ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Advances in Mathematics 194 (2005) 105–140

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

Brunn–Minkowski inequalities for variational functionals and related problems

Andrea Colesanti*

Dipartimento di Matematicà "U. Dini", Universita di Firenze, Viale Morgagni 67/A, 50134 Firenze, Italy

Received 4 August 2003; accepted 8 June 2004

Communicated by Michael Hopkins

Available online 28 July 2004

Abstract

Recently, several inequalities of Brunn–Minkowski type have been proved for well-known functionals in the Calculus of Variations, e.g. the first eigenvalue of the Laplacian, the Newton capacity, the torsional rigidity and generalizations of these examples. In this paper, we add new contributions to this topic: in particular, we establish equality conditions in the case of the first eigenvalue of the Laplacian and of the torsional rigidity, and we prove a Brunn–Minkowski inequality for another class of variational functionals. Moreover, we describe the links between Brunn–Minkowski type inequalities and the resolution of Minkowski type problems.

MSC: 35J25; 39B26; 52A20

Keywords: Convex body; Brunn–Minkowski inequality; Variational functionals; Minkowski problem; Elliptic partial differential equations

1. Introduction

The Brunn–Minkowski inequality, in its classic formulation, states that if K_0 and K_1 are compact, convex sets in $\mathbf{R}^n$ and $t \in [0,1]$, then

$$V((1-t)K_0 + tK_1)^{1/n} \geqslant (1-t)V(K_0)^{1/n} + tV(K_1)^{1/n}, \tag{1}$$

* Fax: +39-055-4222695.

E-mail address: colesant@math.unifi.it (A. Colesanti).

doi:10.1016/j.aim.2004.06.002

where

$$(1-t)K_0 + tK_1 = \{(1-t)x + ty \ : \ x \in K_0\,,\ y \in K_1\}$$

and V denotes the n-dimensional volume (i.e. the Lebesgue measure); moreover, equality holds in (1) if and only if K_0 and K_1 are homothetic, i.e. they coincide up to translation and dilatation.

The Brunn–Minkowski inequality is among the most important and deepest results in the theory of convex bodies, for which the reader is referred to [27], and it is connected with other fundamental inequalities like the isoperimetric inequality, the Sobolev inequality and the Prékopa–Leindler inequality.

In [14], Gardner provides a very detailed presentation of inequality (1), including historical remarks, a description of links to other inequalities, various extensions and so on. Though geometry is the most natural context in which the Brunn–Minkowski inequality has to be situated, the paper by Gardner provides many evidences of the fact that its role has to be fully recognized in analysis as well as in other areas of mathematics.

This paper is concerned with inequalities of the same type as (1), where the volume is replaced by other functionals, arising in the context of the Calculus of Variations and of the theory of elliptic partial differential equations. Firstly, let us explain what do we mean by an inequality of Brunn–Minkowski type.

We will denote by $\mathcal{K}^n$ the family of n-dimensional *convex bodies*, i.e. compact, convex subsets of $\mathbf{R}^n$, with non-empty interior. In $\mathcal{K}^n$ a scalar multiplication for positive numbers and a sum (the Minkowski addition) are defined:

$$sK = \{sx \ : \ x \in K\}\,, \qquad K \in \mathcal{K}^n\,,\ s \geqslant 0\,,$$
$$K_0 + K_1 = \{x + y \ : \ x \in tK_0\,,\ y \in K_1\}\,, \qquad K_0, K_1 \in \mathcal{K}^n\,.$$

Now, inequality (1) can be rephrased as follows: the n-dimensional volume raised to the power $1/n$ is a concave function on $\mathcal{K}^n$. Note that the volume is positively homogeneous and its order of homogeneity is n:

$$V(sK) = s^n V(K)\,, \quad s \geqslant 0\,, \quad K \in \mathcal{K}^n\,.$$

Another familiar geometric functional has a similar concavity property connected with its order of homogeneity. For a given $K \in \mathcal{K}^n$, the $(n-1)$-dimensional measure of ∂K, denoted by $S(K)$, is positively homogeneous of order $(n-1)$ and satisfies the following Brunn–Minkowski type inequality:

$$S((1-t)K_0 + tK_1)^{1/(n-1)} \geqslant (1-t)S(K_0)^{1/(n-1)} + tS(K_1)^{1/(n-1)}\,,$$
$$K_0, K_1 \in \mathcal{K}^n\,, \quad t \in [0,1]\,. \tag{2}$$

Inequalities (1) and (2) are, in turn, included in a class of analogous inequalities regarding the *quermassintegrals* of convex bodies. The quermassintegrals of a convex body K can be defined through the Steiner formula, which claims that the volume of $K + \varepsilon B$, where ε is a nonnegative number and B is the unit ball, is a polynomial of ε,

$$V(K + \varepsilon B) = \sum_{i=0}^{n} \varepsilon^i \binom{n}{i} W_i(K).$$

The (nonnegative) coefficients $W_0(K), \dots, W_n(K)$ are the quermassintegrals of K (see Section 4.2 in [27] for a detailed presentation). Notice that

$$W_0(K) = V(K), \quad W_1(K) = \frac{1}{n} S(K).$$

Each quermassintegral $W_i(\cdot)$ is positively homogeneous of order $(n - i)$ and, if $i < n$, satisfies the inequality

$$W_i((1-t)K_0 + tK_1)^{1/(n-i)} \geqslant (1-t)W_i(K_0)^{1/(n-i)} + tW_i(K_1)^{1/(n-i)},$$
$$K_0, K_1 \in \mathcal{K}^n, \quad t \in [0, 1]$$

(see [27, Theorem 6.4.3]).

These examples suggest to consider the following more general situation. Assume that $\mathbf{F}$ is a functional defined in $\mathcal{K}^n$

$$\mathbf{F} : \mathcal{K}^n \longrightarrow (0, \infty),$$

which is homogeneous of order $\alpha \neq 0$, moreover, assume that $\mathbf{F}$ is invariant under rigid motions, i.e. isometries of $\mathbf{R}^n$ (this property is not needed for the following definition but it is shared by all the examples that we treat). We say that $\mathbf{F}$ satisfies a Brunn–Minkowski inequality if $\mathbf{F}^{1/\alpha}$ is concave in $\mathcal{K}^n$:

$$\mathbf{F}((1-t)K_0 + tK_1)^{1/\alpha} \geqslant (1-t)\mathbf{F}(K_0)^{1/\alpha} + t\mathbf{F}(K_1)^{1/\alpha} \tag{3}$$

for all $K_0, K_1 \in \mathcal{K}^n$ and $t \in [0, 1]$.

The examples that we have seen are all taken from geometry. In recent times, inequalities of Brunn–Minkowski type have been proved for various well-known variational functionals. Brascamp and Lieb in [6] established inequality (3) when $\mathbf{F}(K) = \lambda(K)$ is the first eigenvalue of the Laplace operator of K, in this case $\alpha = -2$. Borell proved the same kind of result for $\mathbf{F}(K) = \operatorname{Cap}(K)$, the Newton capacity of K, $\alpha = n - 2$, $n \geqslant 3$ (see [2]) and for the torsional rigidity $\mathbf{F}(K) = \tau(K)$, $\alpha = n + 2$ (see [4]).

These results have been extended in [3] to the logarithmic capacity (or transfinite diameter) in dimension $n = 2$, in [11] to the p-capacity, $p > 1$, and in [9] to the n-dimensional counterpart of the logarithmic capacity.

In this paper, we make an overview of the present situation and we bring some new contributions to it. We start by describing in some details the Brunn–Minkowski inequality for the functionals λ, Cap and τ, in the next section. We establish equality conditions in the case of the first eigenvalue of the Laplace operator and of the torsional rigidity, i.e. we prove that equality holds in (3) if and only if K_0 and K_1 are homothetic; a similar characterization of equality case was already known for the Newton capacity (see [7]).

How far can the Brunn–Minkowski inequalities for the three main examples be extended to other functionals? To answer this question, we start from the following consideration: for a convex body K, the functionals $\lambda(K)$, $\mathrm{Cap}\,(K)$ and $\tau(K)$ can all be obtained as energy integrals

$$\mathbf{F}(K) = \int_{\Omega} |\nabla u|^2 \, dx ,$$

where Ω is the interior or the complement set of K and u solves a Dirichlet boundary-value problem in Ω, involving an equation of the form $\Delta u = f(u)$ for a suitable function f (see Section 2). Extensions can be obtained: (a) replacing the Laplacian with another elliptic operator, for instance the p-Laplace operator or a fully non-linear operator; (b) choosing other types of function f. In Section 2 we describe the cases in which extensions of this kind have already been achieved (see [3,11,9]) and we establish a new extension, i.e. a new Brunn–Minkowski inequality, for the functional arising when $f(u) = -u^p$, $p \in [0, 1)$, and the operator is the Laplacian. Moreover, throughout the section, we indicate some other possible extensions which are by now open problems.

An important topic in the theory of convex bodies, strongly connected to the Brunn–Minkowski inequality, is the Minkowski problem, which requires to determine (uniquely) a convex body with a prescribed surface area measure (in case of smooth bodies, knowing the surface area measure is equivalent to know the Gauss curvature as a function of the outer unit normal to the body). The Brunn–Minkowski inequality (1) can be used to solve the Minkowski problem in a variational way (see, for instance, [21]), moreover, the equality conditions of (1) imply uniqueness in the Minkowski problem.

Jerison realized that new Minkowski type problems can be posed, replacing the surface area measure by other measures obtained, roughly speaking, as first variations of variational functionals (this concept will be made clearer in Section 4). Furthermore, he observed that Brunn–Minkowski type inequality can be used in the resolution of these Minkowski problems exactly as in the classic case. In [20] he showed existence and uniqueness of the solution to a Minkowski type problem for the Newton capacity; subsequently in [21] he posed a similar problem for the transfinite diameter and for the first eigenvalue of the Laplacian, and he obtained an existence result for both functionals. Uniqueness in the case of transfinite diameter was proved in [9].

In Section 4, after describing in more details Minkowski type problems for variational functionals, we deduce from the characterization of equality conditions in the Brunn–Minkowski inequality for the first eigenvalue of the Laplacian, the uniqueness result also in this case; moreover, we make some remarks about the feasibility of a Minkowski type problem for the torsional rigidity.

