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Nontrivial Periodic Solutions for Asymptotically Linear Hamiltonian Systems with Resonance

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In this paper, we establish some existence results of nontrivial 1-periodic solutions to the first-order asymptotically linear Hamiltonian systems, under the assumptions that the linear operators have different Maslove indices at the origin

1. INTRODUCTION

We consider the periodic solutions of the following problem

$$\dot{z} = JH'(t, z), \tag{1.1}$$

where $H \in C^2([0, 1] \times R^{2N}, R)$ is 1-periodic in t, and $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$.

System (1.1) is called asymptotically linear both at the origin and at infinity, if there are two $2N \times 2N$ symmetric continuous 1-periodic matrix functions $B_0(t)$ and $B_\infty(t)$ such that

$$H'(z, t) = B_0(t)\, z + o(|z|) \qquad \text{as} \quad |z| \to 0 \text{ uniformly in } t \tag{1.2}$$

$$H'(z, t) = B_\infty(t)\, z + o(|z|) \qquad \text{as} \quad |z| \to \infty \text{ uniformly in } t, \tag{1.3}$$

where $|\cdot|$ denotes the norm of R^{2N}.

For system (1.1), the following question is raised: Having found one solution, say θ, which is called the trivial solution, can we conclude the existence of a nontrivial solution by assuming conditions on the two linear systems at θ and at ∞, i.e., on the two matrices $B_0(t)$ and $B_\infty(t)$?

The problem given above was first studied by Amann and Zehnder [1, 2]. They assumed that both $B_0(t)$ and $B_\infty(t)$ are constant matrices and $B_\infty(t)$ is nondegenerate. Since then, in the case that $B_\infty(t)$ is nondegenerate,

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i.e., 1 is not a Floquet multiplier of the linear system $\dot{z} = JB_\infty(t)\,z$, the above problem has been considered by many authors. For example, see Chang [3], Conley and Zehnder [4], Li and Liu [5], Ding and Liu [6], Long and Zehnder [7], Long [8], and the references therein. However, only a few papers have treated the case in which $B_\infty(t)$ is degenerate. For example, in [9, 10], Chang and Szulkin consider the case in which $B_\infty(t)$ is constant and in [11–13], Chang, Fei, and Pisani consider the strong resonant case (the nonlinearities are bounded globally) respectively; in [14] Fei considers the case that $B_\infty(t)$ and $B_0(t)$ are "finitely degenerate" under some stronger assumptions on nonlinearities.

Our approach here is different from those previously applied, as we obtain the nontrivial periodic solutions for system (1.1) as critical points of a strongly indefinite functional. For this kind of functional, we apply a new Morse theory developed in [15]. Moreover, we assume that both $B_0(t)$ and $B_\infty(t)$ are degenerate and the nonlinearities are unnecessary bounded.

Set

$$F(t, z) = H(t, z) - \tfrac{1}{2}(B_0(t)\,z, z) \qquad \forall (t, z) \in R \times R^{2N}$$

$$F(t, z) = H(t, z) - \tfrac{1}{2}(B_\infty(t)\,z, z) \qquad \forall (t, z) \in R \times R^{2N},$$

where $(\cdot, \cdot)$ is the inner product of R^{2N}.

We introduce the following hypotheses for system (1.1).

($H_1^\pm$) There exist constant $M_0 > 0$, $C_0 > 0$, $\sigma \in (1, \infty)$ such that

$$|F_0'(t, z)| \leqslant M_0\,|z|^\sigma, \qquad z \in R^{2N}, \qquad |z| \leqslant C_0$$

$$\lim_{|z| \to 0} \frac{F_0(t, z)}{|z|^{2\sigma}} = \pm\infty \qquad \text{uniformly in } t$$

($H_2^\pm$) There exist constants $M_\infty > 0$, $\alpha \in (0, 1)$ such that

$$|F_\infty'(t, z)| \leqslant M_\infty(1 + |z|^\alpha), \qquad \text{for} \quad z \in R^{2N}$$

$$\lim_{|z| \to \infty} \frac{F_\infty(t, z)}{|z|^{2\alpha}} = \pm\infty \qquad \text{uniformly in } t.$$

Remark 1.1. Evidently, ($H_1^\pm$) and ($H_2^\pm$) imply (1.2) and (1.3), respectively. And under our assumptions $|F_\infty'(t, z)|$ need not be bounded.

Remark 1.2. Our theorems contain the case that $B_0(t) = B_\infty(t)$, which implies that system (1.1) is resonant both at the origin and at infinity with the same matrix. Noting that the assumptions of the theorems in [9, 11–14] imply that $B_0(t) \neq B_\infty(t)$; in this sense, our results are new. In fact, to the best of my knowledge, very little is known for the case that $i_0 = i_\infty$ and $n_0 = n_\infty$.

Remark 1.3. We do not suppose that $B_0(t)$ and $B_\infty(t)$ are finitely degenerate, which is essential to those results in [13, 14].

Our paper is organized as follows. In Section 2, we will recall some results for strongly indefinite functionals developed in [15]. In Section 3, we obtain the precise computations of the critical groups and the nontrivial periodic solutions of system (1.1). It should be mentioned that the method we use to compute the critical group is similar to [16] for the finite dimensional case.

2. MORSE THEORY FOR STRONGLY INDEFINITE FUNCTIONAL

Let H be a real Hilbert space with inner product $\langle\ ,\ \rangle$ and norm $\|\cdot\|$, and $H = \bigoplus_{i=1}^{\infty} H_i$ with all subspaces H_i being mutually orthogonal and of finite dimension. Setting $H^n = \bigoplus_{i=1}^{n} H_i$, we assume

(1) $f \in C^2(H, R)$ with the form $f(x) = \frac{1}{2}\langle Ax, x\rangle + G(x)$,

(2) A is a bounded linear self-adjoint operator with a finite dimensional kernel, and its zero eigenvalue is isolated in the spectrum of A, and

(3) $G'(x) := K(x)$ is compact and global Lipschitz continuous on a bounded set.

