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Theory of fractional hybrid differential equations[☆]

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ABSTRACT

In this paper, we develop the theory of fractional hybrid differential equations involving Riemann–Liouville differential operators of order $0 < q < 1$. An existence theorem for fractional hybrid differential equations is proved under mixed Lipschitz and Carathéodory conditions. Some fundamental fractional differential inequalities are also established which are utilized to prove the existence of extremal solutions. Necessary tools are considered and the comparison principle is proved which will be useful for further study of qualitative behavior of solutions.

1. Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications; see [1–20]. Although the tools of fractional calculus have been available and applicable to various fields of study, there are a few papers on the investigation of the theory of fractional differential equations; see [21–25]. The differential equations involving Riemann–Liouville differential operators of fractional order $0 < q < 1$ are very important in modeling several physical phenomena [26–28] and therefore seem to deserve an independent study of their theory parallel to the well-known theory of ordinary differential equations.

In recent years, quadratic perturbations of nonlinear differential equations have attracted much attention. We call such differential equations hybrid differential equations. There have been many works on the theory of hybrid differential equations, and we refer the readers to the articles [29–32]. Dhage and Lakshmikantham [30] discussed the following first order hybrid differential equation

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad \text{a.e. } t \in J, \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases}$$

where $f \in C(J\times\mathbb{R}, \mathbb{R}\setminus\{0\})$ and $g \in \mathcal{C}(J\times\mathbb{R}, \mathbb{R})$. They established the existence and uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and a comparison result.

From the above works, we develop the theory of fractional hybrid differential equations involving Riemann–Liouville differential operators of order $0 < q < 1$. An existence theorem for fractional hybrid differential equations is proved under

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mixed Lipschitz and Carathéodory conditions. Some fundamental fractional differential inequalities are also established which are utilized to prove the existence of extremal solutions. Necessary tools are considered and the comparison principle is proved which will be useful for further study of qualitative behavior of solutions.

2. Fractional hybrid differential equation

Let $\mathbb{R}$ be the real line and $J = [0, T)$ be a bounded interval in $\mathbb{R}$ for some $T \in \mathbb{R}$. Let $C(J \times \mathbb{R}, \mathbb{R})$ denote the class of continuous functions $f : J \times \mathbb{R} \to \mathbb{R}$ and let $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ denote the class of functions $g : J \times \mathbb{R} \to \mathbb{R}$ such that

(i) the map $t \mapsto g(t, x)$ is measurable for each $x \in \mathbb{R}$, and
(ii) the map $x \mapsto g(t, x)$ is continuous for each $t \in J$.

The class $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R}$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on J.

Definition 2.1 (*[25]*)**.** The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{(n)} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2 (*[25]*)**.** The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s,$$

provided that the right side is pointwise defined on $(0, +\infty)$.

We consider fractional hybrid differential equations (FHDEs) involving Riemann–Liouville differential operators of order $0 < q < 1$,

$$\begin{cases} D^q \left[\dfrac{x(t)}{f(t, x(t))}\right] = g(t, x(t)), \quad \text{a.e. } t \in J, \\ x(0) = 0, \end{cases} \tag{2.1}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$.

By a solution of the FHDE (2.1) we mean a function $x \in C(J, \mathbb{R})$ such that

(i) the function $t \mapsto \frac{x}{f(t,x)}$ is continuous for each $x \in \mathbb{R}$, and
(ii) x satisfies the equations in (2.1).

The theory of strict and nonstrict differential inequalities related to the ODEs and hybrid differential equations is available in the literature (see [30,33]). It is known that differential inequalities are useful for proving the existence of extremal solutions of the ODEs and hybrid differential equations defined on J.

3. Existence result

In this section, we prove the existence results for the FHDE (2.1) on the closed and bounded interval $J = [0, T]$ under mixed Lipschitz and Carathéodory conditions on the nonlinearities involved in it. We place the FHDE (2.1) in the space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. Define a supremum norm $\|\cdot\|$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)|$$

and a multiplication in $C(J, \mathbb{R})$ by

$$(xy)(t) = x(t)y(t)$$

for $x, y \in C(J, \mathbb{R})$. Clearly $C(J, \mathbb{R})$ is a Banach algebra with respect to above norm and multiplication in it. By $L^1(J, \mathbb{R})$ denote the space of Lebesgue integrable real-valued functions on J equipped with the norm $\|\cdot\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^T |x(s)| \mathrm{d}s.$$

We prove the existence of solution for the FHDE (2.1) by a fixed point theorem in Banach algebra due to Dhage [34].

Lemma 3.1 ([34]). *Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $A : X \to X$ and $B : S \to X$ be two operators such that*

(a) *A is Lipschitzian with a Lipschitz constant α,*
(b) *B is completely continuous,*
(c) *$x = AxBy \Rightarrow x \in S$ for all $y \in S$, and*
(d) *$\alpha M < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$.*

Then the operator equation $AxBx = x$ has a solution in S.

We consider the following hypotheses in what follows.

