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Error estimates of the DtN finite element method for the exterior Helmholtz problem

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Abstract

A priori error estimates are established for the DtN (Dirichlet-to-Neumann) finite element method applied to the exterior Helmholtz problem. The error estimates include the effect of truncation of the DtN boundary condition as well as that of the finite element discretization. A property of the Hankel functions which plays an important role in the proof of the error estimates is introduced.

Keywords: Exterior Helmholtz problem; Artificial boundary condition; Drichlet-to-Neumann operator; Finite element method; A priori error estimate

1. Introduction

We consider the exterior Helmholtz problem:

$$\begin{cases} -\Delta u - k^2 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \gamma, \\ \lim\limits_{r \longrightarrow +\infty} r^{(d-1)/2} \left(\dfrac{\partial u}{\partial r} - \mathrm{i}ku \right) = 0 & \text{(the outgoing radiation condition),} \end{cases} \tag{1}$$

where k, called the wave number, is a positive constant, Ω is an unbounded domain of $\mathbb{R}^d$ ($d = 2$ or 3) with sufficiently smooth boundary γ, f is a given datum, $r = |x|$ for $x \in \mathbb{R}^d$, and $\mathrm{i} = \sqrt{-1}$. Assume that $\mathcal{O} \equiv \mathbb{R}^d \backslash \overline{\Omega}$ is a bounded open set and that f has a compact support. Problem (1) arises in models of acoustic scattering by a sound-soft obstacle $\mathcal{O}$ embedded in a homogeneous medium.

To solve numerically problem (1), one often introduces an artificial boundary in order to reduce the computational domain to a bounded domain and imposes an artificial boundary condition on the artificial boundary. Although a variety of artificial boundary conditions have been proposed (see, e.g., [7], for a review), we focus on the exact nonlocal boundary condition based on the *Dirichlet-to-Neumann* (DtN) operator, which is called the exact DtN boundary condition. The comparison of the exact DtN boundary condition with local artificial boundary conditions is described in [8,10]. Imposing the exact DtN boundary condition on the artificial boundary, we can reduce problem (1) equivalently to a problem on the bounded domain between the artificial boundary and the boundary γ. We discretize the reduced problem by using the finite element method. This method is called the DtN finite element method.

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The DtN finite element method for the exterior Helmholtz problem has first been proposed by MacCamy–Marin [23] in 1980, who represented the DtN operator through an integral equation. Feng [4], Masmoudi [24], and Keller–Givoli [18] have derived the Fourier series representation of the DtN operator. Masmoudi [24] and Keller–Givoli [18] incorporated such a representation directly into the finite element method.

The DtN finite element method for other kinds of problems concerning the Helmholtz equation has been investigated by several authors. In 1978, Fix–Marin [6], who are pioneers in the DtN finite element method, studied the under-water acoustic problem. Goldstein [11] established error estimates for the Helmholtz problem on unbounded waveguides. Bao [1] also established error estimates for the problem concerning the diffraction of a time harmonic wave incident on a periodic surface of some inhomogeneous material.

In China, independently of the western world, the DtN finite element method has been suggested and developed first by Feng and Yu in 1980 and 1982, for example, see [5], where they called it the canonical boundary element method or the natural boundary element method. Yu has published many papers [31,32,30,35,17] and a monograph [33,34] (see also its review [9]) in this direction. In Ushijima's paper [28], we can find these words: "Chinese scholars, Feng Kang, Han Houde, Yu De-hao and others should be mentioned among founders of the treatment. In western world, J.B. Keller and D. Givoli are also should be quoted."

In this paper, we establish a priori error estimates in the H^1- and L^2-norms for the exterior Helmholtz problem. The use of the Fourier series representation of the DtN operator requires truncating the series in practical computations. So we analyze the series truncation error as well as the finite element discretization error. To the best knowledge of the author, our error estimates are new, because no error estimate treating both the truncation error and the discretization error simultaneously has been published yet (cf. [10, p. 32]). MacCamy–Marin [23] and Masmoudi [24] have derived an error estimate, but their estimate depends only on the mesh size (see also [3,17]). The counterpart of our error estimates for the Helmholtz problem on unbounded waveguides has been established by Goldstein [11] in 1982. Our error analysis roughly follows his analysis; however, we need some properties of the Hankel functions, which contain a new and important result (Lemma 4); we were inspired to prove Lemma 4 by Han–Bao [12, Lemma 3.1]. We here remark that in the error analysis of ours (and also of Goldstein), the argument of Schatz [27] plays an essential role, since the Helmholtz equation is indefinite.