The following key concepts are due to [11, 19].

DEFINITION 2.1. Let $\Gamma = \{P_n \mid n = 1, 2, ...\}$ be a sequence of orthogonal projections. We call Γ an approximation scheme with respect to A if the following properties hold:

(1) $H_n = P_n H$ is finite dimensional for $\forall n$,

(2) $P_n \to I$ as $n \to \infty$ (strongly), and

(3) $[P_n, A] = P_n A - A P_n \to 0$ (in the operate norm).

DEFINITION 2.2 (Gromoll–Meyer pair). Let f be a C^1 functional on a C^1-Finsler manifold M and let S be a subset of the critical set K for f. A pair of subsets (W, W_-) is said to be a Gromoll–Meyer (G-M) pair for S associated to a pseudo-gradient field X for f by considering the flow η generated by X, and the following conditions hold:

(1) W is a closed neighborhood of S, satisfying $W \cap K = S$ and $W \cap f_\alpha = \varnothing$ for some α,

(2) W_- is an exit set of W, i.e., $\forall x_0 \in W$, $\forall t_1 > 0$ such that $\eta(x_0, t_1) \notin W$, there exists $t_0 \in [0, t_1)$ such that $\eta(x_0, [0, t_0]) \subset W$ and $\eta(x_0, t_0) \in W_-$, and

(3) W is closed and is a union of a finite number of submanifolds that are transversal to the flow η.

DEFINITION 2.3 (Dynamically isolated critical sets). A subset S of the critical set K for f is said to be a dynamically isolated critical set. If there exists a closed neighborhood O of S and regular value $\alpha < \beta$ of f such that

$$O \subset f^{-1}[\alpha, \beta] \qquad \text{and} \qquad cl(\tilde{O}) \cap K \cap f^{-1}[\alpha, \beta] = S$$

we shall then say that (O, α, β) is an isolating triplet for S, where $\tilde{O} = \bigcup_{t \in R} \eta(O, t)$, η is the flow associated with f.

For an isolated set S of f, we define the critical groups $C_*(f, S)$ by (see Guo [15])

$$C_* := H^{* + m(P_n(A+P) P_n)}(W_n, W_{n-}),$$

where (W_n, W_{n-}) is a G-M pair for S_n, which is the critical set of the restriction functional $f_n = f|_{H^n}$, $\{P_n \mid n = 1, 2...\}$ is an approximation scheme with respect to A, and P is the orthogonal projection onto the kernel space of A. It has been shown in [15] that the critical groups are independent of the choice of the isolating triplet, the G-M pair, and the approximation scheme. If f has finite critical value and S is the set consisting of all the critical points of f, we define the critical groups of infinity by $C_*(f, \infty) := C_*(f, S)$.

3. ASYMPTOTICAL LINEAR HAMILTONIAN SYSTEMS

Let $S^1 = [0, 1]/\{0, 1\}$; then the natural space for studying Hamiltonian system (1.1) is the Sobolev space $E = W^{1/2}(S^1, R^{2N})$. Recall that E is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and E consists of those $z(t) \in L^2(S^1, R^{2n})$, whose Fourier series

$$z(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

satisfies

$$\|z\|^2 = |a_0|^2 + \sum_{n=1}^{\infty} n(|a_n|^2 + |b_n|^2) < +\infty,$$

where $a_n, b_n \in R^{2N}$.

An important notion on this line is the Maslov-type index theory. According to [6, 8], for any given continuous 1-periodic and symmetric matrix function $B(t)$, we can assign a pair of integers $(i, n) \in Z \times \{0, 1, \ldots 2N\}$ to it and call the pair (i, n) the Maslov type index of $B(t)$. We denote the Maslov type indices of $B_0(t)$, $B_\infty(t)$ by (i_0, n_0) and (i_∞, n_∞), respectively. Then $n_0 \neq 0$ means that system (1.1) is resonant at origin or that $z = \theta$ is a degenerate trivial solution; $n_\infty \neq 0$ implies that system (1.1) is resonant at infinity.

By using the Morse theory for strongly indefinite functionals and the precise computations of critical groups, we obtain the following main results.

THEOREM 3.1. *Assume* $H \in C^2([0,1] \times R^{2N}, R)$ *and there exist continuous* 1*-periodic and symmetric matrix functions* $B_0(t)$ *and* $B_\infty(t)$ *such that conditions* $(\mathrm{H}_1^\pm)$ *and* $(\mathrm{H}_2^\pm)$ *are satisfied;* $n_0 \neq 0$, $n_\infty \neq 0$. *Then system* (1.1) *possess at least one nontrivial* 1*-periodic solution if one of the following four cases occurs*:

(1) (H_1^-) *and* (H_2^-) *hold,* $i_0 \neq i_\infty$

(2) (H_1^+) *and* (H_2^-) *hold,* $i_0 + n_0 \neq i_\infty$

(3) (H_1^-) *and* (H_2^+) *hold,* $i_0 \neq i_\infty + n_\infty$

(4) (H_1^+) *and* (H_2^+) *hold,* $i_0 + n_0 \neq i_\infty + n_\infty$.

THEOREM 3.2. *In addition to the assumptions of Theorem* 3.1, *suppose that* $x_0 \neq \theta$ *is a nondegenerate periodic solution of system* (1.1). *Then it has at least another solution* $x_1 \neq \theta, x_0$, *if one of the following four cases occurs*:

(1) (H_1^-) *and* (H_2^-) *hold,* $i_0 \neq i_\infty$

(2) (H_1^+) *and* (H_2^-) *hold,* $i_0 + n_0 \neq i_\infty$

(3) (H_1^-) *and* (H_2^+) *hold,* $i_0 \neq i_\infty + n_\infty$

(4) (H_1^+) *and* (H_2^+) *hold,* $i_0 + n_0 \neq i_\infty + n_\infty$.

