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Boundedness of weak solutions to evolutionary partial integro-differential equations with discontinuous coefficients ☆

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ABSTRACT

We investigate linear and quasilinear evolutionary partial integro-differential equations of second order which include time fractional evolution equations of time order less than one. By means of suitable energy estimates and De Giorgi's iteration technique we establish results asserting the global boundedness of appropriately defined weak solutions of these problems. We also show that a maximum principle holds for such equations.

1. Introduction

Let $T>0$, and Ω be a bounded domain in $\mathbb{R}^N$. In this paper we are concerned with linear partial integro-differential equations of the form

$$\partial_t\big(k*(u-u_0)\big)-\mathcal{L}u=f+\operatorname{div} g,\quad t\in(0,T),\ x\in\Omega, \tag{1}$$

as well as related quasilinear problems

$$\partial_t\big(k*(u-u_0)\big)-\operatorname{div} a(t,x,u,Du)=b(t,x,u,Du),\quad t\in(0,T),\ x\in\Omega, \tag{2}$$

where in both cases $k\in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ is a nonnegative kernel that belongs to a certain kernel class (see (H1) and Definition 2.1 below), and $k*v$ denotes the convolution on the positive halfline w.r.t. the time variable, that is $(k*v)(t)=\int_0^t k(t-\tau)v(\tau)\,d\tau$, $t\geqslant 0$.

As to (1), $\mathcal{L}$ is a second order operator w.r.t. the spatial variables in divergence form:

$$\mathcal{L}u=\operatorname{div}\big(A(t,x)Du+b(t,x)u\big)+\big(c(t,x)|Du\big)+d(t,x)u.$$

Here A is $\mathbb{R}^{N\times N}$-valued, b and c take values in $\mathbb{R}^N$, and d is a real-valued function. Further, Du stands for the gradient of u, and $(\cdot|\cdot)$ denotes the scalar product in $\mathbb{R}^N$. The functions $u_0=u_0(x)$, $f=f(t,x)$, and $g=g(t,x)$ are given data; the function u_0 plays the role of the initial data for u.

Concerning the leading coefficients of $\mathcal{L}$ we merely assume measurability, boundedness, and a uniform parabolicity condition. The coefficients of the lower order terms are assumed to belong to certain Lebesgue spaces of mixed type, so they need not be bounded.

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In (2), the functions $a : (0,T) \times \Omega \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ and $b : (0,T) \times \Omega \times \mathbb{R}^{N+1} \to \mathbb{R}$ are measurable and satisfy suitable structure conditions, see (Q1)–(Q5) in Section 4.

An important example for the kernel k we have in mind is given by

$$k(t) = g_{1-\alpha}(t)e^{-\mu t}, \quad t > 0,\ \alpha \in (0,1),\ \mu \geqslant 0, \tag{3}$$

where g_β denotes the Riemann–Liouville kernel

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \quad t > 0,\ \beta > 0. \tag{4}$$

In this case, (1) and (2) amount to time fractional equations of order $\alpha \in (0,1)$. Recall that for a (sufficiently smooth) function v on $\mathbb{R}_+$, the Riemann–Liouville fractional derivative $D_t^\alpha v$ of order $\alpha \in (0,1)$ is defined by $D_t^\alpha v = \frac{d}{dt}(g_{1-\alpha} * v)$.

As to applications, problems of the form (1) and (2) arise for example in mathematical physics when describing dynamic processes in materials with memory, e.g., in the theory of heat conduction with memory, see [13] and the references therein. Time fractional diffusion equations also appear in the context of anomalous diffusion, see e.g. [11].

Letting $\Omega_T = (0,T) \times \Omega$ and $\Gamma_T = (0,T) \times \partial\Omega$ one of the main objectives of this paper is to derive results asserting the boundedness on Ω_T of appropriately defined weak solutions of (1) and (2), respectively, that are bounded on Γ_T. We further establish the analogue of the well-known weak maximum principle for weak solutions of the parabolic equation corresponding to (1), i.e. $\partial_t u - \mathcal{L}u = f + \operatorname{div} g$, see e.g. [12, Theorem 7.2, p. 188].

In the literature one finds many papers where equations of the form (1) and (2), as well as abstract variants of them are studied in a *strong* setting, assuming more smoothness on the coefficients and nonlinearities, see e.g. [1,3,6,8,9,13, 15,16]. The purpose of the present paper is to develop further a theory of *weak* solutions to (1) and (2). In this sense the boundedness results are an important first step towards a De Giorgi–Nash–Moser theory for time fractional evolution equations in divergence form of order $\alpha \in (0,1)$.

Our proofs of the global boundedness results use De Giorgi's iteration technique and are based on suitable a priori estimates for weak solutions of (1) and (2), respectively. These estimates, which by partly standard arguments (cf. [12, Chapters III and V]) lead to suitable Caccioppoli type inequalities, are derived by means of the basic inequality (10) (see below) for nonnegative nonincreasing kernels. We point out that the basic L_2 energy estimate for (1) has already been established in [17], under conditions on the coefficients and data which are slightly more restrictive than the ones assumed in the present paper.

One of the technical difficulties in deriving the desired estimates in the weak setting is to find an appropriate time regularization of the equation. In the classical parabolic theory this is achieved by means of Steklov averages in time. In the case of Eqs. (1) and (2) this method does not work any more, since Steklov average operators and convolution do not commute. It turns out that instead one can use the Yosida approximation of the operator B defined by $Bv = \partial_t(k * v)$, e.g., in $L_2([0,T])$, which leads to a regularization of the kernel k (not of u!). This method has already been used in [14] and [17], we also refer to [9], where a more general class of integro-differential operators (in time) is studied.

The paper is organized as follows. In Section 2 we introduce the class of kernels used in this paper, and explain the approximation method in more detail. We also state the basic inequality (10) and collect some further auxiliary results. In Section 3 we describe the weak formulation of (1) and prove the global boundedness of weak solutions as well as the maximum principle. In Section 4 we extend these results to the quasilinear case.

2. Preliminaries

The following class of kernels has been introduced in [17] and is basic to our treatment of (1).

Definition 2.1. A kernel $k \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ is said to be *of type* $\mathcal{PC}$ if it is nonnegative and nonincreasing, and there exists a kernel $l \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ such that $k * l = 1$ in $(0,\infty)$. In this case, we say that (k,l) is a $\mathcal{PC}$ *pair* and write $(k,l) \in \mathcal{PC}$.

An important example is given by

$$k(t) = g_{1-\alpha}(t)e^{-\mu t} \quad \text{and} \quad l(t) = g_\alpha(t)e^{-\mu t} + \mu\big(1 * \big[g_\alpha(\cdot)e^{-\mu \cdot}\big]\big)(t), \quad t > 0, \tag{5}$$

with $\alpha \in (0,1)$ and $\mu \geqslant 0$. Both kernels are strictly positive and decreasing; observe that $\dot{l}(t) = \dot{g}_\alpha(t)e^{-\mu t} < 0$, $t > 0$. Their Laplace transforms are given by

$$\hat{k}(\lambda) = \frac{1}{(\lambda+\mu)^{1-\alpha}}, \qquad \hat{l}(\lambda) = \frac{1}{(\lambda+\mu)^{\alpha}}\left(1 + \frac{\mu}{\lambda}\right), \quad \operatorname{Re}\lambda > 0,$$

which shows that $k * l = 1$ on $(0,\infty)$. Hence we have both $(k,l) \in \mathcal{PC}$, and $(l,k) \in \mathcal{PC}$.