Analysis of the truncation error in the DtN finite element method is an important topic. For problems of the positive definite type, Yu [32] and Han–Wu [13] have first derived error estimates including the truncation error for the exterior Laplace problem in 1985. After that, such error estimates for other problems are established in many papers, for example, in Han–Wu [14] for the linear elastic problem, and in Givoli–Patlashenko–Keller [10] and Han–Bao [12] for a certain class of the linear elliptic second order boundary value problem on exterior domains and semi-infinite strips (the error estimate in [12] is more sophisticated than that in [10]). For problems of other types, we mention Lenoir–Tounsi [21] (the sea-keeping problem) and Koyama–Tanimoto–Ushijima [20] (the eigenvalue problem of the linear water wave in a water region with a reentrant corner).

The remainder of this paper is organized as follows. In Section 2, we formulate the reduced problem with the DtN boundary condition. In Section 3, we introduce some properties of the Hankel functions, which are employed for establishing the error estimates in Section 4.

2. The DtN formulation

We first introduce a theorem concerning the well-posedness of problem (1).

Theorem 1. *For every compactly supported $f \in L^2(\Omega)$, problem* (1) *has a unique solution in $H^2_{\mathrm{loc}}(\overline{\Omega})$, where*

$$H^m_{\mathrm{loc}}(\overline{\Omega}) = \{u \mid u \in H^m(B) \text{ for all bounded open set } B \subset \Omega\} \quad (m \in \mathbb{N}).$$

Here $L^2(\Omega)$ denotes the usual space of complex-valued square integrable functions on Ω, and $H^m(B)$ denotes the usual complex Sobolev space on B (*see, e.g.*, [22]).

Proof. See [25,26]. □

To seek approximate solutions of problem (1), we introduce an artificial boundary $\Gamma_a = \{x \in \mathbb{R}^d \,|\, |x| = a\}$, where a is a positive number such that $\overline{\mathcal{O}} \cup \operatorname{supp} f \subset B_a \equiv \{x \in \mathbb{R}^d \,|\, |x| < a\}$. Then problem (1) is reduced equivalently to the following problem on the bounded computational domain $\Omega_a = \Omega \cap B_a$:

$$\begin{cases} -\Delta u - k^2 u = f & \text{in } \Omega_a, \\ u = 0 & \text{on } \gamma, \\ \dfrac{\partial u}{\partial r} = -\mathcal{S} u & \text{on } \Gamma_a, \end{cases} \tag{2}$$

where $\mathcal{S}$ is the DtN operator corresponding to the outgoing radiation condition. For its definition, see [24,18].

Now we introduce the circular harmonics Y_n defined by

$$Y_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}},$$

where θ denotes the angular variable of an (r, θ) polar coordinate system, and the spherical harmonics Y_n^m defined by

$$Y_n^m(\theta, \phi) = \sqrt{\frac{(2n+1)}{4\pi} \frac{(n - |m|)!}{(n + |m|)!}} P_n^{|m|}(\cos\theta) e^{im\phi},$$

where θ, ϕ denote the angular variables of an (r, θ, ϕ) spherical coordinate system, and P_n^m are the associated Legendre functions.

We define the Sobolev space $H^s(\Gamma_a)$ $(s > 0)$ by

$$H^s(\Gamma_a) = \{\varphi \in L^2(\Gamma_a) \,|\, \|\varphi\|_{s,\Gamma_a} < \infty\},$$

where $\|\cdot\|_{s,\Gamma_a}$ is the norm of $H^s(\Gamma_a)$ defined by

$$\|\varphi\|^2_{s,\Gamma_a} = \begin{cases} a \displaystyle\sum_{n=-\infty}^{\infty} (1 + n^{2s}) |\varphi_n|^2 & \text{if } d = 2, \\ a^2 \displaystyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} (1 + n^{2s}) |\varphi_n^m|^2 & \text{if } d = 3, \end{cases}$$

where φ_n and φ_n^m, respectively, are the Fourier coefficients of φ defined by

$$\varphi_n = \int_0^{2\pi} \varphi(\theta) \overline{Y_n(\theta)} \, d\theta \tag{3}$$

and

$$\varphi_n^m = \int_0^{2\pi} d\phi \int_0^{\pi} \varphi(\theta, \phi) \overline{Y_n^m(\theta, \phi)} \sin\theta \, d\theta. \tag{4}$$

We here note that the DtN operator $\mathcal{S}$ is a bounded linear operator from $H^{1/2}(\Gamma_a)$ into $H^{-1/2}(\Gamma_a)$ (see [24]), where $H^{-1/2}(\Gamma_a)$ is the set of all bounded semilinear forms on $H^{1/2}(\Gamma_a)$.