2. The main examples

In this section, we focus on the Brunn–Minkowski inequality for the first eigenvalue of the Laplace operator, the Newtonian capacity and the torsional rigidity. As we shall see, each functional can be defined either through a variational problem, posed in a suitable space of functions, or in terms of the solution of a boundary-value problem for an elliptic operator. The first definition is in the spirit of the Calculus of Variations while the second reflects the point of view of elliptic PDEs. The equivalence between the two definitions relies on a well-known principle: under suitable assumptions, the minimizers of a functional are solutions of a differential equation, called the Euler–Lagrange equation of the functional itself.

2.1. The first eigenvalue of the Laplace operator

Throughout, for $K \in \mathcal{K}^n$ we denote by $\mathrm{int}(K)$ its interior. The first eigenvalue of the Laplace operator $\lambda(K)$ can be defined as follows:

$$\lambda(K) = \inf \left\{ \int_K |\nabla v|^2 \, dx \,,\ v \in W_0^{1,2}(\mathrm{int}(K)) \,:\, \int_K v^2 \, dx = 1 \right\} .$$

Here we adopt the standard notation for Sobolev spaces; if Ω is an open subset of $\mathbf{R}^n$, $W^{1,2}(\Omega)$ is the Sobolev space of those functions having weak derivatives up to the second order in $L^2(\Omega)$; $W_0^{1,2}(\Omega)$ is the closure in $W^{1,2}(\Omega)$ of the set of smooth functions with compact support contained in Ω.

Equivalently, $\lambda(K)$ is the smallest positive number for which the Dirichlet boundary-value problem

$$\begin{cases} \Delta u = -\lambda(K)\, u & \text{in } \mathrm{int}(K)\,, \\ u = 0 & \text{on } \partial K\,, \end{cases} \tag{4}$$

admits a nontrivial solution $u \in C^2(\mathrm{int}(K)) \cap C(K)$. The solution of this problem is unique up to a multiplicative factor, i.e. the first eigenvalue has multiplicity one; in particular, if we normalize u so that

$$\int_K u^2 \, dx = 1\,,$$

we obtain (integrating by parts)

$$\lambda(K) = \int_K |\nabla u|^2 \, dx\,.$$

It can be immediately seen from its definition that $\lambda(\cdot)$ is homogeneous of order -2:

$$\lambda(tK) = t^{-2}\lambda(K)\,, \quad K \in \mathcal{K}^n\,,\ t > 0\,.$$

The functional λ satisfies a Brunn–Minkowski inequality.

Theorem 1 (*Brascamp and Lieb*). *Let K_0 and K_1 belong to $\mathcal{K}^n$ and $t \in [0,1]$, then the following inequality holds*:

$$\lambda((1-t)K_0 + tK_1)^{-1/2} \geqslant (1-t)\lambda(K_0)^{-1/2} + t\lambda(K_1)^{-1/2}\,. \tag{5}$$

This result is proved in [6]; in fact, in this paper it proved that the inequality holds for all compact, connected domains having sufficiently regular boundary. Another proof is given by Borell in [5]. In Section 5 of the present paper we provide a new proof of Theorem 1 which can be applied only when K_0 and K_1 are convex, but which allows, in this case, to characterize also the equality conditions of (5).

Theorem 2. *Assume that $K_0, K_1 \in \mathcal{K}^n$ and $t \in [0,1]$ are such that equality holds in* (5), *then K_0 is homothetic to K_1.*

The latter result answers a question posed by Jerison in [21], regarding uniqueness of the solution to the Minkowski problem for the functional λ; see Section 4 for more details.

2.2. The Newtonian capacity

The variational definition of the Newtonian capacity is, for $n \geqslant 3$,

$$\mathrm{Cap}\,(K) = \inf\left\{\int_{\mathbf{R}^n} |\nabla u|^2\,dx\,,\ u \in C_c^\infty(\mathbf{R}^n)\ :\ u \geqslant \chi_K\right\}\,, \tag{6}$$

here $C_c^\infty(\mathbf{R}^n)$ denotes the sets of those function from $C^\infty(\mathbf{R}^n)$ having compact support and χ_K is the characteristic function of K. Equivalently, if u is the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbf{R}^n \setminus K \\ u = 1 & \text{on } \partial K\,, \quad \lim_{|x|\to+\infty} u(x) = 0\,, \end{cases} \tag{7}$$

then the capacity of K is given by

$$\mathrm{Cap}\,(K) = \int_{\mathbf{R}^d\setminus K} |\nabla u|^2\,dx\,.$$

$\mathrm{Cap}\,(\cdot)$ is homogeneous of order $n-2$.

Theorem 3 (*Borell*). *Let* K_0 *and* K_1 *belong to* $\mathcal{K}^n$, $n \geqslant 3$, *and* $t \in [0,1]$, *then the following inequality holds*:

$$\mathrm{Cap}\,((1-t)K_0 + tK_1)^{1/(n-2)} \;\geqslant\; (1-t)\mathrm{Cap}\,(K_0)^{1/(n-2)} + t\,\mathrm{Cap}\,(K_1)^{1/(n-2)}\,. \quad (8)$$

A proof of this result is given in [2]; subsequently, Caffarelli et al. in [7] characterized the equality case.

Theorem 4 (*Caffarelli et al.*). *Assume that* $K_0, K_1 \in \mathcal{K}^n$ *and* $t \in [0,1]$ *are such that equality holds in* (8), *then* K_0 *is homothetic to* K_1.

In dimension $n = 2$ the notion of Newtonian capacity is naturally replaced by the one of *logarithmic capacity*. One way to define the logarithmic capacity $\mathrm{Lcap}\,(K)$ of a two-dimensional convex body K is the following. The boundary value problem

$$\begin{cases} \Delta u = 0 \ \text{ in } \mathbf{R}^2 \setminus K \\ u = 0 \quad \text{on } \partial K\,, \quad u(x) \sim \log|x| \quad \text{as } |x| \to +\infty\,, \end{cases} \quad (9)$$

has a unique solution; the second boundary condition means that there exists a constant $a > 0$ such that

$$\frac{1}{a} \leqslant \frac{u(x)}{\log|x|} \leqslant a\,,$$

when $|x|$ is sufficiently large. Moreover, the following limit

$$\rho = \lim_{|x| \to +\infty} (u(x) - \log|x|)$$

exists and it is known as the *Robin constant* of K. The logarithmic capacity is given by the formula

$$\mathrm{Lcap}\,(K) := e^{-\rho}\,.$$

$\mathrm{Lcap}\,(\cdot)$ is homogeneous of order 1.

The logarithmic capacity of a set coincides with its *transfinite diameter*, with its *conformal radius* and with its *Čebišev constant*; for these notions we refer the reader to [16].

Theorem 5 (*Borell*). *Let* K_0 *and* K_1 *belong to* $\mathcal{K}^2$ *and* $t \in [0,1]$, *then the following inequality holds*:

$$\mathrm{Lcap}\,((1-t)K_0 + tK_1) \geqslant (1-t)\mathrm{Lcap}\,(K_0) + t\,\mathrm{Lcap}\,(K_1)\,. \quad (10)$$

This theorem is contained in [3]; the author and Cuoghi in [9], characterized the corresponding equality conditions.

Theorem 6. *Equality occurs in* (10) *if and only if* K_0 *is homothetic to* K_1.

Remark 7. No extension of the Brunn–Minkowski inequality to classes of nonconvex domains is known either for the Newton capacity or for the transfinite diameter.

2.3. The torsional rigidity

We start with the variational definition: the torsional rigidity $\tau(K)$ of $K \in \mathcal{K}^n$ is given by

$$\frac{1}{\tau(K)} = \inf\left\{ \frac{\int_K |\nabla u|^2\,dx}{\left(\int_K |u|\,dx\right)^2}\,,\ u \in W_0^{1,2}(\mathrm{int}(K))\ :\ \int_K |u|\,dx > 0 \right\}.$$

As in the previous cases, this functional can be expressed in terms of the solution of an elliptic boundary-value problem: let u be the unique solution of

$$\begin{cases} \Delta u = -2 & \text{in } \mathrm{int}(K)\,,\\ u = 0 & \text{on } \partial K\,, \end{cases} \tag{11}$$

then

$$\tau(K) = \int_K |\nabla u|^2\,dx\,.$$

The torsional rigidity is homogeneous of order $(n+2)$.

Theorem 8 (*Borell*). *Let* K_0 *and* K_1 *belong to* $\mathcal{K}^n$ *and* $t \in [0,1]$, *then the following inequality holds*

$$\tau((1-t)K_0 + tK_1)^{1/(n+2)} \geqslant (1-t)\tau(K_0)^{1/(n+2)} + t\tau(K_1)^{1/(n+2)}\,. \tag{12}$$

This theorem is proved in [4]; another proof, together with a generalization, is contained in Theorem 11 of this paper which includes also a characterization of equality conditions.

Theorem 9. *Equality occurs in* (12) *if and only if* K_0 *is homothetic to* K_1.

Inequality (12) can be proved also in the class of compact sets with C^2 boundary.

Theorem 10. *Let C_0 and C_1 be compact sets in $\mathbf{R}^n$ with boundary of class C^2 and let $t \in [0, 1]$, then*

$$\tau((1-t)C_0 + tC_1)^{1/(n+2)} \geqslant (1-t)\tau(C_0)^{1/(n+2)} + t\,\tau(C_1)^{1/(n+2)}\,, \tag{13}$$

moreover equality holds if and only if C_0 and C_1 are convex and homothetic.

Inequality (13), without equality conditions, is contained in [4]; for the proof of Theorem 10 see Remark 22 in Section 6 of the present paper.

3. Extensions

The Brunn–Minkowski inequalities (5), (8), (10) and (12) have been extended in various directions in [11,26,9], a new extension is contained in this paper and further results are contained in [10]. In this section, we shall describe some of these results.

Our first step is to identify some common features of the problems which give rise to the functionals that we have seen in the previous section; they will serve as guidelines for more general results. We recall that we restrict our attention to functionals which are positively homogeneous and invariant under rigid motions.

(1) In the boundary value problems (4), (7), (9) and (11), the differential operator is the Laplacian, which in particular is *isotropic* (invariant under rigid motions) and linear; the resulting equation is of *semi-linear* type. Moreover, the space variable x does not appear explicitly neither in the equation, nor in the boundary conditions. These facts make the relevant functional invariant under rigid motions.

(2) The problems are *homogeneous* in the following sense: if u is the solution in K and s is a positive number, the solution v in the rescaled domain sK is given by

$$v(y) = s^q\, u\left(\frac{y}{s}\right)\,, \quad y \in s\,K\,,$$

for a suitable q; this makes the corresponding functional positively homogeneous.

