x + x = y$$

is uniquely solvable. We here promise that $2^{-1}y$ or $y/2$ stands for the unique solution of the above equation and we inductively define $2^{-(n+1)}y = 2^{-1}(2^{-n}y)$ for each given $y \in G$ and $n \in \mathbf{N}$. We may usually write $x/2^n$ instead of $2^{-n}x$ for each $x \in G$ and $n \in \mathbf{N}$. The second condition in (1) also implies that

$$d\left(\frac{x}{2}, \frac{y}{2}\right) = \frac{1}{2}d(x, y)$$

for $x, y \in G$.

Theorem 1. *Let E be a subset of G with the property that*

$$0 \in E \quad \text{and} \quad 2^k x \in E \quad (\text{for } x \in E \text{ and } k \in \mathbf{N})$$

and let F be a real Hilbert space with the associated inner product $\langle \cdot, \cdot \rangle$. If a mapping $f : E \to F$ satisfies the inequality

$$\big| \|f(x) - f(y)\| - d(x, y) \big| \leqslant \varepsilon d(x, y)^p$$

for some $\varepsilon \geqslant 0$, $0 \leqslant p < 1$ and for all $x, y \in E$, then there exists an isometry $I : E \to F$ that satisfies

$$\|f(x) - I(x) - f(0)\| \leqslant \frac{2^{(1-p)/2}}{2^{(1-p)/2} - 1} \max\{\sqrt{4.5\varepsilon}, 2\varepsilon\} \max\{d(x, 0)^p, d(x, 0)^{(1+p)/2}\} \tag{2}$$

for all $x \in E$. For $0 < p < 1$, the isometry I is uniquely determined.

Proof. If we define a mapping $g : E \to F$ by $g(x) = f(x) - f(0)$, then we have

$$\big| \|g(x) - g(y)\| - d(x, y) \big| \leqslant \varepsilon d(x, y)^p \tag{3}$$

for any $x, y \in E$. With $y = 0$ and $y = 2x$ separately, the inequality (3) together with (1) yields

$$\begin{cases} \big| \|g(x)\| - d(x, 0) \big| \leqslant \varepsilon d(x, 0)^p, \\ \big| \|g(x) - g(2x)\| - d(x, 0) \big| \leqslant \varepsilon d(x, 0)^p, \end{cases} \tag{4}$$

respectively.

It follows from (4) that

$$A(x)^2 \leqslant \|g(x)\|^2 \leqslant \big[d(x, 0) + \varepsilon d(x, 0)^p\big]^2 \tag{5}$$

and

$$\begin{aligned} \|g(x) - g(2x)\|^2 &= \|g(x)\|^2 + \|g(2x)\|^2 - 2\langle g(x), g(2x)\rangle \\ &\leqslant \big[d(x, 0) + \varepsilon d(x, 0)^p\big]^2, \end{aligned} \tag{6}$$

where we set

$$A(x) = \begin{cases} 0 & \text{for } d(x, 0) \leqslant \varepsilon^{1/(1-p)}, \\ d(x, 0) - \varepsilon d(x, 0)^p & \text{for } d(x, 0) > \varepsilon^{1/(1-p)}. \end{cases}$$

If $d(x, 0) > 2^{-1}\varepsilon^{1/(1-p)}$ then $A(2x) = d(2x, 0) - \varepsilon d(2x, 0)^p$ and $A(2x)^2 \leqslant \|g(2x)\|^2$. Hence, it follows from (1), (5) and (6) that

$$\begin{aligned} &2\left\| g(x) - \frac{1}{2} g(2x) \right\|^2 \\ &\quad = 2\|g(x)\|^2 + \frac{1}{2}\|g(2x)\|^2 - 2\langle g(x), g(2x)\rangle \end{aligned}$$

$$
\begin{aligned}
&= \|g(x)\|^2 + \{\|g(x)\|^2 + \|g(2x)\|^2 - 2\langle g(x), g(2x)\rangle\} - \frac{1}{2}\|g(2x)\|^2 \\
&\leqslant 2\big[d(x,0) + \varepsilon d(x,0)^p\big]^2 - \frac{1}{2}\big[d(2x,0) - \varepsilon d(2x,0)^p\big]^2 \\
&= 4\varepsilon\left(1 + \frac{1}{2^{1-p}}\right)d(x,0)^{1+p} + 2\varepsilon^2\left(1 - \frac{1}{2^{2(1-p)}}\right)d(x,0)^{2p} \\
&\leqslant \left(4 + 2^p + \frac{4}{2^p}\right)\varepsilon d(x,0)^{1+p} \leqslant 9\varepsilon d(x,0)^{1+p}.
\end{aligned}
$$