For any continuous 1-periodic and symmetric matrix functional $B(t)$ with the Maslov type index (i, n), we defined two self-adjoint operators A, B by

$$\begin{aligned} \langle Ax, y\rangle &= \int_0^1 (-J\dot{x}, y)\, dt, \\ \langle Bx, y\rangle &= -\int_0^1 (B(t)\, x, y)\, dt \qquad \forall x, y \in E; \end{aligned} \tag{3.1}$$

then B is compact, and its abstract Maslov type index is defined by $(I_-(B), N(B))$ which is equal to the Maslov type index of $B(t)$ (cf. [11]).

Now, we define a functional $f: E \to R$ by

$$f(z) = \tfrac{1}{2}\langle Az, z\rangle - \int_0^1 H(t, z)\, dt, \qquad \forall z \in E. \tag{3.2}$$

Then $f \in C^2(E, R)$ if $H \in C^2([0, 1] \times R^{2N}, R)$, and its gradient is given by

$$\langle f'(z), y\rangle = \langle Az, y\rangle - \int_0^1 (H'(t, z), y)\, dt.$$

Therefore, looking for the nontrivial 1-periodic solution of Hamiltonian system (1.1) is equivalent to looking for the nonzero critical point of f (cf. [10]).

We consider a strongly indefinite functional defined on a Hilbert space E with the form $f(x) = \frac{1}{2}\langle Ax, x\rangle + G(x)$. In order to deal with the system (1.1), we need the following lemmas:

LEMMA 3.1. *Assume that*

(1) *A is a bounded self-adjoint operator,*

(2) *θ is an isolating critical point of f, 0 is the isolating critical value, and $G'(x)$ is compact and global Lipschitz continuous on any bounded set,*

($3^{\pm}$) *there exists a linear, compact, symmetric operator B_0 such that $\|G'(x) - B_0(x)\| \leqslant c(\|x_0\|^{\alpha} + \|x_+ + x_-\|^{\beta})$, and $(G(x_0) - \frac{1}{2}\langle B_0 x, x\rangle)\, \|x_0\|^{-2\alpha} \to \pm\infty$ as $\|x_0\| \to 0$, where $x = x_+ + x_0 + x_- \in H_+ \oplus H_0 \oplus H_-$, which is the orthogonal decomposition corresponding to the spectrum of operator A.*

($4^{\pm}$) *there exists a linear, compact, symmetric operator B_∞ such that $\|G'(x) - B_\infty(x)\| \leqslant c\, \|x\|^{\alpha}$, $\alpha \in (0, 1)$, $c > 0$ and $(G(P_0 x) - \frac{1}{2}\langle B_\infty P_0 x, P_0 x\rangle)\, \|P_0 x\|^{-2\alpha} \to \pm\infty$, as $\|P_0 x\| \to \infty$, where P_0 is the orthogonal projection from H to the kernel space of A. Then*

$$C_q(f, \theta) = \begin{cases} \delta_{q, I_-(B_0) + N(B_0)} G & \text{if } (3^-) \text{ holds} \\ \delta_{q, I_-(B_0)} G & \text{if } (3^+) \text{ holds} \end{cases}$$

and

$$C_q(f, \infty) = \begin{cases} \delta_{q, I_-(B_\infty) + N(B_\infty)} G, & \text{if } (4^-) \text{ holds} \\ \delta_{q, I_-(B_\infty)} G, & \text{if } (4^+) \text{ holds,} \end{cases}$$

where, $(I_-(B_0), N(B_0))$, $(I_-(B_\infty), N(B_\infty))$ is the abstract Maslov index defined in [11], which is equal to the Maslov index of $B_0(t)$, $B_\infty(t)$ according to [11].

Proof. For the compact operator B_0, let $\{e_j \mid j = \pm 1, \pm 2, ...\}$ be the eigenvectors of $A + B_0$ corresponding to its eigenvalues $\{\lambda_j \mid j = \pm 1, \pm 2, ...\}$. For any $n \geqslant 0$, let $H_0 = \ker(A + B_0)$, $H_n = H_0 \oplus \text{span}\{e_1, ..., e_n\} \oplus \text{span}\{e_{-1}, ..., e_{-n}\}$, and $\Gamma = \{P_n \mid n = 1, 2, ...\}$ be the orthogonal projections from H to H_n; then Γ is an approximation scheme with respect to $(A + B_0)$ and also with respect to A.

Case (3^-). Let $C := A + B_0$, $g(x) := G(x) - \frac{1}{2}\langle B_0 x, x\rangle$, and $m = \inf\{|\langle Cx_\pm, x_\pm\rangle| \; \|x_\pm\| = 1, x_\pm \in H_\pm\}$; then we have $f(x) = \frac{1}{2}\langle (A + B_0)\, x, x\rangle + G(x) - \frac{1}{2}\langle B_0 x, x\rangle = \frac{1}{2}\langle C_x, x\rangle + g(x)$. Since θ is an isolated critical point and 0 is an isolating critical value, it is easy to check that $\{\theta\}$ is a dynamically isolated critical set. Recall the critical group

$$C_*(f, \theta) = H^{* + m(P_n(A+P)\, P_n)}(W_n, W_{n-}),$$

where (W_n, W_{n-}) is a G-M pair for a critical set S_n of the restriction functional $f_n = f|_{H_n}$ in H_n, which is associated with the flow generated by $df_n = P_n(A + G')\, P_n$. Let (O, α, β) be an isolating triplet for $\{\theta\}$ and satisfy that $(O \cap H_n, \alpha, \beta)$ is an isolating triplet for S_n (the existence of such a triplet is proved in [15]). For large n, we take a neighborhood N of S_n in $O \cap H_n$ with the following form

$$N = \{x \mid \|x_+\|^2 - d\,\|x_-\|^2 - k\,\|x_0\|^{2\alpha} \leqslant \varepsilon r_0^2,\ \|x_-\|^2 + \|x_0\|^2 \leqslant r_0^2\},$$

where d, k, ε, r_0 are to be determined later, $x = x_+ + x_0 + x_- \in H_n = H_n^+ \oplus H_0 \oplus H_n^-$. The boundary of N consists of two parts, namely

$$\Gamma_1 = \{x \mid \|x_+\|^2 - d\,\|x_-\|^2 - k\,\|x_0\|^{2\alpha} = \varepsilon r_0^2,\ \|x_-\|^2 + \|x_0\|^2 \leqslant r_0^2\}$$

and

$$\Gamma_2 = \{x \mid \|x_+\|^2 - d\,\|x_-\|^2 - k\,\|x_0\|^{2\alpha} \leqslant \varepsilon r_0^2,\ \|x_-\|^2 + \|x_0\|^2 = r_0^2\}.$$

