Certain Relations between Properties of Maps of Tessellation Automata

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This paper considers the relations between various properties of parallel maps of tessellation automata. The properties considered are injectivity, surjectivity, period-preservation, Poisson stability, finite orderedness, and so on for various subsets of configurations. In addition, Sears' result on the denseness of injective maps is extended to multidimensional tessellation spaces.

1. INTRODUCTION

The tessellation automata have been discussed by many authors, Moore [7], Myhill [8], Amoroso and Cooper [1], Yamada and Amoroso [14-16], Richardson [11], Amoroso and Patt [2], Maruoka and Kimura [6], Nasu [9], and Nasu and Honda [10]. Richardson proved the relation shown in Fig. 1, using the Garden of Eden Theorem and the compactness of product topology.

\[
\text{injectivity for } C \implies \text{surjectivity for } C_F
\]

\[
\Downarrow
\]

\[
\text{surjectivity for } C \iff \text{injectivity for } C_F
\]

Fig. 1. Richardson's result.

In a different approach, Hedlund [4], Sears [13], and Ryan [12] defined the shift dynamical system and investigated the properties of the continuous transformations which commute with the shift transformations, and showed many interesting results in one-dimensional tessellation spaces.

Hedlund conjectured that in some sense, the set \( A \) of all parallel maps which are injective on the set \( C \) of all configurations is relatively sparse in the set \( E \) of all parallel maps which are surjective on \( C \). Sears showed that \( A \) is nowhere dense in \( E \) under
the pointwise topology in one-dimensional spaces. In fact, Amoroso and Patt demonstrated
the existence of nontrivial injective parallel maps which appeared to be quite rare.

In this paper, we define a metric on multidimensional tessellation spaces, extending
the metric given by Hedlund, and investigate the properties of parallel maps. The
properties considered are injectivity, surjectivity, period-preservability, Poisson stability,
finite orderedness, and so on for various subsets of configurations. Our results refine
Richardson's results in detail.

In Section 2, notation and definitions are given. In Section 3, we define the set \( C_p \)
of all periodic configurations and period-preserving parallel maps, and show that if a
parallel map is injective on \( C_p \), then it is surjective on \( C_p \). As to one-dimensional
spaces, it is shown that injectivity for \( C \), injectivity for \( C_p \), period-preservability for \( C \),
and period-preservability for \( C_p \) are all equivalent properties.

In Section 4, we define Poisson stable and strongly Poisson stable parallel maps,
and show that a local map \( f \) has \( R \)-property if and only if the parallel map \( f_o \) of \( f \)
is Poisson stable on \( C_p \). In addition, we define the finite orderedness of parallel maps
and investigate the relations between these properties.

In Section 5, we extend the Sears result on the denseness of injective maps to multi-
dimensional spaces. Also we show that, in one-dimensional spaces, the set of all parallel
maps which are surjective on \( C_p \) is nowhere dense in \( E \) under the pointwise topology.
As a corollary, it follows that the set of all strongly \( C_p \) Poisson stable parallel maps
is nowhere dense in \( E \) under the pointwise topology.

Finally, in Section 6, we give examples of various classes of parallel maps which
show nonequivalence between these classes.

2. Preliminaries

Let \( N_0, N \), and \( Z \) denote respectively the set of all nonnegative integers, the set
of all positive integers, and the set of all integers. Let \( s \in N \) and let \( Q \) be a set with
\( |Q| = s \) where \( |Q| \) means the cardinal of \( Q \). A convenient choice of \( Q \) is the set
\( \{0, 1, \ldots, s - 1\} \). Usually 0 is designated as the quiescent symbol.

A configuration over \( Q \) is a function from \( Z^n \) to \( Q \) where \( n \in N \). The set of all configurations over \( Q \) is denoted by \( C(Q) \) or simply \( C \) if it is clear which \( Q \) is involved. The quiescent configuration is defined to be the configuration whose values are all 0 and is denoted
by \( 0 \). Similarly, \( a \) denotes the configuration whose values are all \( a \) where \( a \in Q \).

A pattern is a function from \( Z^n \) to \( Q \) which is defined at finitely many points on \( Z^n \).
Let \( m = (m_1, \ldots, m_n) \in N^n \). An \( m \)-pattern is a function from \( D_{m,t} \) to \( Q \), where

\[
D_{m,t} = \{(r_1, \ldots, r_n) \mid t_i \leq r_i \leq t_i + m_i - 1, 1 \leq i \leq n\}
\]

for some \( t = (t_1, \ldots, t_n) \in Z^n \).

An \( m^* \)-pattern is a function from \( *D_{m,t}(D_{m,t}^*) \) to \( Q \), where

\[
*D_{m,t} = D_{m,t} - \{(t_1, \ldots, t_n)\},
\]

\[
(D_{m,t}^*) = D_{m,t} - \{(t_1 + m_1 - 1, \ldots, t_n + m_n - 1)\},
\]

for some \( t \in Z^n \).
Especially, an \( m \)-pattern on \( D_{m,0} \) is called an \( m_0 \)-pattern. An \(*m_0\)-pattern (\( m_0^* \)-pattern) is similarly defined.

For \( x \in C \), let \( I^*_m x \) denote an \( m_0 \)-pattern which is the restriction of \( x \) to \( D_{m,0} \), and let \( I^*_m x \) denote an \( m_0^* \)-pattern which is the restriction of \( x \) to \( D_{m,0}^* \). Let \( k \) be a nonnegative integer. For \( x \in C \), let \( I_k x \) denote a pattern which is the restriction of \( x \) to \( D_k \) where

\[
D_k = \{ (r_1, \ldots, r_n) \mid r_i \leq k, 1 \leq i \leq n \}.
\]

These notations will be used to describe the concepts of parallel maps in this section, \( R \)-properties in Section 4, rearranged patterns in Section 5, and so on.

Let \( f \) be a mapping from \( Q^m \) to \( Q \) such that \( f(0^m) = 0 \) where \( Q^m \) denotes the set of all \( m_0 \)-patterns and \( 0^m \) denotes the \( m_0 \)-pattern whose values are all 0. The set of all such maps for a given \( m \in \mathbb{N}^n \) and a given symbol set \( Q \) is denoted by \( F(Q, m) \) or simply \( F(m) \) if it is clear which \( Q \) is involved. An element of \( F(Q, m) \) is called a \( local \) map.

For each \( i \) (\( 1 \leq i \leq n \)), the \( shift \) transformation or simply the \( shift \) \( \sigma_i \) is a mapping from \( C \) to \( C \) defined by

\[
[\sigma_i(x)](r_1, \ldots, r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_n) = x(r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_n).
\]

Let \( f \in F(m) \). For \( f \), we define a map \( f_\infty \) from \( C \) to \( C \) as

\[
[f_\infty(x)](r) = f[\Gamma_m^{r'}(x)],
\]

where \( x \in C \), \( r = (r_1, \ldots, r_n) \in \mathbb{Z}^n \), and \( r' = \sigma_1^{r_1} \cdots \sigma_n^{r_n} \). Let \( l \) be any element of \( \mathbb{Z}^n \) and let \( \psi = \sigma_l f_\infty \). \( \psi \) will be called a \( parallel \) map or a \( global \) map.

Now we define a \( metric \) \( d \) on \( C \). Let \( x \) and \( y \) be any element in \( C \). If \( x = y \), define \( d(x, y) = 0 \). If \( x \neq y \), let \( k \) be the least nonnegative integer such that \( \Gamma_k x \neq \Gamma_k y \) and define \( d(x, y) = (1 + k)^{-1} \). It is easily verified that \( d \) is a metric on \( C \) and that the metric topology induced by \( d \) coincides with the product topology induced by the discrete topology of \( Q \) [4].

In the remaining part of this section, we summarize the known definitions and propositions on the metric space which will be used through this paper.

**Definition 2.1.** Let \((X, d)\) be a metric space. Let \( x \in X \) and \( M \subseteq X \). A point \( x \) is an \textit{accumulating point of} \( M \) if for any \( \epsilon > 0 \), \( \bigcup (x, \epsilon) \cap (M - \{x\}) \neq \emptyset \), where

\[
\bigcup (x, \epsilon) = \{ y \mid y \in X \text{ and } d(x, y) < \epsilon \}.
\]

**Definition 2.2.** A metric space \((X, d)\) is \textit{compact} if either \( X \) is a finite set or every infinite subset of \( X \) has at least an accumulating point.

**Proposition 2.1** [4, 11]. \( (C, d) \) is a compact metric space.

**Proposition 2.2** [4, 11]. A map \( \psi \) from \( C \) to \( C \) is a parallel map if and only if (i) \( \psi \)
is continuous, (ii) for each \( i \ (1 \leq i \leq n) \), \( \sigma_i \psi = \psi \sigma_i \), and (iii) \( \psi(\emptyset) = \emptyset \), where \( n \) is the dimension of the tessellation space.

**Proposition 2.3** [5]. Let \( (x, d) \) be a compact metric space and let \( f \) and \( g \) be continuous maps from \( X \) to \( X \).

1. For any subset \( M \) of \( X \), \( f(M) = \overline{f(M)} \) where \( \overline{M} \) and \( \overline{f(M)} \) mean the closures of \( M \) and \( f(M) \), respectively.
2. \( f(X) \) is a compact metric space.
3. Any compact subset of \( X \) is closed.
4. The set \( E = \{ x \mid x \in X \text{ and } f(x) = g(x) \} \) is a closed subset of \( X \).
5. If \( f \) is a bijective map, then \( f^{-1} \) is also a continuous map where \( f^{-1} \) means the inverse map of \( f \).

3. Period-Preservability of Parallel Maps

First, we give the definitions of periodic configurations, period-preservability, and Poisson stability of parallel maps.