To formulate a weak form of problem (2), we introduce the following sesquilinear forms:

$$a(u, v) = \int_{\Omega_a} (\nabla u \cdot \nabla \overline{v} - k^2 u \overline{v}) \, dx + s(u, v) \quad \text{for } u, v \in H^1(\Omega_a),$$

$$s(u, v) = \langle \mathcal{S} u, v \rangle_{H^{-1/2}(\Gamma_a) \times H^{1/2}(\Gamma_a)}$$

$$= \begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} -ka \frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)} u_n(a) \overline{v_n(a)} & \text{if } d = 2, \\ \displaystyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} -ka^2 \frac{h_n^{(1)'}(ka)}{h_n^{(1)}(ka)} u_n^m(a) \overline{v_n^m(a)} & \text{if } d = 3, \end{cases} \tag{5}$$

where $H_n^{(1)}$ and $h_n^{(1)}$ are the cylindrical and spherical Hankel functions of the first kind of order n, respectively, and $u_n(a)$ and $u_n^m(a)$ denote the Fourier coefficients of $u|_{\Gamma_a}$ defined by (3) and (4), respectively. Then a weak form of (2) is written as follows: find $u \in V$ such that

$$a(u, v) = (f, v) \tag{6}$$

for all $v \in V$, where

$$V = \{v \in H^1(\Omega_a) | v = 0 \text{ on } \gamma\},$$

$$(u, v) = \int_{\Omega_a} u\overline{v}\,\mathrm{d}x \quad \text{for } u, v \in L^2(\Omega_a).$$

For every $f \in L^2(\Omega_a)$, problem (6) has a unique solution which is the restriction to Ω_a of the solution of problem (1) (see [24,15,16]).

3. Some properties of the Hankel functions

In this section, we state three lemmas concerning properties of the Hankel functions. These lemmas will be used to establish error estimates in the next section.

Lemma 2. *For each $x > 0$, there exists a positive constant C such that*

$$\left| \frac{1}{1+|n|} \frac{H_n^{(1)'}(x)}{H_n^{(1)}(x)} \right| \leqslant C \quad \text{for all } n \in \mathbb{Z}, \tag{7}$$

$$\left| \frac{1}{1+n} \frac{h_n^{(1)'}(x)}{h_n^{(1)}(x)} \right| \leqslant C \quad \text{for all } n \in \mathbb{N} \cup \{0\}, \tag{8}$$

where C depends on x, but is independent of n.

Proof. For proofs of (7) and (8), see [24] and [16], respectively. □

Lemma 3. *For all $x > 0$, we have*

$$\mathrm{Re}\left\{ \frac{H_\nu^{(1)'}(x)}{H_\nu^{(1)}(x)} \right\} < 0 \quad \text{for all } \nu \in \mathbb{R},$$

$$\mathrm{Re}\left\{ \frac{h_n^{(1)'}(x)}{h_n^{(1)}(x)} \right\} < 0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Proof. See [19]. □

Lemma 4. *For any $r_1 > r_2 > 0$, $|H_\nu^{(1)}(r_1)/H_\nu^{(1)}(r_2)|$ is a decreasing function of ν on $[0, \infty)$, and further $|h_n^{(1)}(r_1)/h_n^{(1)}(r_2)|$ is a decreasing sequence of $n \in \mathbb{N} \cup \{0\}$.*

Proof. We first prove the former assertion. We write

$$F(\nu; r) = |H_\nu^{(1)}(r)|^2.$$

From Nicholson's formula (see [29, p. 444])

$$F(\nu; r) = \frac{8}{\pi^2} \int_0^\infty K_0(2r \sinh t) \cosh(2\nu t)\,\mathrm{d}t \quad (\nu \in \mathbb{R}, r > 0), \tag{9}$$

where K_0 is the modified Bessel function of the second kind of order zero, we see that, for each $r > 0$, $F(\cdot\,; r) \in C^\infty(\mathbb{R})$ and $F(\cdot\,; r) > 0$. Thus, it is sufficient to show that

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{F(v; r_1)}{F(v; r_2)}\right) < 0$$

for all $v \in (0, \infty)$. This inequality holds, if and only if

$$\frac{\mathrm{d}F}{\mathrm{d}v}(v; r_1)F(v; r_2) - F(v; r_1)\frac{\mathrm{d}F}{\mathrm{d}v}(v; r_2) < 0. \tag{10}$$

Hence, let us prove (10) in the following. Differentiating (9) with v leads to

$$\frac{\mathrm{d}F}{\mathrm{d}v}(v; r) = \frac{16}{\pi^2}\int_0^\infty K_0(2r\sinh t)t\sinh(2vt)\,\mathrm{d}t. \tag{11}$$