(A_0) The function $x \mapsto \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$.
(A_1) There exists a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \le L|x-y|$$

for all $t \in J$ and $x, y \in \mathbb{R}$.
(A_2) There exists a function $h \in L^1(J, \mathbb{R})$ such that

$$|g(t,x)| \le h(t) \quad \text{a.e. } t \in J$$

for all $x \in \mathbb{R}$.

Lemma 3.2 ([25]). *Let $0 < q < 1$ and $u \in L^1(0,T)$.*

(H_1) *The equality $D^q I^q u(x) = u(x)$ holds.*
(H_2) *The equality*

$$I^q D^q u(x) = u(x) - \frac{I^{1-q}u(0)}{\Gamma(q)} x^{q-1}$$

holds almost everywhere on J.

The following lemma is useful in what follows.

Lemma 3.3. *Assume that hypothesis (A_0) holds. Then for any $h \in L^1(J, \mathbb{R})$ and $0 < q < 1$, the function $x \in C(J, \mathbb{R})$ is a solution of the FHDE*

$$D^q \left[\frac{x(t)}{f(t, x(t))} \right] = h(t), \quad a.e.\ t \in J, \tag{3.1}$$

and

$$x(0) = 0, \tag{3.2}$$

if and only if x satisfies the hybrid integral equation (HIE)

$$x(t) = \frac{f(t, x(t))}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) \mathrm{d}s, \quad t \in J. \tag{3.3}$$

Proof. Let x be a solution of the Cauchy problem (3.1) and (3.2). Since the Riemann–Liouville fractional integral I^q is a monotone operator, thus, we apply fractional integral I^q on both sides of (3.1), by Lemma 3.2, we have

$$I^q D^q \left[\frac{x(t)}{f(t, x(t))} \right] = \frac{x(t)}{f(t, x(t))} - \frac{I^{1-q} \left. \frac{x(t)}{f(t,x(t))} \right|_{t=0}}{\Gamma(q)} t^{q-1} = I^q h(t),$$

then by (3.2), we get

$$\frac{x(t)}{f(t, x(t))} = I^q h(t) + \frac{I^{1-q} \left. \frac{x(t)}{f(t,x(t))} \right|_{t=0}}{\Gamma(q)} t^{q-1} = I^q h(t),$$

i.e.,

$$x(t) = f(t, x(t)) \cdot I^q h(t) = \frac{f(t, x(t))}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) \mathrm{d}s, \quad t \in J.$$

Thus, (3.3) holds.

Conversely, assume that x satisfies HIE (3.3). Then dividing by $f(t, x(t))$ and applying D^q on both sides of (3.3), so (3.1) is satisfied. Again, substituting $t = 0$ in (3.3) yields

$$\frac{x(0)}{f(0, x(0))} = 0 = \frac{0}{f(0, 0)}.$$

Since the map $x \mapsto \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$, the map $x \mapsto \frac{x}{f(0,x)}$ is injective in $\mathbb{R}$ and $x(0) = 0$. Hence (3.2) also holds. The proof is completed. $\square$

Now we are in a position to prove the following existence theorem for FHDE (2.1).

Theorem 3.1. *Assume that hypotheses* (A_0)–(A_2) *hold. Further, if*

$$\frac{LT^q \|h\|_{L^1}}{\Gamma(q+1)} < 1, \tag{3.4}$$

then the FHDE (2.1) *has a solution defined on* J.

Proof. Set $X = C(J, \mathbb{R})$ and define a subset S of X defined by

$$S = \{x \in X \mid \|x\| \le N\}, \tag{3.5}$$

where $N = \frac{F_0 T^q \|h\|_{L^1}}{\Gamma(q+1) - LT^q \|h\|_{L^1}}$ and $F_0 = \sup_{t\in J} |f(t, 0)|$.

Clearly S is a closed, convex and bounded subset of the Banach space X. By Lemma 3.3, FHDE (2.1) is equivalent to the nonlinear HIE

$$x(t) = \frac{f(t, x(t))}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x(s)) \mathrm{d}s, \quad t \in J. \tag{3.6}$$

Define two operators $A : X \to X$ and $B : S \to X$ by

$$Ax(t) = f(t, x(t)), \quad t \in J, \tag{3.7}$$

and

$$Bx(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x(s)) \mathrm{d}s, \quad t \in J. \tag{3.8}$$

Then the HIE (3.6) is transformed into the operator equation as

$$Ax(t)Bx(t) = x(t), \quad t \in J. \tag{3.9}$$

We shall show that the operators A and B satisfy all the conditions of Lemma 3.1.

First, we show that A is a Lipschitz operator on X with the Lipschitz constant L. Let $x, y \in X$. Then by hypothesis (A_1),

$$|Ax(t) - Ay(t)| = |f(t, x(t)) - f(t, y(t))| \le L|x(t) - y(t)| \le L\|x - y\|,$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|Ax - Ay\| \le L\|x - y\|,$$

for all $x, y \in X$.

Next, we show that B is a compact and continuous operator on S into X. First we show that B is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then by the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x_n(s)) \mathrm{d}s \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \lim_{n\to\infty} g(s, x_n(s)) \mathrm{d}s \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x(s)) \mathrm{d}s \\ &= Bx(t), \end{aligned}$$

for all $t \in J$. This shows that B is a continuous operator on S.