We next discuss an important method of approximating kernels of type $\mathcal{PC}$. Let $1 \leqslant p < \infty$, $(k,l) \in \mathcal{PC}$, $T > 0$, and X be a real Banach space. Then the operator B defined by

$$Bu = \frac{d}{dt}(k * u), \qquad D(B) = \big\{u \in L_p\big([0,T]; X\big):\ k * u \in {}_0H^1_p\big([0,T]; X\big)\big\},$$

where the zero means vanishing at $t = 0$, is known to be m-accretive in $L_p([0,T];X)$, cf. [2,5,9]. Its Yosida approximations B_n, defined by $B_n = nB(n+B)^{-1}$, $n \in \mathbb{N}$, enjoy the property that for any $u \in D(B)$, one has $B_n u \to Bu$ in $L_p([0,T];X)$ as $n \to \infty$. Further, one has the representation

$$B_n u = \frac{d}{dt}(k_n * u), \quad u \in L_p\big([0,T];X\big),\ n \in \mathbb{N}, \tag{6}$$

where $k_n = ns_n$, and s_n is the unique solution of the scalar-valued Volterra equation

$$s_n(t) + n(s_n * l)(t) = 1, \quad t > 0,\ n \in \mathbb{N},$$

see e.g. [14]. Denoting by $h_n \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ the resolvent kernel associated with nl, we have

$$h_n(t) + n(h_n * l)(t) = nl(t), \quad t > 0,\ n \in \mathbb{N}, \tag{7}$$

and hence, by convolving (7) with k,

$$(k * h_n)(t) + n(k * h_n * l)(t) = n, \quad t > 0,\ n \in \mathbb{N},$$

which shows that

$$k_n = ns_n = k * h_n, \quad n \in \mathbb{N}. \tag{8}$$

From $(k,l) \in \mathcal{PC}$ it follows that l is completely positive, see e.g. Theorem 2.2 in [4]. Consequently, l and h_n are nonnegative, and the kernels s_n are nonnegative and nonincreasing for all $n \in \mathbb{N}$, see e.g. [13, Proposition 4.5] and [4, Proposition 2.1]. From $s_n = 1 - 1 * h_n$ we further see that $s_n \in H^1_1([0,T])$. In view of (8) we conclude that the kernels k_n, $n \in \mathbb{N}$, are also nonnegative and nonincreasing, and that they belong to $H^1_1([0,T])$.

Note that for any function $f \in L_p([0,T];X)$, $1 \leqslant p < \infty$, there holds $h_n * f \to f$ in $L_p([0,T];X)$ as $n \to \infty$. In fact, defining $u = l * f$, we have $u \in D(B)$, and

$$B_n u = \frac{d}{dt}(k_n * u) = \frac{d}{dt}(k * l * h_n * f) = h_n * f \to Bu = f \quad \text{in } L_p\big([0,T];X\big)$$

as $n \to \infty$. In particular, $k_n \to k$ in $L_1([0,T])$ as $n \to \infty$.

We next state a fundamental identity for integro-differential operators of the form $\frac{d}{dt}(k * u)$. Suppose $k \in H^1_1([0,T])$ and $H \in C^1(\mathbb{R})$. Then a straightforward computation shows that for a sufficiently smooth function u on $(0,T)$ one has for a.a. $t \in (0,T)$,

$$\begin{aligned} H'\big(u(t)\big)\frac{d}{dt}(k * u)(t) &= \frac{d}{dt}\big(k * H(u)\big)(t) + \big({-H}\big(u(t)\big) + H'\big(u(t)\big)u(t)\big)k(t) \\ &\quad + \int_0^t \big(H\big(u(t-s)\big) - H\big(u(t)\big) - H'\big(u(t)\big)\big[u(t-s) - u(t)\big]\big)\big[{-\dot{k}(s)}\big]\,ds. \end{aligned} \tag{9}$$

We remark that an integrated version of (9) can be found in [10, Lemma 18.4.1].

Define now $H(y) = \frac{1}{2}(y_+)^2$, $y \in \mathbb{R}$, where $y_+ := \max\{y, 0\}$. Evidently, $H \in C^1(\mathbb{R})$ with derivative $H'(y) = y_+$, $y \in \mathbb{R}$. Assume in addition that the kernel $k \in H^1_1([0,T])$ is nonnegative and nonincreasing. Then it follows from (9) and the convexity of H that for any function $u \in L_2([0,T])$,

$$u(t)_+ \frac{d}{dt}(k * u)(t) \geqslant \frac{1}{2}\frac{d}{dt}\big(k * (u_+)^2\big)(t), \quad \text{a.a. } t \in (0,T). \tag{10}$$

The next lemma concerning the geometric convergence of sequences of numbers will be needed for the De Giorgi iteration arguments below. It can be found, e.g., in [12, Chapter II, Lemma 5.6] and [7, Chapter I, Lemma 4.1]. Its proof is by induction.

Lemma 2.1. *Let $\{y_n\}$, $n = 0,1,2,\ldots$, be a sequence of positive numbers, satisfying the recursion inequality*

$$y_{n+1} \leqslant C b^n y_n^{1+\gamma}, \quad n = 0,1,2,\ldots,$$

where $C, b > 1$ and $\gamma > 0$ are given numbers. If

$$y_0 \leqslant C^{-1/\gamma} b^{-1/\gamma^2},$$

then $y_n \to 0$ as $n \to \infty$.

We conclude this preliminary part with an interpolation result which will be frequently used in Sections 3 and 4.

Let $T > 0$ and Ω be a bounded domain in $\mathbb{R}^N$. For $1 < p \leqslant \infty$ we define the space

$$V_p := V_p\big([0,T] \times \Omega\big) = L_{2p}\big([0,T]; L_2(\Omega)\big) \cap L_2\big([0,T]; H^1_2(\Omega)\big), \tag{11}$$

endowed with the norm

$$|u|_{V_p([0,T]\times\Omega)} := |u|_{L_{2p}([0,T];L_2(\Omega))} + |u|_{L_2([0,T];H^1_2(\Omega))}.$$

Suppose that

$$p'\left(1-\frac{2}{r}\right)+N\left(\frac{1}{2}-\frac{1}{q}\right)=1, \tag{12}$$

where $p' = p/(p-1)$, and

$$\left.\begin{aligned} &r\in[2,2p], && q\in\left[2,\frac{2N}{N-2}\right] && \text{for } N>2,\\ &r\in(2,2p], && q\in[2,\infty) && \text{for } N=2,\\ &r\in\left[\frac{4p}{p+1},2p\right], && q\in[2,\infty] && \text{for } N=1. \end{aligned}\right\} \tag{13}$$

Then $V_p \hookrightarrow L_r([0,T];L_q(\Omega))$, and

$$|u|_{L_r([0,T];L_q(\Omega))} \leqslant C(N,q)|u|_{V_p([0,T]\times\Omega)}. \tag{14}$$

This is a consequence of the Gagliardo–Nirenberg and Hölder's inequality. The case $p=\infty$ is contained, e.g., in [12, pp. 74 and 75]. The proof there easily extends to the general case.

3. Linear equations

In this section we study the linear equation (1). Let $T>0$, and Ω be a bounded domain in $\mathbb{R}^N$. In what follows (except for Theorems 3.2 and 3.3) we will assume that

(H1) There exists $l \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ such that $(k,l)\in\mathcal{PC}$. Further, $l\in L_p([0,T])$ for some $p>1$.