From (9) and (11), we have

$$\begin{aligned}
&\frac{\mathrm{d}F}{\mathrm{d}v}(v; r_1)F(v; r_2) - F(v; r_1)\frac{\mathrm{d}F}{\mathrm{d}v}(v; r_2)\\
&\quad = \frac{128}{\pi^4}\int_0^\infty\int_0^\infty [K_0(2r_1\sinh t_1)K_0(2r_2\sinh t_2)t_1\sinh(2vt_1)\cosh(2vt_2)\\
&\qquad - K_0(2r_2\sinh t_1)K_0(2r_1\sinh t_2)t_1\sinh(2vt_1)\cosh(2vt_2)]\,\mathrm{d}t_1\,\mathrm{d}t_2\\
&\quad = \frac{128}{\pi^4}\int_0^\infty \mathrm{d}t_1\int_0^{t_1}[K_0(2r_1\sinh t_1)K_0(2r_2\sinh t_2) - K_0(2r_1\sinh t_2)K_0(2r_2\sinh t_1)]\\
&\qquad \times [t_1\sinh(2vt_1)\cosh(2vt_2) - t_2\sinh(2vt_2)\cosh(2vt_1)]\,\mathrm{d}t_2.
\end{aligned} \tag{12}$$

Here, using Macdnold's formula (see [29, p. 439])

$$K_0(X)K_0(x) = \frac{1}{2}\int_0^\infty \exp\left[-\frac{t}{2} - \frac{X^2 + x^2}{2t}\right]K_0\left(\frac{Xx}{t}\right)\frac{\mathrm{d}t}{t} \quad (X, x > 0)$$

and the fact that

$$(r_1^2\sinh^2 t_1 + r_2^2\sinh^2 t_2) - (r_1^2\sinh^2 t_2 + r_2^2\sinh^2 t_1) = (r_1^2 - r_2^2)(\sinh^2 t_1 - \sinh^2 t_2) > 0$$

for $t_1 > t_2 > 0$, we can conclude that

$$K_0(2r_1\sinh t_1)K_0(2r_2\sinh t_2) - K_0(2r_1\sinh t_2)K_0(2r_2\sinh t_1) < 0 \tag{13}$$

for $t_1 > t_2 > 0$. Further, we have

$$\begin{aligned}
&t_1\sinh(2vt_1)\cosh(2vt_2) - t_2\sinh(2vt_2)\cosh(2vt_1)\\
&\quad = \tfrac{1}{2}\{(t_1 - t_2)\sinh[2v(t_1 + t_2)] + (t_1 + t_2)\sinh[2v(t_1 - t_2)]\} > 0
\end{aligned} \tag{14}$$

for $t_1 > t_2 > 0$. From (12)–(14), we deduce (10).

We next prove the latter assertion. Since

$$h_n^{(1)}(r) = \sqrt{\frac{\pi}{2r}}H_{n+1/2}^{(1)}(r),$$

we have

$$\left|\frac{h_n^{(1)}(r_1)}{h_n^{(1)}(r_2)}\right| = \sqrt{\frac{r_2}{r_1}}\left|\frac{H_{n+1/2}^{(1)}(r_1)}{H_{n+1/2}^{(1)}(r_2)}\right|.$$

Hence, it follows from the above result that $|h_n^{(1)}(r_1)/h_n^{(1)}(r_2)|$ is a decreasing sequence of $n \in \mathbb{N} \cup \{0\}$. $\square$

4. The DtN finite element method and its error estimates

We discretize problem (6) by using the finite element method to obtain approximate solutions to problem (6) (or (1)). We introduce a family $\{V_h | h \in (0, \bar{h}]\}$ of finite dimensional subspaces of V, and assume that this family satisfies the following condition: there exist an integer $p \geqslant 2$ and a constant $C > 0$ such that for all $0 < h \leqslant \bar{h}$ and for every $u \in V \cap H^{p'}(\Omega_a)$ $(2 \leqslant p' \leqslant p)$,

$$\inf_{v^h \in V_h} \|u - v_h\|_{1,\Omega_a} \leqslant C h^{p'-1} \|u\|_{p',\Omega_a}, \tag{15}$$

where C is independent of h and u, and $\|\cdot\|_{p,\Omega_a}$ is the standard norm of $H^p(\Omega_a)$ defined by

$$\|v\|^2_{p,\Omega} = \sum_{|\alpha| \leqslant p} \int_\Omega |D^\alpha v|^2 \, dx \quad (p \in \mathbb{N}).$$

For examples of such a family, see [2,36].