Next we show that B is a compact operator on S. It is enough to show that $B(S)$ is a uniformly bounded and equicontinuous set in X. On the one hand, let $x \in S$ be arbitrary. Then by hypothesis (A_2),

$$\begin{aligned}
|Bx(t)| &= \left|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g(s,x(s))\mathrm{d}s\right| \\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|g(s,x(s))|\mathrm{d}s \\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} h(s)\mathrm{d}s \\
&\le \frac{T^q}{\Gamma(q+1)}\|h\|_{L^1},
\end{aligned}$$

for all $t \in J$. Taking supremum over t,

$$\|Bx\| \le \frac{T^q}{\Gamma(q+1)}\|h\|_{L^1}$$

for all $x \in S$. This shows that B is uniformly bounded on S.

On the other hand, let $t_1, t_2 \in J$, with $t_1 < t_2$. Then for any $x \in S$, one has

$$\begin{aligned}
|Bx(t_1) - Bx(t_2)| &= \left|\frac{1}{\Gamma(q)}\int_0^{t_1} (t_1-s)^{q-1} g(s,x(s))\mathrm{d}s - \frac{1}{\Gamma(q)}\int_0^{t_2} (t_2-s)^{q-1} g(s,x(s))\mathrm{d}s\right| \\
&\le \left|\frac{1}{\Gamma(q)}\int_0^{t_1} (t_1-s)^{q-1} g(s,x(s))\mathrm{d}s - \frac{1}{\Gamma(q)}\int_0^{t_1} (t_2-s)^{q-1} g(s,x(s))\mathrm{d}s\right| \\
&\quad + \left|\frac{1}{\Gamma(q)}\int_0^{t_1} (t_2-s)^{q-1} g(s,x(s))\mathrm{d}s - \frac{1}{\Gamma(q)}\int_0^{t_2} (t_2-s)^{q-1} g(s,x(s))\mathrm{d}s\right| \\
&\le \frac{\|h\|_{L^1}}{\Gamma(q+1)}[|t_2^q - t_1^q - (t_2-t_1)^q| + (t_2-t_1)^q].
\end{aligned}$$

Hence, for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|t_1 - t_2| < \delta \Rightarrow |Bx(t_1) - Bx(t_2)| < \varepsilon,$$

for all $t_1, t_2 \in J$ and for all $x \in S$. This shows that $B(S)$ is an equicontinuous set in X. Now the set $B(S)$ is uniformly bounded and equicontinuous set in X, so it is compact by the Arzela–Ascoli Theorem. As a result, B is a complete continuous operator on S.

Next, we show that hypothesis (c) of Lemma 3.1 is satisfied. Let $x \in X$ and $y \in S$ be arbitrary such that $x = AxBy$. Then, by assumption (A_1), we have

$$\begin{aligned}
|x(t)| &= |Ax(t)||By(t)| \\
&= |f(t,x(t))|\left|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} g(s,y(s))\mathrm{d}s\right| \\
&\le [|f(t,x(t)) - f(t,0)| + |f(t,0)|]\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|g(s,y(s))|\mathrm{d}s\right) \\
&\le [L|x(t)| + F_0]\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} h(s)\mathrm{d}s\right) \\
&\le [L|x(t)| + F_0]\left(\frac{T^q}{\Gamma(q+1)}\|h\|_{L^1}\right).
\end{aligned}$$

Thus,

$$|x(t)| \le \frac{F_0 T^q \|h\|_{L^1}}{\Gamma(q+1) - LT^q\|h\|_{L^1}}.$$

Taking supremum over t,

$$\|x\| \le \frac{F_0 T^q \|h\|_{L^1}}{\Gamma(q+1) - LT^q\|h\|_{L^1}} = N.$$

This shows that hypothesis (c) of Lemma 3.1 is satisfied. Finally, we have

$$M = \|B(S)\| = \sup\{\|Bx\| : x \in S\} \leq \frac{T^q}{\Gamma(q+1)}\|h\|_{L^1}$$

and so,

$$\alpha M \leq L\left(\frac{T^q}{\Gamma(q+1)}\|h\|_{L^1}\right) < 1.$$

Thus, all the conditions of Lemma 3.1 are satisfied and hence the operator equation $AxBx = x$ has a solution in S. As a result, the FHDE (2.1) has a solution defined on J. This completes the proof. □

4. Fractional hybrid differential inequalities

We discuss a fundamental result relative to strict inequalities for the FHDE (2.1).

Lemma 4.1 ([22]). *Let $m : \mathbb{R}_+ \to \mathbb{R}$ be locally Hölder continuous such that for any $t_1 \in (0, +\infty)$, we have*

$$m(t_1) = 0 \quad \text{and} \quad m(t) \leq 0 \quad \text{for } 0 \leq t \leq t_1. \tag{4.1}$$

Then it follows that

$$D^q m(t_1) \geq 0. \tag{4.2}$$

Theorem 4.1. *Assume that hypotheses* (A_0) *holds. Suppose that there exist functions $y, z : [0, T] \to \mathbb{R}$ that are locally Hölder continuous such that*

$$D^q\left[\frac{y(t)}{f(t, y(t))}\right] \leq g(t, y(t)), \quad \text{a.e. } t \in J \tag{4.3}$$

and

$$D^q\left[\frac{z(t)}{f(t, z(t))}\right] \geq g(t, z(t)), \quad \text{a.e. } t \in J, \tag{4.4}$$

one of the inequalities being strict. Then

$$y(0) < z(0) \tag{4.5}$$

implies

$$y(t) < z(t) \tag{4.6}$$

for all $t \in J$.

