Periodic Solutions for $2k$th Order Ordinary Differential Equations with Resonance

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In this paper, a non-variational version of a max-min principle is extended, and some unique existence results are obtained for the periodic boundary value problem of the higher order ordinary differential equations under a resonant condition.

Key Words: Hilbert space; high order differential equation; unique existence; resonance; diffeomorphism.

1. INTRODUCTION AND LEMMAS

During the past 30 years many authors have discussed the periodic boundary value problems for second order ordinary differential equations (see [1, 3–5, 7–11]). In this area, the work developed by Lazer (see [3]) considered the equation

$$u''(t) + \text{grad}G(u(t)) = p(t) = p(t + 2\pi). \quad (1.1)$$

In [6] the following form was considered

$$u^{(2k)}(t) + \text{grad}G(u(t)) = p(t) = p(t + 2\pi). \quad (1.2)$$

Recently, in [2] Cong, with the use of a lemma on bilinear forms [3] and Schauder’s fixed point theorem, deduced a unique existence result of $2\pi$-periodic solution of the following $2k$th order differential equation

$$\sum_{j=1}^{k} \alpha_j u^{(2j)}(t) + (-1)^{k+1}f(t, u(t)) = 0, \quad (1.3)$$
where $u \in \mathbb{R}^n$, $\alpha_j$ are constants and the periodic boundary value problem of the equation is nonresonant. Here the set $S = \{ \tau(n) \mid \tau(n) = \sum_{j=1}^{k} (\tau(n) - 1)^{k-j} \alpha_j n^j, n = 0, 1, \ldots, \infty \}$ is called the set of points of resonance and the periodic boundary value problem of Eq. (1.3) is called the nonresonance if the following conditions hold:

there exist two constant symmetric $n \times n$ matrices $A$ and $B$ such that

$$A \leq f_u \leq B$$

and, if $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ are eigenvalues of $A$ and $B$, respectively, then

$$\bigcup_{i=1}^{n} [\lambda_i, \mu_i] \cap S = \emptyset.$$  

In the present paper, we consider a resonance case of the system (1.3), that is, it is allowed that the nonlinearity $f_u$ (when $u \rightarrow \infty$) interacts with points of resonance, namely here, $\tau(n)$ or $\tau(n + 1)$. With the use of a method inspired by [7, 8, 11], which is different from the one used in [2], we deduce a unique existence result and make the results of [2] as corollaries of our results. Finally, we also give some examples which indicate that the results of [2] are really extended.

We first employ the following lemma from [12].

**LEMMA 1.1** (see [12]). *Assume that $H$ be a Hilbert space. Let $T \in C^1(H, H)$, $T'(u) \in \text{Isom}(H; H)$, $\forall u \in H$. Then $T$ is a global diffeomorphism onto $H$ if there exists a continuous map $\omega : R_+ \rightarrow R_+ \setminus \{0\}$ such that

$$\int_{0}^{+\infty} \frac{ds}{\omega(s)} = +\infty \quad \|T'(u)^{-1}\| \leq \omega(\|u\|).$$  

(1.4)

With this lemma we can prove the following lemma.

**LEMMA 1.2.** *Let $X$ and $Y$ be two closed subspaces of a real Hilbert space $H$, and $H = X \oplus Y$. Suppose $T : H \rightarrow H$ a $C^1$ mapping. If there exist two continuous functions $\alpha : [0, \infty) \rightarrow (0, \infty)$, $\beta : [0, \infty) \rightarrow (0, \infty)$ such that

$$\langle T'(u)x, x \rangle \leq -\alpha(\|u\|)\|x\|^2,$$  

(1.5)

$$\langle T'(u)y, y \rangle \geq \beta(\|u\|)\|y\|^2,$$  

(1.6)

$$\langle T'(u)x, y \rangle = \langle x, T'(u)y \rangle,$$  

(1.7)

for arbitrary $u \in H$, $x \in X$, $y \in Y$, and

$$\int_{0}^{+\infty} \min\{\alpha(s), \beta(s)\} ds = +\infty,$$  

(1.8)

then $T$ is a diffeomorphism from $H$ onto $H$.  