Now since the sesquilinear form s involves the infinite series, we have to truncate it in practice. So we practically solve the following problem: find $u_h^N \in V_h$ such that

$$a^N(u_h^N, v_h) = (f, v_h) \quad \text{for all } v_h \in V_h, \tag{16}$$

where, for $N \in \mathbb{N}$,

$$a^N(u, v) = \int_{\Omega_a} (\nabla u \cdot \nabla \bar{v} - k^2 u \bar{v}) \, dx + s^N(u, v) \quad \text{for } u, v \in H^1(\Omega_a),$$

$$s^N(u, v) = \begin{cases} \displaystyle\sum_{|n|<N} -ka \frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)} u_n(a) \overline{v_n(a)} & \text{if } d = 2, \\ \displaystyle\sum_{n=0}^{N-1} \sum_{m=-n}^{n} -ka^2 \frac{h_n^{(1)'}(ka)}{h_n^{(1)}(ka)} u_n^m(a) \overline{v_n^m(a)} & \text{if } d = 3. \end{cases}$$

Theorem 5. *Let k be an arbitrary positive number and f an arbitrary function of $L^2(\Omega)$ with compact support. Assume that $\overline{\mathcal{O}} \cup \operatorname{supp} f \subset B_{a_0}$ $(a_0 < a)$. Let u be the solution of problem* (1). *Assume that there exists an integer $l \geqslant 2$ such that $u \in H^l(\Omega_a)$. Then there exist a $\gamma_0 > 0$ such that for every $(h, N) \in (0, \bar{h}] \times \mathbb{N}$ satisfying $h + N^{-1} \leqslant \gamma_0$, problem* (16) *has a unique solution u_h^N, and moreover, if $d = 2$, then we have*

$$\|u - u_h^N\|_{1,\Omega_a} \leqslant C \left(h^{m-1} \|u\|_{m,\Omega_a} + N^{-s+1/2} \left| \frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)} \right| R^N(u; s, a_0) \right), \tag{17}$$

$$\|u - u_h^N\|_{0,\Omega_a} \leqslant C(h + N^{-1}) \left(h^{m-1} \|u\|_{m,\Omega_a} + N^{-s+1/2} \left| \frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)} \right| R^N(u; s, a_0) \right), \tag{18}$$

where $\|\cdot\|_{0,\Omega_a}$ is the usual norm of $L^2(\Omega_a)$, $m = \min\{p, l\}$, s is an arbitrary real number $\geqslant \frac{1}{2}$,

$$R^N(u; s, a_0) = \left(a_0 \sum_{|n| \geqslant N} n^{2s} |u_n(a_0)|^2 \right)^2, \tag{19}$$

and positive constants γ_0 and C depend on k, a_0, and Ω_a, but are independent of h, N, s, f, u, and u_h^N. If $d = 3$, then (17) *and* (18) *hold by replacing $H_N^{(1)}$ by $h_N^{(1)}$, and* (19) *by*

$$R^N(u; s, a_0) = \left(a_0^2 \sum_{n \geqslant N} \sum_{m=-n}^{n} n^{2s} |u_n^m(a_0)|^2 \right)^2.$$

Before starting to prove Theorem 5, we introduce the following inequalities associated with the trace theorem:

$$\|v\|_{m-1/2,\Gamma_a} \leqslant C\|v\|_{m,\Omega_a} \quad \text{for all } v \in H^m(\Omega_a) \ (m=1,2), \tag{20}$$

where C is a positive constant depending on Ω_a, but independent of v, and the following sesquilinear form on $H^1(\Omega_a)$:

$$r^N(u,v) = \begin{cases} \displaystyle\sum_{|n|\geqslant N} -ka\frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)} u_n(a)\overline{v_n(a)} & \text{if } d=2, \\ \displaystyle\sum_{n\geqslant N}\sum_{m=-n}^{n} -ka^2\frac{h_n^{(1)'}(ka)}{h_n^{(1)}(ka)} u_n^m(a)\overline{v_n^m(a)} & \text{if } d=3. \end{cases}$$

Note here that we have

$$s(u,v) = s^N(u,v) + r^N(u,v) \quad \text{for } u, v \in H^1(\Omega_a).$$

Proof. We prove only in the case when $d=2$, because the proof of the case when $d=3$ is exactly the same.

We first assume that problem (16) has a solution u_h^N. We postpone proving the well-posedness of problem (16) until completion of the derivation of (17) and (18).