(H2) $A\in L_\infty((0,T)\times\Omega;\mathbb{R}^{N\times N})$, and $\exists\nu>0$ such that

$$\big(A(t,x)\xi|\xi\big)\geqslant\nu|\xi|^2,\quad \text{for a.a. } (t,x)\in\Omega_T, \text{ and all } \xi\in\mathbb{R}^N.$$

(H3) $u_0\in L_2(\Omega)$, and

$$\big||b|^2+|g|^2+|c|^2+|d|+|f|\big|_{L_r([0,T];L_q(\Omega))} =: C_D<\infty,$$

where

$$\frac{p'}{r}+\frac{N}{2q}=1-\beta,$$

and

$$\begin{aligned} &r\in\left[\frac{p'}{(1-\beta)},\infty\right], && q\in\left[\frac{N}{2(1-\beta)},\infty\right], && \beta\in(0,1) \text{ for } N\geqslant 2,\\ &r\in\left[\frac{p'}{(1-\beta)},\frac{2p'}{(1-2\beta)}\right], && q\in[1,\infty], && \beta\in\left(0,\frac{1}{2}\right) \text{ for } N=1. \end{aligned}$$

We say that a function u is a *weak solution* (*subsolution*, *supersolution*) of (1) in Ω_T, if u belongs to the space

$$\tilde V_p := \big\{v\in L_{2p}\big([0,T];L_2(\Omega)\big)\cap L_2\big([0,T];H^1_2(\Omega)\big) \text{ such that } k*v\in C\big([0,T];L_2(\Omega)\big), \text{ and } (k*v)|_{t=0}=0\big\},$$

and for any nonnegative test function

$$\eta\in \mathring{H}^{1,1}_2(\Omega_T) := H^1_2\big([0,T];L_2(\Omega)\big)\cap L_2\big([0,T];\mathring{H}^1_2(\Omega)\big)\quad \big(\mathring{H}^1_2(\Omega) := \overline{C_0^\infty(\Omega)}^{H^1_2(\Omega)}\big)$$

with $\eta|_{t=T}=0$ there holds

$$\int_0^T\int_\Omega\big(-\eta_t\big[k*(u-u_0)\big]+(ADu+bu|D\eta)-(c|Du)\eta-du\eta\big)\,dx\,dt$$
$$=(\leqslant,\geqslant)\int_0^T\int_\Omega\big(f\eta-(g|D\eta)\big)\,dx\,dt. \tag{15}$$

It is not difficult to verify, by means of Hölder's inequality and the interpolation inequality (14), that under conditions (H1)–(H3) the integrals in (15) are finite, cf. the proof of Theorem 3.1 below. We point out that (1) is considered without any boundary conditions, in this sense weak solutions of (1) as defined above are *local* ones. Note that for an energy estimate for weak solutions $u \in \tilde{V}_p$ of (1) one can work with a weaker version of condition (H3), see e.g. Theorem 3.2 below. We further remark that weak solutions of (1) in the class $\tilde{V}_p$ have been constructed in [17] under the assumptions (H1), (H2), and a stronger variant of (H3). Notice also that the function u_0 plays the role of the initial data for u, at least in a weak sense. In case of sufficiently smooth functions u and $k * (u - u_0)$ the condition $(k * u)|_{t=0} = 0$ implies $u|_{t=0} = u_0$, see [17].

The following lemma is basic to deriving a priori estimates for weak (sub-/super-) solutions of (1) as it provides an equivalent weak formulation of (1) where the kernel k is replaced with the more regular kernel k_n ($n \in \mathbb{N}$) defined in (8). In what follows the kernels h_n, $n \in \mathbb{N}$, are as in Section 2.

Lemma 3.1. *Let the assumptions* (H1)–(H3) *be satisfied. Then $u \in \tilde{V}_p$ is a weak solution* (*subsolution, supersolution*) *of* (1) *if and only if for any nonnegative function $\psi \in \mathring{H}_2^1(\Omega)$ one has*

$$\int_\Omega \big(\psi\, \partial_t\big[k_n * (u - u_0)\big] + \big(h_n * [ADu + bu]|D\psi\big) - \big(h_n * \big[(c|Du) + du\big]\big)\psi\big)\, dx$$
$$= (\leqslant, \geqslant) \int_\Omega \big([h_n * f]\psi - (h_n * g|D\psi)\big)\, dx, \quad a.a.\ t \in (0,T),\ n \in \mathbb{N}. \tag{16}$$

Proof. We may restrict ourselves to the subsolution case as the remaining cases can be treated analogously.

The 'if' part is readily seen as follows. Given an arbitrary nonnegative $\eta \in \mathring{H}_2^{1,1}(\Omega_T)$ satisfying $\eta|_{t=T} = 0$, we take in (16) $\psi(x) = \eta(t,x)$ for any fixed $t \in (0,T)$, integrate from $t = 0$ to $t = T$, and integrate by parts w.r.t. the time variable. Relation (15) then follows by sending $n \to \infty$; here we use the approximating properties of the kernels h_n described in Section 2.

To show the 'only–if' part, we choose the test function

$$\eta(t,x) = \int_t^T h_n(\sigma - t)\varphi(\sigma, x)\, d\sigma = \int_0^{T-t} h_n(\sigma)\varphi(\sigma + t, x)\, d\sigma, \quad t \in (0,T),\ x \in \Omega, \tag{17}$$

with arbitrary $n \in \mathbb{N}$ and nonnegative $\varphi \in \mathring{H}_2^{1,1}(\Omega_T)$ satisfying $\varphi|_{t=T} = 0$; η is nonnegative since φ and h_n are so (see Section 2). Then

$$\eta_t(t,x) = \int_t^T h_n(\sigma - t)\varphi_\sigma(\sigma, x)\, d\sigma, \quad \text{a.a. } (t,x) \in \Omega_T.$$

By Fubini's theorem, we have

$$\int_0^T \left(\int_t^T h_n(\sigma - t)\psi_1(\sigma)\, d\sigma\right)\psi_2(t)\, dt = \int_0^T \psi_1(t)\left(\int_0^t h_n(t - \sigma)\psi_2(\sigma)\, d\sigma\right) dt,$$

for all $\psi_1, \psi_2 \in L_2([0,T])$. So it follows from (15) and $k_n = h_n * k$ (cf. (8)) that

$$\int_0^T \int_\Omega \big(-\varphi_t\big[k_n * (u - u_0)\big] + \big(h_n * [ADu + bu]|D\varphi\big) - \big(h_n * \big[(c|Du) + du\big]\big)\varphi\big)\, dx\, dt$$
$$\leqslant \int_0^T \int_\Omega \big([h_n * f]\varphi - (h_n * g|D\varphi)\big)\, dx\, dt, \quad n \in \mathbb{N}.$$

Observe that $k_n * (u - u_0) \in {}_0H_2^1([0,T]; L_2(\Omega))$. Therefore, integrating by parts and using $\varphi|_{t=T} = 0$ yields

$$\int_0^T \int_\Omega \big(\varphi \partial_t\big[k_n * (u - u_0)\big] + \big(h_n * [ADu + bu]|D\varphi\big) - \big(h_n * \big[(c|Du) + du\big]\big)\varphi\big)\, dx\, dt$$
$$\leqslant \int_0^T \int_\Omega \big([h_n * f]\varphi - (h_n * g|D\varphi)\big)\, dx\, dt \tag{18}$$

for all $n \in \mathbb{N}$ and $\varphi \in \mathring{H}_2^{1,1}(\Omega_T)$ with $\varphi|_{t=T} = 0$. By means of a simple approximation argument, we infer that (18) holds true for any φ of the form $\varphi(t,x) = \chi_{(t_1,t_2)}(t)\psi(x)$, where $\chi_{(t_1,t_2)}$ denotes the characteristic function of the time-interval (t_1,t_2), $0 < t_1 < t_2 < T$, and $\psi \in \mathring{H}_2^1(\Omega)$ is nonnegative. Appealing to the Lebesgue differentiation theorem, we then obtain the desired relation (16). □

Theorem 3.1. *Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let further the assumptions* (H1)–(H3) *be satisfied. Suppose $K \geqslant 0$ is such that $u_0 \leqslant K$ a.e. in Ω. Then there exists a constant $C = C(p,q,r,|l|_{L_p([0,T])},T,N,\nu,\Omega,C_D)$ such that for any weak subsolution $u \in \tilde{V}_p$ of* (1) *in Ω_T satisfying $u \leqslant K$ a.e. on Γ_T there holds $u \leqslant C(1+K)$ a.e. in Ω_T.*

Remarks 3.1.

(i) There is a corresponding result for weak supersolutions u of (1) in the situation where $u_0 \geqslant K$ a.e. in Ω, and $u \geqslant K$ a.e. on Γ_T, for some $K \leqslant 0$. This follows immediately from Theorem 3.1 by replacing u with $-u$, and u_0 with $-u_0$.