(3) In all problems the functional $\mathbf{F}$ coincides with the *energy integral* of the solution, i.e.

$$\mathbf{F}(K) = \int_\Omega |\nabla u|^2\, dx\,,$$

where Ω is the interior or the complement of K.

We shall see three types of extensions of Brunn–Minkowski inequalities for variational functionals. The distinction is made accordingly to the second-order differential operator which appears in the boundary-value problem.

3.1. The linear case

If we restrict ourselves to the case of functionals coming from problems where the second-order differential operator is linear, then invariance under rigid motions implies that the operator must be the Laplacian.

We consider the following situation: for $K \in \mathcal{K}^n$, $p \geqslant 0$ and $c \in \mathbf{R}$, we pose the boundary-value problem

$$\begin{cases} \Delta u = c\, u^p, \quad u \geqslant 0 \quad \text{in } \operatorname{int}(K)\,, \\ u = 0 \quad \text{on } \partial K\,. \end{cases} \tag{14}$$

The question is whether the energy integral of the solution satisfies a Brunn–Minkowski inequality. For $c \geqslant 0$ the above problem admits only the trivial solution $u \equiv 0$, this is a simple consequence of the maximum principle; then we take $c < 0$. Except for the case $p = 1$, that we will consider later, we may always reduce to the case $c = -1$ by multiplying the solution for a suitable constant. So we are dealing with

$$\begin{cases} \Delta u = -u^p, \quad u > 0 \quad \text{in } \operatorname{int}(K)\,, \\ u = 0 \quad \text{on } \partial K\,. \end{cases} \tag{15}$$

For $p = 0$ we have the problem that gives rise to the torsional rigidity. For $p \in (0, 1)$ problem (15) is well posed, i.e. we have existence and uniqueness of the solution in $C^2(\operatorname{int}(K)) \cap C(K)$; this fact will be proved in Section 6. The energy integral of the solution

$$\mathbf{F}(K) = \int_K |\nabla u|^2\, dx\,,$$

is homogeneous of order $\alpha_p = n + \frac{2+p}{1-p}$. In Section 6 we prove the following

Theorem 11. *The functional* $\mathbf{F}$ *satisfies a Brunn–Minkowski inequality*:

$$\mathbf{F}((1-t)K_0 + tK_1)^{1/\alpha_p} \geqslant (1-t)\mathbf{F}(K_0)^{1/\alpha_p} + t\mathbf{F}(K_1)^{1/\alpha_p} \tag{16}$$

for all $K_0, K_1 \in \mathcal{K}^n$ *and* $t \in [0, 1]$. *Moreover, equality holds if and only if* K_0 *is homothetic to* K_1.

The same remark as for the functional $\tau(\cdot)$ applies: the inequality can be proved in the class of compact sets with boundary of class C^2 and equality holds only for *convex* homothetic sets.

For $p = 1$ we have existence of at least one nontrivial solution of problem (14) provided $c = -\lambda_k(K)$ for some $k \in \mathbf{N}$, where $\lambda_k(K)$, $k = 1, 2, \ldots$, are the eigenvalues of $-\Delta$ in K; in this case, if $k \geqslant 2$, we have to drop the requirement $u \geqslant 0$ in $\operatorname{int}(K)$.

For $k = 1$ the energy integral is $\lambda_1(K) = \lambda(K)$ and it satisfies the Brunn–Minkowski inequality (5). For $k > 1$, if u is any solution, normalized so that

$$\int_K u^2 \, dx = 1 \, ,$$

its energy integral coincides with $\lambda_k(K)$, so that the question is whether the functionals $\lambda_k(\cdot)$, $k = 2, 3, \ldots$, satisfy a Brunn–Minkowski inequality (note that the order of homogeneity of all these functionals is -2). This is an open problem; the available proofs of inequality (5) (see [6,5] and Section 5 of this paper) do not seem to be adaptable to the other eigenvalues.

For $1 < p < \frac{n+2}{n-2}$ (and $c < 0$), existence of at least one solution to problem (14) continues to hold while uniqueness is not guaranteed; nevertheless a variational definition of $\mathbf{F}$ could still be given, coherently with the case $p < 1$ (see the proof of Proposition 19 in Section 6). Anyway our proof of the Brunn–Minkowski inequality for $\mathbf{F}$ does not extend to this case. Finally for $p \geqslant \frac{n+2}{n-2}$, problem (14) admits no positive solution (see, for instance, [23,24]).

3.2. Quasi-linear operators

In [11] Salani and the author proved that the p-capacity of convex bodies satisfies a Brunn–Minkowski inequality. For an arbitrary compact set A in $\mathbf{R}^n$ and for $p \in [1, n)$, the p-capacity is defined in a similar way as for $p = 2$:

$$\mathrm{Cap}_p\,(A) = \inf \left\{ \int_{\mathbf{R}^n} |\nabla u|^p \, dx \, , \; u \in C_c^\infty(\mathbf{R}^n) \, : \, u \geqslant \chi_A \right\} \tag{17}$$

(where χ_A is the characteristic function of A). When $A = K$ is a convex body (but also under much less restrictive assumptions on K), an equivalent definition can be given, based as usual on a boundary value problem; indeed

$$\mathrm{Cap}_p\,(K) = \int_{\mathbf{R}^n \setminus K} |\nabla u|^p \, dx, \tag{18}$$

where u is the unique solution of

$$\begin{cases} \mathrm{div}\left(|\nabla u|^{p-2} \nabla u\right) = 0 \quad \text{in } \mathbf{R}^n \setminus K \, , \\ u = 1 \quad \text{on } \partial K \, , \quad \lim_{|x| \to +\infty} u(x) = 0 \, . \end{cases} \tag{19}$$

The second-order differential operator involved in the above problem is called the p-Laplacian and the corresponding equation is quasi-linear. Clearly, for $p = 2$, $n > 2$, we get the Newtonian capacity. $\mathrm{Cap}_p\,(\cdot)$ is homogeneous of order $(n - p)$.

Theorem 12. *The p-capacity satisfies the following Brunn–Minkowski inequality*:

$$\begin{aligned}&\mathrm{Cap}_p\,((1-t)K_0+tK_1)^{1/(n-p)} \geqslant (1-t)\mathrm{Cap}_p\,(K_0)^{1/(n-p)} + t\,\mathrm{Cap}_p\,(K_1)^{1/(n-p)}\\ &\quad \forall K_0,\ \ K_1 \in \mathcal{K}^n\,,\ \ t \in [0,1]\,;\end{aligned} \tag{20}$$

moreover, equality holds if and only if K_0 and K_1 are homothetic.

For the proof see [11].

The limit case $p = n$, which has been treated in [9], is similar to the case $p = n = 2$ that we have described in Section 2.2. Firstly, the notion of n-dimensional logarithmic capacity is defined for a convex body in $\mathcal{K}^n$; the definition is completely analogous to the one valid in the two-dimensional case. This quantity turns out to be positively homogeneous of order one, it satisfies a Brunn–Minkowski inequality and the equality case is characterized as usual.

The author together with Cuoghi and Salani (see [10]) studied Brunn–Minkowski type inequalities for the functionals analogous to the first eigenvalue of the Laplacian and the torsional rigidity, when the Laplace operator is replaced by the p-Laplacian.

3.3. Fully non-linear operators

Recently, Salani (see [26]) proved a Brunn–Minkowski type inequality for the eigenvalue of the Monge–Ampère operator. This quantity, that we shall denote by $\Lambda(\cdot)$, like all the ones that we have seen until now, admits either a variational definition or a definition based on a boundary value problem. A difference with respect to the previous examples is that $\Lambda(K)$ can be defined only for those convex bodies having boundary of class C^2, with strictly positive Gauss curvature at each point of the boundary; we will denote this class of sets by $\mathcal{K}^n_r$.

We have, for $K \in \mathcal{K}^n_r$,

$$\Lambda(K) = \inf\left\{-\frac{\int_K u\det(D^2u)\,dx}{\int_K |u|^{n+1}\,dx}\right\}$$

where the infimum is taken over the functions $u \in C^2(\mathrm{int}(K)) \cap C(K)$, convex and such that $u = 0$ on ∂K.

Equivalently, $\Lambda(K)$ is the unique (positive) number such that the problem

$$\begin{cases}\det(D^2u) = \Lambda(K)\,(-u)^n, & u < 0 \quad \text{in } K\,,\\ u = 0 \quad \text{on } \partial K\end{cases} \tag{21}$$

admits a (convex) solution. The existence of a number $\Lambda(K)$ such that (21) can be solved, was proved by Lions in [22]. The equivalence between the two definitions is due to Tso, see [28]. The Monge–Ampère operator belongs to the class of *fully non-linear* elliptic operators; it is, of course, invariant under translations and rotations. The eigenvalue Λ is homogeneous of order $-2n$. The following result is proved in [26].

Theorem 13 (*Salani*). *The functional Λ satisfies the inequality*:

$$\Lambda((1-t)K_0 + tK_1)^{-1/2n} \geqslant (1-t)\Lambda(K_0)^{-1/2n} + t\Lambda(K_1)^{-1/2n}\,,$$
$$K_0\,,\ K_1 \in \mathcal{K}^n_r\,,\ t \in [0,1]\,.$$

Moreover equality holds if and only if K_0 is homothetic to K_1.

It would be interesting to see whether the same kind of result holds for another class of fully non-linear elliptic operators, i.e. the so-called Hessian operators. For $k \in \{1, 2, \dots n\}$, the kth Hessian operator $S_k(D^2u)$ applied to a C^2 function u is the kth elementary symmetric function of the eigenvalues of the Hessian matrix of u; note that this class includes the Laplace operator (corresponding to $k = 1$) and the Monge–Ampère operator ($k = n$). Wang (see [29]) proved that for $k > 1$, like in the case of the Monge–Ampère operator, S_k admits exactly one positive eigenvalue in a convex domain with smooth boundary (in fact, the result by Wang is true for a larger class of domains); does this eigenvalue satisfy a Brunn–Minkowski type inequality?

4. Minkowski type problems

4.1. The Minkowski problem for the volume

The area measure σ_K of a convex body K in $\mathbf{R}^n$ is a nonnegative Borel measure σ_K defined on the unit sphere S^{n-1}, characterized by the following property: for a Borel set $\omega \subset S^{n-1}$, $\sigma_K(\omega)$ is the $(n-1)$-dimensional measure of the set

$$\{x \in \partial K \ :\ v(x) \ \subset\ \omega\}\,,$$

where $v(x)$ is the set of outer unit normal vectors to ∂K at x (see [27, Chapter 4]).