**Definition 3.1.** Let \( n \) be the dimension of a tessellation space and let \( x \in C(Q) \). We define the period vector of \( x \), denoted by \( \omega(x) \in (N \cup \{\infty\})^n \), as follows: For each \( i \ (1 \leq i \leq n) \), let \( \omega_i(x) = r \) if there exists the least positive integer \( r \), such that \( \sigma^r_i(x) = x \) and \( \omega_i(x) = \infty \) if not. Then \( \omega(x) = (\omega_1(x), \ldots, \omega_n(x)) \). A configuration \( x \in C(Q) \) is called a periodic configuration if \( \sum_{i=1}^{n} \omega_i(x) < \infty \) and the set of all periodic configurations is denoted by \( C_p(Q) \) or simply \( C_p \).

**Definition 3.2.** Let \( \psi \) be a parallel map.

1. \( \psi \) is called period-preserving on \( C \) if for all \( x \in C \), \( \omega[\psi(x)] = \omega(x) \).
2. \( \psi \) is called period-preserving on \( C_p \) if for all \( x \in C_p \), \( \omega[\psi(x)] = \omega(x) \).
3. Let \( M \subseteq C \). \( \psi \) is called surjective on \( M \) if \( \psi(M) = M \). \( \psi \) is called injective on \( M \) if for all \( x, y \in M \), \( \psi(x) = \psi(y) \) implies \( x = y \).

**Remark 3.1.** \( C_p \) is a countable dense subset of \( C \).

**Remark 3.2.** Let \( x \in C_p \) and \( \psi \) be a parallel map. Then \( \omega_i[\psi(x)] \mid \omega_i(x) \) for each \( i \ (1 \leq i \leq n) \) where \( a \mid b \) means that \( a \) divides \( b \).

**Definition 3.3.** Let \( x \in C \) and \( \psi \) be a parallel map. \( x \) is called Poisson stable w.r.t. \( \psi \) if there exists a sequence of nonnegative integers \( n_1 < n_2 < \cdots \) such that \( \lim_{i \to \infty} \psi^{n_i}(x) = x \). Let \( M \subseteq C \). \( \psi \) is called \( M \)-Poisson stable if any point in \( M \) is Poisson stable w.r.t. \( \psi \).

Also, \( x \) is called strongly Poisson stable w.r.t. \( \psi \) if there exists \( n_x \in N \) such that \( \psi^{n_x}(x) = \infty \) (\( n_x \) may depend on \( x \)); \( \psi \) is called strongly \( M \)-Poisson stable if any point in \( M \)
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is strongly Poisson stable w.r.t. $\psi$. By definition, strongly $M$ Poisson stability implies $M$ Poisson stability. The concept of Poisson stability is introduced in dynamical systems [3]. Intuitively, Poisson stability means that the sequence of configurations $\psi^n(x)$'s ($n = 0, 1, \ldots$) enters infinitely many times in arbitrary neighborhoods of $x$, i.e., for any $\epsilon > 0$, $\bigcup (x, \epsilon) \cap \{\psi^i(x) \mid i \in \mathbb{N}\} \neq \emptyset$. Poisson stability and strongly Poisson stability are properties concerned with the dynamical behavior of tessellation automata.

In the following, the relations among injectivity, surjectivity, period-preservability, and (strongly) Poisson stability are considered. The results are summarized in Proposition 3.1 and Fig. 2.

**FIG. 2.** Summary of the results in Section 3.

**LEMMA 3.1.** Let $\psi$ be a parallel map. $\psi$ is injective on $C_p$ if and only if $\psi$ is period-preserving and surjective on $C_p$.

**Proof.** Let $\psi$ be injective on $C_p$. Suppose that there exists $x \in C_p$ such that $\omega[\psi(x)] \neq \omega(x)$. Let $E_1 = \{y \mid y \in C_p \text{ and } \omega_i(y) = \omega_i(x) \text{ for each } i \ (1 \leq i \leq n)\}$. The set $E_1$ is of finite cardinality. Hence it follows from Remark 3.2 that $\psi$ is not injective on $E_1 \subseteq C_p$. Thus injectivity for $C_p$ implies period-preservability for $C_p$.

Let $E_x = \{u \mid u \in C_p \text{ and } \omega(x) = \omega(u)\}$. Since $E_x$ is of finite cardinality for any $x \in C_p$, there exists $n_x \in \mathbb{N}$ such that $\psi^n_x(x) = x$. Thus $\psi$ is surjective on $C_p$.

Conversely, let $\psi$ be period-preserving and surjective on $C_p$. Suppose that $\psi$ is not injective on $C_p$. Since $\psi$ is period-preserving on $C_p$, it contradicts the assumption that $\psi$ is surjective on $C_p$.

**LEMMA 3.2.** Let $\psi$ be a parallel map. Then the following statements are equivalent.

1. $\psi$ is injective on $C_p$.
2. $\psi$ is strongly $C_p$ Poisson stable.
3. $\psi$ is $C_p$ Poisson stable.

**Proof.** Proofs for (1) $\implies$ (2) and (2) $\implies$ (3) are obvious. It is sufficient to show that (3) implies (1). Suppose that $\psi$ is not injective on $C_p$. Then it follows from Lemma 3.1 that $\psi$ is not period-preserving or not surjective on $C_p$. Clearly $\psi$ is not $C_p$ Poisson stable.

**PROPOSITION 3.1.** Let $\psi$ be a parallel map. The relations shown in Fig. 2 hold. In Fig. 2,
\( \Rightarrow \) means implication, which holds for multidimensional tessellation spaces, whereas \( \rightarrow \) means implication, which holds for one-dimensional spaces, and \( \leftrightarrow \) means nonimplication.

**Proof.** (1) Relations for multidimensional spaces. It is easy to show that if \( \psi \) is injective on \( C \), then \( \psi \) is period-preserving on \( C \). Suppose that \( \psi \) is period-preserving on \( C \) and not injective on \( C_p \). Then for some \( x, y \in C_p \), \( x \neq y \), \( \psi(x) = \psi(y) \), and \( \omega(x) = \omega[\psi(x)] = \omega[\psi(y)] = \omega(y) \). Let \( E_\alpha \) be the subset of \( C_p \) given in Lemma 3.1. Since \( |E_\alpha| < \infty \), \( \psi \) is not surjective on \( C \). By the Garden of Eden Theorem, there exist mutually erasable patterns \( A_1 \) and \( B_1 \). Let \( A_2 \) (\( B_2 \)) be a pattern obtained by surrounding \( A_1 \) (\( B_1 \)) with layers of 0 symbols. \( A_2 \) and \( B_2 \) are \( m \)-patterns for some \( m \in \mathbb{N} \). Let \( u \) be a periodic configuration obtained by periodically continuing \( A_1 \). Let \( D \) be a pattern constructed from \( A_2 \) and \( B_2 \) as shown in Fig. 3, and let \( v \) be a periodic configuration obtained by periodically continuing \( D \). Clearly \( \omega(u) \neq \omega(v) \) but \( \psi(u) = \psi(v) \). This contradicts the assumption that \( \psi \) is period-preserving on \( C \). This proof also implies that if \( \psi \) is period-preserving on \( C_p \), then \( \psi \) is surjective on \( C \).

![Fig. 3. Construction of D from A_2 and B_2.](image)

Next, we show that surjectivity for \( C_p \) implies surjectivity for \( C \). By the hypothesis, \( \psi(C_p) = C_p \). By Proposition 2.3(1), \( \psi(C_p) = \psi(C_p) \). Since \( C_p = C \), it follows that \( C = \psi(C_p) = \psi(C_p) = \psi(C) \). Hence with Lemmas 3.1 and 3.2, the relations \( \Rightarrow \) in Fig. 2 are proved to hold.

(2) Relations for one-dimensional spaces. Nasu and Honda [10] proved that in one-dimensional spaces, \( \psi \) is injective on \( C \) if and only if \( \psi \) is surjective on \( C \) and injective on \( C_p \). Then it follows from (1) that \( \psi \) is injective on \( C \) if and only if \( \psi \) is injective on \( C_p \). Hedlund [4] showed that if \( \psi \) is surjective on \( C \), then for \( x \in C_p \), \( \psi^{-1}(x) \) is a subset of \( C_p \). Then it follows from Lemma 3.1 that period-preservability for \( C_p \) implies injectivity for \( C_p \).

In one-dimensional spaces, it is well known that there exists a parallel map which is surjective on \( C \) but not injective on \( C \).

Thus the relations \( \rightarrow \) and \( \leftrightarrow \) in Fig. 2 are proved to hold.
4. Poisson Stability of Parallel Maps

This section is concerned with Poisson stability (strongly Poisson stability) and finite orderedness of parallel maps. Especially, the relations between these properties of parallel maps and R-property of local maps are considered.

**Definition 4.1.** For any $a, b \in Q$, we define $\rho_0(a, b)$ as follows: If $a \neq b$ then $\rho_0(a, b) = 1$, else $\rho_0(a, b) = 0$. For any $x, y \in C$, let $\rho(x, y) = \sum_{r \in \mathbb{Z}} \rho_0(x(r), y(r))$. Then the set $C_F$ of all finite configurations is defined to be $\{x \mid x \in C$ and $\rho(x, 0) < \infty\}$. For a parallel map $\psi$, let $\text{fix } \psi = \{x \mid x \in C$ and $\psi(x) = x\}$ and $C_F(\psi) = \{x \mid x \in C, y \in \text{fix } \psi$ and $\rho(x, y) < \infty\}$. Fix $\psi$ is the set of all fixed points of $\psi$, and $C_F(\psi)$ is the set of all configurations which differ from some fixed point of $\psi$ at finitely many points. If $\overline{0}$ is the only fixed point of $\psi$, then $C_F(\psi) = C_F$. Properties of parallel maps on $C_F$ and $C_F(\psi)$ will be considered.