(ii) The statement of Theorem 3.1 remains true if r and q in (H3) are different for different coefficients and data, that is when $|b|^2 \in L_{r_1}([0,T]; L_{q_1}(\Omega))$, $|g|^2 \in L_{r_2}([0,T]; L_{q_2}(\Omega))$, and so forth with r_i and q_i satisfying the same conditions as r and q in (H3). This can been seen by working with several functions $\mu_{\kappa,i}$ and by generalizing the iteration argument for the function ϕ, see below. In the classical parabolic case this issue is discussed in [12, Chapter III, Remark 7.2].

Proof of Theorem 3.1. Suppose $u \in \tilde{V}_p$ is a weak subsolution of (1) in Ω_T. Then, by Lemma 3.1, for any nonnegative function $\psi \in \mathring{H}_2^1(\Omega)$ relation (16) holds with the '$\leqslant$' sign. For $t \in (0,T)$ we take in (16) the test function $\psi = u_\kappa^+ := (u_\kappa)_+$, where $u_\kappa := u - \kappa$, and $\kappa \in \mathbb{R}$ satisfying the condition

$$\kappa \geqslant \kappa_0 := \max\Big\{0, \operatorname*{ess\,sup}_{\Omega} u_0, \operatorname*{ess\,sup}_{\Gamma_T} u\Big\}. \tag{19}$$

The resulting inequality can be written in the form

$$\begin{aligned} &\int_\Omega \big(u_\kappa^+ \partial_t(k_n * u_\kappa) + \big(h_n * [ADu + bu] | Du_\kappa^+\big) - \big(h_n * \big[(c|Du) + du\big]\big) u_\kappa^+\big)\, dx \\ &\quad \leqslant \int_\Omega \big([h_n * f] u_\kappa^+ - \big(h_n * g | Du_\kappa^+\big) + u_\kappa^+ (u_0 - \kappa) k_n\big)\, dx, \quad \text{a.a. } t \in (0,T). \end{aligned} \tag{20}$$

Clearly,

$$\int_\Omega u_\kappa^+ (u_0 - \kappa) k_n \, dx \leqslant 0, \quad \text{a.a. } t \in (0,T),$$

by positivity of k_n and (19). Thanks to (10) we further have

$$u_\kappa^+ \partial_t(k_n * u_\kappa) \geqslant \frac{1}{2} \partial_t\big(k_n * \big(u_\kappa^+\big)^2\big), \quad \text{a.a. } (t,x) \in \Omega_T. \tag{21}$$

Using these relations it follows from (20) that

$$\begin{aligned} &\int_\Omega \bigg(\frac{1}{2}\partial_t\big[k_n * \big(u_\kappa^+\big)^2\big] + \big(h_n * [ADu + bu] | Du_\kappa^+\big) - \big(h_n * \big[(c|Du) + du\big]\big) u_\kappa^+\bigg)\, dx \\ &\quad \leqslant \int_\Omega \big([h_n * f] u_\kappa^+ - \big(h_n * g | Du_\kappa^+\big)\big)\, dx, \quad \text{a.a. } t \in (0,T). \end{aligned} \tag{22}$$

Next we convolve (22) with the nonnegative kernel l from assumption (H1), and observe that in view of

$$k_n * \big(u_\kappa^+\big)^2 \in {}_0H_1^1\big([0,T]; L_1(\Omega)\big)$$

and $k_n = k * h_n$ we have

$$l * \partial_t\big(k_n * \big(u_\kappa^+\big)^2\big) = \partial_t\big(l * k_n * \big(u_\kappa^+\big)^2\big) = h_n * \big(u_\kappa^+\big)^2.$$

Sending then $n \to \infty$, and selecting an appropriate subsequence, if necessary, we thus arrive at

$$\frac{1}{2}\int_\Omega \big(u_\kappa^+\big)^2 dx + l * \int_\Omega \big(ADu | Du_\kappa^+\big)\, dx \leqslant l * F, \quad \text{a.a. } t \in (0,T), \tag{23}$$

where

$$F(t) = \int_\Omega \left(-\left(bu + g|Du_\kappa^+\right) + \left[(c|Du) + du + f\right]u_\kappa^+\right) dx.$$

By (H2), we have

$$\int_\Omega \left(ADu|Du_\kappa^+\right) dx = \int_\Omega \left(ADu_\kappa^+|Du_\kappa^+\right) dx \geqslant \nu \int_\Omega \left|Du_\kappa^+\right|^2 dx, \tag{24}$$

and thus

$$\int_\Omega \left(u_\kappa^+\right)^2 dx \leqslant 2l * F, \quad \text{a.a. } t \in (0,T).$$

Young's inequality for convolutions then gives

$$\begin{aligned} \left|u_\kappa^+\right|^2_{L_{2p}([0,t_1];L_2(\Omega))} &= \left|\left(u_\kappa^+\right)^2\right|_{L_p([0,t_1];L_1(\Omega))} \\ &\leqslant 2|l|_{L_p([0,t_1])}|F|_{L_1([0,t_1])} \leqslant 2|l|_{L_p([0,T])}|F|_{L_1([0,t_1])} \end{aligned} \tag{25}$$

for all $t_1 \in (0,T]$.

Returning to (23), we may also drop the first term, convolve the resulting inequality with k, and use $k * l = 1$ as well as (24), thereby obtaining

$$\nu\left|Du_\kappa^+\right|^2_{L_2([0,t_1];L_2(\Omega))} \leqslant |F|_{L_1([0,t_1])}. \tag{26}$$

In order to estimate $|F|_{L_1([0,t_1])}$, which appears on the right side of both (25) and (26), we proceed similarly as in [12, p. 184]. We denote the Lebesgue measure in $\mathbb{R}^N$ by λ_N and set

$$A_\kappa(t) = \left\{x \in \Omega\colon\ u(t,x) > \kappa\right\}, \quad t \in (0,T).$$

Then

$$\begin{aligned} |F|_{L_1([0,t_1])} &\leqslant \varepsilon\left|Du_\kappa^+\right|^2_{L_2([0,t_1];L_2(\Omega))} \\ &\quad + C(\varepsilon)\int_0^{t_1}\int_{A_\kappa(t)} \left(|b|^2u^2 + |g|^2 + |c|^2(u_\kappa)^2 + |du|u_\kappa + |f|u_\kappa\right) dx\,dt, \end{aligned}$$

for all $\varepsilon > 0$. Selecting ε sufficiently small and assuming $\kappa \geqslant 1$, this together with (25), and (26) gives

$$\left|u_\kappa^+\right|^2_{V_p([0,t_1]\times\Omega)} \leqslant C\left(\nu, |l|_p, T, p\right)\int_0^{t_1}\int_{A_\kappa(t)} \mathcal{D}(t,x)\left((u_\kappa)^2 + \kappa^2\right) dx\,dt, \tag{27}$$

where $|l|_p := |l|_{L_p([0,T])}$, and

$$\mathcal{D}(t,x) = \left|b(t,x)\right|^2 + \left|g(t,x)\right|^2 + \left|c(t,x)\right|^2 + \left|d(t,x)\right| + \left|f(t,x)\right|,$$

and $V_p([0,t_1]\times\Omega)$ is defined as in (11). Using Hölder's inequality and (H3) we thus have with $1/r + 1/r' = 1$ and $1/q + 1/q' = 1$ that

$$\left|u_\kappa^+\right|^2_{V_p([0,t_1]\times\Omega)} \leqslant C|\mathcal{D}|_{L_r([0,t_1];L_q(\Omega))}\left|\left(u_\kappa^+\right)^2 + \kappa^2\chi_{\{u>\kappa\}}\right|_{L_{r'}([0,t_1];L_{q'}(\Omega))}; \tag{28}$$

here C is as in (27), and $\chi_{\{u>\kappa\}}$ denotes the characteristic function of the set of points $(t,x) \in (0,t_1)\times\Omega$ at which $u(t,x) > \kappa$. We may then estimate, using again Hölder's inequality,

$$\left|\left(u_\kappa^+\right)^2\right|_{L_{r'}([0,t_1];L_{q'}(\Omega))} \leqslant \left|u_\kappa^+\right|^2_{L_{2r'(1+\delta)}([0,t_1];L_{2q'(1+\delta)}(\Omega))}\mu_\kappa^{\frac{\delta}{r'(1+\delta)}}, \tag{29}$$