Proof. Denote \( \gamma(s) = \min\{\alpha(s), \beta(s)\} \). Clearly \( \gamma(s) > 0 \). Set \( w = x + y \), where \( x \in X \), \( y \in Y \), and \( w \in H \). Since

\[
\langle T'(u)w, y - x \rangle = \langle T'(u)x, y \rangle - \langle T'(u)x, x \rangle + \langle T'(u)y, y \rangle - \langle T'(u)y, x \rangle \geq \alpha(\|u\|)\|x\|^2 + \beta(\|u\|)^2\|y\|^2,
\]

we have that

\[
\|T'(u)w\|\|y - x\| \geq \gamma(\|u\|)(\|x\|^2 + \|y\|^2); \tag{1.9}
\]

therefore,

\[
\gamma(\|u\|)\|w\| \leq 2\|T'(u)w\|. \tag{1.10}
\]

From (1.10) it follows that \( T'(u) \) is a one to one mapping. If not, suppose \( w_1 \neq w_2 \) (\( w_1, w_2 \in H \)), such that \( T'(u)w_1 = T'(u)w_2 \). Then

\[
0 = \|T'(u)w_1 - T'(u)w_2\| = \|T'(u)(w_1 - w_2)\| \geq \frac{\gamma(\|u\|)}{2}\|w_1 - w_2\| > 0,
\]

which is a contradiction. It also follows from (1.10) that \( T'(u)H \) is a closed subspace of \( H \). In fact, let \( \{y_m\} \subseteq T'(u)H \) and \( y_m \to y \), as \( m \to \infty \). Then there exist \( w_m \) such that

\[
\|y_n - y_m\| = \|T'(u)w_n - T'(u)w_m\| = \|T'(u)(w_n - w_m)\| \geq \frac{\gamma(\|u\|)}{2}\|w_n - w_m\|
\]

and \( \|y_n - y_m\| \to 0 \), as \( n, m \to \infty \). It implies that \( \{w_n\} \) is a Cauchy sequence and consequently, converges in \( H \); thus there exists \( w \in H \) satisfying \( w_n \to w \). By the continuity of \( T'(u) \), we have

\[
T'(u)w_n \to T'(u)w, \quad \text{as} \quad w_n \to w.
\]

Hence \( y = T'(u)w \), i.e., \( y \in T'(u)H \). This proves that \( T'(u)H \) is a closed subspace of \( H \).

Next, we prove that \( T'(u)H = H \). For this purpose, let us assume that there exists a \( z \in [T'(u)H]^\perp \) and \( z \neq 0 \). Then \( \langle z, T'(u)w \rangle = 0, \forall w \in H \).
Let $z = z_X + z_Y$, $z_X \in X$, $z_Y \in Y$ and set $w = z_Y - z_X$. Then
\[ 0 = \langle z, T'(u)w \rangle \geq \alpha(\|u\|)\|z_X\|^2 + \beta(\|u\|)\|z_Y\|^2 \geq \frac{\gamma(\|u\|)}{2}\|w\|^2, \]
again a contradiction. Hence $T'(u)H = H$. By (1.10), we get that
\[ \|T'(u)^{-1}\| \leq \frac{2}{\gamma(\|u\|)}. \quad (1.11) \]
Applying Lemma 1.1, the proof of this lemma is completed.

Remark. Lemma 1.2 is a generalization of Proposition 2.1 in [8]. It is similar to but slightly different from Theorem 2 in [7], where $x$ and $y$ are replaced by $u$ in $\alpha(\|x\|)$ and $\beta(\|y\|)$. Tersian [11] proved a generalization of Proposition 2.1 in [8] when $T$ is hemicontinuous on $H$ and his proof rested upon two theorems on $\alpha$-convex functionals.

The following lemma is required in the proof of our main theorem.