On the other hand, for $d(x,0) \leqslant 2^{-1}\varepsilon^{1/(1-p)}$, it analogously follows from (5) and (6) that

$$
2\left\|g(x) - \frac{1}{2}g(2x)\right\|^2 \leqslant 2\big[d(x,0) + \varepsilon d(x,0)^p\big]^2 \leqslant 8\varepsilon^2 d(x,0)^{2p}.
$$

Hence, we have

$$
\left\|g(x) - \frac{1}{2}g(2x)\right\| \leqslant C \max\big\{d(x,0)^p, d(x,0)^{(1+p)/2}\big\} \tag{7}
$$

for all $x \in E$, where we set $C = \max\{\sqrt{4.5\varepsilon}, 2\varepsilon\}$.

The last inequality implies the validity of the following inequality

$$
\begin{aligned}
&\left\|g(x) - \frac{1}{2^n}g\big(2^n x\big)\right\| \\
&\quad\leqslant C \max\big\{d(x,0)^p, d(x,0)^{(1+p)/2}\big\} \sum_{i=0}^{n-1} 2^{-i(1-p)/2}
\end{aligned} \tag{8}
$$

for $n = 1$. Assume now that the inequality (8) is true for some $n \in \mathbf{N}$. It then follows from (1), (7) and (8) that

$$
\begin{aligned}
&\left\|g(x) - \frac{1}{2^{n+1}}g\big(2^{n+1}x\big)\right\| \\
&\quad\leqslant \left\|g(x) - \frac{1}{2^n}g\big(2^n x\big)\right\| + \left\|\frac{1}{2^n}g\big(2^n x\big) - \frac{1}{2^{n+1}}g\big(2^{n+1}x\big)\right\| \\
&\quad\leqslant C \max\big\{d(x,0)^p, d(x,0)^{(1+p)/2}\big\} \sum_{i=0}^{n-1} 2^{-i(1-p)/2} \\
&\qquad + \frac{1}{2^n} C \max\big\{d\big(2^n x,0\big)^p, d\big(2^n x,0\big)^{(1+p)/2}\big\} \\
&\quad\leqslant C \max\big\{d(x,0)^p, d(x,0)^{(1+p)/2}\big\} \sum_{i=0}^{n} 2^{-i(1-p)/2},
\end{aligned}
$$

which implies the validity of (8) for all $x \in E$ and $n \in \mathbf{N}$.

For given $m, n \in \mathbf{N}$ with $n > m$, we use (1) and (8) to verify

$$\begin{aligned}
&\left\| \frac{1}{2^m} g\left(2^m x\right) - \frac{1}{2^n} g\left(2^n x\right) \right\| \\
&\quad = \frac{1}{2^m} \left\| g\left(2^m x\right) - \frac{1}{2^{n-m}} g\left(2^{n-m} \cdot 2^m x\right) \right\| \\
&\quad \leqslant C \max\left\{ d(x,0)^p, d(x,0)^{(1+p)/2} \right\} \sum_{i=m}^{n-1} 2^{-i(1-p)/2} \\
&\quad \to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

Thus, $\{2^{-n} g(2^n x)\}$ is a Cauchy sequence for any $x \in E$. Let us define a mapping $I : E \to F$ by

$$I(x) = \lim_{n \to \infty} \frac{1}{2^n} g\left(2^n x\right). \tag{9}$$

If we substitute $2^n x$ and $2^n y$ for x and y in (3) and divide the resulting inequality by 2^n, and if we consider the case that n goes to the infinity, then we see that I is an isometry. The inequality (8), together with (9), shows that the inequality (2) holds true for any $x \in E$.