with

$$\mu_\kappa = \begin{cases} \int_0^{t_1}\lambda_N(A_\kappa(t))^{\frac{r'}{q'}}\,dt\colon & q > 1, \\ \lambda_1(\{t \in (0,t_1)\colon\ \lambda_N(A_\kappa(t)) > 0\})\colon & q = 1, \end{cases}$$

and

$$\delta = \frac{2\beta}{2(p'-1)+N}. \tag{30}$$

It is not difficult to verify that, by virtue of (H3), the numbers $\tilde{r} := 2r'(1+\delta)$ and $\tilde{q} := 2q'(1+\delta)$ are subject to conditions (12) and (13) with (r,q) being replaced by $(\tilde{r},\tilde{q})$. Therefore, using inequality (14), it follows from (29) that

$$\left|\left(u_\kappa^+\right)^2\right|_{L_{r'}([0,t_1];L_{q'}(\Omega))} \leqslant C(N,q)\left|u_\kappa^+\right|^2_{V_p([0,t_1]\times\Omega)}\mu_\kappa^{\frac{\delta}{r'(1+\delta)}}. \tag{31}$$

We may further write

$$\left|\kappa^2\chi_{\{u>\kappa\}}\right|_{L_{r'}([0,t_1];L_{q'}(\Omega))} = \kappa^2\mu_\kappa^{\frac{1}{r'}}. \tag{32}$$

Combining (28), (31), and (32) we obtain

$$\left|u_\kappa^+\right|^2_{V_p([0,t_1]\times\Omega)} \leqslant C_1|\mathcal{D}|_{L_r([0,t_1];L_q(\Omega))}\left(\left|u_\kappa^+\right|^2_{V_p([0,t_1]\times\Omega)}\mu_\kappa^{\frac{\delta}{r'(1+\delta)}} + \kappa^2\mu_\kappa^{\frac{1}{r'}}\right), \tag{33}$$

with $C_1 = C_1(\nu, |l|_p, T, p, N, q)$.

We now choose $t_1 = T/n$ where $n\in\mathbb{N}$ is so large that

$$C_1|\mathcal{D}|_{L_r([0,T];L_q(\Omega))}t_1^{\frac{\delta}{r'(1+\delta)}}\lambda_N(\Omega)^{\frac{\delta}{q'(1+\delta)}} \leqslant \frac{1}{2}. \tag{34}$$

Setting $C_2^2 = 2C_1|\mathcal{D}|_{L_r([0,T];L_q(\Omega))}$, inequality (33) then implies

$$\left|u_\kappa^+\right|^2_{V_p([0,t_1]\times\Omega)} \leqslant C_2^2\kappa^2\mu_\kappa^{\frac{1}{r'}}, \quad \kappa \geqslant \tilde{\kappa}_0 := \max\{\kappa_0, 1\}. \tag{35}$$

Define the function

$$\phi(\kappa) = \mu_\kappa^{\frac{1}{\tilde{r}}}, \quad \kappa \geqslant \tilde{\kappa}_0.$$

We will show that $\phi(2M) = 0$ provided $M \geqslant \tilde{\kappa}_0$ is sufficiently large. The argument is analogous to the proof of Theorem 6.1 in Chapter II of [12]. For the sake of completeness we give the details.

By virtue of inequalities (14) and (35), we have for any $\kappa_2 > \kappa_1 \geqslant \tilde{\kappa}_0$

$$(\kappa_2-\kappa_1)\phi(\kappa_2) \leqslant \left|u_{\kappa_1}^+\right|_{L_{\tilde{r}}([0,t_1];L_{\tilde{q}}(\Omega))} \leqslant C(N,q)\left|u_{\kappa_1}^+\right|_{V_p([0,t_1]\times\Omega)} \leqslant C_3\kappa_1\phi(\kappa_1)^{1+\delta}, \tag{36}$$

where $C_3 = CC_2$. We take $\kappa_2 = \xi_{n+1}$ and $\kappa_1 = \xi_n$ with $\xi_n = M(2-2^{-n})$, $n = 0,1,2,\ldots,$ and $M \geqslant \tilde{\kappa}_0$ being fixed. This gives

$$\phi(\xi_{n+1}) \leqslant \frac{C_3\xi_n}{\xi_{n+1}-\xi_n}\phi(\xi_n)^{1+\delta} \leqslant 4C_3 2^n\phi(\xi_n)^{1+\delta},$$

which, together with Lemma 2.1, shows that the sequence $y_n = \phi(\xi_n)$, $n = 0,1,\ldots,$ will go to zero as $n\to\infty$, provided $\phi(\xi_0)$ is sufficiently small, namely

$$\phi(\xi_0) = \phi(M) \leqslant (4C_3)^{-1/\delta}2^{-1/\delta^2}. \tag{37}$$

By taking in (36) $\kappa_2 = M = m\tilde{\kappa}_0$ and $\kappa_1 = \tilde{\kappa}_0$, we obtain

$$\phi(M) \leqslant \frac{C_3}{m-1}\phi(\tilde{\kappa}_0)^{1+\delta} \leqslant \frac{C_3}{m-1}t_1^{(1+\delta)/\tilde{r}}\lambda_N(\Omega)^{(1+\delta)/\tilde{q}}.$$

Hence (37) is satisfied for

$$m = 1 + C_3 t_1^{(1+\delta)/\tilde{r}}\lambda_N(\Omega)^{(1+\delta)/\tilde{q}}(4C_3)^{1/\delta}2^{1/\delta^2}.$$

It follows that for this m

$$\operatorname*{ess\,sup}_{[0,t_1]\times\Omega} u \leqslant 2M = 2m\tilde{\kappa}_0. \tag{38}$$

To obtain a bound on the whole time-interval $[0,T]$, we proceed by induction. Using (38) we next derive an estimate on $[t_1, 2t_1]$, which together with (38) is then employed to find an upper bound on $[2t_1, 3t_1]$, and so forth until we reach T after finitely many steps. Due to the nonlocalness of the integro-differential operator in time, in each step we have to use the bounds established in all of the previous steps, that is up to $t = 0$.

Let $T_0 \in (0,T)$ and suppose that $u \in \tilde{V}_p$ is a weak subsolution of (1) in Ω_T which is bounded above on $[0,T_0]\times\Omega$. Then as above we have

$$\begin{aligned}&\int_\Omega \left(\psi\,\partial_t(k_n * u_\kappa) + \left(h_n * [ADu + bu]|D\psi\right) - \left(h_n * \left[(c|Du) + du\right]\right)\psi\right)dx \\ &\quad \leqslant \int_\Omega \left([h_n * f]\psi - (h_n * g|D\psi) + \psi(u_0-\kappa)k_n\right)dx, \quad \text{a.a. } t \in (T_0,T),\end{aligned} \tag{39}$$

for any nonnegative $\psi \in \mathring{H}^1_2(\Omega)$, $\kappa \in \mathbb{R}$, and $n \in \mathbb{N}$. Recall that $k_n \in H^1_1([0,T])$ with derivative $\dot{k}_n \leqslant 0$. We define

$$H_{\kappa,n}(t,x) = \int_0^{T_0} \big[-\dot{k}_n(t-\tau)\big] u_\kappa(\tau,x)\,d\tau, \quad t \in (T_0,T),\ x \in \Omega. \tag{40}$$