Proof. Suppose that inequality (4.4) is strict. Assume that the claim is false. Then there exists a $t_1 \in J, t_1 > 0$ such that $y(t_1) = z(t_1)$ and $y(t) < z(t)$ for $0 \leq t < t_1$.

Define

$$Y(t) = \frac{y(t)}{f(t, y(t))} \quad \text{and} \quad Z(t) = \frac{z(t)}{f(t, z(t))}.$$

Then we have $Y(t_1) = Z(t_1)$ and by virtue of hypothesis (A_0), we get $Y(t) < Z(t)$ for all $0 \leq t < t_1$. Setting $m(t) = Y(t) - Z(t), 0 \leq t \leq t_1$, we find that $m(t) \leq 0, 0 \leq t \leq t_1$ and $m(t_1) = 0$. Then by Lemma 4.1, we have $D^q m(t_1) \geq 0$. By (4.3) and (4.4), we obtain

$$g(t_1, y(t_1)) \geq D^q Y(t_1) \geq D^q Z(t_1) > g(t_1, z(t_1)).$$

This is a contradiction with $y(t_1) = z(t_1)$. Hence the conclusion (4.6) is valid and the proof is complete. □

The next result is concerned with nonstrict fractional differential inequalities which requires a kind of one sided Lipschitz condition.

Theorem 4.2. *Assume that the conditions of Theorem 4.1 hold with inequalities (4.3) and (4.4). Suppose that there exists a real number $M > 0$ such that*

$$g(t, x_1) - g(t, x_2) \leq \frac{M}{1 + t^q}\left(\frac{x_1}{f(t, x_1)} - \frac{x_2}{f(t, x_2)}\right), \quad \text{a.e. } t \in J \tag{4.7}$$

for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \geq x_2$. Then $y(0) \leq z(0)$ implies, provided $MT^q \leq \frac{1}{\Gamma(1-q)}$,

$$y(t) \leq z(t) \tag{4.8}$$

for all $t \in J$.

Proof. We set

$$\frac{z_\varepsilon(t)}{f(t, z_\varepsilon(t))} = \frac{z(t)}{f(t, z(t))} + \varepsilon(1 + t^q),$$

for small $\varepsilon > 0$, so that we have

$$\frac{z_\varepsilon(t)}{f(t, z_\varepsilon(t))} > \frac{z(t)}{f(t, z(t))} \Rightarrow z_\varepsilon(t) > z(t). \tag{4.9}$$

Let $Z_\varepsilon(t) = \frac{z_\varepsilon(t)}{f(t, z_\varepsilon(t))}$ so that $Z(t) = \frac{z(t)}{f(t, z(t))}$ for $t \in J$. Since

$$g(t, z_\varepsilon) - g(t, z) \leq \frac{M}{1 + t^q}\left(\frac{z_\varepsilon}{f(t, z_\varepsilon)} - \frac{z}{f(t, z)}\right)$$

for all $t \in J$ and $MT^q \leq \frac{1}{\Gamma(1-q)}$, one has

$$\begin{aligned} D^q Z_\varepsilon(t) &= D^q Z(t) + \varepsilon D^q (1 + t^q) \\ &\geq g(t, z(t)) + \varepsilon\left(\frac{1}{t^q \Gamma(1-q)} + \Gamma(1+q)\right) \\ &> g(t, z_\varepsilon(t)) - M\varepsilon + \varepsilon \frac{1}{t^q \Gamma(1-q)} \\ &> g(t, z_\varepsilon(t)). \end{aligned}$$

Also, we have $z_\varepsilon(0) > z(0) \geq y(0)$. Hence, by an application of Theorem 4.1 with $z = z_\varepsilon$ yields that $y(t) < z_\varepsilon(t)$ for all $t \in J$. By the arbitrariness of $\varepsilon > 0$, taking the limits as $\varepsilon \to 0$, we have $y(t) \leq z(t)$ for all $t \in J$. This completes the proof. □

Remark 4.1. Let $f(t, x) \equiv 1$ and $g(t, x) = x$. We can easily verify that f and g satisfy the condition (4.7).

Remark 4.2. The conclusion of Theorems 4.1 and 4.2 also remains true if we replace the derivatives in the inequalities (4.1) and (4.2) by the Dini-derivative $D_{\pm}^{\pm}$ of the function $\frac{x(t)}{f(t, x(t))}$ on the bounded interval J.

5. Existence of maximal and minimal solutions

In this section, we shall prove the existence of maximal and minimal solutions for the FHDE (2.1) on $J = [0, T]$. We need the following definition in what follows.