**Remark 4.1.** $C_F$ and $C_F(\psi)$ are dense subsets of $C$.

**Proposition 4.1.** Let $\psi$ be a parallel map and let $M$ be a dense subset of $C$. If $\psi$ is $M$ Poisson stable, then $\psi$ is surjective on $C$.

**Proof.** Suppose that $\psi$ is not surjective, that is, $\psi(C) \subsetneq C$. $\psi(C)$ is a compact subset of $C$ by Proposition 2.3(2) and then a closed subset by Proposition 2.3(3). Hence $C - \psi(C)$ is an open subset of $C$. Let $x \in C - \psi(C)$. There exists $\delta > 0$ such that $\bigcup (x, \delta) \subseteq C - \psi(C)$. Since $M = C$ by the assumption, there exist $y \in M$ and $\epsilon > 0$ such that $\bigcup (y, \epsilon) \subseteq \bigcup (x, \delta)$. Since $\psi$ is $M$ Poisson stable, there exists $n_i \in \mathbb{N}$ such that $\psi^{n_i}(y) \in \bigcup (y, \epsilon)$. Thus $\psi^{n_i}(y) \in C - \psi(C)$. This is a contradiction.

Next, we consider the relation between $R$-property of a local map and Poisson stability of the corresponding parallel map.

**Definition 4.2.** For $f \in F(Q, m)$, $f$ is said to have $R$-property if, for any $m_\sigma$-pattern $A$ and any $a, b \in Q$, $f(\overline{a} \ A) = f(\overline{b} \ A)$ implies $a = b$ where $\overline{a} \ A$ is the $m_\sigma$-pattern consisting of $A$ and $a$ as shown in Fig. 4.
LEMMA 4.1. Let \( f \in F(Q, m) \) have R-property. Let \( t \in \mathbb{N} \) and \( A_i \in Q^{m-1} \) for \( i = 1, 2, \ldots, t \). We consider a sequence of \( m \)-patterns shown in (1).

\[
a^{(1)}_1 A_1, a^{(2)}_2 A_2, \ldots, a^{(t)}_t A_t, a^{(t+1)}_{t+1} A_1, \ldots, a^{(2t)}_{2t+1} A_1, \ldots
\]

where \( a^{(i)}_j \in Q \) are defined as

\[
a^{(i)}_j = f(a^{(i-1)}_{j-1} A_{j-1}) \quad (2 \leq i \leq t \text{ and } j \geq 1),
\]

\[
a^{(i)}_j = f(a^{(i-2)}_{j-2} A_{j-2}) \quad (j \geq 2),
\]

and \( a^{(i)}_1 \) is given. Then \( a^{(t)}_1 A_1 \) appears at a period of at most \( st \) (\( s = |Q| \)) in the sequence (1).

Proof. Consider the sequence of symbols \( a^{(1)}_1, a^{(2)}_2, \ldots, a^{(t)}_t \). Since \( Q \) is a finite set, there exist positive integers \( j_1 \) and \( j_2 \) (\( 1 \leq j_1 < j_2 \)) such that \( a^{(2)}_j = a^{(1)}_j \). Then \( f(a^{(i-1)}_{j-1} A_i) = f(a^{(i-2)}_{j-2} A_i) \). By the R-property of \( f \), \( a^{(i-2)}_{j-2} A_i \) appears at a period of \( 1 \). This process can be continued to obtain \( a^{(j)}_j = a^{(i)}_i \) for some positive integer \( j > 1 \). Clearly \( j \leq s + 1 \).

Since \( A_1 \) appears at a period of \( t \) in (1), \( a^{(t)}_1 A_1 \) appears at a period of at most \( st \) in (1).

THEOREM 4.1. Let \( f \in F(Q, m) \). Then the following statements are equivalent.

1. \( f \) has R-property.

2. \( f \) is \( C_F \) Poisson stable.

3. \( f \) is \( C_F(f \circ \alpha) \) Poisson stable.

Proof. First, we prove that (1) implies (2). We show that for given \( x \in C_F \) and \( \varepsilon > 0 \), there exists \( l \in \mathbb{N} \) such that \( f^l(x) \in \bigcup \{ x \pm \varepsilon \} \).

For simplicity, we first consider one-dimensional spaces. Without loss of generality, we can assume that \( x(i) = 0 \) for \( i \geq 0 \) and \( x(-1) \neq 0 \). It is sufficient to show that, for any \( n \in \mathbb{N} \), there exists \( l \in \mathbb{N} \) such that \( x(i) = x^l(i) \) for \( -n \leq i \leq -1 \) where \( f^l(x) \) is abbreviated as \( x^l \).

We apply Lemma 4.1 to \( x(-n) \cdots x(-1) 0_r \) where \( 0_r \) means that the symbols in the right semi-infinite space from \( (-1) \) are all zero. Let \( A_{i+1} = x^{(i)}(0) \cdots x^{(i)}(m - 2) \) for \( i \geq 0 \) and let \( a^{(i)}_i = x(-1) \). Since \( f(0^m) = 0 \) and \( x(-1) \neq 0 \), \( A_1 \) appears at a period of \( 1 \) and \( x(-1) 0_r \) appears at a period of at most \( s - 1 \). Next, let \( A_{i+1} = x^{(i)}(-1) \cdots x^{(i)}(m - 3) \) for \( i \geq 0 \), and let \( a^{(1)}_1 = x(-2) \). Then \( x(-2) x(-1) 0_r \) appears at a period of at most \( s(s - 1) \). Similarly \( x(-n) \cdots x(-1) 0_r \) appears at a period of at most \( s^{n-1}(s - 1) \).

Next we consider two-dimensional spaces. Let \( x \in C_F \). We can assume that \( x(r_1, r_2) = 0 \) for \( r_1 \geq 0 \) or \( r_2 \geq 0 \). By an argument similar to that for one-dimensional spaces, it is shown that an \( m \)-pattern defined on \( D_{m-1, m} \) appears at a period of at most \( s^{m_1 m_2 - 1} \), where \( m = (m_1, m_2) \). Thus any \( (m_1, m_2) \)-pattern will be regenerated within \( s^{m_1 m_2 - 1}(s - 1) \) steps. The proofs for spaces of more than two dimensions are similar.

The above arguments are similarly applied for \( C_F(f \circ \alpha) \). Hence R-property implies \( C_F \) Poisson stability and \( C_F(f \circ \alpha) \) Poisson stability. Clearly \( C_F(f \circ \alpha) \) Poisson stability implies \( C_F \) Poisson stability. Then the proof will be completed by showing that \( C_F \) Poisson stability implies R-property.
We prove that (2) implies (1). For simplicity, we consider one-dimensional spaces. The proofs for multidimensional spaces are similar. Suppose that $f \in (Q, m)$ does not have $R$-property, that is, there exist $A \in Q^{m-1}$ and $a, b \in Q$ ($a \neq b$) such that $f(aA) = f(bA)$. Let

$$P_1 = \{ A \mid A \in Q^{m-1} \text{ and } f(aA) = f(bA) \text{ for some } a, b \in Q \ (a \neq b) \}.$$ 

We assign a degree to each element of $P_1$ as follows. Let $A \in P_1$. If $A = 0^{m-1}$, then the degree of $A$ is $m - 1$. If $A = a_1 \cdots a_{t-1} 0^{m-1-t}$ ($a_t \neq 0$), then the degree of $A$ is $m - 1 - t$. Let $A_1$ be an element with the greatest degree in $P_1$. Let

$$A_1 = a_1 \cdots a_k 0^{m-1-k} = A_1(1) \cdots A_1(k) 0^{m-1-k} \quad (a_k \neq 0),$$

and

$$f(aA_1) = f(bA_1) \quad \text{for} \quad a \neq b. \quad (2)$$

We consider semi-infinite spaces $D(j) = \{ i \mid i \in \mathbb{Z} \text{ and } i > j \}$ for some $j \in \mathbb{Z}$ and semi-infinite configurations defined on $D(j)$. Let $f_{\omega j}$ be the restriction of $f_\omega$ on the set of semi-infinite configurations defined on $D(j)$. Let $A_1\overline{\omega}_r$ be a semi-infinite configuration on $D(1)$ and let

$$f_{\omega 1}(A_1\overline{\omega}_r) = A_{i+1}\overline{\omega}_r = A_{i+1}(1) A_{i+1}(2) \cdots A_{i+1}(k) 0^{m-1-k}\overline{\omega}_r.$$

Since $A_{i+1}(2) \cdots A_{i+1}(k) 0^{m-k} \notin P_1$ by the definition of $A_1$, $A_1\overline{\omega}_r$ appears periodically in the sequence of $A_{i+1}\overline{\omega}_r$'s ($i = 0, 1, \ldots$). Let its period be $t$. Then for any $l \in \mathbb{N}_0$,

$$A_i = A_{tl+i} \quad (1 \leq i \leq t).$$

We define $a_1^{(j)}$ as

$$a_1^{(1)} = a \text{ or } b,$$

$$a_1^{(j)} = f(a_1^{(j-1)}A_{i-1}) \quad (j \geq 1, 2 \leq i \leq t),$$

$$a_1^{(j+1)} = f(a_1^{(j)}A_i) \quad (j \geq 1).$$

Then by operating $f_{\omega 0}$ successively to $a_1^{(1)}A_1\overline{\omega}_r$, we obtain a sequence of semi-infinite configurations

$$a_1^{(1)}A_1\overline{\omega}_r, a_1^{(2)}A_2\overline{\omega}_r, \ldots, a_1^{(i)}A_i\overline{\omega}_r, a_1^{(o)}A_1\overline{\omega}_r, \ldots, a_1^{(o)}A_1\overline{\omega}_r, a_1^{(o)}A_1\overline{\omega}_r, \ldots. \quad (3)$$

By virtue of (2), at least one of the terms $aA_1$ and $bA_1$ is never regenerated in sequence (3). Thus $f_\omega$ is not $C_\omega$ Poisson stable.