**Lemma 1.3** (see [3]). Let $H$ be a vector space such that for subspaces $X$ and $Y$, $H = X \oplus Y$. If $Y$ is finite-dimensional and $Z$ is a subspace of $H$ such that $X \cap Z = \{0\}$ and dimension $Y =$ dimension $Z$, then $H = X \oplus Z$.

2. UNIQUE EXISTENCE

We will be concerned about the unique existence of periodic solutions to system (1.3) and assume throughout the section that the following conditions hold.

(H$_1$) $f \in C^1(R \times R^n)$, $f(t + 2\pi, x) = f(t, x)$, the Jacobian matrix $f_u = (f_{ui})$ is a symmetric $n \times n$ matrix, and $\alpha_j$ are constants for $j = 1, 2, \ldots, k$;

(H$_2$) there exist two constant symmetric $n \times n$ matrices $A$ and $B$ such that
\[ A + \alpha(\|u\|) \leq f_u \leq B - \beta(\|u\|) \]
on $R \times R^n$ and the eigenvalues of $A$ and $B$ are $\tau(N_i) = \sum_{j=1}^{k} (-1)^{k-j} \alpha_j N_i^{2j}$ and $\tau(N_i + 1) = \sum_{j=1}^{k} (-1)^{k-j} \alpha_j (N_i + 1)^{2j}$, $i = 1, \ldots, n$, respectively ($\alpha_k \neq 0$), where the $N_i$ are nonnegative integers. $\alpha(s)$ and $\beta(s)$ satisfy the conditions of Lemma 1.2. For simplicity, without the loss of generality, we always assume that $\tau(m) = \sum_{j=1}^{k} (-1)^{k-j} \alpha_j m^{2j}$ is a nonnegative nondecreasing sequence about $m$ and $\tau(N_i) < \tau(N_i + 1)$. (In fact, the proof in [2] implied them.)
THEOREM 2.1. Assume that the conditions $(H_1)$ and $(H_2)$ hold. Then there exists a unique $2\pi$-periodic solution to systems (1.3).

Proof. From $(H_2)$, let $a_i$ and $b_i$ be the eigenvectors of $A$ and $B$, respectively, corresponding to $\tau(N_i)$ and $\tau(N_i + 1)$ and satisfying

$$a_i^T a_j = b_i^T b_j = \delta_{ij}, \quad i, j = 1, 2, \ldots, n,$$

where $\delta_{ij} = 0, i \neq j$, $\delta_{ii} = 1$. Define

$$H = \{v(t) = (v_1(t), \ldots, v_n(t))^T \mid v^{(i)}(0) = v^{(i)}(2\pi),$$

$$i = 0, 1, \ldots, 2k - 1;$$

$$v^{(2k-1)}(t)$$

absolutely continuous and $v^{(2k)}(t) \in L^2[0, 2\pi] \}.$$

It is easy to know $H$ is a Hilbert space about the inner product

$$\langle u, v \rangle = \int_0^{2\pi} \left\{ \sum_{j=1}^{k} (-1)^{k-j} a_j^T u^{(j)}(t) v^{(j)}(t) + u(t)^T v(t) \right\} dt. \quad (2.1)$$

Denote by $\| \cdot \|_H$ the norm induced by this inner product and define subspaces of $H$ as

$$X = \left\{ x(t) = \sum_{i=1}^{n} f_i(t) a_i \mid f_i(t) = c_{i0} + \sum_{m=1}^{N_i} (c_{im} \cos mt + d_{im} \sin mt) \right\};$$

$$Y = \left\{ y(t) = \sum_{i=1}^{n} g_i(t) a_i \mid g_i(t) = \sum_{m=N_i+1}^{\infty} (c_{im} \cos mt + d_{im} \sin mt) \right\};$$

$$Z = \left\{ z(t) = \sum_{i=1}^{n} h_i(t) b_i \mid h_i(t) = \sum_{m=N_i+1}^{\infty} (p_{im} \cos mt + q_{im} \sin mt) \right\},$$

where the $N_i$ are as in $(H_2)$ and $c_{im}$, $d_{im}$, $p_{im}$, $q_{im}$ are constants. Obviously, $H = X \oplus Y$. Next using the Riesz representation theorem, we define a mapping $T : H \to H$ by

$$\langle T(u), v \rangle = \int_0^{2\pi} \left\{ \sum_{j=1}^{k} (-1)^{k-j} a_j^T u^{(j)}(t) v^{(j)}(t) - f^T(t, u) v(t) \right\} dt,$$