Since the normal vector on Γ_1 is $n = x_+ - dx_- - k\alpha\,\|x_0\|^{2\alpha-2}\, x_0$, we have

$$\begin{aligned}
(df_n(x), n) &= (P_n(C + g')\, P_n(x), n) \\
&= (P_n C P_n x_+, x_+) - d(P_n C P_n x_-, x_-) + (P_n g' P_n(x), n) \\
&\geqslant m\,\|x_+\|^2 + dm\,\|x_-\|^2 \\
&\quad - c(\|x_0\|^\alpha + \|x_+ + x_-\|^\beta) \cdot (\|x_+\|^2 + d\,\|x_-\| + k\alpha\,\|x_0\|^{2\alpha-1}) \\
&\geqslant m\,\|x_+\|^2 + dm\,\|x_-\|^2 - c\,\|x_0\|^\alpha\,(\|x_+\| + d\,\|x_-\|) - L(x) \\
&\geqslant \frac{m}{2}(\|x_+\|^2 - d\,\|x_-\|^2 - k\,\|x_0\|^{2\alpha}) = \frac{1}{2} m\varepsilon r_0^2 > 0
\end{aligned} \tag{3.3}$$

as k large and r_0 small, where $L(x)$ consists of some higher term with respect to $\|x_+\|^2$, $\|x_-\|^2$ and $\|x_0\|^\alpha$. Next we study the behavior of f_n near the boundary Γ_2:

$$\begin{aligned} f_n(x) &= \tfrac{1}{2}\langle Cx_+, x_+\rangle + \tfrac{1}{2}\langle Cx_-, x_-\rangle + g(x) \\ &\leqslant \tfrac{1}{2}\|C\|\,\|x_+\|^2 - \tfrac{1}{2}m\,\|x_-\| + g(x_0) \\ &\quad + c(\|x_0\|^\alpha + \|x_+ + x_-\|^\beta)(\|x_+\| + \|x_-\|) \\ &\leqslant \|C\|\,\|x_+\|^2 - \tfrac{1}{4}m\,\|x_-\|^2 + \tfrac{1}{2}g(x_0) \\ &\leqslant \|C\|\,\varepsilon r_0^2 + (\|C\|\,d - \tfrac{1}{4}m)\,\|x_-\|^2 + \tfrac{1}{4}g(x_0). \end{aligned} \tag{3.4}$$

Take d satisfying $\|C\|\,d - \frac{1}{4}m < 0$. For given r_0 and ε small enough, we choose two constants r_1, r_2, $\delta > 0$, $r_1 < r_2 < r_0$ such that

$$\begin{aligned} f_n(x) &\geqslant -\frac{\delta}{2} &&\text{if } x \in N, && \|x_0 + x_-\| \leqslant r_1 \\ f_n(x) &< 0 &&\text{if } x \in N, && \|x_0 + x_-\| > r_1 \\ f_n(x) &\geqslant -\frac{3}{4}\delta &&\text{if } x \in N, && \|x_0 + x_-\| \leqslant r_2 \\ f_n(x) &< -\frac{\delta}{2} &&\text{if } x \in N, && \|x_0 + x_-\| = r_2 \\ f_n(x) &< -\delta &&\text{if } x \in N, && \|x_0 + x_-\| = r_0. \end{aligned}$$

Let

$$\begin{aligned} N_i &= \{x \in N \mid \|x_0 + x_-\| \leqslant r_i\} \\ \Gamma_{r_i} &= \{x \in N \mid \|x_0 + x_-\| = r_i\} \qquad i = 1, 2. \end{aligned}$$

Set $W_n = \{\eta_n(t, u) \mid t \geqslant 0,\ u \in N_2,\ f_n(\eta_n(t, u)) \geqslant -\frac{3}{4}\delta\}$, $W_{n-} = W_n \cap f_n^{-1}(-\frac{3}{4}\delta)$ where η_n is the negative gradient flow generated by df_n in H_n. Then (W_n, W_{n-}) is a G-M pair for $N \cap K_{f_n}$ associated with flow η_n. If $A_1 = \{\eta_n(t, u) \mid t \geqslant 0, u \in \Gamma_{r_2}, f(\eta_n(t, u)) \geqslant -\frac{3}{4}\delta\}$, then $W_n = N_2 \cup (A_1 \cup W_{n-})$. Since $\eta_n(t, u)$ cannot enter N_1 whenever $u \in \Gamma_{r_2}$, it follows that if $\eta_n(t, u) \in W_n$, then there exists a unique t_1 such that $\eta_n(t_1, u) \in W_{n-}$. Hence, we can use negative gradient flow η_n to deform $\Gamma_{r_2} \cup W_{n-}$ onto W_{n-} and $A_1 \cup W_{n-}$ onto W_{n-}. So

$$\begin{aligned} H^*(W_n, W_{n-}) &= H^*(N_2 \cup (A_1 \cup W_{n-}), W_{n-}) \\ &\cong H^*(N_2 \cup W_{n-}, W_{n-}) \\ &\cong H^*(N_2 \cup W_{n-}, \Gamma_{r_2} \cup W_{n-}) \\ &= H^*(N_2, \Gamma_{r_2}) \qquad \text{(excitation)} \\ &= \delta_{*, m(P_n(A+B_0) P_n) + \dim(\ker(A+B_0))} G. \end{aligned}$$