**COROLLARY 4.1.** Let $m, l \in \mathbb{N}^n$ and let $f \in F(m)$, $g \in F(l)$.

1. If both of $f_\omega$ and $g_\omega$ are $C_\omega$ Poisson stable, then the composite map $f_\omega g_\omega$ is also $C_\omega$ Poisson stable.

2. If $f_\omega$ is $C_\omega(f_\omega)$ Poisson stable and $g_\omega$ is $C_\omega(g_\omega)$ Poisson stable, then $f_\omega g_\omega$ is $C_\omega(f_\omega g_\omega)$ Poisson stable.
Proof. It follows from Theorem 4.1 that both of the local maps \( f \) and \( g \) have \( R \)-property. It is easily shown that the composite map \( fg \) belongs to \( F(m + l - e) \) where \( e = (1, \ldots, 1) \in \mathbb{N}^n \) and has \( R \)-property. (It was proved by Hedlund [4] for one-dimensional spaces.) Noting that \( (fg)_\infty = f_\infty g_\infty \), the statements are easily proved.

We next consider the Poisson stability for the reflection and the inverse maps of parallel maps.

**Definition 4.3.** For each \( i \) \((1 \leq i \leq n)\), the reflection function \( \tau_i \) from \( C \) to \( C \) is defined as

\[
[\tau_i(x)](r_1, \ldots, r_{i-1}, r_i, r_{i+1}, \ldots, r_n) = x(r_1, \ldots, r_{i-1}, -r_i, r_{i+1}, \ldots, r_n).
\]  

(4)

\( \tau_i(x) \) is the reflection of \( x \) along the \( i \)-th axis around the origin.

Clearly, \( \tau_i^2 = I \). Hence \( \tau_i \) is a homeomorphism from \( C \) onto \( C \).

**Property 4.1.** \( \tau_i \sigma_i = \sigma_i \tau_i \) for \( i \neq j \) \((1 \leq i, j \leq n)\), and \( \tau_i \sigma_i = \sigma_i^{-1} \tau_i \).

**Lemma 4.2.** For each \( i \) \((1 \leq i \leq n)\) and for any \( f \in F(m) \), there exists \( g \in F(m) \) such that \( \tau_i f \sigma_i = \tau_i g \sigma_i \), where \( m = (m_1, \ldots, m_i, \ldots, m_n) \).

Proof. Let \( \theta_i \) denote the reverse function along the \( i \)-th axis from \( Q^m \) to \( Q^n \). For example, \( \theta_1(a_1 \cdots a_m) = a_m \cdots a_1 \) in one-dimensional spaces. Then it is easily shown that for any \( x \in C \),

\[
\theta_i \Gamma_m \sigma_i^{-(m_i-1)}(x) = \Gamma_m \tau_i(x).
\]  

(5)

Let \( g = f \theta_i \). For any \( r = (r_1, \ldots, r_i, \ldots, r_n) \in \mathbb{Z}^n \) and \( x \in C \),

\[
[\tau_i f \sigma_i \tau_i(x)](r) = [f \sigma_i \tau_i(x)](r_1, \ldots, -r_i, \ldots, r_n)
\]  

by (4)

\[
= f[\Gamma_m \sigma_i \tau_i(x)]
\]  

\[
= f[\Gamma_m \tau_i \sigma_i(x)]
\]  

by Property (4.1)

\[
= f[\theta_i \Gamma_m \sigma_i^{-(m_i-1)} \sigma_i(x)]
\]  

by (5)

\[
= g[i \sigma_i^{-(m_i-1)}(x)]
\]  

\[
= [g_\infty \sigma_i^{-(m_i-1)}(x)](r).
\]

Hence \( \tau_i f \sigma_i \tau_i = g_\infty \sigma_i^{-(m_i-1)} \).

**Lemma 4.3.** Let \( f \) be a local map and let \( j_1, \ldots, j_l \) be any positive integers such that \( 1 \leq j_t \leq n \) \((1 \leq t \leq l)\) and the \( j_t \)'s \((t = 1, \ldots, l)\) are distinct from each other. Then

(1) \( \tau_{j_t} \tau_{j_1} \cdots \tau_{j_i} f \sigma_{j_1} \tau_{j_1} \cdots \tau_{j_i} f \sigma_{j_1} \cdots \tau_{j_i} f \sigma_{j_1} \cdots \tau_{j_i} = C_F \) Poisson stable if and only if \( f_\infty \) is \( C_F \) Poisson stable;

(2) \( \tau_{j_t} \tau_{j_1} \cdots \tau_{j_i} f \sigma_{j_1} \tau_{j_1} \cdots \tau_{j_i} f \sigma_{j_1} \cdots \tau_{j_i} f \sigma_{j_1} \cdots \tau_{j_i} f \sigma_{j_1} \cdots \tau_{j_i} = C_F \) Poisson stable if and only if \( f_\infty \) is \( C_F(f_\infty) \) Poisson stable.
PARALLEL MAPS OF TESSELLATION AUTOMATA

Proof. Let \( \tau_0 \) denote \( \tau_{i_1} \tau_{i_2} \cdots \tau_{i_1} \). Assume that \( f_\infty \) is \( C_F \) Poisson stable. Then for any \( x \in C_f \), there exists a sequence of nonnegative integers \( n_1 < n_2 < \cdots \) such that \( \lim_{j \to \infty} f_\infty^n(x) = x \). Since \( \tau_0 \) is a bijective map on \( C_F \), there exists \( y \in C_F \) such that \( \tau_0(y) = x \). Hence \( \lim_{j \to \infty} f_\infty^n \tau_0(y) = \tau_0(y) \). Since \( \tau_0 \) is continuous and \( \tau_0^I = I \),

\[
\lim_{j \to \infty} (\tau_0 f_\infty \tau_0)^n(y) = \tau_0 \lim_{j \to \infty} f_\infty^n \tau_0(y) = \tau_0 \tau_0(y) = y.
\]

Thus \( \tau_0 f_\infty \tau_0 \) is \( C_F \) Poisson stable. The converse is similarly shown.

The proof for statement (2) is similar to the proof for (1).

Theorem 4.2. Let \( f \in F(m) \) where \( m = (m_1, \ldots, m_n) \) and let \( j_1, \ldots, j_t \) be any positive integers similar to those in Lemma 4.3. The following statements are equivalent.

1. \( f \) has R-property.
2. \( \sigma_{j_1}(m_{j_1}^{-1}) \cdots \sigma_{j_t}(m_{j_t}^{-1})(f \theta_{j_1} \cdots \theta_{j_t})_\infty \) is \( C_F \) Poisson stable.
3. \( \sigma_{j_1}(m_{j_1}^{-1}) \cdots \sigma_{j_t}(m_{j_t}^{-1})(f \theta_{j_1} \cdots \theta_{j_t})_\infty \) is \( C_F(\tau_{i_1} \cdots \tau_{i_f}) \) Poisson stable.

Proof. By Lemma 4.2, it is shown that \( \sigma_{j_1}(m_{j_1}^{-1}) \cdots \sigma_{j_t}(m_{j_t}^{-1})(f \theta_{j_1} \cdots \theta_{j_t})_\infty = \tau_{j_1} \cdots \tau_{j_t} f_\infty \tau_{j_1} \cdots \tau_{j_t} \).

Then the theorem follows from Lemma 4.3 and Theorem 4.1.

Proposition 4.2. Let \( f \) be a local map such that \( f_\infty \) is injective on \( C \). Then if \( f_\infty \) is \( C_t(C_F(f_\infty)) \) Poisson stable, then \( f_\infty^{-1} \) is also \( C_t(C_F(f_\infty)) \) Poisson stable.

Proof. Let \( x \in C_F \) and let \( x^i \) denote \( f_\infty^i(x) \) for \( i \in Z \). Without loss of generality, we can assume that \( x(r) \) is nonquiescent only on \( D = \{(r_1, \ldots, r_n) | r_1 < 0, \ldots, r_n < 0 \} \). For any \( k \in N \), let \( k e = (k, \ldots, k) \in N^n \) and let us observe \( k \)-patterns on \( D_{ke, -ke} = \{(r_1, \ldots, r_n) | -k \leq r_1 \leq -1, \ldots, -k \leq r_n \leq -1 \} \) which are the restriction of \( x^i \)'s on \( D_{ke, -ke} \). Let \( A_i \) denote the restriction of \( x^i \) on \( D_{ke, -ke} \).

Since \( f_\infty \) is \( C_F \) Poisson stable, there exists \( l \in N \) such that \( A_0 = A_1 \). It is clear that the sequence \( A_0, A_1, A_2, \ldots \) is a periodic sequence with period \( l \). Moreover, it is easily seen that the \( x^{-i} \)'s (\( i = 1, 2, \ldots \)) are Poisson stable w.r.t. \( f_\infty \) and \( x^{-i} \)'s are nonquiescent only on \( D \). Thus the sequence \( \ldots, A_{-i}, \ldots, A_{-1}, A_0, A_1, \ldots, A_i, \ldots \) is a periodic sequence with period \( l \). Since \( k \) is an arbitrary positive integer, we conclude that there exists a sequence of nonnegative integers \( n_1 < n_2 < \cdots \) such that \( \lim_{i \to \infty} f_\infty^{-n_i}(x) = x \). Since \( x \) is an arbitrary element of \( C_F \), \( f_\infty^{-1} \) is \( C_F \) Poisson stable. The statement for \( C_F(f_\infty) \) Poisson stability is similarly proved.