Definition 5.1. A solution r of the FHDE (2.1) is said to be maximal if for any other solution x to the FHDE (2.1) one has $x(t) \leq r(t)$, for all $t \in J$. Similarly, a solution ρ of the FHDE (2.1) is said to be minimal if $\rho(t) \leq x(t)$, for all $t \in J$, where x is any solution of the FHDE (2.1) on J.

We discuss the case of maximal solution only, as the case of minimal solution is similar and can be obtained with the same arguments with appropriate modifications. Given an arbitrary small real number $\varepsilon > 0$, consider the following initial value problem of FHDE of order $0 < q < 1$,

$$\begin{cases} D^q\left[\dfrac{x(t)}{f(t, x(t))}\right] = g(t, x(t)) + \varepsilon, & \text{a.e. } t \in J, \\ x(0) = 0, \end{cases} \tag{5.1}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$.

An existence theorem for the FHDE (5.1) can be stated as follows.

Theorem 5.1. *Assume that hypotheses* (A_0)–(A_2) *hold. Suppose that inequality* (3.4) *holds. Then for every small number $\varepsilon > 0$, the FHDE* (5.1) *has a solution defined on J.*

Proof. By hypothesis, since

$$\frac{LT^q\|h\|_{L^1}}{\Gamma(q+1)} < 1,$$

there exists an $\varepsilon_0 > 0$ such that

$$\frac{LT^q(\|h\|_{L^1} + \varepsilon T)}{\Gamma(q+1)} < 1,$$

for all $0 < \varepsilon \le \varepsilon_0$. Now the rest of the proof is similar to Theorem 3.1. □

Our main existence theorem for maximal solution for the FHDE (2.1) is

Theorem 5.2. *Assume that hypotheses* (A_0)–(A_2) *hold. Furthermore, if condition* (3.4) *holds, then the FHDE* (2.1) *has a maximal solution defined on* J.

Proof. Let $\{\varepsilon_n\}_0^\infty$ be a decreasing sequence of positive real numbers such that $\lim_{n\to\infty}\varepsilon_n = 0$, where ε_0 is a positive real number satisfying the inequality

$$\frac{LT^q(\|h\|_{L^1} + \varepsilon_0 T)}{\Gamma(q+1)} < 1, \tag{5.2}$$

The number ε_0 exists in view of inequality (3.4). By Theorem 5.1, then there exists a solution $r(t,\varepsilon_n)$ defined on J of the FHDE,

$$\begin{cases} D^q\left[\dfrac{x(t)}{f(t,x(t))}\right] = g(t,x(t)) + \varepsilon_n, \quad \text{a.e. } t \in J, \\ x(0) = 0. \end{cases} \tag{5.3}$$

Then for any solution u of the FHDE (2.1) satisfies

$$D^q\left[\frac{u(t)}{f(u,u(t))}\right] \le g(t,u(t)),$$

and any solution of auxiliary problem (5.3) satisfies

$$D^q\left[\frac{r(t,\varepsilon_n)}{f(t,r(t,\varepsilon_n))}\right] = g(t,r(t,\varepsilon_n)) + \varepsilon_n > g(t,r(t,\varepsilon_n)),$$

where $u(0) = 0 \le \varepsilon_n = r(0,\varepsilon_n)$. By Theorem 4.2, we infer that

$$u(t) \le r(t,\varepsilon_n) \tag{5.4}$$

for all $t \in J$ and $n \in \mathbb{N} \cup \{0\}$.

Since $\varepsilon_2 = r(0,\varepsilon_2) \le r(0,\varepsilon_1) = \varepsilon_1$, then by Theorem 4.2, we infer that $r(t,\varepsilon_2) \le r(t,\varepsilon_1)$. Therefore, $r(t,\varepsilon_n)$ is a decreasing sequence of positive real numbers, the limit

$$r(t) = \lim_{n\to\infty} r(t,\varepsilon_n) \tag{5.5}$$

exists. We show that the convergence in (5.5) is uniform on J. To finish, it is enough to prove that the sequence $r(t,\varepsilon_n)$ is equicontinuous in $C(J,R)$. Let $t_1, t_2 \in J$ with $t_1 < t_2$ be arbitrary. Then,

$$\begin{aligned} |r(t_1,\varepsilon_n) - r(t_2,\varepsilon_n)| = &\left|[f(t_1,r(t_1,\varepsilon_n))]\left(\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)\mathrm{d}s\right)\right. \\ &\left.-[f(t_2,r(t_2,\varepsilon_n))]\left(\frac{1}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)\mathrm{d}s\right)\right| \\ \le &\left|[f(t_1,r(t_1,\varepsilon_n))]\left(\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)\mathrm{d}s\right)\right. \\ &\left.-[f(t_2,r(t_2,\varepsilon_n))]\left(\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)\mathrm{d}s\right)\right| \\ &+\left|[f(t_2,r(t_2,\varepsilon_n))]\left(\frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)\mathrm{d}s\right)\right. \\ &\left.-[f(t_2,r(t_2,\varepsilon_n))]\left(\frac{1}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)\mathrm{d}s\right)\right| \end{aligned}$$

$$\leq |f(t_1, r(t_1, \varepsilon_n)) - f(t_2, r(t_2, \varepsilon_n))| \frac{(\|h\|_{L^1} + \varepsilon_n)T^q}{\Gamma(q+1)} + F\frac{(\|h\|_{L^1} + \varepsilon_n)}{\Gamma(q+1)}[|t_2^q - t_1^q - (t_2 - t_1)^q| + (t_2 - t_1)^q],$$

where $F = \sup_{(t,x)\in J\times[-N,N]} |f(t, x)|$.