Minkowski problem. *Given a nonnegative Borel measure σ on S^{n-1}, find a convex body K whose area measure is σ.*

What properties must σ have so that this problem can be solved? A consequence of the invariance of the volume under translations is the following property of area measures:

$$\int_{S^{n-1}} X\, d\sigma_K(X) = 0, \quad \forall K \in \mathcal{K}^n\,. \tag{22}$$

Moreover, as a convex body has non-empty interior, from the definition of area measure it is clear that its support cannot be contained in a great sub-sphere of S^{n-1}. These two properties are sufficient to characterize area measures.

Theorem 14. *Let σ be a nonnegative Borel measure on S^{n-1}, such that its support is not contained in any great sub-sphere and*

$$\int_{S^{n-1}} X \, d\sigma(X) = 0 \, .$$

Then there exists $K \in \mathcal{K}^n$ such that $\sigma_K = \sigma$. Moreover K is uniquely determined up to a translation.

For a proof, see for instance [27, Chapter 7]. We want to describe a connection between the Minkowski problem and the Brunn–Minkowski inequality. We begin with the following simple formula relating the volume of K to its area measure:

$$V(K) = \frac{1}{n} \int_{S^{n-1}} h_K(X) \, d\sigma_K(X) \, , \tag{23}$$

where $h_K : S^{n-1} \to \mathbf{R}$, is the support function of K:

$$h_K(X) = \sup_{x \in K} (X, x), \quad X \in S^{n-1} \, .$$

The validity of (23) is rather intuitive (especially for polyhedra); for a proof, see Chapter 4 in [27]. There is another formula which, roughly speaking, expresses the first variation of the volume of K, when K is perturbed by another convex body L:

$$\frac{d}{dt} V(K + tL)|_{t=0} = \int_{S^{n-1}} h_L(X) \, d\sigma_K(X), \quad \forall L \in \mathcal{K}^n \tag{24}$$

(note that (23) follows from (24), and the homogeneity of the volume, when we choose $K = L$). Formula (24) follows immediately from the well-known expansion of $V(K + tL)$ as a polynomial of t whose coefficients are the *mixed volumes* of K and L; we refer again to [27] for the details. A consequence of (24) is equality (22), which is obtained letting L be the set formed by a single point x and then letting x vary in $\mathbf{R}^n$.

From the Brunn–Minkowski inequality we have that, for $K, L \in \mathcal{K}^n$,

$$V(K + tL)^{1/n} \geqslant V(K)^{1/n} + tV(L)^{1/n} \, .$$

The two terms of this inequality, as functions of $t > 0$, coincide when $t = 0$, then

$$\frac{d}{dt} V(K + tL)^{1/n}|_{t=0} \geqslant \frac{d}{dt} [V(K)^{1/n} + tV(L)^{1/n}]|_{t=0} = V(L)^{1/n} \, .$$

Using (24), we obtain

$$\int_{S^{n-1}} h_L(X)\, d\sigma_K(X) \geqslant nV(L)^{1/n} V(K)^{(n-1)/n}\,, \quad \forall L \in \mathcal{K}^n,$$

which becomes an equality when $K = L$ and more generally when $K = sL$ for any $s > 0$. This fact can be rephrased in the following way:

Let K be fixed in $\mathcal{K}^n$ and let $L \in \mathcal{K}^n$ be such that $V(L) \geqslant 1$; then the quantity

$$\int_{S^{n-1}} h_L(X)\, d\sigma_K(X)$$

attains its minimum when

$$L = \frac{1}{V(K)^{1/n}} K\,.$$

This fact suggests an argument to solve the Minkowski problem (the existence part): given a nonnegative Borel measure σ on S^{n-1}, satisfying the assumptions of Theorem 14, consider the variational problem

$$\inf\left\{\int_{S^{n-1}} h_L(X)\, d\sigma(X),\ \ L \in \mathcal{K}^n\,, V(L) \geqslant 1\right\}\,.$$

By the previous considerations, any solution is a good candidate to solve the Minkowski problem for σ. Indeed, this approach can be successfully applied. The original proof of Minkowski uses this argument in the special case of polyhedra (in this case σ is the sum of point masses on S^{n-1}, see for instance, [1]), but the same can be done in the general case, as shown in [21] (see also the historical note at the end of Section 7.1 in [27]).

Regarding the uniqueness part of Theorem 14, once again this depends on the Brunn–Minkowski inequality and in particular on the characterization of the equality conditions. The argument is quite standard, we describe it here since it will be used in the proof of another uniqueness result presented in the sequel of this section. Assume that there exist two convex bodies K and L such that

$$\sigma = \sigma_K = \sigma_L\,.$$

Consider the function

$$m(s) = [V(sK + (1-s)L)]^{1/n}\,, \quad s \in [0, 1]\,.$$

By the Brunn–Minkowski inequality, $m(s)$ is concave in $[0,1]$; its derivative at $s=0$ is given by

$$\begin{aligned} m'(0) &= \frac{1}{n}\,[V(L)]^{-\frac{n-1}{n}}\,\frac{d}{ds}V(sK+(1-s)L)|_{s=0} \\ &= \frac{1}{n}\,[V(L)]^{-\frac{n-1}{n}}\int_{S^{n-1}}(h_K(\xi)-h_L(\xi))\,d\sigma(\xi) \\ &= [V(L)]^{-\frac{n-1}{n}}\,[V(K)-V(L)] \\ &= [m(0)]^{1-n}\left(m(1)^n-m(0)^n\right). \end{aligned}$$

Since $m(s)$ is concave

$$m'(0) \geqslant m(1)-m(0)\,.$$

This fact, together with the above inequality, gives $m(1)^{n-1} \geqslant m(0)^{n-1}$, i.e. $V(K) \geqslant V(L)$. Interchanging the roles of K and L we conclude that $V(L)=V(K)$. This implies at once $m(0)=m(1)$ and $m'(0)=0$, so that m must be constant in $[0,1]$ and consequently K and L render the Brunn–Minkowski inequality an equality. Then K coincides with L up to a translation (since $\sigma_K=\sigma_L$, no dilatation can occur).

We might conclude that formulas (23) and (24) (together with the Brunn–Minkowski inequality) are the starting point for a variational solution of existence part of the Minkowski problem, while the uniqueness part can be deduced from characterization of equality cases in the Brunn–Minkowski inequality.

Jerison observed (see [20,21]) that if we replace the volume by either the Newton capacity or the first eigenvalue of the Laplace operator, we find ourselves in a similar situation.

4.2. The Minkowski problem for the Newton capacity

In the paper [20], a Minkowski type problem for the Newton capacity is solved. The starting point is a formula similar to (23). Let $K \in \mathcal{K}^n$ $(n>2)$, and let u_K be the solution of problem (7), then $|\nabla u|^2$ is defined almost everywhere on ∂K, with respect to the $(n-1)$-dimensional Hausdorff measure so that one can define the measure σ_K^{Cap} through the formula

$$\sigma_K^{\mathrm{Cap}}(\omega) := \int_{g_K^{-1}(\omega)} |\nabla u(x)|^2\,d\mathcal{H}^{n-1}(x)$$

for every Borel subset ω of S^{n-1}. Here $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure and $g_K : \partial K \to S^{n-1}$ is the Gauss map of K i.e., for $x \in \partial K$, $g_K(x)$ is the set of outer unit normal vectors to ∂K at x. In particular, when the boundary of K is of class C^2 with positive Gaussian curvature at every point, we can write

$$d\sigma_K^{\mathrm{Cap}}(X) := |\nabla u(g_K^{-1}(X))|^2 \, d\sigma_K(X),$$

where σ_K is the area measure of K introduced in the previous section. The relevant formula is

$$\mathrm{Cap}\,(K) = \frac{1}{n-2} \int_{S^{n-1}} h_K(X) \, d\sigma_K^{\mathrm{Cap}}(X). \tag{25}$$

In the case of convex bodies with sufficiently smooth boundary, this formula comes from a clever use of the divergence theorem and the conditions contained in (7). A further step is to prove that for an arbitrary convex body L

$$\frac{d}{dt}\mathrm{Cap}\,(K+tL)|_{t=0} = \int_{S^{n-1}} h_L(X) \, d\sigma_K^{\mathrm{Cap}}(X), \tag{26}$$

in this case the proof is much more delicate. Comparing (25) and (26) with (23) and (24), it becomes clear that when we replace the volume with the capacity, correspondingly the measure σ_K^{Cap} have to play the role of area measure. The following result is the counterpart of Theorem 14.

Theorem 15 (*Minkowski problem for capacity*). *Let σ be a nonnegative Borel measure on S^{n-1}, such that its support is not contained in any great sub-sphere and*

$$\int_{S^{n-1}} X \, d\sigma(X) = 0\,.$$

Then there exists $K \in \mathcal{K}^n$ such that $\sigma_K^{\mathrm{Cap}} = \sigma$. Moreover K is uniquely determined up to a translation.

The existence part of this theorem is proved in [20]; in [21] another proof is given, which makes use of the variational method that we described before in the case of the volume. The uniqueness part follows from the characterization of equality cases in the Brunn–Minkowski inequality for capacity proved in [7], i.e. Theorem 4 in Section 2.2 of the present paper.

4.3. The Minkowski problem for λ

In [21], the same problem has been studied for the first eigenvalue of the Laplacian. For a convex body K, let u_K be the solution of problem (4) normalized so that

$$\int_K |u_K|^2\,dx = 1$$

and define the measure σ_K^λ on S^{n-1} through the formula

$$\sigma_K^\lambda(\omega) := \int_{g_K^{-1}(\omega)} |\nabla u(x)|^2\,d\mathcal{H}^{n-1}(x)$$

for every Borel subset ω of S^{n-1}. Then we have (see [21])

$$\lambda(K) = \frac{1}{2}\int_{S^{n-1}} h_K(X)\,d\sigma_K^\lambda(X) \tag{27}$$

and

$$\frac{d}{dt}\lambda(K+tL)|_{t=0} = -\int_{S^{n-1}} h_L(X)\,d\sigma_K^\lambda(X)\,, \quad \forall L \in \mathcal{K}^n \tag{28}$$

(note that the order of homogeneity of λ is negative). Using these formulas and the variational approach, the following result can be shown (see Theorem 7.5 in [21])

Theorem 16 (*Jerison*). *Let σ be a nonnegative Borel measure on S^{n-1}, such that its support is not contained in any great sub-sphere and*

$$\int_{S^{n-1}} X\,d\sigma(X) = 0\,.$$

Then there exists $K \in \mathcal{K}^n$ such that $\sigma_K^\lambda = \sigma$.