Therefore (for large n)

$$\begin{aligned} C_*(f, \theta) &= H^{* + m(P_n(A+P) P_n)}(W_n, W_{n-}) \\ &= \delta_{*, m(P_n(A+B_0) P_n) - m(P_n(A+P) P_n) + \dim(\ker(A+B_0))} G \\ &= \delta_{*, I_-(B_0) + N(B_0)} G. \end{aligned}$$

Case 3^+. Now we define the neighborhood of S_n as follows:

$$N = \{x \mid -d \|x_+\|^2 + \|x_-\|^2 - k \|x_0\|^{2\alpha} \leqslant \varepsilon r_0^2, \|x_+\|^2 + \|x_0\|^2 \leqslant r_0^2\}.$$

Then the boundaries of N are

$$\begin{aligned} \Gamma_1 &= \{x \mid -d \|x_+\|^2 + \|x_-\|^2 - k \|x_0\|^{2\alpha} = \varepsilon r_0^2, \|x_+\|^2 + \|x_0\|^2 \leqslant r_0^2\} \\ \Gamma_2 &= \{x \mid -d \|x_+\|^2 + \|x_-\|^2 - k \|x_0\|^{2\alpha} \leqslant \varepsilon r_0^2, \|x_+\|^2 + \|x_0\|^2 = r_0^2\}. \end{aligned}$$

We define

$$\Gamma_r = \{x \mid -d \|x_+\|^2 + \|x_-\|^2 - k \|x_0\|^{2\alpha} = \varepsilon r_0^2, \|x_+\|^2 + \|x_0\|^2 \leqslant r^2\}.$$

Then the normal vector on Γ_1 is $n = -dx_+ + x_- - k\alpha \|x_0\|^{2\alpha - 2} x_0$. Similar to the case (3^-), we have

$$(df_n(x), n) \leqslant \frac{m}{2} (d \|x_+\|^2 - \|x_-\|^2 + k \|x_0\|^{2\alpha}) = -\frac{1}{2} m\varepsilon r_0^2 < 0; \quad (3.5)$$

this implies that the negative gradient of f_n is outward on Γ_1.

Since

$$\begin{aligned} f_n(x) &= \tfrac{1}{2}\langle Cx_+, x_+\rangle + \tfrac{1}{2}\langle Cx_-, x_-\rangle + g(x) \\ &\geqslant \tfrac{1}{2}m \|x_+\|^2 - \tfrac{1}{2} \|C\| \|x_-\|^2 + g(x_0) \\ &\quad - c(\|x_0\|^\alpha + \|x_+ + x_-\|^\beta)(\|x_+\| + \|x_-\|) \\ &\geqslant -\|C\| \|x_0\|^2 + \tfrac{1}{4}m \|x_+\|^2 + \tfrac{1}{2} g(x_0) \\ &\geqslant -\|C\| \varepsilon r_0^2 - (\|C\| d - \tfrac{1}{4}m) \|x_+\|^2 + \tfrac{1}{4} g(x_0) \end{aligned} \quad (3.6)$$

choose d satisfying $\|C\|\,d-\frac{1}{4}m<0$. For given r_0 and small enough ε, we take positive constants r_1, r_2, $\delta>0$, $r_1<r_2<r_0$ such that

$$f_n(x)\leqslant\frac{\delta}{2} \quad \text{if} \quad x\in N, \quad \|x_0+x_+\|\leqslant r_1$$

$$f_n(x)>0 \quad \text{if} \quad x\in N, \quad \|x_0+x_+\|\geqslant r_1$$

$$f_n(x)\geqslant\delta, \quad \text{if} \quad x\in N, \quad \|x_0+x_+\|\geqslant r_2.$$

First, we deform N to $\Gamma_{r_0}\cup N_2=\{x\in N\mid \|x_0+x_+\|\leqslant r_2\}$ by a geometric deformation σ_2. We define $W_n=N_2\cap f_{n(\delta/2)}\cup\Gamma_{r_0}$ $W_{n-}=\Gamma_{r_0}\cap f_{n(\delta/2)}$. Then, it is easy to check that (W_n, W_{n-}) is a G-M we want.

Second, we use the negative gradient flow η_n generated by df_n to make a deformation. Let t_1 be the time of reaching the set $f_{n(\delta/2)}$, let t_2 be the time of reaching the boundary Γ_{r_0}, take $t=\min\{t_1, t_2\}$, and define

$$\sigma_1(s,u)=\begin{cases}\eta_n(st,u), & u\in N_2\cup\Gamma_{r_0}, \quad t>0\\ u, & u\in N_2\cup\Gamma_{r_0}, \quad t=0.\end{cases}$$

Then $\sigma=\sigma_2\circ\sigma_1$ is a deformation retraction of $N_2\cup\Gamma_{r_0}$ onto W_n. Hence

$$H^*(W_n, W_{n-})\cong H(N_2\cup\Gamma_{r_0},\Gamma_{r_0})=\delta_{*,\,m(P_n(A+B_0)\,P_n)}G.$$

So (for n large enough)

$$\begin{aligned}C_*(f,\theta)&=H^{*\,+m(P_n(A+P)\,P_n)}(W_n, W_{n-})\\&=\delta_{*,\,m(P_n(A+B_0)\,P_n)-m(P_n(A+P)\,P_n)}G\\&=\delta_{*,\,I_-(B_0)}G.\end{aligned}$$

Now, we compute the critical group at infinity. For the compact operator B_∞, let $\cdots\gamma_{-2}\leqslant\gamma_{-1}<0<\gamma_1\leqslant\gamma_2\cdots$ be the eigenvalues of $A+B_\infty$ and let $\{q_j\mid j=\pm1,\pm2,...\}$ be the eigenvectors of $A+B_\infty$ corresponding to $\{\gamma_j\mid j=\pm1,\pm2,...\}$. For any $n\geqslant0$, set $H_0=\ker(A+B_\infty)$, $H_n=H_0\oplus\operatorname{span}\{e_1,...,e_m\}\oplus\operatorname{span}\{q_{-1},...,q_{-n}\}$, and let $\Gamma=\{P_n\mid n=1,2,...\}$ be the orthogonal projection from H to H_n; then $\Gamma=\{P_m\mid m=1,2,...\}$ is an approximation scheme with respect to $(A+B_0)$. It is very easy to show that Γ is also an approximation scheme with respect to A.