By Jensen's inequality,

$$\big|H_{\kappa,n}(t,x)\big|^2 \leqslant \big(k_n(t-T_0)-k_n(t)\big) \int_0^{T_0} \big[-\dot{k}_n(t-\tau)\big]\big|u_\kappa(\tau,x)\big|^2\,d\tau, \tag{41}$$

which shows that $H_{k,n} \in L_2([T_0,T]\times\Omega)$. Therefore we may use the decomposition

$$(k_n * u_\kappa)(t,x) = \int_{T_0}^{t} k_n(t-\tau)u_\kappa(\tau,x)\,d\tau + \int_0^{T_0} k_n(t-\tau)u_\kappa(\tau,x)\,d\tau, \quad t \in (T_0,T),$$

to rewrite (39) as

$$\begin{aligned} &\int_\Omega \left(\psi\,\partial_t \int_{T_0}^{t} k_n(t-\tau)u_\kappa(\tau,x)\,d\tau + \big(h_n * [ADu+bu]|D\psi\big) - \big(h_n * \big[(c|Du)+du\big]\big)\psi\right)dx \\ &\quad\leqslant \int_\Omega \big([h_n * f]\psi - (h_n * g|D\psi) + \psi(u_0-\kappa)k_n + \psi H_{\kappa,n}\big)\,dx, \quad \text{a.a. } t \in (T_0,T). \end{aligned} \tag{42}$$

We then shift the time according to $s = t - T_0$. Employing the notation $\tilde{v}(s) = v(s+T_0)$, $s \in (0, T-T_0)$, for functions v defined on (T_0,T), (42) becomes

$$\begin{aligned} &\int_\Omega \big(\psi\,\partial_s(k_n * \tilde{u}_\kappa) + \big((h_n * [ADu+bu])\tilde{}\,|D\psi\big) - \big(h_n * \big[(c|Du)+du\big]\big)\tilde{}\,\psi\big)\,dx \\ &\quad\leqslant \int_\Omega \big([h_n * f]\tilde{}\,\psi - \big((h_n * g)\tilde{}\,|D\psi\big) + \psi(u_0-\kappa)\tilde{k}_n + \psi \tilde{H}_{\kappa,n}\big)\,dx, \quad \text{a.a. } s \in (0, T-T_0). \end{aligned} \tag{43}$$

Setting $T_0 = t_1$, we can now argue as above to get an upper bound for u on $[t_1, 2t_1]\times\Omega$. We restrict κ to

$$\kappa \geqslant \tilde{\kappa}_1 := \max\Big\{\tilde{\kappa}_0, \operatorname*{ess\,sup}_{[0,t_1]\times\Omega} u\Big\} = \max\{\tilde{\kappa}_0, 2m\tilde{\kappa}_0\} = 2m\tilde{\kappa}_0,$$

which entails that $u_0 - \kappa \leqslant 0$ as well as $\tilde{H}_{\kappa,n} \leqslant 0$. Consequently, the terms involving these functions can be dropped in (43). We take $\psi = \tilde{u}^+_\kappa$ and use the analogue of (21). Convolving the resulting inequality with l, and sending $n \to \infty$ then yields

$$\frac{1}{2}\int_\Omega \big(\tilde{u}^+_\kappa\big)^2\,dx + l * \int_\Omega \big(\tilde{A}D\tilde{u}|D\tilde{u}^+_\kappa\big)\,dx \leqslant l * \tilde{F}, \quad \text{a.a. } s \in (0, T-t_1),$$

which is the time shifted version of (23). We conclude that

$$\operatorname*{ess\,sup}_{[t_1,t_2]\times\Omega} u \leqslant 2m\tilde{\kappa}_1 = 4m^2\tilde{\kappa}_0. \tag{44}$$

These arguments can now be repeated for the time-intervals $[jt_1,(j+1)t_1]$, $j = 2,\ldots,n-1$, thereby obtaining a bound

$$\operatorname*{ess\,sup}_{\Omega_T} u \leqslant C\tilde{\kappa}_0,$$

with a constant $C = C(p,q,r,|l|_p,T,N,\nu,\lambda_N(\Omega),C_D)$. □

As an immediate consequence of Theorem 3.1 and Remark 3.1(i) we obtain the global boundedness of weak solutions of (1) that are bounded on the parabolic boundary of Ω_T.

Corollary 3.1. *Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Assume that the assumptions* (H1)–(H3) *are satisfied. Suppose $K \geqslant 0$ is such that $|u_0| \leqslant K$ a.e. in Ω. Then there exists a constant $C = C(p,q,r,|l|_{L_p([0,T])},T,N,\nu,\Omega,C_D)$ such that for any weak solution $u \in \tilde{V}_p$ of* (1) *in Ω_T satisfying $|u| \leqslant K$ a.e. on Γ_T there holds $|u| \leqslant C(1+K)$ a.e. in Ω_T.*

For weak subsolutions (supersolutions) of (1) the maximum (minimum) principle is valid in the subsequent form. Let (H3′) stand for

$$u_0 \in L_2(\Omega), \qquad \big||c|^2 + |d|\big| \in L_r\big([0,T]; L_q(\Omega)\big),$$

where

$$\frac{p'}{r} + \frac{N}{2q} = 1,$$

and

$$r \in [p', \infty), \qquad q \in \left[\frac{N}{2}, \infty\right] \quad \text{for } N \geqslant 2,$$
$$r \in [p', 2p'], \qquad q \in [1, \infty] \quad \text{for } N = 1.$$

Theorem 3.2. *Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose the conditions* (H1), (H2), *and* (H3′) *are fulfilled, and assume that $b \equiv g \equiv 0$, $f \equiv 0$, and $d \leqslant 0$ in Ω_T. Then for any weak subsolution* (*supersolution*) *$u \in \tilde{V}_p$ of* (1), *we have for a.a. $(t,x) \in \Omega_T$*

$$u(t,x) \leqslant \max\Big\{0, \operatorname*{ess\,sup}_{\Omega} u_0, \operatorname*{ess\,sup}_{\Gamma_T} u\Big\} \qquad \Big(u(t,x) \geqslant \min\Big\{0, \operatorname*{ess\,inf}_{\Omega} u_0, \operatorname*{ess\,inf}_{\Gamma_T} u\Big\}\Big),$$

provided this maximum (*minimum*) *is finite.*

Proof. It suffices to consider the subsolution case. Note first that Lemma 3.1 also holds under the conditions of Theorem 3.2. We take

$$\kappa = \max\Big\{0, \operatorname*{ess\,sup}_{\Omega} u_0, \operatorname*{ess\,sup}_{\Gamma_T} u\Big\}$$

in (23), assuming that this quantity is finite. By the assumptions on the coefficients and data, we have

$$F(t) \leqslant G(t) := \int_{\Omega} (c|Du) u_\kappa^+ \, dx, \quad \text{a.a. } t \in (0,T).$$

We may then argue similarly as in the proof of Theorem 3.1 to find that for any $t_1 \in (0,T]$

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leqslant C\big(\nu, |l|_{L_p([0,T])}, p, T\big)|G|_{L_1([0,t_1])},$$

and thus

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leqslant \tilde{C}\big(\nu, |l|_p, p, T\big)\big||c|^2\big|_{L_r([0,t_1];L_q(\Omega))} |u_\kappa^+|^2_{L_{2r'}([0,t_1];L_{2q'}(\Omega))}. \tag{45}$$

By (H3′), the numbers $2r'$ and $2q'$ are subject to the conditions (12) and (13). Therefore, using inequality (14), we deduce that

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leqslant C_0\big||c|^2\big|_{L_r([0,t_1];L_q(\Omega))} |u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)},$$

with a positive constant $C_0 = C_0(\nu, |l|_p, p, T, N, q)$. For t_1 satisfying the condition

$$C_0\big||c|^2\big|_{L_r([0,t_1];L_q(\Omega))} < 1$$

we then obtain

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leqslant 0,$$

that is $u \leqslant \kappa$ a.e. in $(0,t_1)\times\Omega$. To establish this inequality on Ω_T we proceed by induction as in the proof of Theorem 3.1, using the fact that the function $H_{\kappa,n}$ defined in (40) is nonpositive on (T_0,T) whenever $u \leqslant \kappa$ a.e. in $(0,T_0)\times\Omega$. □

In all of the previous results we assumed that the kernel l belongs to $L_p([0,T])$ for some $p > 1$. It turns out that the maximum principle still holds when this assumption is dropped and in addition we have $c \equiv 0$.