Now, assume that $0 < p < 1$ and $J : E \to F$ is an isometry satisfying the inequality (2). Because of our assumption $p > 0$, it then follows from (2) that $J(0) = 0$. Since

$$\left\| J(x) - J(y) \right\| = d(x, y)$$

for all $x, y \in E$, it follows from (1) that

$$\left\| J(2x) - J(x) \right\| = d(2x, x) = d(x, 0) = \left\| J(x) \right\|$$

and

$$\left\| J(2x) \right\| = d(2x, 0) = 2d(x, 0) = 2\left\| J(x) \right\|.$$

Hence, we have

$$\left\| J(2x) - J(x) \right\|^2 = \left\| J(2x) \right\|^2 - 2\left\langle J(2x), J(x) \right\rangle + \left\| J(x) \right\|^2 = \left\| J(x) \right\|^2.$$

Thus, we get

$$\left\| J(2x) \right\| \left\| J(x) \right\| = \left\langle J(2x), J(x) \right\rangle,$$

i.e.,

$$J(2x) = 2J(x) \tag{10}$$

for any $x \in E$. Assume that

$$\frac{1}{2^k} J\left(2^k x\right) = J(x) \tag{11}$$

for all $x \in E$ and some $k \in \mathbf{N}$. Then, by (10) and (11), we obtain

$$\frac{1}{2^{k+1}} J\bigl(2^{k+1}x\bigr) = \frac{1}{2^k} J\bigl(2^k x\bigr) = J(x),$$

which implies that, for $0 < p < 1$, the equality (11) is true for all $x \in E$ and all $k \in \mathbf{N}$.

For some $0 < p < 1$ and for an arbitrary $x \in E$, it follows from (2) and (11) that

$$\begin{aligned}\bigl\|I(x) - J(x)\bigr\| &= \frac{1}{2^k}\bigl\|I\bigl(2^k x\bigr) - J\bigl(2^k x\bigr)\bigr\| \\ &\leqslant \frac{1}{2^k}\frac{2 \cdot 2^{(1-p)/2}}{2^{(1-p)/2} - 1} C \max\bigl\{d\bigl(2^k x, 0\bigr)^p, d\bigl(2^k x, 0\bigr)^{(1+p)/2}\bigr\} \\ &\leqslant 2^{-k(1-p)/2}\frac{2^{(3-p)/2}}{2^{(1-p)/2} - 1} C \max\bigl\{d(x, 0)^p, d(x, 0)^{(1+p)/2}\bigr\} \\ &\to 0 \quad \text{as } k \to \infty.\end{aligned}$$

This implies the uniqueness of I for the case $0 < p < 1$. $\square$

It would be interesting to compare the previous theorem with Theorem 1 of [21] because the mapping f involved in the former theorem is a kind of asymptotic isometry. The following corollary may be proven by use of Theorem 1 or in a straight forward manner. Indeed, it is an immediate consequence of Proposition 4 in [21].

Corollary 1. *Let G be a real normed space and let F be a real Hilbert space. Assume that a mapping $f : G \to F$ satisfies $f(0) = 0$ and $f(2^k x) = 2^k f(x)$ for all $x \in G$ and $k \in \mathbf{N}$. The mapping f is a linear isometry if and only if there exists a $0 < p < 1$ such that*

$$\bigl|\bigl\|f(x) - f(y)\bigr\| - \|x - y\|\bigr| = O\bigl(\|x - y\|^p\bigr)$$

as $\|x\| \to \infty$ and $\|y\| \to \infty$.

According to a theorem of Baker [1], F may be assumed to be real normed space which is strictly convex.

3. Local stability of isometries on bounded domains

Let $(G, +)$ be the abelian metric group with the metric $d(\cdot, \cdot)$ which satisfies all the conditions given in the previous section.

Theorem 2. *Let E be a subset of G with the property that*

$$0 \in E \quad \text{and} \quad 2^{-k}x \in E \quad (\text{for } x \in E \text{ and } k \in \mathbf{N})$$

and let F be a real Hilbert space with the associated inner product $\langle \cdot, \cdot \rangle$. If a mapping $f : E \to F$ satisfies the inequality

$$\big| \| f(x) - f(y) \| - d(x, y) \big| \leqslant \varepsilon d(x, y)^p$$

for some $\varepsilon \geqslant 0$, $p > 1$ and for all $x, y \in E$, then there exists a unique isometry $I : E \to F$ such that

$$\begin{aligned} &\| f(x) - I(x) - f(0) \| \\ &\quad \leqslant \frac{2^{(p-1)/2}}{2^{(p-1)/2} - 1} \max\{2\sqrt{\varepsilon}, 2\varepsilon\} \max\{d(x,0)^p, d(x,0)^{(1+p)/2}\} \end{aligned} \tag{12}$$

for any $x \in E$.