Since f is continuous on compact set $J \times [-N, N]$, it is uniformly continuous there. Hence,

$$|f(t_1, r(t_1, \varepsilon_n)) - f(t_2, r(t_2, \varepsilon_n))| \to 0 \quad \text{as } t_1 \to t_2$$

uniformly for all $n \in \mathbb{N}$.

Therefore, from the above inequality, it follows that

$$|r(t_1, \varepsilon_n) - r(t_2, \varepsilon_n)| \to 0 \quad \text{as } t_1 \to t_2$$

uniformly for all $n \in \mathbb{N}$. Therefore,

$$r(t, \varepsilon_n) \to r(t) \quad \text{as } n \to \infty$$

for all $t \in J$.

Next, we show that the function $r(t)$ is a solution of the FHDE (2.1) defined on J. Now, since $r(t, \varepsilon_n)$ is a solution of the FHDE (5.3), we have

$$r(t, \varepsilon_n) = [f(t, r(t, \varepsilon_n))]\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(g(s, r(s, \varepsilon_n)) + \varepsilon_n)\mathrm{d}s\right) \tag{5.6}$$

for all $t \in J$. Taking the limit as $n \to \infty$ in the above Eq. (5.6) yields

$$r(t) = [f(t, r(t))]\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}g(s, r(s))\mathrm{d}s\right)$$

for all $t \in J$. Thus, the function r is a solution of the FHDE (2.1) on J. Finally, from inequality (5.3), it follows that $u(t) \leq r(t)$ for all $t \in J$. Hence, the FHDE (2.1) has a maximal solution on J. This completes the proof. □

6. Comparison theorems

The main problem of the differential inequalities is to estimate a bound for the solution set for the differential inequality related to the FHDE (2.1). In this section, we prove that the maximal and minimal solutions serve as bounds for the solutions of the related differential inequality to FHDE (2.1) on $J = [0, T]$.

Theorem 6.1. *Assume that hypotheses* (A_0)–(A_2) *and condition* (3.4) *hold. Suppose that there exists a real number* $M > 0$ *such that*

$$g(t, x_1) - g(t, x_2) \leq \frac{M}{1+t^q}\left(\frac{x_1}{f(t, x_1)} - \frac{x_2}{f(t, x_2)}\right), \quad \text{a.e. } t \in J$$

for all $x_1, x_2 \in \mathbb{R}$ *with* $x_1 \geq x_2$, *where* $MT^q \leq \frac{1}{\Gamma(1-q)}$. *Furthermore, if there exists a function* $u \in C(J, \mathbb{R})$ *such that*

$$\begin{cases} D^q\left[\dfrac{u(t)}{f(t, u(t))}\right] \leq g(t, u(t)), & \text{a.e. } t \in J, \\ u(0) \leq 0. \end{cases} \tag{6.1}$$

Then

$$u(t) \leq r(t) \tag{6.2}$$

for all $t \in J$, *where* r *is a maximal solution of the FHDE* (2.1) *on* J.

Proof. Let $\varepsilon > 0$ be arbitrary small. By Theorem 5.2, $r(t, \varepsilon)$ is a maximal solution of the FHDE (5.1) and that the limit

$$r(t) = \lim_{\varepsilon\to 0} r(t, \varepsilon) \tag{6.3}$$

is uniform on J and the function r is a maximal solution of the FHDE (2.1) on J. Hence, we obtain

$$\begin{cases} D^q\left[\dfrac{r(t, \varepsilon)}{f(t, r(t, \varepsilon))}\right] = g(t, r(t, \varepsilon)) + \varepsilon, & \text{a.e. } t \in J, \\ r(0, \varepsilon) = 0. \end{cases}$$

From the above inequality it follows that

$$\begin{cases} D^q\left[\dfrac{r(t,\varepsilon)}{f(t,r(t,\varepsilon))}\right] > g(t,r(t,\varepsilon)), \quad \text{a.e. } t\in J,\\ r(0,\varepsilon)=0. \end{cases} \tag{6.4}$$

Now we apply Theorem 4.2 to the inequalities (6.1) and (6.4) and conclude that $u(t) < r(t,\varepsilon)$ for all $t \in J$. This further in view of limit (6.3) implies that inequality (6.2) holds on J. This completes the proof. □

Theorem 6.2. *Assume that hypotheses* (A_0)–(A_2) *and condition* (3.4) *hold. Suppose that there exists a real number* $M > 0$ *such that*

$$g(t,x_1)-g(t,x_2) \le \frac{M}{1+t^q}\left(\frac{x_1}{f(t,x_1)}-\frac{x_2}{f(t,x_2)}\right), \quad \text{a.e. } t\in J$$

for all $x_1, x_2 \in \mathbb{R}$ *with* $x_1 \ge x_2$, *where* $MT^q \le \frac{1}{\Gamma(1-q)}$. *Furthermore, if there exists a function* $u \in C(J,\mathbb{R})$ *such that*

$$\begin{cases} D^q\left[\dfrac{v(t)}{f(t,v(t))}\right] \ge g(t,v(t)), \quad \text{a.e. } t\in J,\\ v(0) > 0. \end{cases}$$

Then

$$\rho(t) \le v(t)$$

for all $t \in J$, *where* ρ *is a minimal solution of the FHDE* (2.1) *on* J.