Case (4^-). In this case, we write f as $f(x)=\frac{1}{2}\langle(A+B_\infty)\,x,x\rangle+G(x)-\frac{1}{2}\langle B_\infty x,x\rangle=:\frac{1}{2}\langle Cx,x\rangle+g(x)$. Let $m=\inf\{|\langle Cx_\pm,x_\pm\rangle|\ \|x_\pm\|=1,\ x_\pm\in H_\pm\}$. According to the definition, to compute $C_*(f,\infty)$, we need only to compute $H^{*\,+m(P_n(A+P)\,P_n)}(W_n, W_{n-})$, where (W_n, W_{n-}) is a G-M pair for S_n associated with the flow generated by df_n, and S_n is a set consisting of all the critical points of the restriction function $f_n = f|_{H_n}$. In the following, we define a "cylinder" in H_n (for n large enough) by

$$C_1 = \{x \mid \|x_+\|^2 - d\,\|x_-\|^2 - kh(\|x_0\|) \leqslant M\},$$

where d, k, $M > 0$ will be determined later and

$$h(t) = \begin{cases} |t|^{2\alpha} & \alpha > \frac{1}{2} \\ |t|^{2\alpha} & \alpha \leqslant \frac{1}{2}, \quad |t| \geqslant 2 \\ |t|^2 & \alpha \leqslant \frac{1}{2}, \quad |t| \leqslant 1 \\ \text{smooth} & \alpha \leqslant \frac{1}{2}, \quad 1 \leqslant |t| \leqslant 2. \end{cases} \tag{3.7}$$

The normal vector on ∂C_1 is $n = x_+ - dx_- - kh'(\|x_0\|)(x_0/\|x_0\|)$. For n large enough, we have

$$\begin{aligned} (df_n(x), n) &= (P_n C P_n x_+, x_+) - d(P_n C P_n x_-, x_-) + (P_n g' P_n(x), n) \\ &\geqslant m\,\|x_+\|^2 - dm\,\|x_-\|^2 - c(\|x\|^\alpha + 1) \\ &\quad \times \left(\|x_+\| + d\,\|x_-\| + kh'(\|x_0\|)\,\frac{x_0}{\|x_0\|} \right) \\ &\geqslant m\,\|x_+\|^2 + dm\,\|x_-\|^2 - c\,\|x_0\|^\alpha\,(\|x_+\| + d\,\|x_-\|) - L(x) - c, \end{aligned}$$

where $L(x)$ consists of lower terms with respect to $\|x_+\|^2$, $\|x_-\|^2$ and $\|x_0\|^{2\alpha}$. By choosing k large enough, we have

$$\begin{aligned} (df_n(x), n) &\geqslant \frac{m}{2}\,(\|x_+\|^2 - d\,\|x_-\|^2 - k\,\|x_0\|^{2\alpha}) - c \\ &= \frac{m}{2}\,M - c > 0, \qquad \text{if} \quad M > \frac{2c}{m}. \end{aligned}$$

So the negative gradient $-df_n(x)$ points inward to C_1 on ∂C_1 and f_n has no critical point outside C_1.

Now we prove that $\forall x \in C_1$

$$f_n(x) \to -\infty \Leftrightarrow \|x_- + P_0 x\| \to \infty, \qquad \text{uniformly in } x_+. \tag{3.8}$$

In fact, for $\forall x \in C_1$

$$\begin{aligned} f_n(x) &= \tfrac{1}{2}(Cx_+, x_+) + \tfrac{1}{2}(Cx_-, x_-) + g(x) \\ &\leqslant \tfrac{1}{2}\,\|C\|\,\|x_+\|^2 - \tfrac{1}{2}m\,\|x_-\|^2 + g(x_0) + c(\|x\|^\alpha + 1)(\|x_+ + x_-\|) \\ &\leqslant \|C\|\,\|x_+\|^2 - \tfrac{1}{4}m\,\|x_-\|^2 + \tfrac{1}{2}\,g(x_0) + c \\ &\leqslant (-\tfrac{1}{4}m + \|c\|\,d)\,\|x_-\|^2 + \tfrac{1}{4}\,g(x_0) + c. \end{aligned}$$

Choose d such that $-\frac{1}{4}m+\|c\|\,d<0$, then

$$f_n(x)\to -\infty, \qquad \text{as} \quad \|x_-+P_0x\|\to\infty \qquad \text{uniformly in } x_+. \quad (3.9)$$

On the other hand, by (4^-)

$$\begin{aligned} f_n(x) &\geqslant -\tfrac{1}{2}\|C\|\,\|x_-\|^2+g(x_0)-c(\|x\|^\alpha+1)(\|x_++x_-\|)\\ &\geqslant -\|C\|\,\|x_-\|^2+2g(x_0)-c. \end{aligned} \quad (3.10)$$

From (3.9) and (3.10), (3.8) is proved.