A consequence of Theorem 2 is the following uniqueness result.

Theorem 17. *Let $K, L \in \mathcal{K}^n$ be such that*

$$\sigma_K^\lambda = \sigma_L^\lambda\,,$$

then K and L coincide up to a translation.

Proof. The argument is exactly the same as in the uniqueness proof for the Minkowski problem for the volume, Section 4.1, with

$$m(s) = \left[\lambda(sK + (1-s)L)\right]^{-1/2}, \quad s \in [0,1].$$

Clearly, Theorems 1 and 2 have to be used, instead of the classic Brunn–Minkowski inequality and corresponding equality conditions.

4.4. The case of torsional rigidity

The aim of this section is to present a couple of formulas regarding the torsional rigidity of convex bodies *with smooth boundary*, corresponding to (23) and (24), which indicates the feasibility of a Minkowski type problem for the functional τ.

Proposition 18. *Let K and L be convex bodies with boundary of class C^2 such that the Gauss curvature is positive at every point of their boundary*; *let u_K be the solution of problem* (11) *in K. The following formulas hold*:

$$\tau(K) = \frac{1}{n+2} \int_{S^{n-1}} h_K(X) |\nabla u(g_K^{-1}(X))|^2 \, d\sigma_K(X), \tag{29}$$

$$\frac{d}{dt} \tau(K + tL)|_{t=0} = \int_{S^{n-1}} h_L(X) |\nabla u(g_K^{-1}(X))|^2 \, d\sigma_K(X). \tag{30}$$

Proof. We start proving (29). By the divergence theorem

$$\begin{aligned} \tau(K) &= \int_K |\nabla u(x)|^2 \, dx = \int_K [\mathrm{div}(u(x)\nabla u(x)) - u(x)\Delta u(x)] dx \\ &= 2 \int_K u(x) \, dx. \end{aligned} \tag{31}$$

As the boundary of K is C^2, $u \in C^2(K)$, this follows from standard regularity results for solutions of elliptic equations (see for instance Theorem 6.14 in [15]). Moreover by the Hopf Lemma, ∇u does not vanish on ∂K. As $u > 0$ in $\mathrm{int}(K)$ (by the maximum principle), for every $x \in \partial K$ we have

$$\frac{\nabla u(x)}{|\nabla u(x)|} = -\nu(x),$$

where ν is the outer unit normal to ∂K. Hence the support function of K can be written in the following way:

$$h_K(X) = -\frac{1}{|\nabla u(g_K^{-1}(X))|}\left(g_K^{-1}(X)), \nabla u(g_K^{-1}(X))\right), \quad X \in S^{n-1}$$

(we recall that g_K^{-1} is the inverse of the Gauss map of K). We define

$$w(x) = (x, \nabla u(x)), \quad x \in K .$$

We have

$$\begin{aligned}
\int_{S^{n-1}} h_K(X)|\nabla u(g_K^{-1}(X))|^2\, d\sigma_K(X) &= \int_{\partial K} h_k(\nu(x))|\nabla u(x)|^2\, d\mathcal{H}^{n-1}(x)\\
&= -\int_{\partial K} w(x)|\nabla u(x)|\, d\mathcal{H}^{n-1}(x)\\
&= \int_{\partial K} w(x)(\nabla u(x), \nu(x))\, d\mathcal{H}^{n-1}(x)\\
&= \int_K \operatorname{div}(w(x)\nabla u(x))\, dx\\
&= \int_K [(\nabla w(x), \nabla u(x)) + w(x)\Delta u(x)]\, dx\\
&= -\int_K [u(x)\Delta w(x) + 2w(x)]\, dx ,
\end{aligned}$$

where we have used the divergence theorem, the equation and the boundary condition of problem (11). Now

$$\Delta w(x) = 2\Delta u(x) + (x, \nabla(\Delta u(x))) = -4$$

and

$$\int_K w(x)\, dx = \int_K (x, \nabla u(x))\, dx = -n \int_K u(x)\, dx$$

again by the divergence theorem. Consequently,

$$\int_{S^{n-1}} h_K(X)|\nabla u(g_K^{-1}(X))|^2\, d\sigma_K(X) = 2(n+2)\int_K u(x)\, dx = (n+2)\tau(K) ,$$

where we have used (31). Thus (29) is proved. Formula (30) can be proved with the help of (29), applying the same argument used in the proof of formula (a) in Proposition 2.10 in [20]; for brevity we omit the proof. $\square$

Starting from the last proposition, the strategy to solve a Minkowski type problem for τ should be the same described in the previous sections:

(1) Extend formulas (29) and (30) to all convex bodies;
(2) apply the variational method proposed in [21] to prove the existence of a solution;
(3) establish uniqueness of the solution using the characterization of equality conditions in the Brunn–Minkowski inequality for τ, i.e. Theorem 9 in this paper.

5. Proof of Theorems 1 and 2

In this section, we give a new proof of the Brunn–Minkowski inequality for the first eigenvalue of the Laplace operator, in the class of convex bodies, which allows to determine equality conditions. Let $K_0, K_1 \in \mathcal{K}^n$ and $t \in [0,1]$. For $i = 0, 1$, let u_i be a solution of

$$\begin{cases} \Delta u_i = -\lambda(K_i) u_i\,, \quad u_i > 0 \quad \text{in } \mathrm{int}(K_i)\,, \\ \\ u_i = 0 \quad \text{on } \partial K_i \end{cases}$$

and consider the function

$$v_i(x) = -\log u_i(x)\,, \quad x \in \mathrm{int}(K_i)\,,$$

v_i solves

$$\begin{cases} \Delta v_i = \lambda(K_i) + |\nabla v_i|^2 \quad \text{in } \mathrm{int}(K_i)\,, \\ \lim_{x \to \partial K_i} v_i(x) = +\infty\,. \end{cases} \tag{32}$$

The functions v_0 and v_1 are convex (equivalently, u_0 and u_1 are log-concave); this fact is proved in [6] and a different proof can be found in [8]. Moreover, it follows from (32) and Remark 1 in Section 5 of [19], that the rank of the Hessian matrix $D^2 v_i$ is maximum, i.e. equal to n, all over $\mathrm{int}(K)$, so that

$$\det(D^2 v_i(x)) > 0, \quad \forall x \in \mathrm{int}(K_i)\,. \tag{33}$$

In particular, this implies that v_i is strictly convex. Note that, by the boundary condition verified by v_i, we have that the gradient of v_i maps K_i onto $\mathbf{R}^n$:

$$\nabla v_i(\mathrm{int}(K_i)) = \mathbf{R}^n\,. \tag{34}$$

We will need to consider the conjugate function v_i^* of v_i:

$$v_i^*(\rho) = \sup_{x \in K_i} [(x, \rho) - v_i(x)], \quad \rho \in \mathbf{R}^n .$$

We refer to [25] for the basic properties of this function; v_i^* is defined on the image of K_i through the gradient map of v_i, which is, by (34), the whole $\mathbf{R}^n$; moreover v_i is convex. As v_i is strictly convex, $v_i \in C^1(\mathbf{R}^n)$, and ∇v_i^* is the inverse map of ∇v_i:

$$x = \nabla v_i^*(\nabla v_i(x)), \quad \forall x \in K_i .$$

In particular this identity and (33) imply that $v_i^* \in C^2(\mathbf{R}^n)$ and

$$D^2 v_i(x) = [D^2 v_i^*(\nabla v_i(x))]^{-1}, \quad \forall x \in K_i . \tag{35}$$

We construct a new function defined in K_t:

$$w(z) = \inf\{(1-t)v_0(x) + t v_1(y) \,:\, x \in K_0,\ y \in K_1,\ (1-t)x + ty = z\}, \quad z \in K_t . \tag{36}$$

The function w is called the *infimal convolution* of v_0 and v_1 (see [25]). It is a convex function and, from the boundary conditions in problem (32) it can be deduced that

$$\lim_{z \to \partial K_t} w(z) = +\infty . \tag{37}$$

Moreover, w verifies the following identity (see Theorem 16.4 in [25])

$$w^* = (1-t)v_0^* + t v_1^* \quad \text{in } \mathbf{R}^n . \tag{38}$$

Now, (33), (35) and (38) implies that w^* is $C^2(\mathbf{R}^n)$, is strictly convex and

$$D^2 w^* > 0 \quad \text{in } \mathbf{R}^n .$$

Consequently, $w \in C^2(\text{int}(K_t))$. Let us fix $z \in K_t$. By the definition of w and since, for $i = 0, 1$, v_i tends to $+\infty$ at the boundary of K_i, there exist $x \in \text{int}(K_0)$ and $y \in \text{int}(K_1)$ such that $z = (1-t)x + ty$ and

$$w(z) = (1-t)v_0(x) + t v_1(y) . \tag{39}$$

By the Lagrange multipliers Theorem one deduces immediately that

$$\nabla v_0(x) = \nabla v_1(y) = \rho ; \tag{40}$$

but then

$$\nabla w^*(\rho) = (1-t)\nabla v_0(\rho) + t\nabla v_1(\rho) = (1-t)x + ty = z = \nabla w^*(\nabla w(z)) \tag{41}$$

and by the injectivity of ∇w we have

$$\nabla w(z) = \rho \,.$$

Therefore,

$$\begin{aligned} D^2 w(z) &= [D^2 w^*(\rho)]^{-1} = [(1-t)D^2 v_0^*(\rho) + tD^2 v_1^*(\rho)]^{-1} \\ &= [(1-t)(D^2 v_0(x))^{-1} + t(D^2 v_1(y))^{-1}]^{-1} \,. \end{aligned} \tag{42}$$

Now we use the convexity of the application

$$M \longrightarrow \mathrm{trace}\,(M^{-1})$$

in the family of positive definite matrices M (see, for instance, Lemma 4.2 in [11]), to infer

$$\Delta w(z) \leqslant (1-t)\Delta v_0(x) + t\Delta v_1(y) \tag{43}$$

and consequently

$$\Delta w(z) \leqslant (1-t)\lambda(K_0) + t\lambda(K_1) + |\nabla w(z)|^2 \,, \quad \forall z \in \mathrm{int}(K_t) \,. \tag{44}$$

The function

$$\overline{u}(z) := e^{-w(z)}, \quad z \in K_t \,,$$

has the following properties:

$$\begin{cases} \Delta \overline{u} \geqslant -[(1-t)\lambda(K_0) + t\lambda(K_1)]\,\overline{u} \,, \ \overline{u} > 0 \quad \text{in } \mathrm{int}(K_t) \,, \\ \overline{u} = 0 \quad \text{on } \partial K_t \,. \end{cases}$$