Next, we give the definition of finite orderedness of parallel maps and consider the relation between finite orderedness and Poisson stability.

Definition 4.4. Let \( \psi \) be a parallel map. \( \psi \) is said to have finite order if there exists a positive integer \( l \) such that \( \psi^l = I \) where \( I \) denotes the identity map. The least such positive integer \( l \) is called the order of \( \psi \). \( \psi \) is said to have infinite order if \( \psi \) is injective on \( C \) and for any \( l \in N, \psi^l \neq I \).
Let $M \subseteq C$. $\psi$ is said to have finite order on $M$ if $\psi^l | M = I | M$ for some $l \in \mathbb{N}$ where $\psi^l | M$ and $I | M$ denote respectively the restriction of $\psi^l$ and $I$ on $M$.

**Lemma 4.4.** Let $\psi$ be a parallel map and let $M$ be a subset of $C$ such that $M = C$. Then $\psi I = I$ if and only if $\psi^l | M = I | M$.

**Proof.** The "only if" part is obvious. We prove the "if" part. Suppose that $\psi^l | M = I | M$. Then fix $\psi^l \geq M$. It follows from Proposition 2.3(4) that fix $\psi^l$ is closed. Then fix $\psi^l = \text{fix } \psi^l \geq M = C$. Hence $\psi I = I$.

**Lemma 4.5.** Let $\psi$ be a parallel map.

1. $\psi$ is strongly $C_P$ Poisson stable if and only if $\psi$ is $C_P$ Poisson stable and, for any $x \in C_P$, there exists $P_x \in \mathbb{N}$ such that $\rho(\psi^l(x), 0) \leq P_x$ for all $l \geq 0$.

2. $\psi$ is strongly $C_{C_P}(\psi)$ Poisson stable if and only if $\psi$ is $C_{C_P}(\psi)$ Poisson stable and, for any $x \in C(\psi)$, there exist $y \in \text{fix } \psi$ and $P(x, y) \in \mathbb{N}$ such that $\rho(\psi^l(x), y) \leq P(x, y)$ for all $l \geq 0$.

**Proof.** (1) The "only if" part is obvious. We prove the "if" part. Let $x$ be any element in $C_P$. By the hypothesis, there exists $P_x \in \mathbb{N}$ such that $P_x = \max_{l \geq 0} \rho(\psi^l(x), 0)$. Let $l_x \in \mathbb{N}$ such that $P_x = \rho(\psi^{l_x}(x), 0)$ and let $u = \psi^{l_x}(x) \in C_F$. There exists $k \in \mathbb{N}$ such that $u(r) = 0$ for any $r \notin D_k$. Since $\psi$ is $C_P$ Poisson stable, there exists $n \in \mathbb{N}$ such that $\psi^n(u) \in \bigcup \{u, 1/1 + k\}$, that is, $[\psi^n(u)](r) = u(r)$ for $r \in D_k$. Since $\rho(u, 0) \geq \rho(\psi^n(u), 0)$, $[\psi^n(u)](r) = 0$ for $r \notin D_k$. Thus $\psi^n(u) = u$ and hence $\psi^{l_x+n}(x) = \psi^{l_x+n}(x)$. Since $\psi$ is injective on $C_F$, $\psi^n(x) = x$. Thus (1) is proved. Statement (2) is proved similarly.

**Theorem 4.3.** Let $\psi$ be a parallel map. The following statements are equivalent.

1. $\psi$ has finite order.
2. $\psi$ has finite order on $C_P$.
3. $\psi$ has finite order on $C_F$.
4. $\psi$ has finite order on $C_{C_P}(\psi)$.
5. $\psi$ is strongly $C$ Poisson stable.

**Proof.** By Lemma 4.4, statements (1) to (4) are all equivalent. Then it is sufficient to show that (5) implies (1). Hedlund [4] showed the existence of positively transitive points of $C$ for one-dimensional spaces. A point $z \in C$ is called positively transitive if, for any $x \in C$, there exists a sequence of integers $m_1 < m_2 < \cdots$ such that $\lim_{j \to \infty} \sigma_1^m(z) = x$. The definition and the existence of positively transitive points are similarly extended to multidimensional spaces.

For simplicity, we consider one-dimensional spaces. Let $z$ be a positively transitive point. By the assumption, there exists $n_x \in \mathbb{N}$ such that $\psi^n(x) = z$. Since $\psi$ is continuous,

$$
\psi^n(x) = \psi^n(\lim_{j \to \infty} \sigma_1^m(z)) = \lim_{j \to \infty} \psi^n \sigma_1^m(z) = \lim_{j \to \infty} \sigma_1^m \psi^n(z) = x.
$$

Thus $\psi$ has finite order.
Finally, some closure properties of the classes of parallel maps are given.

**Proposition 4.3.** Let \( \psi \) be a parallel map. Each class of parallel maps of finite order, strongly \( C_F(\psi) \) Poisson stability, \( C_F(\psi) \) Poisson stability, strongly \( C_F \) Poisson stability, and \( C_F \) Poisson stability has the following properties.

1. Each class is not closed under composition.
2. Let \( \lambda \) be a parallel map which is injective on \( C \). Let \( A_\lambda \) be an operation on the set of parallel maps such that
   \[ A_\lambda(\psi) = \lambda \psi \lambda^{-1}. \]

Then each class is closed under \( A_\lambda \).

**Proof.** (1) By definition, we have the relations shown in Fig. 5. Then it is sufficient to show that there exist \( \psi_1 \) and \( \psi_2 \) such that both of \( \psi_1 \) and \( \psi_2 \) have finite order but \( \psi_2 \psi_1 \) is not \( C_F \) Poisson stable.

![Diagram showing relations between Poisson stability and finite orderedness.](image)

Let \( f_1 \) and \( f_2 \) be the local maps in \( F(\{0, 1, 2\}, 2) \) given in Table I and let \( \psi_1 = f_1 \sigma \) and \( \psi_2 = \sigma^{-1} f_2 \sigma \). Clearly \( \psi_1^2 = I \) and \( \psi_2^2 = I \). Let \( x = \bar{0}_1 \bar{1}_0 \in C_F \). It is easy to show that \( x \) is not Poisson stable w.r.t. \( \psi_2 \psi_1 \).

**Table I**

<table>
<thead>
<tr>
<th></th>
<th>Local Maps ( f_1 ) and ( f_2 )</th>
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<tbody>
<tr>
<td>( f_1 )</td>
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<td>21</td>
<td>0</td>
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<tr>
<td>22</td>
<td>2</td>
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</table>
(2) The proof is given for the class of \(C_\rho(\psi)\) Poisson stability. The proofs for other classes are similar. For any \(x \in C_\rho(\psi)\), there exists a sequence of nonnegative integers \(n_1 < n_2 < \cdots\) such that \(\lim_{i \to \infty} \psi^{n_i}(x) = x\). Let \(y \in \text{fix } \psi\) such that \(\rho(x, y) < \infty\).

Let \(x = \lambda(x)\). Since \(\lambda\) is bijective on \(C\), \(x = \lambda^{-1}(z)\). Hence \(\lim_{i \to \infty} \psi^{n_i}(\lambda^{-1}(z)) = \lambda^{-1}(x)\).

Since \(\lambda\) is continuous,

\[
z = \lambda\lambda^{-1}(z) = \lambda \lim_{i \to \infty} \psi^{n_i}(\lambda^{-1}(z)) = \lim_{i \to \infty} \lambda \psi^{n_i}(\lambda^{-1}(z)) = \lim_{i \to \infty} (\lambda \psi \lambda^{-1})^{n_i}(z).
\]

Thus \(\lambda \psi \lambda^{-1}\) is \(C_\rho(\psi)\) Poisson stable.

Now, it is easily seen that \(y \in \text{fix } \psi\) if and only if \(\lambda(y) \in \text{fix } \lambda \psi \lambda^{-1}\). Clearly \(\rho(\lambda(x), \lambda(y)) < \infty\). Then \(\lambda(C_\rho(\psi)) = C_\rho(\lambda \psi \lambda^{-1})\). Thus \(\lambda \psi \lambda^{-1}\) is \(C_\rho(\lambda \psi \lambda^{-1})\) Poisson stable.

5. Sparseness of the Set ofInjective Maps

Let \(A(Q)\) denote the set of all parallel maps which are injective on \(C(Q)\) and \(E(Q)\) denote the set of all parallel maps which are surjective on \(C(Q)\). \(A(Q)\) and \(E(Q)\) are simply written as \(A\) and \(E\), respectively, if it is clear which \(Q\) is involved. This section shows that \(A\) is nowhere dense in \(E\) under the pointwise topology (Theorem 5.1) and thus extends Sears' result to multidimensional spaces as a corollary (Corollary 5.2).

**Definition 5.1.** A subset \(M\) of a topological space is said to be **nowhere dense** if \(\overline{M}\) has empty interior.

The next proposition is a known result and will be used in the sequel.

**Proposition 5.1 [5].** A set \(M\) is nowhere dense if and only if every nonempty open set has a nonempty open subset disjoint from \(M\).

Let \(m = (m_1, \ldots, m_n) \in \mathbb{N}^n\) and let

\(C_p(m) = \{x \mid x \in C_p\ and \ \omega_i(x) = m_i \ for \ each \ 1 \leq i \leq n\} \).