Note that Theorem 6.1 is useful to prove the boundedness and uniqueness of the solutions for the FHDE (2.1) on J. A result in this direction is

Theorem 6.3. *Assume that hypotheses* (A_0)–(A_2) *and condition* (3.4) *hold. Suppose that there exists a real number* $M > 0$ *such that*

$$g(t,x_1)-g(t,x_2) \le \frac{M}{1+t^q}\left(\frac{x_1}{f(t,x_1)}-\frac{x_2}{f(t,x_2)}\right), \quad \text{a.e. } t\in J$$

for all $x_1, x_2 \in \mathbb{R}$ *with* $x_1 \ge x_2$, *where* $MT^q \le \frac{1}{\Gamma(1-q)}$. *If identically zero function is the only solution of the differential equation*

$$D^q m(t) = \frac{M}{1+t^q} m(t) \quad \text{a.e. } t\in J, \quad m(0)=0, \tag{6.5}$$

then the FHDE (2.1) *has a unique solution on* J.

Proof. By Theorem 3.1, the FHDE (2.1) has a solution defined on J. Suppose that there are two solutions u_1 and u_2 of the FHDE (2.1) existing on J with $u_1 > u_2$. Define a function $m : J \to \mathbb{R}$ by

$$m(t) = \frac{u_1(t)}{f(t,u_1(t))} - \frac{u_2(t)}{f(t,u_2(t))}.$$

In view of hypothesis (A_0), we conclude that $m(t) > 0$. Then we have

$$\begin{aligned} D^q m(t) &= D^q\left[\frac{u_1(t)}{f(t,u_1(t))}\right] - D^q\left[\frac{u_2(t)}{f(t,u_2(t))}\right]\\ &= g(t,u_1) - g(t,u_2)\\ &\le \frac{M}{1+t^q}\left(\frac{u_1}{f(t,u_1)} - \frac{u_2}{f(t,u_2)}\right)\\ &= \frac{M}{1+t^q} m(t) \end{aligned}$$

for almost everywhere $t \in J$, and that $m(0) = 0$.

Now, we apply Theorem 6.1 with $f(t,x) \equiv 1$ to get that $m(t) \le 0$ for all $t \in J$, where identically zero function is the only solution of the differential equation (6.5). $m(t) \le 0$ is a contradiction with $m(t) > 0$. Then we can get $u_1 = u_2$. This completes the proof. □

7. Existence of extremal solutions in vector segment

Sometimes it is desirable to have knowledge of the existence of extremal positive solutions for the FHDE (2.1) on J. In this section, we shall prove the existence of maximal and minimal positive solutions for the FHDE (2.1) between the given upper and lower solutions on $J = [0, T]$. We use a hybrid fixed point theorem of Dhage [31] in ordered Banach spaces for establishing our results. We need the following preliminaries in what follows.

A non-empty closed set K in a Banach algebra X is called a cone with vertex 0, if

(i) $K + K \subseteq K$,
(ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R},\ \lambda \geq 0$,
(iii) $(-K) \bigcap K = 0$, where 0 is the zero element of X,
(iv) a cone K is called to be positive if $K \circ K \subseteq K$, where $\circ$ is a multiplication composition in X.

We introduce an order relation $\leq$ in X as follows. Let $x, y \in X$. Then $x \leq y$ if and only if $y - x \in K$. A cone K is said to be normal if the norm $\|\cdot\|$ is semi-monotone increasing on K, that is, there is a constant $N > 0$ such that $\|x\| \leq N\|y\|$ for all $x, y \in K$ with $x \leq y$. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. The details of cones and their properties appear in Heikkilä and Lakshmikantham [35].

Lemma 7.1 *([31]). Let K be a positive cone in a real Banach algebra X and let $u_1,\ u_2,\ v_1,\ v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. Then $u_1u_2 \leq v_1v_2$.*

For any $a,\ b \in X$, the order interval $[a, b]$ is a set in X given by

$$[a, b] = \{x \in X :\ a \leq x \leq b\}.$$

Definition 7.1. A mapping $Q :\ [a, b] \to X$ is said to be nondecreasing or monotone increasing if $x \leq y$ implies $Qx \leq Qy$ for all $x, y \in [a, b]$.

We use the following fixed point theorems of Dhage [32] for proving the existence of extremal solutions for the IVP (2.1) under certain monotonicity conditions.