We multiply both sides of the differential inequality by $\overline{u}$ and we integrate by parts, taking the boundary condition into account; in this way we get

$$(1-t)\lambda(K_0) + t\lambda(K_1) \geqslant \frac{\int_{K_t} |\nabla \overline{u}|^2 \, dx}{\int_{K_t} |\overline{u}|^2 \, dx} \geqslant \lambda(K_t) \,, \tag{45}$$

where the last inequality follows from the definition of first eigenvalue of the Laplacian. We have proved that $\lambda(\cdot)$ is convex in $\mathcal{K}^n$ and as a further consequence of this property we obtain

$$\lambda((1-t)K_0 + tK_1) \leqslant \max\{\lambda(K_0), \lambda(K_1)\}, \quad \forall K_0, K_1 \in \mathcal{K}^n, \quad t \in [0,1]. \tag{46}$$

In order to deduce the Brunn–Minkowski inequality from (46), we use a standard argument: for arbitrary K_0 and K_1 in $\mathcal{K}^n$ and $t \in [0,1]$, let

$$\begin{aligned} K_0' &= [\lambda(K_0)]^{1/2} K_0, \quad K_1' = [\lambda(K_1)]^{1/2} K_1, \\ t' &= \frac{t[\lambda(K_1)]^{-1/2}}{(1-t)[\lambda(K_0)]^{-1/2} + t[\lambda(K_1)]^{-1/2}} \end{aligned} \tag{47}$$

and apply (46) to K_0', K_1' and t'. The proof of Theorem 1 is complete.

Assume now that K_0, K_1 and t are such that there is equality in (5); let K_0', K_1' and t' be as in (47) and

$$K_{t'} = (1-t')K_0' + t'K_1'.$$

Then clearly

$$\lambda(K_{t'}) = \lambda(K_0') = \lambda(K_1') = 1.$$

Hence we may reduce ourselves to the case in which the bodies K_0, K_1 and K_t have the same eigenvalue and this is equal to 1. Repeating the construction made in the first part of the proof, we obtain from (45)

$$1 = (1-t)\lambda(K_0) + t\lambda(K_1) \geqslant \frac{\int_{K_t} |\nabla \overline{u}|^2 \, dx}{\int_{K_t} |\overline{u}|^2 \, dx} \geqslant \lambda(K_t) = 1,$$

so that all the inequalities have to be equalities. In particular this implies that $\overline{u}$ must be an eigenfunction corresponding to $\lambda(K_t)$. Then

$$\Delta \overline{u} = -\overline{u} \quad \text{in } K_t \Rightarrow \Delta w = 1 + |\nabla w|^2 \quad \text{in } K_t$$

i.e. equality holds in (44), but the latter is a consequence of the previous inequality (43), hence

$$\begin{aligned} &\operatorname{trace}\left[(1-t)(D^2 v_0(x))^{-1} + t(D^2 v_1(y))^{-1}\right]^{-1} \\ &\quad = (1-t)\operatorname{trace}(D^2 v_0(x)) + t\operatorname{trace}(D^2 v_1(y)). \end{aligned}$$

In this situation we can apply again Lemma 4.2 in [11] (the equality case) and conclude that

$$D^2 v_0(x) = D^2 v_1(y) \Rightarrow D^2 v_0^*(\rho) = D^2 v_1^*(\rho) \quad \forall \rho \in \mathbf{R}^n .$$

A further consequence is that

$$\nabla v_0^*(\rho) = \nabla v_1^*(\rho) + \overline{\rho} \quad \forall \rho \in \mathbf{R}^n$$

for some fixed vector $\overline{\rho}$. Finally

$$K_0 = \nabla v_0^*(\mathbf{R}^n) = \nabla v_1^*(\mathbf{R}^n) + \overline{\rho} = K_1 + \overline{\rho} .$$

6. Proof of Theorem 11

Let $K \in \mathcal{K}^n$; throughout this section, $p \in [0, 1)$ is fixed. We consider the boundary value problem

$$\begin{cases} \Delta u = -u^p , \; u > 0 \quad \text{in } \mathrm{int}(K) , \\ \\ u = 0 \quad \text{on } \partial K \end{cases} \tag{48}$$

and we denote the energy integral of its solution by

$$\mathbf{F}(K) = \int_K |\nabla u|^2 \, dx .$$

Our final goal is to prove that $\mathbf{F}$ satisfies a Brunn–Minkowski inequality. Our first issue is an existence an uniqueness result for problem (48).

Proposition 19. *There exists a unique solution* $u \in C^2(\mathrm{int}(K)) \cap C(K)$ *of problem* (48).

Proof. For simplicity, throughout the proof Ω will denote the interior of K. Consider the functional

$$\mathfrak{F}(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \frac{1}{p+1} \int_\Omega |v|^{p+1} \, dx , \quad v \in W_0^{1,2}(\Omega) . \tag{49}$$

We shall find our solution u as a minimizer of $\mathfrak{F}$. By the Sobolev inequality (see for instance (7.26) in [15]),

$$\left(\int_\Omega |v|^{p+1}\,dx\right)^{\frac{1}{p+1}} \leqslant C(n,p)\left(\int_\Omega |\nabla v|^q\,dx\right)^{\frac{1}{q}}, \quad q = \frac{n(p+1)}{n+p+1}.$$

Note that, as $p<1$ we have $q<2$ so that, by the Hölder inequality,

$$\int_\Omega |v|^{p+1}\,dx \leqslant C(n,p,\Omega)\left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{1+p}{2}}.$$

Hence

$$\mathfrak{F}(v) \geqslant \frac{t}{2} + Ct^a, \tag{50}$$

where $t = \int_\Omega |\nabla v|^2\,dx$ and $a = \frac{p+1}{2} \in [0,1)$, so that $\mathfrak{F}$ is bounded from below. We set

$$m = \inf\{\mathfrak{F}(v) : v \in W_0^{1,2}(\Omega)\}.$$

Note that $m<0$; indeed, an easy computation shows that when K is a ball m is strictly negative. On the other hand, for an arbitrary K,

$$m \leqslant \inf\{\mathfrak{F}(v) : v \in W_0^{1,2}(\Omega')\} < 0,$$

where Ω' is any open ball contained in Ω.

Let $v_j \in W_0^{1,2}(\Omega)$, $j \in \mathbf{N}$, be a minimizing sequence for $\mathfrak{F}$:

$$\lim_{j\to+\infty} \mathfrak{F}(v_j) = m.$$

From (50) we deduce that the sequence

$$\int_\Omega |\nabla v_j|^2\,dx$$

is bounded and from Poincaré inequality (see [15, (7.44)]):

$$\int_\Omega v_j^2\,dx \leqslant C(n,p,\Omega)\int_\Omega |\nabla v_j|^2\,dx,$$

consequently, the sequence v_j is bounded in $W_0^{1,2}(\Omega)$ and then, up to a subsequence, it converges weakly to a function $u \in W_0^{1,2}(\Omega)$. Now we apply a standard semi-continuity result in the Calculus of Variations (see Theorem 4.1, Chapter 3, in [12]) to infer that $\mathfrak{F}$ is lower semi-continuous, this implies

$$\mathfrak{F}(u) = m \,,$$

i.e. u is a minimizer of $\mathfrak{F}$; note that, since $\mathfrak{F}(v) = \mathfrak{F}(|v|)$ for every v, we may assume that u is nonnegative. By Theorem 4.4, Chapter 3, in [12], u is a weak solution of the equation

$$\Delta u = -u^p \quad \text{in } \Omega \,.$$

Now prove the regularity of u. As $u \in L^2(\Omega)$ and $p < 1$, the above equation implies that $\Delta u \in L^2(\Omega)$; by the regularity theory for solutions of elliptic partial differential equations, this property improves the regularity of u which turns out to belong to $W_0^{2,2}(\Omega)$ (see, for instance, Theorem 8.8 in [15]). Applying the Sobolev inequality we obtain $u \in L^{p'}(\Omega)$ for some $p' > 2$ and consequently, using the equation again, $u \in W_0^{2,p'}(\Omega)$. This regularizing procedure can be iterated until it is proved that u is Hölder continuous and then, again by regularity results, $u \in C^2(\Omega) \cap C(K)$.

Note that u cannot be identically equal to zero (since $m < 0$); more precisely u is strictly positive in Ω by the strong maximum principle.

Regarding uniqueness, if u is a solution of problem (48) then the function

$$v(x) = u^{\frac{1}{q}}(x) \,, \quad q = \frac{2}{1-p} \,,$$

solves the problem

$$\begin{cases} \Delta v = -\dfrac{1}{v}[A|\nabla v|^2 + B] = 0 \,, \quad v > 0 \quad \text{in } \Omega \,, \\[2ex] v = 0 \quad \text{on } \partial\Omega \,, \end{cases} \tag{51}$$

where

$$A = \frac{1+p}{1-p} \,, \quad B = \frac{2}{1+p} \,.$$

As A and B are positive, the left-hand side of the differential equation is an increasing function of v, so that the comparison principle holds; for this reason, problem (51) admits only one positive solution, and the same conclusion holds for problem (48).

The functional $\mathbf{F}$ is positively homogeneous of order

$$\alpha(p) = n + \frac{2+2p}{1-p} .$$

For brevity, in the sequel we will write α instead of $\alpha(p)$. The homogeneity can be proved as follows. If u is the solution of problem (48) in K and $s > 0$, then the function

$$v(y) = s^{\frac{2}{1-p}} u\left(\frac{y}{s}\right) , \quad y \in sK$$

is the unique solution of problem (48) in *sK*. Hence

$$\begin{aligned} \mathbf{F}(sK) &= \int_{s\Omega} |\nabla v(y)|^2 \, dy = s^{\frac{2+2p}{1-p}} \int_{s\Omega} \left|\nabla u\left(\frac{y}{s}\right)\right|^2 \, dy \\ &= s^{\alpha} \int_{\Omega} |\nabla u(x)|^2 \, dx = s^{\alpha} \mathbf{F}(K) . \end{aligned}$$

The proof of Theorem 11 is based on the following comparison result for solutions of problem (48).