(2.2)

for arbitrary $v \in H$. We observe that $T$ above is defined implicitly. From (2.2) and the fact that $f$ is $C^1$ it can be proved that $T$ is a $C^1$ mapping and
that
\[
\langle T'(u)w, v \rangle = \int_0^{2\pi} \left( \sum_{j=1}^{k} (-1)^{k-j} \alpha_j \left[ w^{(j)}(t) \right]^T v^{(j)}(t) - w^T(t) f_u(t, u(t)) v(t) \right) dt,
\]
\[ (2.3) \]
for all \( v(t), u(t), w(t) \in H \). It can be proved that \( u \) is a 2\( \pi \)-periodic solution of (1.3) if and only if \( u \) satisfies the operator equation
\[
T(u) = 0.
\]
(2.4)
We will next show that \( T \) satisfies the conditions of Lemma 1.2. This in turn will imply that (1.3) has a unique 2\( \pi \)-periodic solution. For any \( x \in X \) and \( u \in H \), we have that
\[
\langle T'(u)x, x \rangle = \int_0^{2\pi} \left( \sum_{j=1}^{k} (-1)^{k-j} \alpha_j \left[ x^{(j)}(t) \right]^T x^{(j)}(t) - x^T(t) f_u(t, u(t)) x(t) \right) dt
\]
\[ \leq \int_0^{2\pi} x^T(t) Ax(t) - x^T(t) f_u(t, u(t)) x(t) \right) dt
\]
\[ \leq -\alpha(\|u\|_H) \int_0^{2\pi} x^T(t) x(t) dt
\]
\[ \leq -\alpha(\|u\|_H) \frac{1}{1 + \min_{1 \leq i \leq N} \tau(N_i)} \|x\|_H. \]
(2.5)
Similarly, for any \( z \in Z \) and \( u \in H \), the following inequality is true
\[
\langle T'(u)z, z \rangle = \int_0^{2\pi} \left( \sum_{j=1}^{k} (-1)^{k-j} \alpha_j \left[ z^{(j)}(t) \right]^T z^{(j)}(t) - z^T(t) f_u(t, u(t)) z(t) \right) dt
\]
\[ \geq \beta(\|u\|_H) \frac{1}{1 + \max_{1 \leq i \leq N} \tau(N_i + 1)} \|z\|_H. \]
(2.6)
From the symmetry of \( f_u \), it is easy to prove that
\[
\langle T'(u)w, v \rangle = \langle T'(u)v, w \rangle,
\]
(2.7)
for $\forall u, v, w \in H$. Therefore, let

$$\tilde{\alpha}(s) = \frac{\alpha(s)}{1 + \min_{1 \leq i \leq n} \tau(N_i)}, \quad \tilde{\beta}(s) = \frac{\beta(s)}{1 + \max_{1 \leq i \leq n} \tau(N_i + 1)}.$$ 

Then

$$\gamma(s) = \min\{\tilde{\alpha}(s), \tilde{\beta}(s)\} \geq \frac{\min\{\alpha(s), \beta(s)\}}{1 + \max_{1 \leq i \leq n} \tau(N_i + 1)}. \quad (2.8)$$

It is obvious that $\int_0^{+\infty} \gamma(s) \, ds = +\infty$.