Theorem 3.3. *Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose there exists $l \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ such that $(k,l) \in \mathcal{PC}$. Let further* (H2) *be satisfied, and assume that $u_0 \in L_2(\Omega)$, $b \equiv c \equiv g \equiv 0$, $f \equiv 0$, and $0 \geqslant d \in L_\infty([0,T]; L_q(\Omega))$, where $q \in [N/2, \infty]$ for $N \geqslant 3$, $q \in (1,\infty]$ for $N = 2$, and $q \in [1,\infty]$ for $N = 1$. Then for any weak subsolution* (*supersolution*) *$u \in \tilde{V}_1$ of* (1), *we have for a.a. $(t,x) \in \Omega_T$*

$$u(t,x) \leqslant \max\Big\{0, \operatorname*{ess\,sup}_{\Omega} u_0, \operatorname*{ess\,sup}_{\Gamma_T} u\Big\} \qquad \Big(u(t,x) \geqslant \min\Big\{0, \operatorname*{ess\,inf}_{\Omega} u_0, \operatorname*{ess\,inf}_{\Gamma_T} u\Big\}\Big),$$

provided this maximum (*minimum*) *is finite.*

Proof. We proceed as in the proof of the preceding theorem. Observe that the assumptions on d ensure that $duu_\kappa^+ \in L_1(\Omega_T)$. Since $c \equiv 0$, we have this time $F \leqslant 0$ a.e. in $(0,T)$, and hence $(V_1(\Omega_T) = L_2([0,T]; H_2^1(\Omega)))$

$$\left|u_\kappa^+\right|^2_{V_1([0,T]\times\Omega)} \leqslant 0, \quad \text{with } \kappa = \max\Big\{0, \operatorname*{ess\,sup}_{\Omega} u_0, \operatorname*{ess\,sup}_{\Gamma_T} u\Big\},$$

which immediately implies the assertion. □

We conclude this section with an example showing that the case $p = 1$ can occur.

Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\sum_{n=1}^\infty \gamma_n < \infty$. Let further $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of numbers in $(0,1)$ that converges to 0 as $n \to \infty$. We then set

$$l(t) = \sum_{n=1}^{\infty} \gamma_n g_{\alpha_n}(t) e^{-t}, \quad t > 0,$$

see (4) for the definition of g_{α_n}. By Euler's integral for the Gamma function,

$$\left|g_{\alpha_n}(\cdot)e^{-\cdot}\right|_{L_1(\mathbb{R}_+)} = 1, \quad n \in \mathbb{N},$$

and therefore $l \in L_1(\mathbb{R}_+)$ with $|l|_{L_1(\mathbb{R}_+)} = \sum_{n=1}^\infty \gamma_n$. Moreover, for every $n \in \mathbb{N}$, $g_{\alpha_n}(t)e^{-t}$ is completely monotone, that is $(-1)^j (g_{\alpha_n} e^{-\cdot})^{(j)}(t) \geqslant 0$, $t > 0$, for $j = 0, 1, 2, \ldots$. Consequently, l enjoys the same property. Furthermore, by Theorem 5.4 in Chapter 5 of [10], the kernel l has a resolvent $k \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ of the first kind, that is $k * l = 1$ on $(0,\infty)$, and this resolvent is completely monotone as well. In particular, k is nonnegative and nonincreasing, and so $(k,l) \in \mathcal{PC}$. Since $\alpha_n \to 0$, there do not exist $p > 1$ and $T > 0$ such that $l \in L_p([0,T])$.

4. Quasilinear equations

In this section we extend the previous results to quasilinear equations of the form (2) with suitable structure conditions. This is possible, as also known from the elliptic and parabolic case, since the test function method used above does not depend so much on the linearity of the operator $\mathcal{L}$ but on a certain nonlinear structure.

Let (H1) hold, and $u_0 \in L_2(\Omega_T)$. We will assume that the functions $a : \Omega_T \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ and $b : \Omega_T \times \mathbb{R}^{N+1} \to \mathbb{R}$ are measurable and satisfy

(Q1) $\quad \big(a(t,x,\xi,\eta)|\eta\big) \geqslant C_0|\eta|^2 - c_0|\xi|^\gamma - \varphi_0(t,x),$

(Q2) $\quad \big|a(t,x,\xi,\eta)\big| \leqslant C_1|\eta| + c_1|\xi|^{\tilde\gamma} + \varphi_1(t,x),$

(Q3) $\quad \big|b(t,x,\xi,\eta)\big| \leqslant C_2|\eta|^{\frac{2(\gamma-1)}{\gamma}} + c_2|\xi|^{\gamma-1} + \varphi_2(t,x),$

for a.a. $(t,x) \in \Omega_T$, and all $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}^N$. Here C_i, c_i, $i = 0,1,2$, are positive constants, and

(Q4) The parameter γ lies in the range

$$2 \leqslant \gamma < 2\tilde\gamma, \quad \text{with } \tilde\gamma := \frac{2p' + N}{2p' + N - 2}.$$

(Q5) The functions φ_i, $i = 0,1,2$, defined on Ω_T are nonnegative, $\varphi_1 \in L_2(\Omega_T)$, and $\varphi_0, \varphi_2 \in L_{\hat q}(\Omega_T)$, where

$$\frac{1}{\hat q}\left(p' + \frac{N}{2}\right) = 1 - \hat\beta, \quad \hat\beta \in (0,1].$$

A function $u \in \tilde V_p$ is called a *weak solution* (*subsolution*, *supersolution*) of (2) in Ω_T, if $a(t,x,u,Du)$ and $b(t,x,u,Du)$ are measurable, and for any nonnegative test function $\eta \in \mathring{H}_2^{1,1}(\Omega_T)$ with $\eta|_{t=T} = 0$ there holds

$$\int_0^T \int_\Omega \left(-\eta_t\big[k*(u-u_0)\big] + \big(a(t,x,u,Du)|D\eta\big) - b(t,x,u,Du)\eta\right) dx\,dt = (\leqslant, \geqslant)\ 0. \tag{46}$$

One verifies using (14), which shows $V_p \hookrightarrow L_{2\tilde\gamma}(\Omega_T)$, and Hölder's inequality that under the above structure conditions this definition makes sense, see also the estimates below.

Theorem 4.1. *Let $T > 0$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $u_0 \in L_2(\Omega)$, and assume that* (H1), (Q1)–(Q5) *are satisfied. Let q be a fixed positive number such that*

$$(\gamma - 2)\left(p' + \frac{N}{2}\right) < q \leqslant 2\tilde\gamma.$$

Suppose further that $K \geqslant 0$ is such that $u_0 \leqslant K$ a.e. in Ω. Then any weak subsolution $u \in \tilde{V}_p$ of (2) satisfying $u \leqslant K$ a.e. on Γ_T is essentially bounded above in Ω_T by a constant C depending only on the data, q, and $|u|_{L_q(\Omega_T)}$. In the case $\gamma = 2$, the constant C depends only on the data.

An analogous result holds for supersolutions that are bounded below on the parabolic boundary, cf. Remark 3.1(i) in the linear case.