Now, we take $T>0$ large enough such that there are no any critical points in set f_{-T}. By (3.2), there exist $\alpha_1<\alpha_2<-T$, $R_1>R_2>0$ such that

$$\begin{aligned} f_n(x) &\geqslant \alpha_2, &&\text{if } x\in C_1 && \|x_-+x_0\|<R_2\\ f_n(x) &< \alpha_2, &&\text{if } x\in C_1 && \|x_-+x_0\|>R_1\\ f_n(x) &\geqslant \alpha_1, &&\text{if } x\in C_1 && \|x_-+x_0\|<R_1. \end{aligned}$$

Set

$$W_n=\{x\in C_1 \mid f_n(x)\geqslant\alpha_1\}, \qquad W_{n-}=\{x\in C_1 \mid f_n(x)=\alpha_1\}.$$

Then, it is easy to check that (W_n, W_{n-}) is a G-M pair for the critical set which includes all the critical points of f_n. Let $A_1=\{x\in C_1 \mid f_n(x)\geqslant\alpha_1$ and $\|x_-+x_0\|\geqslant R_2\}$. First, we deform $C_2:=\{x\in C_1 \mid \|x_-+x_0\|\geqslant R_2\}$ onto $C_3:=\{x\in C_1 \mid \|x_-+x_0\|\geqslant R_1\}$ by a geometric deformation σ_1; then we us the negative gradient flow $\eta_n(t,x)$ to deform A_1. Let t be the time of reaching the set f_{α_1}. We define a deformation retraction σ_2 by the negative gradient flow

$$\sigma_2(s,x)=\begin{cases}\eta_n(st,x), & t>0\\ x, & t=0.\end{cases}$$

Then $\sigma=\sigma_2\circ\sigma_1$ is a deformation retraction of A_1 onto W_{n-}.

Set

$$\begin{aligned} D_1 &:=\{x\in C_1 \mid \|x_-+x_0\|\leqslant R_1\},\\ D_{12} &:=\{x\in C_1 \mid R_2\leqslant\|x_-+x_0\|\leqslant R_1\}. \end{aligned}$$

Then

$$H^*(W_n, W_{n-})\cong(H^*(W_n, A_1))\cong H^*(D_2, D_{12})=\delta_{*,\,I_-(B_\infty)+N(B_\infty)}G.$$

Case (4^+). In this case, we define

$$C_1 = \{x \mid \|x_-\|^2 - d\,\|x_+\|^2 - kh(\|x_0\|) \leqslant M\};$$

the normal vector on ∂C_1 is $n = x_-\, dx_+ - kh'(\|x_0\|)(x_0/\|x_0\|)$. Similar to case (4^-), we have

$$(df_n(x), n) \leqslant \frac{m}{2}(d\,\|x_+\|^2 - \|x_-\|^2 + k\,\|x_0\|^{2\alpha}) + c \leqslant -\frac{m}{2}M + c < 0.$$

Here, we choose k, M large enough; this implies that f_n has no critical point outside C_1, and the negative gradient $-df_n$ points outside on ∂C_1. Now we prove that $\forall x \in C_1$

$$f_n(x) \to +\infty \Leftrightarrow \|x_+ + x_0\| \to \infty, \qquad \text{uniformly in } x_-. \tag{3.11}$$

In fact, for $\forall x \in C_1$,

$$\begin{aligned} f_n(x) &= \tfrac{1}{2}(Cx_+ x_+) + \tfrac{1}{2}(Cx_-, x_-) + g(x) \\ &\geqslant \tfrac{1}{2}m\,\|x_+\|^2 - \tfrac{1}{2}\,\|C\|\,\|x_-\|^2 + g(x_0) - c(\|x\|^\alpha + 1)(\|x_+\| + \|x_-\|) \\ &\geqslant \tfrac{1}{4}m\,\|x_+\|^2 - \|C\|\,\|x_-\|^2 + \tfrac{1}{2}g(x_0) - c \\ &\geqslant (\tfrac{1}{4}m - \|C\|\,d)\,\|x_+\|^2 + \tfrac{1}{4}g(x_0) - c. \end{aligned} \tag{3.12}$$

On the other hand,

$$\begin{aligned} f_n(x) &\leqslant \tfrac{1}{2}\,\|C\|\,\|x_+\|^2 + g(x_0) + c(\|x\|^\alpha + 1)(\|x_+\| + \|x_-\|) \\ &\leqslant \|C\|\,\|x_+\|^2 + 2g(x_0) + c. \end{aligned} \tag{3.13}$$

(3.12) and (3.13) imply (3.11).

So, for $\forall T > 0$, $\exists M_2 > M_1 > T$ and $R_2 > R_1 > 0$ such that

$$\begin{aligned} &f_n(x) \leqslant M_1, && \text{if} \quad x \in C_1 && \|x_0 + x_+\| < R_1 \\ &f_n(x) > M_1, && \text{if} \quad x \in C_1 && \|x_0 + x_-\| > R_2 \\ &f_n(x) \leqslant M_2, && \text{if} \quad x \in C_1 && \|x_0 + x_+\| \leqslant R_2. \end{aligned}$$

Set

$$\begin{aligned} W_n &= \{x \in C_1 \mid f_n(x) \leqslant M_1\}, \\ W_{n-} &= \{x \in C_1 \mid f_n(x) \leqslant M_1 \text{ and } x \in \partial C_1\}. \end{aligned}$$

Let

$$N_i = \{x \in C_1 \mid \|x_0 + x_1\| \leqslant R_i\},$$
$$\Gamma_i = \{x \in \partial C_1 \mid \|x_0 + x_+\| \leqslant R_i\}, \qquad i = 1, 2.$$

Similar to the case (4^-), we deform N_2 to $N_1 \cup \Gamma_2$ by a geometric deformation σ_1. Let s_1 be the time of reaching ∂C_1 along the negative gradient flow and s_2 be the time of reaching the set $f_n^{-1}(M_1)$. We take $s = \min(s_1, s_2)$ and define

$$\sigma_2(t, x) = \begin{cases} \eta(ts, x), & s > 0 \\ x, & s = 0. \end{cases}$$

Then $\sigma = \sigma_2 \circ \sigma_1$ is a deformation retraction of W_n onto $N_1 \cup \Gamma_2$. Thus

$$\begin{aligned} H^*(W_n, W_{n-}) &\cong H^*(N_1 \cup \Gamma_2, W_{n-}) \\ &\cong H^*(N_1, \Gamma_1) \qquad \text{(excision)} \\ &= \delta_{*,\, m(P_n(A + B_\infty) P_n)} G; \end{aligned}$$

hence

$$\begin{aligned} H^{* + m(P_n(A+P) P_n)}(W_n, W_{n-}) &= \delta_{* + m(P_n(A+P) P_n),\, m(P_n(A+B_\infty) P_n)} G \\ &= \delta_{*,\, m(P_n(A+B_\infty) P_n) - m(P_n(A+P) P_n)} G. \end{aligned}$$

So (for large n)

$$C_*(f, \infty) := C_*(f, S) = \delta_{*,\, I\_(B_\infty)} G.$$

The theorem is proved.