Since $H$ is positive definite on $X$ and negative definite on $Z$, we see that $X \cap Z = \{0\}$. Moreover, it is readily seen that

$$\text{dimension } Y = \text{dimension } Z = \sum_{i=1}^{n} (2N_i + 1).$$

Thus since it was shown above that $H = X \oplus Y$ it follows by an application of Lemma 1.3 that $H = X \oplus Z$. We may therefore apply Lemma 1.2 to get the conclusion of the theorem.

### 3. RELATED RESULTS AND EXAMPLES

Using the similar technique of Theorem 2.1, we can prove the following conclusion about boundary value problem.

**Theorem 3.1.** Suppose that the continuous function $f(t, u)$ is $C^1$ mapping about $u$, $\forall t \in [0, \pi], u \in \mathbb{R}^n$ and that the Jacobian matrix $f_u = (f_{ui})$ is a symmetric $n \times n$ matrix, and $\alpha_i$ are constants for $j = 1, 2, \ldots, k$; the conditions $(H_2)$ hold. Then there exists a unique solution to the system (1.3) which satisfies

$$u^{(2i)}(0) = u^{(2i)}(\pi) = 0, \quad i = 0, 1, \ldots, k - 1.$$ 

Based on Theorem 2.1, the results of Cong [2] then become a special case of this theorem.

**Corollary 3.2.** If the inequality about $f_u$ in the condition $(H_2)$ is replaced by

$$A \leq f_u \leq B, \quad (3.1)$$

where the eigenvalues of $A$ and $B$ are $\lambda_i$ and $\mu_i$, respectively, and $\tau(N_i) < \lambda_i \leq \mu_i < \tau(N_i + 1), i = 1, \ldots, n$, then the conclusion of Theorem 2.1 is valid.
Proof. In fact, let $\alpha(s) = \min_{1 \leq i \leq n} (\lambda_i - \tau(N_i))$, $\beta(s) = \min_{1 \leq i \leq n} (\tau(N_i) - \mu_i)$. Then the conditions of theorem 2.1 are satisfied. Therefore, there exists a unique $2\pi$-periodic solution to (1.3).

When $n = 1$, the system (1.3) becomes a single equation, and we have the following conclusion. To our best knowledge, the following result also seems to be new.

**COROLLARY 3.3.** Assume that the continuous function $f(t,u)$ is a $C^1$ mapping about $u$ and that the following inequality holds,

$$\tau(m) + \alpha(|u|) \leq f_u(t,u) \leq \tau(m + 1) - \beta(|u|). \quad (3.2)$$

where $m$ is a nonnegative integer. If $\alpha(s)$ and $\beta(s)$ are two continuous positive functions and $\int_0^{+\infty} \min\{\alpha(s), \beta(s)\} \, ds = +\infty$, then there exists a unique $2\pi$-periodic solution to the single equation (1.3).

When $k = 1$ and $\alpha_k = 1$, the system (1.3) turns into the Duffing System

$$u''(t) + f(t,u(t)) = 0. \quad (3.3)$$

Applying Theorem 2.1, we get the following corollary which is the main result of [10].

**COROLLARY 3.4.** Suppose that $f \in C^1(R \times R^n)$, $f(t + 2\pi, x) = f(t, x)$, that the Jacobian matrix $f_u = (f_u)_{ij}$ is a symmetric $n \times n$ matrix, and its eigenvalues are $\gamma_1(t,u(t)), \ldots, \gamma_n(t,u(t))$. If there exist integers $N_k$, $k = 1, \ldots, n$ such that for all $t \in [0, 2\pi]$, $u(t) \in R^n$,

$$N_k^2 < \gamma_k(t,u(t)) < (N_k + 1)^2, \quad (3.4)$$

and for each fixed $t \in [0, 2\pi]$,

$$\int_0^{+\infty} \min_{1 \leq k \leq n} \left\{ \min_{1 \leq k \leq n} \left( \gamma_k(t,u(t)) - N_k^2, (N_k + 1)^2 - \gamma_k(t,u(t)) \right) \right\} \, ds$$

$$= +\infty, \quad (3.5)$$

then there exists a unique $2\pi$-periodic solution to (3.3).