Proof of Theorem 4.1. We proceed as in the linear case. Note first that one can easily prove a result analogous to Lemma 3.1. Following the lines in the proof of Theorem 3.1 we obtain for $\kappa \geqslant \kappa_0$ (see (19)), by means of the assumed structure conditions,

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leqslant C \int_0^{t_1} \int_{A_\kappa(t)} \big(\big[|Du|^{\frac{2(\gamma-1)}{\gamma}} + |u|^{\gamma-1} + \varphi_2\big] u_\kappa^+ + |u|^\gamma + \varphi_0 \big)\, dx\, dt, \tag{47}$$

where the constant C depends only on $|l|_p, T, p$ and the constants appearing in (Q1) and (Q3). The first term on the right is estimated using Young's inequality,

$$|Du|^{\frac{2(\gamma-1)}{\gamma}} u_\kappa^+ \leqslant \varepsilon |Du|^2 + C(\varepsilon)\big(u_\kappa^+\big)^\gamma, \quad \varepsilon > 0.$$

Hence, choosing ε sufficiently small, the gradient term can be absorbed by the left-hand side in (47). Setting $\mu_\kappa := |\lambda_N(A_\kappa(\cdot))|_{L_1([0,t_1])}$,

$$\beta := 1 - \frac{1}{q}(\gamma-2)\left(p' + \frac{N}{2}\right) \in (0,1], \quad \text{and} \quad \delta := \frac{2\beta}{2(p'-1)+N},$$

we further have (cf. [12, pp. 425, 426])

$$\begin{aligned}\int_0^{t_1} \int_{A_\kappa(t)} |u|^\gamma\, dx\, dt &\leqslant |u|^{\gamma-2}_{L_q(\Omega_T)} |u\chi_{\{u>\kappa\}}|^2_{L_{\frac{2q}{q-(\gamma-2)}}([0,t_1]\times\Omega)} \\ &\leqslant C(N,q)|u|^{\gamma-2}_{L_q(\Omega_T)} \Big(|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \mu_\kappa^{\frac{\delta(q-\gamma+2)}{(1+\delta)q}} + \kappa^2 \mu_\kappa^{\frac{q-(\gamma-2)}{q}} \Big).\end{aligned} \tag{48}$$

Recall that $V_p \hookrightarrow L_{2\tilde{\gamma}}(\Omega_T)$, so $|u|_{L_q(\Omega_T)}$ is finite.

As in the proof of Theorem 3.1 we may estimate, with the aid of (Q5),

$$\int_0^{t_1} \int_{A_\kappa(t)} \big(\varphi_2 u_\kappa^+ + \varphi_0\big)\, dx\, dt \leqslant C(N,\hat{q})|\varphi_2+\varphi_0|_{L_{\hat{q}}(\Omega_T)} \Big(|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \mu_\kappa^{\frac{\hat{\delta}}{\hat{q}'(1+\hat{\delta})}} + \kappa^2 \mu_\kappa^{\frac{1}{\hat{q}'}} \Big), \tag{49}$$

provided that $\kappa \geqslant 1$; here $\hat{\delta}$ is defined as δ with β replaced by $\hat{\beta}$. From (47)–(49) and the trivial inequality $\mu_\kappa \leqslant t_1 \lambda_N(\Omega)$ we then infer that

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leqslant C\Big(|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} t_1^\rho + \kappa^2 \mu_\kappa^{\min\{\frac{q-(\gamma-2)}{q}, \frac{1}{\hat{q}'}\}} \Big), \tag{50}$$

where

$$\rho = \min\left\{ \frac{\delta(q-\gamma+2)}{(1+\delta)q}, \frac{\hat{\delta}}{\hat{q}'(1+\hat{\delta})} \right\},$$

and C depends on the data (including $\lambda_N(\Omega)$), q, and on $|u|_{L_q(\Omega_T)}$; in the case $\gamma = 2$ the constant C depends only on the data. Choose t_1 so small that $Ct_1^\rho \leqslant \frac{1}{2}$. Then

$$|u_\kappa^+|^2_{V_p([0,t_1]\times\Omega)} \leqslant 2C\kappa^2 \mu_\kappa^{\min\{\frac{q-(\gamma-2)}{q}, \frac{1}{\hat{q}'}\}}, \quad \kappa \geqslant \tilde{\kappa}_0 = \max\{\kappa_0, 1\}.$$

Defining $\phi(\kappa) = \mu_\kappa^{1/\tilde{q}}$, $\kappa \geqslant \tilde{\kappa}_0$, with

$$\tilde{q} = \begin{cases} \frac{2(1+\delta)q}{q-(\gamma-2)}: & \frac{q-(\gamma-2)}{q} < \frac{1}{\hat{q}'}, \\ 2\hat{q}'(1+\hat{\delta}): & \frac{q-(\gamma-2)}{q} \geqslant \frac{1}{\hat{q}'}, \end{cases}$$

we may then proceed exactly as in the proof of Theorem 3.1, thereby establishing first an upper bound on $(0,t_1)\times\Omega$, and then also on Ω_T, by an analogous induction argument. □

The maximum principle holds in the following form.

Theorem 4.2. *Let $T>0$ and $\Omega\subset\mathbb{R}^N$ be a bounded domain. Suppose there exists $l\in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ such that $(k,l)\in\mathcal{PC}$. Suppose further $u_0\in L_2(\Omega)$, (Q1) with $c_0=0$ and $\varphi_0\equiv 0$, as well as (Q2) with $\varphi_1\in L_2(\Omega_T)$, and assume that $b\equiv 0$. Then for any weak subsolution (supersolution) $u\in\tilde V_1$ of (2), we have for a.a. $(t,x)\in\Omega_T$*

$$u(t,x)\leqslant \max\Big\{0,\operatorname*{ess\,sup}_{\Omega}u_0,\operatorname*{ess\,sup}_{\Gamma_T}u\Big\}\qquad \Big(u(t,x)\geqslant \min\Big\{0,\operatorname*{ess\,inf}_{\Omega}u_0,\operatorname*{ess\,inf}_{\Gamma_T}u\Big\}\Big),$$

provided this maximum (minimum) is finite.

Proof. The proof is analogous to that of Theorem 3.3. □

Finally we consider the case of 'natural' or Hadamard growth conditions with respect to $|Du|$. Suppose for simplicity that

$$\text{(Q)}\qquad \big(a(t,x,\xi,\eta)|\eta\big)\geqslant C_0|\eta|^2,\qquad \big|a(t,x,\xi,\eta)\big|\leqslant C_1|\eta|,\qquad \big|b(t,x,\xi,\eta)\big|\leqslant C_2|\eta|^2,$$

for a.a. $(t,x)\in\Omega_T$, and all $\xi\in\mathbb{R}$, $\eta\in\mathbb{R}^N$, where C_i, $i=0,1,2$ are positive constants. In the classical parabolic case one knows that weak solutions of the corresponding problem under the conditions (Q) are in general not bounded. However there exist results (also in a more general situation) providing L_∞ bounds in terms of the data under the additional assumption that the weak solution is bounded, see e.g. [12, Chapter V, Theorem 2.2]. It turns out that analogous results can be proved for (2). Here we only formulate such a result in the case where (Q) holds.

Theorem 4.3. *Let $T>0$ and $\Omega\subset\mathbb{R}^N$ be a bounded domain. Suppose there exists $l\in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ such that $(k,l)\in\mathcal{PC}$. Suppose further $u_0\in L_\infty(\Omega)$, and that (Q) is satisfied. Then for any bounded weak solution $u\in\tilde V_1$ of (2),*

$$|u|_{L_\infty(\Omega_T)}\leqslant \max\Big\{|u_0|_{L_\infty(\Omega)},\operatorname*{ess\,sup}_{\Gamma_T}|u|\Big\}.$$

Proof. We proceed as in the proof of [7, Theorem 17.1]. Set

$$\kappa_0=\Big\{|u_0|_{L_\infty(\Omega)},\operatorname*{ess\,sup}_{\Gamma_T}|u|\Big\},$$

and assume that $K:=\operatorname{ess\,sup}_{\Omega_T}u>\kappa_0$. We then take test functions u_κ^+ where $\kappa=K-\varepsilon\geqslant\kappa_0$, $\varepsilon>0$, and estimate as above. By (Q) we obtain

$$\big|u_\kappa^+\big|^2_{V_1(\Omega_T)}\leqslant C(C_0,C_2)\big|\big|Du_\kappa^+\big|^2u_\kappa^+\big|_{L_1(\Omega_T)}\leqslant \varepsilon C(C_0,C_2)\big|\big|Du_\kappa^+\big|^2\big|_{L_1(\Omega_T)}.$$

Thus if ε is sufficiently small, we have $|u_\kappa^+|^2_{V_1(\Omega_T)}\leqslant 0$, that is $u\leqslant\kappa<K$ a.e. in Ω_T, a contradiction. Hence, $u\leqslant\kappa_0$ a.e. in Ω_T. The lower bound is proved analogously. □

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