The cyclic shift \(S_j\) is defined to be a map from \(Q^m\) to \(Q^m\) such that the diagram in Fig. 6 is commutative.
Let $A, B \in \mathbb{Q}^n$. We say that $A$ is a rearranged pattern of $B$ if $A = S_{i_1} \cdots S_{i_n} B$ for some $i_1, \ldots, i_n \in N_0$.

**Lemma 5.1.** Let $p$ be a prime number and let $m = (p, \ldots, p)$. Let $a, b \in \mathbb{Q}$ and $A \mid a, A \mid b \in \mathbb{Q}^m$. $A \mid a$ is shown in Fig. 7 and $A$ is an $m_o^*$-pattern. If $A \mid a = \epsilon_{i_1}^{(p)} \cdots \epsilon_{i_n}^{(p)} A \mid b$, then $a = b$ where $0 \leq r_j \leq p - 1 \ (1 \leq j \leq n)$.

**Proof.** Let $r = (r_1, \ldots, r_n)$ and $e = (1, \ldots, 1) \in N^n$. Without loss of generality, we can assume that $r \neq (0, \ldots, 0)$. Let $S^r = S_{i_1}^{(p)} \cdots S_{i_n}^{(p)}$. By hypothesis $A \mid a = S^r A \mid b$. Then $A \mid b(r - e) = a$. For a vector $t = (t_1, \ldots, t_n)$, let $t^{(p)} = (t_1^{(p)}, \ldots, t_n^{(p)})$ mean the vector such that for each $i (1 \leq i \leq n) t_i^{(p)} = t_i (\text{mod } p)$ and $0 \leq t_i^{(p)} < p$. Clearly $A \mid b(r - e) = A \mid b((2r - e)^{(p)})$. Proceeding inductively, we obtain

$$a = A \mid b(r - e) = A \mid b((2r - e)^{(p)}) = \cdots = A \mid b((kr - e)^{(p)}).$$

There exist $k_1$ and $k_2 \in N \ (1 \leq k_1 < k_2)$ such that $(k_1 r - e)^{(p)} = (k_2 r - e)^{(p)}$. Then there exists $k \in N$ such that $(kr)^{(p)} = (0, \ldots, 0)$. Let $k_0$ be the least such positive integer. Since $0 \leq r_i < p$, we conclude that $k_0 = p$. Then $A \mid b((pr - e)^{(p)}) = b$. Hence $a = b$.

**Lemma 5.2.** Let $x_1, \ldots, x_k$ be points in $C$. Then for a sufficiently large integer $m$, we can find two $(m_e)^0$-patterns $M_1$ and $M_2$ such that

1. for each $a \in Q$, $(I_{m_e}^{x_j} a_j) \mid a$ is not equal to any rearranged pattern of either $M_1$ or $M_2$ where $j = 1, \ldots, k$;
2. if a rearranged pattern of $M_1$ is $M_1 \mid a$, then no rearranged pattern of $M_2$ is $M_2 \mid b$ for any $b \in Q$.

**Proof.** There are at most $skm^n$ rearranged patterns of $(I_{m_e}^{x_j} a_j)$'s. Since $|Q_{m_e}| = |Q|^{m^n}$, the number of $(m_e)^0$-patterns satisfying (1) is at least $|Q|^{m^n} - |Q| \cdot km^n$ for sufficiently large $m$. Let $M_1$ be an $(m_e)^0$-pattern satisfying (1). Let the rearranged patterns of
$M_1$ be $x_{k-1}, \ldots, x_{k+m^n}$. Using the same argument as that above for $x_1, x_2, \ldots, x_{k+m^n}$, we can choose $M_2$ satisfying (2).

**Proposition 5.2.** The subset of $E(Q)$ consisting of all noninjective maps on $C_\rho(Q)$ is dense in $E(Q)$ under the pointwise topology. Here the pointwise topology means the topology based on the neighborhoods 

$$\bigcup (\psi, x_1, \ldots, x_k, \epsilon) = \{ \lambda \in E(Q) \mid \max_{1 \leq i \leq k} d(\psi(x_i), \lambda(x_i)) < \epsilon \} \quad (\epsilon > 0).$$

**Proof.** Let $\psi = \sigma^g_\epsilon$ where $f \in F(m_1)$ and $q \in \mathbb{Z}^n$. Let $y_1, \ldots, y_s$ be arbitrary elements of $C$ and $\epsilon > 0$. We show that, for some $m_2 \in \mathbb{N}^n$, there exists $g \in F(m_2)$ such that $g_\omega$ is surjective on $C$, noninjective on $C_\rho$, and $\sigma^\omega g_\omega \in \bigcup (\psi, y_1, \ldots, y_s, \epsilon)$. Choose an integer $t$ such that $(1 + t)^{-1} < \epsilon$. Now, consider the set

$$\{\sigma^{r+i}y_j \mid j = 1, \ldots, s, r = (r_1, \ldots, r_n), \text{ and } r_i \in \{-(t - 1), 0, \ldots, (t - 1)\}\}$$

and denote the points in this set by $x_1, \ldots, x_k$. Then there exists $l \in \mathbb{N}$ such that if $\Gamma_l^x x_i = \Gamma_l^x x_i \ (i \neq j)$, then $x_i(r) = x_j(r)$ for all $r \in D$, and if $\Gamma_l^x x_i = 0^{(te)r}(i = 1, \ldots, k)$, then $x_i(r) = 0$ for all $r \in D$ where $D = \{(r_1, \ldots, r_n) \mid r_1 > 0, \ldots, r_n > 0\}$ and $0^{(te)r}$ denotes the $(le)^r$-pattern whose values are all 0.

We apply Lemma 5.2 to $x_1, \ldots, x_k$. Choosing a prime $p$ such that $p \geq \max\{l, m\}$, we have two $pe$-patterns $M_1$ and $M_2$ of the lemma.

We define a map $g \in F(pe)$ as

$$g \Gamma_{pe} x_j = f \Gamma_{pe} x_j \quad \text{for } j = 1, \ldots, k,$$

$$g(S^r M_1) = g(S^r M_2) = a, \quad \text{where } a \text{ is an arbitrary element of } Q$$

$$g(0_{pe}) = 0.$$

The above equations define $g$ on $(2p^n + k + 1)$ $pe$-patterns. By the choice of $M_1$ and $M_2$, and by Lemma 5.1, the above $pe$-patterns have the form $B [a]$ such that for any pair $B_i [a_i]$ and $B_j [a_j]$ if $B_i = B_j$, then $a_i = a_j$. We can extend the domain of $g$ to $Q^{pe}$ as follows. For any $(pe)^r$-pattern $M$, $g(M [a]) = g(M [b])$ implies $a = b$. Since $g\theta_1 \cdots \theta_n$ has $R$-property, $g_\omega$ is surjective on $C$.

Let $u$ and $v$ be periodic configurations obtained periodically continuing $M_1$ and $M_2$, respectively. Then $g_\omega(u) = g_\omega(v)$ ($u \neq v$). We now show that $\sigma^g g_\omega \in \bigcup (\psi, y_1, \ldots, y_s, \epsilon)$. It is sufficient to show that

$$[\sigma^g g_\omega(y_j)](r) = [\psi(y_j)](r),$$

or equivalently

$$[g_\omega(y_j)](q + r) = [f_\omega(y_j)](q + r)$$

for $j = 1, \ldots, s$, and $r \in \{(r_1, \ldots, r_n) \mid r_i \in \{-(t - 1), \ldots, 0, \ldots, (t - 1)\}\}$.

We have

$$[g_\omega(y_j)](r + q) = g[\Gamma_{pe} \sigma^{r+i} y_j] = g[\Gamma_{pe} x_1]$$
and

$$[f_x(y_j)](r + q) = f^r[T_{m_1} \sigma^y_j y_j]$$

for some \( l \in \{1, 2, \ldots, k\} \). From the definition of \( g \),

$$g[T_{m_1} x_i] = f[T_{m_1} x_i].$$

Then the proof is completed.

**Theorem 5.1.** \( A \) is nowhere dense in \( E \) under the pointwise topology.

**Proof.** From Proposition 5.1, it is sufficient to show that every map defined in Proposition 5.2 has a neighborhood disjoint from \( A \). Let \( \lambda \) be one of these maps and let \( x, y \in C_p \) such that \( \lambda(x) = \lambda(y) \). Let \( \omega(x) = (\omega_1(x), \ldots, \omega_n(x)) \) and \( \omega(y) = (\omega_1(y), \ldots, \omega_n(y)) \) be the period vectors of \( x \) and \( y \), respectively, and let \( l \) be the least common multiple of \( \omega_1(x), \ldots, \omega_n(x), \omega_1(y), \ldots, \omega_n(y) \). Then \( \sigma^l_i(x) = x \) and \( \sigma^l_i(y) = y \) for \( i = 1, \ldots, n \).

Now we consider \( \bigcup \{ \lambda, x, y, (1 + l)^{-1} \} \). If \( \psi \in \bigcup \{ \lambda, x, y, (1 + l)^{-1} \} \), then \( \Gamma_i \lambda(x) = \Gamma_i \psi(x) \) and \( \Gamma_i \lambda(y) = \Gamma_i \psi(y) \). Hence \( \lambda(x) = \psi(x) \) and \( \lambda(y) = \psi(y) \). Since \( \lambda(x) = \lambda(y) \), \( \psi(x) = \psi(y) \). The proof is completed.

In the remainder of this section, we consider one-dimensional spaces. Let \( E_p \) denote the set of all surjective parallel maps on \( C_f \). It will be shown that \( E_p \) is nowhere dense in \( E \) under the pointwise topology.