Set $e_h^N = u - u_h^N$. Then we have

$$a^N(e_h^N, v_h) + r^N(u, v_h) = 0 \tag{21}$$

for all $v_h \in V_h$. Note the following identical equation:

$$\|e_h^N\|_{1,\Omega_a}^2 = a^N(e_h^N, e_h^N) + (k^2+1)\|e_h^N\|_{0,\Omega_a}^2 - s^N(e_h^N, e_h^N).$$

Taking the real part of this identity, we can get

$$\|e_h^N\|_{1,\Omega_a}^2 = \mathrm{Re}\{a^N(e_h^N, e_h^N)\} + (k^2+1)\|e_h^N\|_{0,\Omega_a}^2 - \mathrm{Re}\{s^N(e_h^N, e_h^N)\}.$$

By virtue of Lemma 3, we have

$$\|e_h^N\|_{1,\Omega_a}^2 \leqslant \mathrm{Re}\{a^N(e_h^N, e_h^N)\} + (k^2+1)\|e_h^N\|_{0,\Omega_a}^2. \tag{22}$$

Step 1. We show that for an arbitrary $\varepsilon > 0$, there exists a positive constant $C_3(\varepsilon)$ such that

$$|a^N(e_h^N, e_h^N)| \leqslant \varepsilon\|e_h^N\|_{1,\Omega_a}^2 + C_3(\varepsilon)\left\{h^{2m-2}\|u\|_{m,\Omega_a}^2 + N^{-2s+1}\left|\frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)}\right|^2 [R^N(u; s, a_0)]^2\right\}, \tag{23}$$

where s is an arbitrary number $\geqslant \frac{1}{2}$ and $C_3(\varepsilon)$ depends on k, a_0, and Ω_a, but is independent of h, N, s, u, and u_h^N. By (21), we have, for all $v_h \in V_h$,

$$a^N(e_h^N, e_h^N) = a^N(e_h^N, u - v_h) + r^N(u, u_h^N - v_h) = a^N(e_h^N, u - v_h) + r^N(u, u - v_h) - r^N(u, e_h^N).$$

Thus, by using the trigonometric inequality, the Schwarz inequality, Lemma 2, and the trace inequality (20), we get

$$\begin{aligned} |a^N(e_h^N, e_h^N)| &\leqslant |e_h^N|_{1,\Omega_a}|u - v_h|_{1,\Omega_a} + k^2\|e_h^N\|_{0,\Omega_a}\|u - v_h\|_{0,\Omega_a} \\ &\quad + C(k,a)\|e_h^N\|_{1/2,\Gamma_a}\|u - v_h\|_{1/2,\Gamma_a} + |r^N(u, u - v_h)| + |r^N(u, e_h^N)| \\ &\leqslant C(k,\Omega_a)\|e_h^N\|_{1,\Omega_a}\|u - v_h\|_{1,\Omega_a} + |r^N(u, u - v_h)| + |r^N(u, e_h^N)|. \end{aligned} \tag{24}$$

Let us estimate the second term on the right-hand side of (24). Since $\overline{\mathcal{O}} \cup \mathrm{supp}\, f \subset B_{a_0}$, the solution u can be analytically represented as follows

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka_0)} u_n(a_0) Y_n(\theta) \quad \text{on } \mathbb{R}^2 \backslash \overline{B_{a_0}}.$$

This implies

$$u_n(a) = \frac{H_n^{(1)}(ka)}{H_n^{(1)}(ka_0)} u_n(a_0)$$

for all $n \in \mathbb{Z}$. Moreover, we can see from the usual regularity argument that $u|_{\Gamma_{a_0}} \in H^s(\Gamma_{a_0})$ for all $s > 0$. Thus, by the trigonometric inequality, Lemmas 2 and 4, the Schwarz inequality, and the trace inequality (20), we have, for every $s \geqslant \frac{1}{2}$,

$$\begin{aligned}
|r^N(u, u - v_h)| &\leqslant \sum_{|n| \geqslant N} \left| ka \frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)} \right| |u_n(a)||(u - v_h)_n(a)| \\
&= \sum_{|n| \geqslant N} |n|^{-s+1/2} \left| \frac{ka}{n} \frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)} \right| \left| \frac{H_n^{(1)}(ka)}{H_n^{(1)}(ka_0)} \right| |n|^s |u_n(a_0)||n|^{1/2}|(u - v_h)_n(a)| \\
&\leqslant C(k, a, a_0) N^{-s+1/2} \left| \frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)} \right| R^N(u; s, a_0) \|u - v_h\|_{1/2, \Gamma_a} \\
&\leqslant C(k, a_0, \Omega_a) N^{-s+1/2} \left| \frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)} \right| R^N(u; s, a_0) \|u - v_h\|_{1, \Omega_a},
\end{aligned} \tag{25}$$

where $(u - v_h)_n(a)$ are the Fourier coefficients of $u - v_h$. In exactly the same way, we can estimate the third term on the right-hand side of (24) as follows:

$$|r^N(u, e_h^N)| \leqslant C(k, a_0, \Omega_a) N^{-s+1/2} \left| \frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)} \right| R^N(u; s, a_0) \|e_h^N\|_{1, \Omega_a}. \tag{26}$$