By analogy to Corollary 3.4, we have the following corollary.

**COROLLARY 3.5.** Let $f(t,u)$ be as in Corollary 3.4. Suppose there exist two real constant symmetric matrices $A$ and $B$ such that for all $(t,u) \in [0, 2\pi] \times R^n$,

$$A \leq f_u \leq B$$

and such that if $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$ and $\mu_1 \leq \mu_2 \leq \cdots \mu_n$ denote the eigenvalues of $A$ and $B$, respectively, then exist integers $N_k \geq 0$, $k = 1, 2, \ldots, n$,
such that

\[ N_k^2 < \lambda_k \leq \mu_k < (N_k + 1)^2. \]

Then the differential equation (3.3) has a unique \(2\pi\)-periodic solution.

Corollary 3.5 was proved by many methods; see [1, 3–5, 7–11], etc. The following examples indicate that the results of Cong [2] are really extended.

**Example 1.** Consider the differential equation

\[
\frac{1}{6}u^{(6)} + \frac{1}{2}u^{(4)} + \frac{1}{3}u'' + f(t, u) = 0, \tag{3.6}
\]

where

\[ f(t, u) = 4u + \ln\left(\sqrt{u^2 + 1} + u\right) + 2\sin t. \]

It is clear that

\[
\tau(2) = 4 < f_u = 4 + \frac{1}{\sqrt{u^2 + 1}} \leq 5 < 84 = \tau(3),
\]

\[
\int_0^{+\infty} \frac{ds}{\sqrt{s^2 + 1}} = +\infty,
\]

where \(\tau(m) = \frac{1}{2}m^6 - \frac{1}{2}m^4 + \frac{1}{4}m^2\). It is easy to see that the conditions of [2] are not satisfied. But based on Corollary 3.3, there exists a unique \(2\pi\)-periodic solution to Eq. (3.6).

**Example 2.** If we set \(f(t, u) = \nabla G(t, u)\) and

\[
G(t, u) = \frac{5}{4}\left(1 + \frac{4}{5}\sin^2 t\right)(u_1^2 + u_2^2) + \frac{3}{2}\left(1 + \frac{2}{3}\sin^2(2t)\right)u_1u_2
\]

\[ + u_1\ln(u_1 + \sqrt{1 + u_1^2}) + u_2\ln(u_2 + \sqrt{1 + u_2^2})
\]

\[ - \sqrt{1 + u_1^2} - \sqrt{1 + u_2^2} + c_1u_1 + c_2u_2, \]
then
\[
f_u(t, u) = \nabla^2 G(t, u) = \begin{pmatrix}
\frac{\sin t}{\pi} & \frac{\sin 2t}{\pi} \\
\frac{\sin 2t}{\pi} & \frac{\sin^2 t}{\pi} + \frac{1}{\sqrt{1 + u^2}}
\end{pmatrix}
\]
and hence
\[
\left(\begin{array}{cc}
\frac{5}{\pi} & \frac{3}{\pi} \\
\frac{3}{\pi} & \frac{5}{\pi}
\end{array}\right) + \left(\begin{array}{cc}
\frac{1}{\sqrt{1 + u^2}} & 0 \\
0 & \frac{1}{\sqrt{1 + u^2}}
\end{array}\right)
\leq f_u \leq \left(\begin{array}{cc}
\frac{13}{\pi} & \frac{5}{\pi} \\
\frac{5}{\pi} & \frac{13}{\pi}
\end{array}\right) + \left(\begin{array}{cc}
1 - \frac{1}{\sqrt{1 + u^2}} & 0 \\
0 & 1 - \frac{1}{\sqrt{1 + u^2}}
\end{array}\right).
\]

It is easy to prove that \( f_u \) satisfies the conditions of Theorem 2.1, so there exists a unique \( 2\pi \)-periodic solution to \( u'' + \nabla G(t, u) = 0 \).

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REFERENCES