**Proposition 5.3.** Let \( G \) be a set of local maps such that \( g \in G \) if and only if both \( g \) and \( g \circ 1 \) have \( R \)-property. Then \( \{ \sigma^p_i g \mid p \in \mathbb{Z} \text{ and } g \in G \} \) is dense in \( E \) under the pointwise topology.

**Proof.** Let \( \psi \) be \( \sigma^p_i \phi \) where \( f \in F(n) \). Let \( y^{(1)}, \ldots, y^{(s)} \) be arbitrary elements of \( C \) and \( \epsilon \) be any positive number. We show that there exists \( g \in F(2m + 1) \cap G \) such that \( \sigma^p_i g \in \bigcup \{ \psi, y^{(1)}, \ldots, y^{(s)}, \epsilon \} \) for some \( p \in \mathbb{Z} \). Let \( t \) be an integer such that \( (1 + t)^{-1} < \epsilon \).

We form the set of configurations

$$\{ \sigma^r_i x^{(j)} \mid j = 1, \ldots, s \text{ and } r = -(t - 1), \ldots, 0, \ldots, (t - 1) \}$$

and denote the points in this set as \( x^{(1)}, \ldots, x^{(s)} \). Then there exists \( m \in \mathbb{N} \) such that if \( \Gamma_{m-1}^{(j)} x^{(j)}(l) = \Gamma_{m-1}^{(j)} x^{(j)}(l) \), i.e., \( x^{(j)}(l) = x^{(j)}(l) \) for \( l = -(m - 1), \ldots, 0, \ldots, (m - 1) \), then \( x^{(j)}(l) = x^{(j)}(l) \) for all \( l \in \mathbb{Z} \), and if \( x^{(j)}(l) = 0 \) for \( l = -(m - 1), \ldots, 0, \ldots, (m - 1) \), then \( x^{(j)}(l) = 0 \) for all \( l \in \mathbb{Z} \). For simplicity, we denote \( x^{(j)}(l) \) by \( x^{(j)}_l \) in the following.

We now define a local map \( g \in F(2m + 1) \) as

$$g(x^{(j)}_m, \ldots, x^{(j)}_0, \ldots, x^{(j)}_m) = f(x^{(j)}_0, \ldots, x^{(j)}_m)$$

for each \( i = 1, \ldots, k \). We can extend the domain of \( g \) to \( Q^{2m+1} \) so as for \( g \) and \( g \circ 1 \) to have \( R \)-property. Then it is easy to see that

$$\sigma^p_i g \in \bigcup \{ \sigma^p_1 g, y^{(1)}, \ldots, y^{(s)}, \epsilon \}$$

where \( p = q - m \).
Theorem 5.2. \( E_F \) is nowhere dense in \( E \) under the pointwise topology.

Proof. It is sufficient to show that every mapping \( \sigma_1 g \) defined in Proposition 5.3 has a neighborhood disjoint from \( E_F \). Let \( \lambda \) be one of these maps. Then \( \lambda \) is a many-to-one map from \( C \) onto \( C \) [4, Theorem 17.2]. Hence, there exists \( x \in C \) (\( x \neq \emptyset \)) such that \( \lambda(x) = \emptyset \). Let \( l \) be the period of \( x \). Now we consider \( \bigcup (\lambda, x, (1 + l)^{-1}) \). If \( \psi \in \bigcup (\lambda, x, (1 + l)^{-1}) \), then \( \Gamma_\lambda(x) = \Gamma_\psi(x) \). Hence, \( \lambda(x) = \psi(x) = \emptyset \). Then \( \psi \) is not surjective on \( C_F \) [9, Corollary 1.1].

Corollary 5.1. The set of all strongly \( C_F \) Poisson stable parallel maps is nowhere dense in \( E \) under the pointwise topology.

Proof. The set of all strongly \( C_F \) Poisson stable parallel maps is included in \( E_F \).

Corollary 5.2. \( A \) is nowhere dense in \( E \) under the pointwise topology [13].

6. Examples of Maps with Various Properties

This section gives examples of maps with various properties. These examples are used to show nonequivalence between various properties.

First we give an algorithm to obtain the set of all fixed points of a parallel map. Let \( f_\theta \in F(\{0, 1, 2\}, 3) \) be a local map given in Table II. \( \sigma_1^{-1} f_{3\alpha} \) is used as an example for obtaining the set of all fixed points.

| TABLE II |
|---|---|---|
| \( f_\theta \) | \( f_\theta \) | \( f_\theta \) |
| 000 | 0 | 100 | 0 | 200 | 0 |
| 001 | 2 | 101 | 2 | 201 | 2 |
| 002 | 0 | 102 | 0 | 202 | 0 |
| 010 | 1 | 110 | 2 | 210 | 2 |
| 011 | 1 | 111 | 2 | 211 | 2 |
| 012 | 1 | 112 | 0 | 212 | 0 |
| 020 | 2 | 120 | 1 | 220 | 1 |
| 021 | 2 | 121 | 1 | 221 | 1 |
| 022 | 0 | 122 | 1 | 222 | 1 |

Algorithm for \( \text{fix} \sigma_1^{-1} f_{3\alpha} \). Let

\[
N(\sigma_1^{-1} f_{3\alpha}) = \{abc | f_\theta(abc) = b \text{ where } a, b, c \in Q\},
\]

and

\[
\rho = \{ab | abc \in N(\sigma_1^{-1} f_{3\alpha})\} \cup \{bc | abc \in N(\sigma_1^{-1} f_{3\alpha})\},
\]

\[
q = \{(ab, bc) | abc \in N(\sigma_1^{-1} f_{3\alpha})\}.
\]
Generally, if we consider $f \in F(Q, m)$ and $\sigma_1^k f_\infty$ for $k \geq 0$, then we put

$$N(\sigma_1^k f_\infty) = \{a_1 \cdots a_m \mid f(a_1 \cdots a_{k+1} \cdots a_m) = a_{k+1} \text{ where } a_1, \ldots, a_m \in Q\},$$

and

$$p = \{a_1 \cdots a_{m-1} \mid a_1 \cdots a_m \in N(\sigma_1^{-k} f_\infty)\} \cup \{a_2 \cdots a_m \mid a_1 \cdots a_m \in N(\sigma_1^{-k} f_\infty)\},$$

$$q = \{(a_1 \cdots a_{m-1}, a_2 \cdots a_m) \mid a_1 \cdots a_m \in N(\sigma_1^{-k} f_\infty)\}.$$

Let $M(\sigma_1^{-1} f_\infty^\infty) = \langle p, q \rangle$ be the directed graph where the set of all nodes of $M$ is $p$ and the set of all directed branches of $M$ is $q$. $M(\sigma_1^{-1} f_\infty^\infty)$ is shown in Fig. 8.

Let $M'(\sigma_1^{-1} f_\infty^\infty)$ be the graph obtained from $M(\sigma_1^{-1} f_\infty^\infty)$ by removing all nodes and branches which do not form cycles. $M'(\sigma_1^{-1} f_\infty^\infty)$ is shown in Fig. 9. Then fix $\sigma_1^{-1} f_\infty^\infty$ is obtained from the set of all infinite pathes on $M'(\sigma_1^{-1} f_\infty^\infty)$.
PROPOSITION 6.1. There exists a parallel map which has infinite order and is strongly $C_r(\psi)$ Poisson stable.

Proof. $\sigma_1^{-1} f_{3x}$ has the desired properties. We first show that $\sigma_1^{-1} f_{3x}$ is strongly $C_r(\sigma_1^{-1} f_{3x})$ Poisson stable. It is easily shown that $\sigma_1^{-1} f_{3x}$ is injective on $C$. By Lemma 4.5(2), it is sufficient to show that, for any $y \in C_r(\sigma_1^{-1} f_{3x})$, there exist $x \in \text{fix } \sigma_1^{-1} f_{3x}$ and $P(a, y) \in N$ such that $\rho((\sigma_1^{-1} f_{3x})^l(y), x) \leq P(x, y)$ for all $l \geq 0$.

Let $y \in C_r(\sigma_1^{-1} f_{3x})$. By definition, there exists $x \in \text{fix } \sigma_1^{-1} f_{3x}$ such that $\rho(x, y) < \infty$. Therefore, without loss of generality, we can assume that there exists $t \in N$ such that $x(i) = y(i)$ for $i \leq 0$ or $i > t$. Denoting $x(1) \cdots x(t)$ by $A$ and $y(1) \cdots y(t)$ by $B$, $x$ and $y$ are written as

$$x = \cdots x(0) x(1) x(t + 1) x(t + 2) x(t + 3) \cdots,$$

$$y = \cdots y(0) y(1) y(t + 1) y(t + 2) y(t + 3) \cdots.$$

From Fig. 9, we have the following five possible cases for $x(0) x(1) x(2) x(3)$:

(1) 000, (2) 002, (3) 020, (4) 200, (5) 202.

Let $(\sigma_1^{-1} f_{3x})^l(x)$ and $(\sigma_1^{-1} f_{3x})^l(y)$ be abbreviated as $x^l$ and $y^l$, respectively.

**Cases 1 and 2.**

$$y = \cdots 000B \cdots, \quad \text{or} \quad y = \cdots 002B \cdots,$$

$$y = \cdots B000 \cdots, \quad \text{or} \quad y = \cdots B002 \cdots.$$

From Table II, it is easily seen that $y^l(-2) = 0$ and $y^l(-1) = 0$ for all $l \geq 0$. Thus $y^l(i) = x^l(i)$ for all $l \geq 0$ and all $i \leq 1$. Similarly, it is shown that $y^l(t + 1) = 0$ and $y^l(t + 2) = 0$ for all $l \geq 0$ and thus $y^l(i) = x^l(i)$ for all $l \geq 0$ and all $i \geq t + 1$.