Lemma 7.2 *([32]). Let K be a cone in a Banach algebra X and let $a,\ b \in X$ be such that $a \leq b$. Suppose that $A,\ B :\ [a, b] \to K$ are two nondecreasing operators such that*

(a) *A is Lipschitzian with a Lipschitz constant α,*
(b) *B is complete,*
(c) *$AxBx \in [a, b]$ for each $x \in [a, b]$.*

Further, if the cone K is positive and normal, then the operator equation $AxBx = x$ has a least and a greatest positive solution in $[a, b]$, whenever $\alpha M < 1$, where $M = \|B([a, b])\| = \sup\{\|Bx\| : x \in [a, b]\}$.

We equip the space $C(J, \mathbb{R})$ with the order relation $\leq$ with the help of cone K defined by

$$K = \{x \in C(J, \mathbb{R}) :\ x(t) \geq 0, \forall\, t \in J\}. \tag{7.1}$$

It is well known that the cone K is positive and normal in $C(J, \mathbb{R})$. We need the following definitions in what follows.

Definition 7.2. A function $a \in C(J, \mathbb{R})$ is called a lower solution of the FHDE (2.1) defined on J if it satisfies (4.3). Similarly, a function $a \in C(J, \mathbb{R})$ is called an upper solution of the FHDE (2.1) defined on J if it satisfies (4.4). A solution to the FHDE (2.1) is a lower as well as an upper solution for the FHDE (2.1) defined on J and vice versa.

We consider the following set of assumptions:

(B_0) $f :\ J \times \mathbb{R} \to \mathbb{R}^+ - \{0\},\ g :\ J \times \mathbb{R} \to \mathbb{R}^+$.
(B_1) The FHDE (2.1) has a lower solution a and an upper solution b defined on J with $a \leq b$.
(B_2) The function $x \to \frac{x}{f(t,x)}$ is increasing in the interval $[\min_{t\in J} a(t), \max_{t\in J} b(t)]$ almost everywhere for $t \in J$.
(B_3) The functions $f(t, x)$ and $g(t, x)$ are nondecreasing in x almost everywhere for $t \in J$.
(B_4) There exists a function $k \in L^1(J, \mathbb{R})$ such that $g(t, b(t)) \leq k(t)$.

We remark that hypothesis (B_4) holds in particular if f is continuous and g is L^1-Carathéodory on $J \times \mathbb{R}$.

Theorem 7.1. *Suppose that assumptions (A_1) and (B_0)–(B_4) hold. Furthermore, if*

$$\frac{LT^q}{\Gamma(q+1)}\|k\|_{L^1} < 1, \tag{7.2}$$

then the FHDE (2.1) has a minimal and a maximal positive solution defined on J.

Proof. Now, the FHDE (2.1) is equivalent to integral equation (3.6) defined on J. Let $X = C(J, \mathbb{R})$. Define two operators A and B on X by (3.7) and (3.8) respectively. Then the integral equation (3.6) is transformed into an operator equation $Ax(t)Bx(t) = x(t)$ in the Banach algebra X. Notice that hypothesis (B_0) implies $A, B : [a, b] \to K$. Since the cone K in X is normal, $[a, b]$ is a norm-bounded set in X. Now it is shown, as in the proof of Theorem 3.1, that A is a Lipschitzian with the Lipschitz constant L and B is completely continuous operator on $[a, b]$. Again, hypothesis (B_3) implies that A and B are nondecreasing on $[a, b]$. To see this, let $x, y \in [a, b]$ be such that $x \le y$. Then, by hypothesis (B_3),

$$Ax(t) = f(t, x(t)) \le f(t, y(t)) = Ay(t)$$

for all $t \in J$. Similarly, we have

$$\begin{aligned} Bx(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x(s))\mathrm{d}s \\ &\le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, y(s))\mathrm{d}s \\ &= By(t) \end{aligned}$$

for all $t \in J$. So A and B are nondecreasing operators on $[a, b]$. By Lemma 7.1 and hypothesis (B_3) together imply that

$$\begin{aligned} a(t) &\le \frac{f(t, a(t))}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x(s))\mathrm{d}s \\ &\le \frac{f(t, x(t))}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x(s))\mathrm{d}s \\ &\le \frac{f(t, b(t))}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s, x(s))\mathrm{d}s \\ &\le b(t) \end{aligned}$$

for all $t \in J$ and $x \in [a, b]$. As a result $a(t) \le Ax(t)Bx(t) \le b(t)$, for all $t \in J$ and $x \in [a, b]$. Hence, $AxBx \in [a, b]$ for all $x \in [a, b]$. Again,

$$M = \|B([a, b])\| = \sup\{\|Bx\| : x \in [a, b]\} \le \frac{T^q}{\Gamma(q+1)} \|k\|_{L^1}$$

and so,

$$\alpha M \le \frac{LT^q}{\Gamma(q+1)} \|k\|_{L^1} < 1.$$

Now, we apply Lemma 7.2 to the operator equation $AxBx = x$ to yield that the FHDE (2.1) has a minimal and a maximal positive solution in $[a, b]$ defined on J. This completes the proof. □

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