Theorem 20. *Let* K_0, $K_1 \in \mathcal{K}^n$, $t \in [0,1]$ *and* $K_t = (1-t)K_0 + tK_1$. *Let* u_i *be the solution of problem* (48) *in* K_i, $i = 0, 1, t$. *Then*

$$[u_t((1-t)x + ty)]^{\frac{1-p}{2}} \geqslant (1-t)[u_0(x)]^{\frac{1-p}{2}} + t[u_1(y)]^{\frac{1-p}{2}} \quad \forall x \in K_0 \, , y \in K_1 \, .$$

Proof. The argument is an adaptation of the technique introduced by Korevaar in [18], and developed by many other authors, which was used to prove quasi-concavity of solutions of elliptic equations. Here we follow an improved version of such technique presented by Kennington in [17].

Firstly, we prove the theorem under the additional assumption that the boundary of K_0 and K_1 are of class C^2. For $t = 0$ and $t = 1$ the theorem is trivial; in the sequel we assume $t \in (0,1)$. For simplicity let $q = \frac{2}{1-p}$. For $i = 0, 1$, we define the function $v_i(x) = u_i^{1/q}(x)$. As we already saw in the proof of Proposition 19, we have

$$\begin{cases} \Delta v_i + \dfrac{1}{v_i}[A|\nabla v_i|^2 + B] = 0 \, , & v_i > 0 \quad \text{in } \mathrm{int}(K_i) \, , \\ v_i = 0 \quad \text{on } \partial K_i \, , & \end{cases} \tag{52}$$

with

$$A = \frac{1+p}{1-p}\,, \qquad B = \frac{2}{1+p}\,.$$

For $x \in K_0$ and $y \in K_1$ define

$$c(x, y) = v_t((1-t)x + ty) - [(1-t)v_0(x) + tv_1(y)]$$

(for $K_0 = K_1 = K$ this is the Korevaar *concavity function*). The assert of the theorem is equivalent to the inequality

$$\min_{K_0 \times K_1} c(x, y) \geqslant 0\,. \tag{53}$$

The function $c(x, y)$ is continuous in $K_0 \times K_1$ and hence attains its minimum at some point $(\bar{x}, \bar{y})$. We consider separately the cases $(\bar{x}, \bar{y}) \in \mathrm{int}(K_0) \times \mathrm{int}(K_1)$ and $(\bar{x}, \bar{y}) \in \partial(K_0 \times K_1)$.

Case I: $(\bar{x}, \bar{y}) \in \mathrm{int}(K_0) \times \mathrm{int}(K_1)$. Let $\bar{z} = t\bar{x} + (1-t)\bar{y}$. We have

$$\nabla_x c(\bar{x}, \bar{y}) = t\nabla v_t(\bar{z}) - t\nabla v_0(\bar{x}) = 0\,,$$
$$\nabla_y c(\bar{x}, \bar{y}) = (1-t)\nabla v_t(\bar{z}) - (1-t)\nabla v_1(\bar{y}) = 0\,.$$

Consequently,

$$\nabla v_t(\bar{z}) = \nabla v_0(\bar{x}) = \nabla v_1(\bar{y})\,.$$

The Hessian matrix of c has the following form:

$$D^2 c(\bar{x}, \bar{y}) = \left(\begin{array}{c|c} (1-t)^2 D^2 v_t(\bar{z}) - (1-t) D^2 v_0(\bar{x}) & t(1-t) D^2 v_t(\bar{z}) \\ \hline t(1-t) D^2 v_t(\bar{z}) & t^2 D^2 v_t(\bar{z}) - t D^2 v_1(\bar{y}) \end{array}\right) \tag{54}$$

Let

$$\left(\begin{array}{c|c} a^2\, I_n & ab\, I_n \\ \hline ab\, I_n & b^2\, I_n \end{array}\right)$$

where I_n is the identity $n \times n$ matrix and $a, b \in \mathbf{R}$. $D^2 c(\bar{x}, \bar{y})$ is positive semidefinite, as $(\bar{x}, \bar{y})$ is a minimum point, and the same holds for M (this is straightforward). Now, the trace of the product of positive semidefinite matrices is nonnegative (see [17, Appendix]), so that we have

$$\begin{aligned} \mathrm{trace}(D^2 c(\bar{x}, \bar{y})\, M) = \; & a^2[(1-t)^2 \Delta v_t(\bar{z}) - (1-t)\Delta v_0(\bar{x})] \\ & +2ab[t(1-t)\Delta v_t(\bar{z})] \\ & +b^2[t^2 \Delta v_t(\bar{z}) - t\Delta v_1(\bar{y})] \geqslant 0\,, \quad \forall a, b \in \mathbf{R}\,. \end{aligned}$$

It follows

$$(1-t)\Delta v_t(\bar{z}) \geqslant \Delta v_0(\bar{x})\,, \quad t\Delta v_t(\bar{z}) \geqslant \Delta v_1(\bar{y}) \tag{55}$$

and

$$[(1-t)^2 \Delta v_t(\bar{z}) - (1-t)\Delta v_0(\bar{x})][t^2 \Delta v_t(\bar{z}) - t\Delta v_1(\bar{y})] \geqslant [t(1-t)\Delta v_t(\bar{z})]^2\,.$$

After some computations, the last inequality yields

$$\Delta v_t(\bar{z})[t\Delta v_0(\bar{x}) + (1-t)\Delta v_1(\bar{y})] \leqslant \Delta v_0(\bar{x})\Delta v_1(\bar{y})\,. \tag{56}$$

In view of the differential equations satisfied by v_0 and v_1 (problem (52)) we must have

$$t\Delta v_0(\bar{x}) + (1-t)\Delta v_1(\bar{y}) < 0\,. \tag{57}$$

Let $\beta = |\nabla v_s(\bar{z})| = |\nabla v_0(\bar{x})| = |\nabla v_1(\bar{y})|$. From (56) and (57) we have

$$\Delta v_t(\bar{z}) \geqslant \Delta v_0(\bar{x})\Delta v_1(\bar{y})[(1-t)\Delta v_0(\bar{x}) + t\Delta v_1(\bar{y})]^{-1},$$

which (using the differential equations) is equivalent to

$$\frac{A\beta^2+B}{v_t(\bar z)} \leqslant \frac{(A\beta^2+B)^2}{v_0(\bar x)v_1(\bar y)}\left[t\frac{A\beta^2+B}{v_0(\bar x)}+(1-t)\frac{A\beta^2+B}{v_1(\bar y)}\right]^{-1}$$

and then

$$\frac{1}{v_t(\bar z)} \leqslant \frac{1}{(1-t)v_0(\bar x)+tv_1(\bar y)} \quad \Rightarrow \quad c(\bar x,\bar y)\geqslant 0\,.$$

Case II: $(\bar x,\bar y)\in\partial(K_0\times K_1)$. Notice that if $\bar x\in\partial K_0$ and $\bar y\in\partial K_1$, we have trivially $c(\bar x,\bar y)\geqslant 0$. So we have to deal with the case: $\bar x\in\mathrm{int}(K_0)$ and $\bar y\in\partial K_1$ (the symmetric case can be treated exactly in the same way). Let v be the outer unit normal to ∂K_1 at $\bar y$; the function

$$\phi(r)=c(\bar x+rv,\bar y+rv)=v_t(\bar z+rv)-[(1-t)v_0(\bar x+rv)+tv_1(\bar y+rv)]\,,$$

is defined in $r\in[-\delta,0]$ for some positive δ; moreover, if c attains its absolute minimum at $(\bar x,\bar y)$, then ϕ attains its absolute minimum at 0. We compute the left-side derivative of ϕ at 0:

$$\phi'(0^-)=\frac{\partial v_t}{\partial v}(\bar z)-\left[(1-t)\frac{\partial v_0}{\partial v}(\bar x)+t\frac{\partial v_1}{\partial v}(\bar y)\right].$$

By the Hopf Lemma, which can be applied as ∂K_1 is of class C^2,

$$\frac{\partial u_1}{\partial v}(\bar y)<0 \quad \text{whence} \quad \frac{\partial v_1}{\partial v}(\bar y)=-\infty\,.$$

Consequently $\phi'(0^-)=\infty$ which contradicts the fact that 0 is a minimum point for ϕ.

Next, we consider the general case, i.e. without assumptions on the regularity of ∂K_0 and ∂K_1. For a convex body K, let u be the solution of problem (48) in $\Omega=\mathrm{int}(K)$. There exists a sequence of convex open sets $\{\Omega_j\}_j$, $j\in\mathbf{N}$, with boundary of class C^2, such that

$$\bar\Omega_j\subset\Omega_{j+1},\quad \bigcup_{j=1}^{+\infty}\Omega_j=\Omega\,.$$

For every $j\in\mathbf{N}$, let u_j be the unique solution of (48) in Ω_j; by Proposition 19, we know that u_j minimizes (49) in Ω_j. Setting

$$u_j(x)=0 \quad \text{in } \Omega\backslash\Omega_j\,,$$

we get $u_j \in W_0^{1,2}(\Omega)$, so that, for the minimizing properties of u in Ω,

$$\mathfrak{F}(u_j) \geqslant \mathfrak{F}(u)\,, \quad \forall\, j \in \mathbf{N}\,.$$

From (48), the Gauss–Green formula and the definition of $\mathfrak{F}$ it follows that

$$\mathfrak{F}(u_j) = \frac{p-1}{2(p+1)} \int_{\Omega_j} |\nabla u_j|^2\, dx\,,$$

so that, as $p < 1$,

$$\int_{\Omega_j} |\nabla u_j|^2\, dx \leqslant \int_{\Omega} |\nabla u|^2\, dx\,. \tag{58}$$

The Poincaré inequality together with (58) imply that the sequence u_j is bounded in $W_0^{1,2}(\Omega)$, therefore we can find a subsequence $u_{j'}$ and a function $\tilde{u} \in W_0^{1,2}(\Omega)$ satisfying $u_{j'} \rightharpoonup \tilde{u}$ in $W_0^{1,2}(\Omega)$ as $j' \to +\infty$. In particular $\tilde{u}$ must be a solution of (48) in Ω and then $\tilde{u} = u$; this implies that the whole sequence u_j converges to u. From (58) and the lower semi-continuity of

$$w \to \int_{\Omega} |\nabla w|^2\, dx, \quad w \in W_0^{1,2}(\Omega)\,,$$

it follows that

$$\lim_{j\to+\infty} \int_{\Omega} |\nabla u_j|^2\, dx = \int_{\Omega} |\nabla u|^2\, dx\,.$$

Using this fact and the weak convergence we obtain that u_j tends to u in $W_0^{1,2}(\Omega)$ and, up to a subsequence, we may assume that the convergence is almost everywhere.