**Case 3.**

$$y = \cdots 020B \cdots \quad \text{or} \quad y = \cdots B020 \cdots.$$

From Fig. 9, it is seen that $y(-3) = 0$ or $y(-3) = 2$. The former case is reduced to Case 2 and the latter is reduced to Case 5. Similarly, $y(t + 4) = 0$ or $y(t + 4) = 2$. The former is reduced to Case 4 and the latter is reduced to Case 5.

**Case 4.**

$$y = \cdots 200B \cdots \quad \text{or} \quad y = \cdots B200 \cdots.$$

From Fig. 9, it is seen that $y(-3) = 0$. This case is reduced to Case 3. Similarly, $y(t + 4) = 0$ or $y(t + 4) = 2$. The former is reduced to Case 1 and the latter is reduced to Case 2.

**Case 5.**

$$y = \cdots 202B \cdots \quad \text{or} \quad y = \cdots B202 \cdots.$$

From Table II, it is seen that $y^l(-1) = 0$, $y^l(-2) = 2$, $y^l(t + 2) = 0$, and $y^l(t + 3) = 2$ for all $l \geq 0$. Thus $y^l(i) = x^l(i)$ for all $l \geq 0$ and all $i \leq -1$ or $i \geq t + 2$. 
Then it follows from Lemma 4.5(2) that $\sigma_1^{-1}f_{3\infty}$ is strongly $C_F(\sigma_1^{-1}f_{3\infty})$ Poisson stable. We now show that $\sigma_1^{-1}f_{3\infty}$ has infinite order.

Let $x \in C_F$ for which

$$x(i) = 0 \quad \text{for} \quad i \leq 0 \quad \text{or} \quad i > n,$$

$$x(i) = 1 \quad \text{for} \quad 1 \leq i \leq n.$$  

$x$ is expressed as $\bar{0}_11^n\bar{0}_r$. Since $\sigma_1^{-1}f_{3\infty}$ is strongly $C_F$ Poisson stable, $(\sigma_1^{-1}f_{3\infty})^n(x) = x$ for some $n \in N$. It is sufficient to show that $n_x$ is not bounded. It is seen that

$$(\sigma_1^{-1}f_{3\infty})^4(\bar{0}_11^n\bar{0}_r) = \bar{0}_1101^{n-2}\bar{0}_r,$$

and

$$(\sigma_1^{-1}f_{3\infty})^5(\bar{0}_110^k1^t\bar{0}_r) = \bar{0}_1(10)^{k+1}1^{t-k}\bar{0}_r \quad (k \geq 1).$$

Then it is easily shown that

$$(\sigma_1^{-1}f_{3\infty})^{4t}(\bar{0}_11^n\bar{0}_r) = \bar{0}_1(10)^t1^{n-2t}\bar{0}_r$$

for any $t \in N$ such that $n - 2t > 0$. Thus, it is proved that $n_x$ is not bounded.

**Proposition 6.2.** There exists a parallel map $\psi$ which is strongly $C_F(\psi)$ Poisson stable but not injective on $C$.

Proof. Let $f_\delta \in F(\{0, 1, 2\}, 3)$ be the local map given in Table III. We show that $f_\delta$ has the desired properties. Since $f_\delta$ has $R$-property, $f_{\delta\infty}$ is $C_F(f_{\delta\infty})$ Poisson stable. It is seen that $\text{fix } f_{\delta\infty} = \{0\}$. Using methods similar to those used in proving Proposition 6.1, it can be shown that $f_{\delta\infty}$ is strongly $C_F(f_{\delta\infty})$ Poisson stable. Since $f_{\delta\infty}(\cdots 1212 \cdots) = f_{\delta\infty}(\cdots 2121 \cdots) = 1$, $f_{\delta\infty}$ is not injective on $C$.

<table>
<thead>
<tr>
<th></th>
<th>$f_\delta$</th>
<th>$f_\delta$</th>
<th>$f_\delta$</th>
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</thead>
<tbody>
<tr>
<td>000</td>
<td>0</td>
<td>100</td>
<td>200</td>
</tr>
<tr>
<td>001</td>
<td>0</td>
<td>101</td>
<td>201</td>
</tr>
<tr>
<td>002</td>
<td>0</td>
<td>102</td>
<td>202</td>
</tr>
<tr>
<td>010</td>
<td>0</td>
<td>110</td>
<td>210</td>
</tr>
<tr>
<td>011</td>
<td>0</td>
<td>111</td>
<td>211</td>
</tr>
<tr>
<td>012</td>
<td>0</td>
<td>112</td>
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<tr>
<td>022</td>
<td>0</td>
<td>122</td>
<td>222</td>
</tr>
</tbody>
</table>

**Proposition 6.3.** There exists a parallel map $\psi$ which is strongly $C_F$ Poisson stable but not surjective on $C_F(\psi)$. 


Proof. Let $f_5 \in F\{0, 1, 2\}, 3$ be the local map given in Table IV. We show that $f_{5\infty}$ has the desired properties. From Table IV, it is easily shown that $f_5$ has $R$-property and then $f_{5\infty}$ is strongly $C_F$ Poisson stable. It is seen that $\text{fix} f_{5\infty} = \{0, 1, 2\}$. Let $x = 121$. Then $x \in C_F(f_{5\infty})$. It is easy to show that $(f_{5\infty})^{-1}(x) \notin C_F(f_{5\infty})$. Thus $f_{5\infty}$ is not surjective on $C_F(f_{5\infty})$.

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<thead>
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<th>TABLE IV</th>
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<td>$f_5$</td>
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<td>021</td>
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<td>022</td>
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</table>

Proposition 6.4. (1) There exists a parallel map $\psi$ which is strongly $C_F$ Poisson stable but not $C_F(\psi)$ Poisson stable.

(2) Conversely, there exists a parallel map $\psi$ which is $C_F(\psi)$ Poisson stable but not strongly $C_F$ Poisson stable.

Proof. Let $f_6 \in F\{0, 1, 2\}, 3$ be the local map given in Table V. We show that $\sigma^{-1}_1 f_{6\infty}$ has the properties of (1). It is seen that $\sigma^{-1}_1 f_{6\infty}$ is injective on $C$. Using methods similar to those used in proving Proposition 6.1, it is shown that $\sigma^{-1}_1 f_{6\infty}$ is strongly $C_F$ Poisson stable.

<table>
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<tr>
<td>$f_6$</td>
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Let $x = 110$. It is easily shown that $x$ is not Poisson stable w.r.t. $\sigma^{-1}_1 f_{6\infty}$.

Let $f_7 \in F\{0, 1\}, 3$ be the local map given in Table VI. We show that $f_{7\infty}$ has the properties of (2). Since $f_7$ has $R$-property, $f_{7\infty}$ is $C_F(f_{7\infty})$ Poisson stable. Since $f_7(001) = 1$, it is shown that $f_{7\infty}$ is not strongly $C_F$ Poisson stable.
TABLE VI

<table>
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<th>$f_3$</th>
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<tr>
<td>110 0</td>
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<tr>
<td>111 1</td>
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</table>

PROPOSITION 6.5. There exists a parallel map which is injective on $C$ but not $C_F$ Poisson stable.

Proof. $\sigma_1$ is a parallel map. $\sigma_1$ is injective on $C$ but not $C_F$ Poisson stable.

PROPOSITION 6.6. There exists a parallel map $\psi$ which is $C_F(\psi)$ Poisson stable but not surjective on $C_F$.

Proof. Let $f_8$ be the local map given in Table VII. We show that $f_{80}$ has the desired properties. Since $f_8$ has $R$-property, $f_{80}$ is $C_F(f_{80})$ Poisson stable. Let $x = 01010$. It is seen that $f_{80}^{-1}(x) \notin C_F$ and hence $f_{80}$ is not surjective on $C_F$.

TABLE VII

<table>
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<tr>
<td>11 0</td>
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</table>

THEOREM 6.1. The relations between properties of parallel maps shown in Fig. 10 hold.
Proof. Let $\psi$ be a parallel map. By definition, strongly $C_f$ Poisson stability implies surjectivity for $C_f$ and strongly $C_f(\psi)$ Poisson stability implies surjectivity for $C_f(\psi)$. And if $\psi$ is injective on $C$, then $\psi$ is surjective on $C_f(\psi)$ because of $\text{fix } \psi = \text{fix } \psi^{-1}$. By the Garden of Eden Theorem, it is shown that $\psi$ is surjective on $C$ if and only if $\psi$ is injective on $C_f(\psi)$. Then by Propositions 6.1–6.6 and the relations in Fig. 5, we obtain the relations shown in Fig. 10.

Summarizing the results obtained so far, we show the relations among the various properties of parallel maps in Fig. 11.

<table>
<thead>
<tr>
<th>strongly $C_f$ Poisson stability</th>
<th>finite orderedness for $C_f$</th>
<th>finite orderedness for $C$</th>
<th>finite orderedness for $C_f(\psi)$</th>
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</thead>
<tbody>
<tr>
<td>strongly $C_f(\psi)$ Poisson stability</td>
<td>injectivity for $C_f$</td>
<td>injectivity for $C_f(\psi)$</td>
<td>injectivity for $C$</td>
</tr>
<tr>
<td>$C_f$ Poisson stability</td>
<td>period-preservability for $C_f$</td>
<td>surjectivity for $C_f$</td>
<td>surjectivity for $C_f(\psi)$</td>
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<tr>
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<td>period-preservability for $C_f$</td>
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<tr>
<td>injectivity for $C$</td>
<td>injectivity for $C_f$</td>
<td>injectivity for $C_f(\psi)$</td>
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</table>

FIG. 11. Relations among properties of parallel maps of tessellation automata.

ACKNOWLEDGMENTS

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