Combining (24)–(26) and (15) leads to

$$\begin{aligned}
|a^N(e_h^N, e_h^N)| \leqslant C(k, a_0, \Omega_a) \Bigg[& h^{m-1} \|e_h^N\|_{1, \Omega_a} \|u\|_{m, \Omega_a} + h^{m-1} N^{-s+1/2} \left| \frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)} \right| R^N(u; s, a_0) \|u\|_{m, \Omega_a} \\
& + N^{-s+1/2} \left| \frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)} \right| \|e_h^N\|_{1, \Omega_a} R^N(u; s, a_0) \Bigg].
\end{aligned}$$

Applying the arithmetic–geometric mean inequality to each term on the right-hand side of the above inequality, we obtain (23).

Step 2. We show that there exists a positive constant C_4 such that

$$\|e_h^N\|_{0, \Omega_a} \leqslant C_4 \left[(h + N^{-1}) \|e_h^N\|_{1, \Omega_a} + (h N^{-s+1/2} + N^{-s-1/2}) \left| \frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)} \right| R^N(u; s, a_0) \right], \tag{27}$$

where s is an arbitrary number $\geqslant \frac{1}{2}$ and C_4 depends on k, a_0, and Ω_a, but is independent of h, N, s, u, and u_h^N. Suppose that $w \in V$ satisfies

$$a(v, w) = (v, e_h^N) \tag{28}$$

for all $v \in V$. Then w is the *incoming* solution, that is, w is the restriction to Ω_a of the solution of problem (1) where the outgoing radiation condition is replaced by the incoming radiation condition:

$$\lim_{r \longrightarrow +\infty} r^{(d-1)/2} \left(\frac{\partial u}{\partial r} + \mathrm{i} k u \right) = 0$$

and $f = e_h^N$. Note here that the sesquilinear form s corresponding to the incoming radiation condition is represented by replacing $H_n^{(1)}$ by $H_n^{(2)}$ (the Hankel function of the second kind) in (5). Since Theorem 1 also holds for the incoming problem, we have $w \in H^2(\Omega_a)$ and the following a priori estimate:

$$\|w\|_{2,\Omega_a} \leqslant C\|e_h^N\|_{0,\Omega_a}, \tag{29}$$

where C is a positive constant independent of e_h^N. Taking $v = e_h^N$ in (28), we obtain

$$\|e_h^N\|_{0,\Omega_a}^2 = a(e_h^N, w) = a^N(e_h^N, w) + r^N(e_h^N, w). \tag{30}$$

Subtracting (21) from (30) gives

$$\begin{aligned}\|e_h^N\|_{0,\Omega_a}^2 &= a^N(e_h^N, w - v_h) + r^N(e_h^N, w) - r^N(u, v_h)\\ &= a^N(e_h^N, w - v_h) + r^N(e_h^N, w) + r^N(u, w - v_h) - r^N(u, w).\end{aligned}$$

Employing the argument leading to (24), we can get

$$\|e_h^N\|_{0,\Omega_a}^2 \leqslant C(k, \Omega_a)\|w - v_h\|_{1,\Omega_a}\|e_h^N\|_{1,\Omega_a} + |r^N(e_h^N, w)| + |r^N(u, w - v_h)| + |r^N(u, w)|. \tag{31}$$

Employing an argument similar to the one used in (25), we can estimate the last three terms on the right-hand side of (31) as follows:

$$|r^N(e_h^N, w)| \leqslant C(k, a)N^{-1}\|e_h^N\|_{1/2,\Gamma_a}\|w\|_{3/2,\Gamma_a} \leqslant C(k, \Omega_a)N^{-1}\|e_h^N\|_{1,\Omega_a}\|w\|_{2,\Omega_a}, \tag{32}$$

$$|r^N(u, w - v_h)| \leqslant C(k, a_0, \Omega_a)N^{-s+1/2}\left|\frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)}\right| R^N(u; s, a_0)\|w - v_h\|_{1,\Omega_a}, \tag{33}$$

$$\begin{aligned}|r^N(u, w)| &\leqslant C(k, a, a_0)N^{-s-1/2}\left|\frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)}\right| R^N(u; s, a_0)\|w\|_{3/2,\Gamma_a}\\ &\leqslant C(k, a_0, \Omega_a)N^{-s-1/2}\left|\frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)}\right| R^N(u; s, a_0)\|w\|_{2,\Omega_a}.\end{aligned} \tag{34}$$