Proof. Let us define a mapping $g : E \to F$ by $g(x) = f(x) - f(0)$. Then, the inequality (3) holds for $p > 1$ and for all $x, y \in E$. Put $y = 0$ and $y = 2^{-1}x$ in (3) separately and consider the conditions in (1) to get

$$\begin{cases} \big| \| g(x) \| - d(x,0) \big| \leqslant \varepsilon d(x,0)^p, \\ \big| \| g(x) - g(2^{-1}x) \| - 2^{-1} d(x,0) \big| \leqslant 2^{-p} \varepsilon d(x,0)^p. \end{cases} \tag{13}$$

By (13), we have

$$A(x)^2 \leqslant \| g(x) \|^2 \leqslant \big[d(x,0) + \varepsilon d(x,0)^p \big]^2 \tag{14}$$

and

$$\begin{aligned} \left\| g(x) - g\left(\frac{x}{2}\right) \right\|^2 &= \| g(x) \|^2 + \left\| g\left(\frac{x}{2}\right) \right\|^2 - 2\left\langle g(x), g\left(\frac{x}{2}\right) \right\rangle \\ &\leqslant \left[\frac{1}{2} d(x,0) + \frac{\varepsilon}{2^p} d(x,0)^p \right]^2, \end{aligned} \tag{15}$$

where we define

$$A(x) = \begin{cases} 0 & \text{for } d(x,0) \geqslant \varepsilon^{-1/(p-1)}, \\ d(x,0) - \varepsilon d(x,0)^p & \text{for } d(x,0) < \varepsilon^{-1/(p-1)}. \end{cases}$$

If $d(x,0) < \varepsilon^{-1/(p-1)}$, then it follows from (1), (14) and (15) that

$$\begin{aligned} &\frac{1}{2} \left\| g(x) - 2g\left(\frac{x}{2}\right) \right\|^2 \\ &\quad = \frac{1}{2} \| g(x) \|^2 + 2 \left\| g\left(\frac{x}{2}\right) \right\|^2 - 2\left\langle g(x), g\left(\frac{x}{2}\right) \right\rangle \\ &\quad = -\frac{1}{2} \| g(x) \|^2 + \left\| g\left(\frac{x}{2}\right) \right\|^2 + \| g(x) \|^2 + \left\| g\left(\frac{x}{2}\right) \right\|^2 - 2\left\langle g(x), g\left(\frac{x}{2}\right) \right\rangle \end{aligned}$$

$$\leqslant -\frac{1}{2}\big[d(x,0)-\varepsilon d(x,0)^p\big]^2 + 2\left[\frac{1}{2}d(x,0)+\frac{\varepsilon}{2^p}d(x,0)^p\right]^2$$
$$= \left(\frac{2}{2^p}+1\right)\varepsilon d(x,0)^{1+p} + \frac{1}{2}\left(\frac{4}{2^{2p}}-1\right)\varepsilon^2 d(x,0)^{2p}$$
$$\leqslant \left(\frac{2}{2^p}+\frac{2}{2^{2p}}+\frac{1}{2}\right)\varepsilon d(x,0)^{1+p} \leqslant 2\varepsilon d(x,0)^{1+p}.$$

For $d(x,0) \geqslant \varepsilon^{-1/(p-1)}$, it analogously follows from (14) and (15) that

$$\frac{1}{2}\left\| g(x) - 2g\left(\frac{x}{2}\right)\right\|^2$$
$$= \frac{1}{2}\|g(x)\|^2 + 2\left\|g\left(\frac{x}{2}\right)\right\|^2 - 2\left\langle g(x), g\left(\frac{x}{2}\right)\right\rangle$$
$$\leqslant \left\|g\left(\frac{x}{2}\right)\right\|^2 + \|g(x)\|^2 + \left\|g\left(\frac{x}{2}\right)\right\|^2 - 2\left\langle g(x), g\left(\frac{x}{2}\right)\right\rangle$$
$$\leqslant 2\left[\frac{1}{2}d(x,0)+\frac{\varepsilon}{2^p}d(x,0)^p\right]^2 \leqslant 2\varepsilon^2 d(x,0)^{2p}.$$

Hence, we have

$$\left\| g(x) - 2g\left(\frac{x}{2}\right)\right\| \leqslant C\max\big\{d(x,0)^p, d(x,0)^{(1+p)/2}\big\} \tag{16}$$

for any $x \in E$, where we set $C = \max\{2\sqrt{\varepsilon}, 2\varepsilon\}$.