LEMMA 3.2. *Assume* ($H_1^\pm$); *then*

(1) $\|J_0'(z)\| \leqslant c\, \|z\|^\sigma$ *for* $z \in E$ *and* $\|z\| \leqslant c$, $\sigma > 1$

(2) $J_\infty(z^0)\ \|z^0\|^2\ \sigma$, *as* $\|z^0\| \to 0$, $z^0 \in \ker(A + B_0)$,

where $J_0(z) = -\int_0^1 F_0(t, z)\, dt$. B_0 *is the operator defined by* $B_0(t)$ *according to* (3.1).

Proof. (1) is a straightforward consequence of the condition ($H_1^\pm$).

Now we prove (2). Note that the finite dimensionality and the unique continuation property of kernel space $\ker(A + B_0)$ imply that $|z^0(t)| \to 0$ if

$z^0 \in \ker(A+B_0)$, $\|z^0\| \to 0$; and similar to the proof of Lemma 3.2 in [20], for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{meas } \Omega_1 := \{t \in [0,1], |z^0(t)| < \delta \, \|z^0\|\} < \varepsilon$$

for $z^0 \in \ker(A+B_0)$, $z^0 \neq 0$. Therefore, if (H_1^-) holds and $\|z^0\| \to 0$, we have

$$\frac{J_0(z^0)}{\|z^0\|^{2\sigma}} = -\int_{\Omega_1} \frac{F_0(t,z^0)}{\|z^0\|^{2\sigma}}\,dt - \inf_{[0,1]\backslash\Omega_1} \frac{F_0(t,z^0)}{\|z^0\|^{2\sigma}}\,dt$$

$$\geqslant -\delta^{1\sigma} \int_{[0,1]\backslash\Omega} \frac{F_0(t,z^0)}{|z^0|^{2\sigma}}\,dt \to +\infty, \qquad \|z^0\| \to 0.$$

Similarly, $J_0(z^0)/\|z^0\|^{2\sigma} \to -\infty$ if (H_2^+) holds and $\|z^0\| \to 0$.

LEMMA 3.3. *Assume* $(\mathrm{H}_2^\pm)$. *Then*

(1) $\|J'_\infty(z)\| \leqslant c(\|z\|^\alpha + 1)$, $z \in E$

(2) $J_\infty(z^0)\,\|z^0\|^{2\alpha} \to \pm\infty$, *as* $z^0 \in \ker(A+B_0)$ *and* $\|z^0\| \to \infty$.

Proof. Note that the finite dimensionality and the unique continuation property of kernel space $\ker(A+B_\infty)$ imply that $|z^0(t)| \to \infty$ for a.e. $t \in [0,1]$ if $\|z^0\| \to \infty$. Using the same method as in Lemma 3.2 ([20]), we get the proof of Lemma 3.3.

Based on the preceding lemmas, and combining with Morse inequality, we get the proof of Theorems 3.1 and 3.2.

Proof of Theorem 3.1. We only prove the case (1); other cases are similar. Set $G(z) = -\int_0^1 H(t,z)\,dt$, and let B_0, B_∞ be linear, symmetric compact operators defined by

$$\langle B_0 x, y \rangle = -\int_0^1 (B_0(t)\,x, y)\,dt \qquad \forall x, y \in E$$

$$\langle B_\infty x, y \rangle = -\int_0^1 (B_\infty(t)\,x, y)\,dt \qquad \forall x, y \in E.$$

Then the corresponding functional of system (1.1) can be written as

$$\begin{aligned} f(z) &= \tfrac{1}{2}\langle Az, z\rangle + G(z) \\ &= \tfrac{1}{2}\langle (A+B_0)\,z, z\rangle + J_0(z) \\ &= \tfrac{1}{2}\langle (A+B_\infty)\,z, z\rangle + J_\infty(z). \end{aligned}$$

By Lemmas 3.2, 3.3, and 3.1, we have

$$C_q(f, \theta) = \delta_{q, I_-(B_0)} G, \qquad \forall q$$
$$C_q(f, \infty) = \delta_{q, I_-(B_\infty)} G, \qquad \forall q.$$

If $I_-(B_\infty) \neq I_-(B_0)$, then

$$C_q(f, \theta) \not\cong C_q(f, \infty) \qquad \forall q.$$

By $I_-(B_\infty)$th Morse inequality, f has at least one nontrivial critical point x_1 satisfying $C_{i_\infty}(f, x_1) \not\cong 0$. The proof is complete.

Proof of Theorem 3.2. If f only has critical points x_0, θ, then

$$C_q(f, \theta) = \delta_{q, j} G,$$

where

$$j = \begin{cases} I_-(B_0) = i_0 & \text{if } (\mathrm{H}_1^+) \text{ holds} \\ I_-(B_0) + N(B_0) = i_0 + n_0 & \text{if } (\mathrm{H}_1^-) \text{ holds} \end{cases}$$

since $C_q(f, x_0) = \delta_{q, \mu} G$, where μ is the Morse index of x_0.

By the last Morse equality,

$$(-1)^r = (-1)^j + (-1)^\mu$$

where

$$r = \begin{cases} I_-(B_\infty) & \text{if } (\mathrm{H}_2^+) \text{ holds} \\ I_-(B_\infty) + N(B_\infty) & \text{if } (\mathrm{H}_2^-) \text{ holds} \end{cases}$$

a contradiction! Our theorems are proved.

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