Collecting (31)–(34) yields

$$\begin{aligned}\|e_h^N\|_{0,\Omega_a}^2 \leqslant C(k, a_0, \Omega_a)\Bigg\{&\left[\|e_h^N\|_{1,\Omega_a} + N^{-s+1/2}\left|\frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)}\right| R^N(u; s, a_0)\right]\|w - v_h\|_{1,\Omega_a}\\ &+ \left[N^{-1}\|e_h^N\|_{1,\Omega_a} + N^{-s-1/2}\left|\frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)}\right| R^N(u; s, a_0)\right]\|w\|_{2,\Omega_a}\Bigg\}.\end{aligned}$$

Using (15) and (29), we get

$$\begin{aligned}\|e_h^N\|_{0,\Omega_a}^2 \leqslant C(k, a_0, \Omega_a)\Bigg\{&\left[\|e_h^N\|_{1,\Omega_a} + N^{-s+1/2}\left|\frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)}\right| R^N(u; s, a_0)\right]h\|e_h^N\|_{0,\Omega_a}\\ &+ \left[N^{-1}\|e_h^N\|_{1,\Omega_a} + N^{-s-1/2}\left|\frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)}\right| R^N(u; s, a_0)\right]\|e_h^N\|_{0,\Omega_a}\Bigg\},\end{aligned}$$

and further dividing by $\|e_h^N\|_{0,\Omega_a}$, we obtain (27).

Step 3. Let us collect the results above to get (17) and (18).

Squaring both sides of (27) and using arithmetic-geometric mean inequality, we have

$$\|e_h^N\|_{0,\Omega_a}^2 \leqslant 2C_4^2\left\{(h + N^{-1})^2\|e_h^N\|_{1,\Omega_a}^2 + (hN^{-s+1/2} + N^{-s-1/2})^2\left|\frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)}\right|^2[R^N(u; s, a_0)]^2\right\}. \tag{35}$$

Combining (22), (23), and (35), we get

$$\|e_h^N\|_{1,\Omega_a}^2 \leqslant \varepsilon\|e_h^N\|_{1,\Omega_a}^2 + C_3(\varepsilon)\left\{h^{2m-2}\|u\|_{m,\Omega_a}^2 + N^{-2s+1}\left|\frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)}\right|^2 [R^N(u;s,a_0)]^2\right\}$$
$$+ C_5\left\{(h+N^{-1})^2\|e_h^N\|_{1,\Omega_a}^2 + (hN^{-s+1/2}+N^{-s-1/2})^2\left|\frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)}\right|^2 [R^N(u;s,a_0)]^2\right\},$$

where $C_5 = 2(k^2+1)C_4^2$. This implies

$$\{1-\varepsilon - C_5(h+N^{-1})^2\}\|e_h^N\|_{1,\Omega_a}^2 \leqslant C_6(\varepsilon)\left(h^{2m-2}\|u\|_{m,\Omega_a}^2 + N^{-2s+1}\left|\frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)}\right|^2 [R^N(u;s,a_0)]^2\right),$$

where $C_6(\varepsilon) = C_3(\varepsilon) + (\bar{h}+1)^2$, and further, by taking $\varepsilon = \frac{1}{2}$,

$$\left\{\frac{1}{2} - C_5(h+N^{-1})^2\right\}\|e_h^N\|_{1,\Omega_a}^2 \leqslant C_7\left(h^{2m-2}\|u\|_{m,\Omega_a}^2 + N^{-2s+1}\left|\frac{H_N^{(1)}(ka)}{H_N^{(1)}(ka_0)}\right|^2 [R^N(u;s,a_0)]^2\right),$$

where $C_7 = C_6(\frac{1}{2})$. For every $\{h, N\} \in (0, \bar{h}] \times \mathbb{N}$ satisfying

$$\tfrac{1}{2} - C_5(h+N^{-1})^2 \geqslant \tfrac{1}{4},$$

which is equivalent to

$$h + N^{-1} \leqslant \frac{1}{\sqrt{4C_5}} \equiv \gamma_0,$$

we have (17). Further, from (27) and (17), we can derive (18).

Step 4. We finally show the well-posedness of problem (16). For this purpose, it is sufficient to prove uniqueness of the solution of problem (16) since V_h is finite dimensional. Thus, assume now that $u_h^N \in V_h$ is a solution of problem (16) with $f=0$. Since the solution u of problem (6) with $f=0$ is identically zero, it follows from (17) (or (18)) that $u_h^N = 0$. Therefore we can conclude that problem (16) is well-posed when $h + N^{-1} \leqslant \gamma_0$. □

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