Given K_0 and K_1 in $\mathcal{K}_n$, let $\Omega_{0,j}$ and $\Omega_{1,j}$ be two sequences of open sets approximating the interior of K_0 and K_1, respectively, constructed as above, and let

$$\Omega_{t,j} = (1-t)\Omega_{0,j} + t\Omega_{1,j}\,.$$

With obvious extension of notation, for $i = 0, 1, t$, let $u_{i,j}$ be the solution of problem (48) in $\Omega_{i,j}$, and $v_{i,j} = u_{i,j}^{(p-1)/p}$. For the previous part of the proof,

$$v_{t,j}((1-t)x + ty) \geqslant (1-t)v_{0,j}(x) + tv_{1,j}(y)\,, \quad x \in \Omega_{0,j}\,, \quad y \in \Omega_{1,j}\,. \tag{59}$$

As j tends to $+\infty$ (up to subsequences), for $i = 0, 1, t$,

$$v_t((1-t)x + ty) \geqslant (1-t)v_0(x) + tv_1(y)$$

for almost every $x \in \Omega_0$ and almost every $y \in \Omega_1$; as all the involved functions are continuous, we obtain the claim of the theorem. $\square$

Another result that we shall use is the following theorem, containing the *Prékopa–Leindler* inequality and including a *necessary* equality condition.

Theorem 21 (*Prékopa–Leindler inequality*). *Let f, g and h be measurable, nonnegative functions defined in* $\mathbf{R}^n$ *and let* $t \in [0,1]$. *Assume that*

$$h((1-t)x+ty) \geqslant f^{1-t}(x)g^t(y), \quad \forall x, y \in \mathbf{R}^n. \tag{60}$$

Then

$$\int_{\mathbf{R}^n} h(z)\,dz \geqslant \left(\int_{\mathbf{R}^n} f(x)\,dx\right)^{1-t} \left(\int_{\mathbf{R}^n} g(y)\,dy\right)^t. \tag{61}$$

Moreover, if

$$0 < \int_{\mathbf{R}^n} f(x)\,dx, \int_{\mathbf{R}^n} g(y)\,dy,$$

and equality holds in (61), *then f coincides almost everywhere with a log-concave function and there exist C,* $a > 0$, *and a vector* $\overline{x} \in \mathbf{R}^n$ *such that*

$$f(x) = Cg(ax+\overline{x}), \quad \forall x \in \mathbf{R}^n. \tag{62}$$

For the proof of inequality (61) we refer, for instance, to [14]; the equality condition follows from Theorem 12 in [13].

Proof of Theorem 11. Firstly, we consider the *multiplicative form* of the inequality contained in Theorem 11:

$$\mathbf{F}((1-t)K_0+tK_1) \geqslant \mathbf{F}(K_0)^{1-t}\mathbf{F}(K_1)^t, \quad \forall K_0, \ K_1 \in \mathcal{K}^n, \ t \in [0,1]. \tag{63}$$

We remark that, for arbitrary $K_0, K_1 \in \mathcal{K}^n$ and $t \in [0,1]$, (16) follows from (63) applied to

$$K_0' = (\mathbf{F}(K_0))^{-1/\alpha} K_0, \quad K_1' = (\mathbf{F}(K_1))^{-1/\alpha} K_1,$$
$$t' = \frac{t(\mathbf{F}(K_1))^{1/\alpha}}{(1-t)(\mathbf{F}(K_0))^{1/\alpha} + t(\mathbf{F}(K_1))^{1/\alpha}}.$$

Moreover, if $K_0, K_1 \in \mathcal{K}^n$ and $t \in [0,1]$ render (16) an equality, then K_0', K_1' and t' defined as above give equality in (63). Hence, it suffices to prove (63) and to characterize the corresponding equality conditions.

For $i = 0, 1, t$, let u_i be the solution u of problem (48) for $K = K_i$; we have, by the divergence theorem:

$$\mathbf{F}(K_i) = \int_{K_i} |\nabla u_i|^2 \, dx = \int_{K_i} (\mathrm{div}(u_i \nabla u_i) - u \Delta u_i) \, dx = \int_{K_i} u_i^{p+1} \, dx \, .$$

Let $x \in K_0$, $y \in K_1$ and $z = (1-t)x + ty \in K_t$; from Theorem 20 we know that

$$[u_t(z)]^{\frac{1-p}{2}} \geqslant (1-t)[u_0(x)]^{\frac{1-p}{2}} + t[u_1(y)]^{\frac{1-p}{2}} \, .$$

Let us extend u_i as zero outside K_i, $i = 0, 1, s$, and define

$$f = u_0^{p+1} \, , \quad g = u_1^{p+1} \, , \quad h = u_t^{p+1} \, .$$

We have, for $x \in K_0$ and $y \in K_1$,

$$h((1-t)x + ty) \geqslant [(1-t)f(x)^r + tg(y)^r]^{1/r} \, , \quad \text{where } r = \frac{1-p}{2(p+1)} > 0 \, .$$

By the arithmetic–geometric mean inequality

$$h((1-t)x + ty) \geqslant f(x)^{1-t} g(y)^t \, , \quad \forall x \in K_0 \, , \; y \in K_1,$$

in fact, this inequality holds for all $x, y \in \mathbf{R}^n$: indeed, if either $x \notin K_0$ or $y \notin K_1$, then the right-hand side vanishes. Hence we can apply the Prékopa–Leindler inequality (Theorem 21) to obtain

$$\int_{\mathbf{R}^n} h(x) \, dx \geqslant \left(\int_{\mathbf{R}^n} f(x) \, dx \right)^{1-t} \left(\int_{\mathbf{R}^n} g(x) \, dx \right)^t \, , \tag{64}$$

i.e. (63). Moreover, if equality holds in (63), then (64) becomes an equality and in particular f and g render the Prékopa–Leindler inequality an equality. By Theorem 21 and the fact that $f(x)$ is positive if and only if $x \in K_0$, and $g(y)$ is positive if and only if $y \in K_1$, we conclude that K_0 and K_1 coincide up to a translation. $\square$

Remark 22. Theorem 10 can be proved along the lines of the proof of Theorem 11 and taking the following considerations into account. Firstly, neither the proof of Proposition 19, nor the ones of Theorems 20 and 11 require the convexity of the involved sets, once that they are assumed to have boundary of class C^2; indeed, the only assumption on the boundary that is necessary is to have the interior sphere property in order to apply the Hopf Lemma. Moreover, concerning equality conditions, in Theorem 10 it has to be used the fact that functions giving equality in the Prékopa–Leindler inequality are necessarily log-concave, this implies that if C_0 and C_1 give equality in (13), then they are convex.

Acknowledgment

I would like to thank R. Gardner for directing my attention to the subject of this paper.

References

[1] T. Bonnesen, W. Fenchel, Theory of Convex Bodies (transl. from German ed. 1934), BCS Associates, Moscow, Idaho, 1987.
[2] C. Borell, Capacitary inequalities of the Brunn–Minkowski type, Math. Ann. 263 (1983) 179–184.
[3] C. Borell, Hitting probability of killed Brownian motion: a study on geometric regularity, Ann. Sci. Ecole Norm. Sup. Paris 17 (1984) 451–467.
[4] C. Borell, Greenian potentials and concavity, Math. Ann. 272 (1985) 155–160.
[5] C. Borell, Diffusion equation and geometric inequalities, Potential Anal. 12 (2000) 49–71.
[6] H. Brascamp, E. Lieb, On extension of the Brunn–Minkowski and Prékopa–Leindler inequality, including inequalities for log concave functions, and with an application to diffusion equation, J. Funct. Anal. 22 (1976) 366–389.
[7] L. Caffarelli, D. Jerison, E. Lieb, On the case of equality in the Brunn–Minkowski inequality for capacity, Adv. Math. 117 (1996) 193–207.
[8] L. Caffarelli, J. Spruck, Convexity property of solutions to some classical variational problems, Comm. Partial Differential Equations 7 (1982) 1337–1379.
[9] A. Colesanti, P. Cuoghi, The Brunn–Minkowski inequality for the *n*-dimensional logarithmic capacity of convex bodies, Potential Anal., to appear.
[10] A. Colesanti, P. Cuoghi, P. Salani, Brunn–Minkowski inequalities for two functionals involving the *p*-Laplace operator, preprint, 2003.
[11] A. Colesanti, P. Salani, The Brunn–Minkowski inequality for *p*-capacity of convex bodies, Math. Ann. 327 (2003) 459–479.
[12] B. Dacorogna, Direct methods in the Calculus of Variations, Springer, Berlin, 1989.
[13] S. Dubuc, Critères de convexité et inégalités intégrales, Ann. Inst. Fourier Grenoble 27 (1) (1977) 135–165.
[14] R. Gardner, The Brunn–Minkowski inequality, Bull. Amer. Math. Soc. (N.S.) 39 (2002) 355–405.
[15] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, 1977.
[16] G.M. Goluzin, Geometric theory of Functions of a Complex Variable, American Mathematical Society, Providence, RI, 1969.
[17] A. Kennington, Power concavity and boundary value problems, Indiana Univ. Math. J. 34 (1985) 687–704.
[18] N. Korevaar, Capillary surface convexity over convex domains, Indiana Univ. Math. J. 32 (1983) 73–81.
[19] N. Korevaar, J. Lewis, Convex solutions of certain elliptic equations have constant rank Hessians, Arch. Rational Mech. Anal. 97 (1987) 19–32.
[20] D. Jerison, A Minkowski problem for electrostatic capacity, Acta Math. 176 (1996) 1–47.
[21] D. Jerison, The direct method in the calculus of variations for convex bodies, Adv. Math. 122 (1996) 262–279.
[22] P.L. Lions, Two remarks on Monge–Ampère equations, Ann. Mat. Pura Appl. 142 (1985) 262–275.
[23] S.I. Pohožaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Soviet Math. Dokl. 6 (1965) 1408–1411 (English transl. of Dokl. Akad. Nauk. SSSR 165 (1965) 33–36).

[24] P. Pucci, J. Serrin, A general variational identity, Indiana Univ. Math. J. 35 (1986) 681–703.
[25] T. Rockafellar, Convex analysis, Princeton University Press, Princeton, 1970.
[26] P. Salani, The Brunn-Minkowski inequality for the Monge–Ampère eigenvalue, Adv. Math., to appear.
[27] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, Cambridge University Press, Cambridge, 1993.
[28] K.-T. Tso, On a real Monge–Ampère functional, Invent. Math. 101 (1990) 425–448.
[29] X.-J. Wang, A class of fully nonlinear elliptic equations and related functionals, Indiana Univ. Math. J. 43 (1994) 25–54.