The last inequality means the validity of the inequality

$$\left\| g(x) - 2^n g\left(\frac{x}{2^n}\right)\right\|$$
$$\leqslant C\max\big\{d(x,0)^p, d(x,0)^{(1+p)/2}\big\}\sum_{i=0}^{n-1} 2^{-i(p-1)/2} \tag{17}$$

for $n = 1$. Assume that the inequality (17) holds true for some $n \in \mathbf{N}$. Then, by (1), (16) and (17), we get

$$\left\| g(x) - 2^{n+1} g\left(\frac{x}{2^{n+1}}\right)\right\|$$
$$\leqslant \left\| g(x) - 2^n g\left(\frac{x}{2^n}\right)\right\| + \left\| 2^n g\left(\frac{x}{2^n}\right) - 2^{n+1} g\left(\frac{x}{2^{n+1}}\right)\right\|$$
$$\leqslant C\max\big\{d(x,0)^p, d(x,0)^{(1+p)/2}\big\}\sum_{i=0}^{n-1} 2^{-i(p-1)/2}$$
$$+ 2^n C\max\left\{d\left(\frac{x}{2^n},0\right)^p, d\left(\frac{x}{2^n},0\right)^{(1+p)/2}\right\}$$

$$\leqslant C \max\{d(x,0)^p, d(x,0)^{(1+p)/2}\} \sum_{i=0}^{n} 2^{-i(p-1)/2},$$

which implies the validity of (17) for all $x \in E$ and $n \in \mathbf{N}$.

Let $m, n \in \mathbf{N}$ be given with $n > m$. By (1) and (17), we obtain

$$\begin{aligned}
&\left\| 2^m g\left(\frac{x}{2^m}\right) - 2^n g\left(\frac{x}{2^n}\right) \right\| \\
&\quad = 2^m \left\| g\left(\frac{x}{2^m}\right) - 2^{n-m} g\left(\frac{1}{2^{n-m}} \frac{x}{2^m}\right) \right\| \\
&\quad \leqslant C \max\{d(x,0)^p, d(x,0)^{(1+p)/2}\} \sum_{i=m}^{n-1} 2^{-i(p-1)/2} \\
&\quad \to 0 \quad \text{as } m \to \infty,
\end{aligned}$$

which means that $\{2^n g(2^{-n}x)\}$ is a Cauchy sequence for every $x \in E$. Since F is complete, we may define a mapping $I : E \to F$ by

$$I(x) = \lim_{n\to\infty} 2^n g\left(\frac{x}{2^n}\right). \tag{18}$$

Hence, the inequality (3), together with (18), implies that I is an isometry. The inequality (12) is an immediate consequence of (17) and (18).

Now, let $J : E \to F$ be an isometry satisfying the inequality (12). It then follows from (12) that $J(0) = 0$. Similarly as in the proof of Theorem 1, we may verify

$$2^k J\left(\frac{x}{2^k}\right) = J(x) \tag{19}$$

for all $x \in E$ and all $k \in \mathbf{N}$.

Finally, it follows from (12) and (19) that

$$\begin{aligned}
\|I(x) - J(x)\| &= 2^k \left\| I\left(\frac{x}{2^k}\right) - J\left(\frac{x}{2^k}\right) \right\| \\
&\leqslant 2^k \frac{2 \cdot 2^{(p-1)/2}}{2^{(p-1)/2} - 1} C \max\left\{ d\left(\frac{x}{2^k}, 0\right)^p, d\left(\frac{x}{2^k}, 0\right)^{(1+p)/2} \right\} \\
&\leqslant 2^{-k(p-1)/2} \frac{2^{(p+1)/2}}{2^{(p-1)/2} - 1} C \max\{d(x,0)^p, d(x,0)^{(1+p)/2}\} \\
&\to 0 \quad \text{as } k \to \infty,
\end{aligned}$$

for any $x \in E$, which implies the uniqueness of I. □

The following corollary may be proven by use of Theorem 2. However, the assumptions are so strong that the corollary can easily be proven in a straight forward manner. Hence we omit the proof.

Corollary 2. *Let G be a real normed space and let F be a real Hilbert space. Suppose a mapping $f : G \to F$ satisfies $f(0) = 0$ and $f(2^k x) = 2^k f(x)$ for all $x \in G$ and $k \in \mathbf{N}$. The mapping f is a linear isometry if and only if there exists a $p > 1$ such that*

$$\big| \| f(x) - f(y) \| - \|x - y\| \big| = O\big(\|x - y\|^p\big)$$

as $\|x\| \to 0$ and $\|y\| \to 0$.

In the above corollary, we may assume that F is a strictly convex real normed space (see [1]).

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